



Algebraically generated groups and their Lie algebras

Hanspeter Kraft, M Zaidenberg

► To cite this version:

Hanspeter Kraft, M Zaidenberg. Algebraically generated groups and their Lie algebras. 2022. hal-03615849v2

HAL Id: hal-03615849

<https://hal.science/hal-03615849v2>

Preprint submitted on 24 May 2022 (v2), last revised 31 Oct 2023 (v3)

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ALGEBRAICALLY GENERATED GROUPS AND THEIR LIE ALGEBRAS

HANSPETER KRAFT AND MIKHAIL ZAIDENBERG

ABSTRACT. The automorphism group $\text{Aut}(X)$ of an affine variety X is an ind-group. Its Lie algebra is canonically embedded into the Lie algebra $\text{Vec}(X)$ of vector fields on X . We study the relations between subgroups of $\text{Aut}(X)$ and Lie subalgebras of $\text{Vec}(X)$.

We show that a subgroup $G \subseteq \text{Aut}(X)$ generated by a family of connected algebraic subgroups G_i of $\text{Aut}(X)$ is algebraic if and only if the Lie algebras $\text{Lie } G_i \subseteq \text{Vec}(X)$ generate a finite dimensional Lie subalgebra of $\text{Vec}(X)$.

Extending a result by COHEN-DRAISMA [CD03] we prove that a locally finite Lie algebra $L \subseteq \text{Vec}(X)$ generated by locally nilpotent vector fields is algebraic, i.e. $L = \text{Lie } G$ for an algebraic subgroup $G \subseteq \text{Aut}(X)$.

Along the same lines we prove that if a subgroup $G \subseteq \text{Aut}(X)$ generated by finitely many connected algebraic groups is solvable, then it is a solvable algebraic group.

We also show that a unipotent algebraic subgroup $U \subseteq \text{Aut}(X)$ has derived length $\leq \dim X$. This result is based on the following triangulation theorem: *Every unipotent algebraic subgroup of $\text{Aut}(\mathbb{A}^n)$ with a dense orbit in \mathbb{A}^n is conjugate to a subgroup of the DE JONQUIÈRES subgroup.*

Furthermore, we give an example of a free subgroup $F \subseteq \text{Aut}(\mathbb{A}^2)$ generated by two algebraic elements such that the Zariski closure \overline{F} is a free product of two nested commutative closed unipotent ind-subgroups.

To any ind-group \mathfrak{G} one can associate a canonical ideal $L_{\mathfrak{G}} \subseteq \text{Lie } \mathfrak{G}$ generated by the Lie algebras of algebraic subgroups of \mathfrak{G} , and another one associated to the set of all algebraic subvarieties of \mathfrak{G} . We study functorial properties of these ideals, ind-subgroups of finite codimension in \mathfrak{G} and the corresponding Lie subalgebras of $\text{Lie } \mathfrak{G}$.

2020 *Mathematics Subject Classification.* 14J50 (primary), 14L40, 17B66, 22E65 (secondary).

Key words and phrases. ind-group, automorphism group, Lie algebra, algebraically generated subgroup, group action.

A part of the paper was written during a stay of the second author at the Max Planck Institute of Mathematics in Bonn. He thanks this institution for its support and excellent working conditions.

CONTENTS

1. Introduction and main results	3
1.1. Notation	3
1.2. Ind-groups and vector fields	3
1.3. Algebraically generated groups	4
1.4. Solvability and triangulation	5
1.5. Locally finite subsets	8
1.6. Locally finite vector fields	8
1.7. The adjoint action on $\text{Vec}(X)$	10
1.8. Toral subalgebras	11
1.9. Integration of Lie algebras	12
1.10. A free subgroup of $\text{Aut}(\mathbb{A}^2)$ and its closure	12
1.11. Ind-groups and their Lie algebras	13
2. Algebraically generated subgroups	13
2.1. Orbits of algebraically generated subgroups	13
2.2. Vector fields associated to subvarieties of $\text{Aut}(X)$	14
2.3. Lie algebras of algebraically generated groups	16
2.4. Ramanujam's alternative	17
2.5. The finite dimensional case	17
3. Integration of vector fields	18
3.1. Integration of locally finite Lie algebras	18
3.2. JORDAN-saturated subspaces	19
4. Nested subgroups	21
4.1. Nested subgroups and nested Lie algebras	21
4.2. Integration of nested Lie algebras	22
5. Solvability and triangulation	22
5.1. Solvable subgroups	22
5.2. Triangulation in $\text{Aut}(\mathbb{A}^n)$	25
5.3. Nested unipotent subgroups are solvable	27
6. An interesting example	28
6.1. Notation and first results	28
6.2. The Lie algebra of \mathfrak{F}	29
6.3. De Jonquières subgroups and amalgamated products	30
6.4. The fixed points under S	32
7. Homomorphisms of ind-groups and of their Lie algebras	34
7.1. A canonical Lie subalgebra of $\text{Lie } \mathfrak{G}$	34
7.2. Surjective homomorphisms	36
7.3. Subgroups of finite codimension	36
References	37

1. INTRODUCTION AND MAIN RESULTS

The introduction contains the necessary preliminaries and definitions, gives some background material and describes the main results of the paper. The details then follow in the subsequent sections.

1.1. Notation. Our base field \mathbb{K} is algebraically closed and of characteristic zero. \mathbb{A}^n stands for affine n -space over \mathbb{K} , and $\mathbb{G}_a := \mathbb{K}^+$ and $\mathbb{G}_m := \mathbb{K}^* := \mathbb{K} \setminus \{0\}$ denote the additive and the multiplicative group of \mathbb{K} . If G is an algebraic group, then a G -variety X is an affine variety with an action of G such that the corresponding map $G \times X \rightarrow X$ is a morphism.

For every G -variety X there is a canonical (anti-) homomorphism of Lie algebras $\xi: \text{Lie } G \rightarrow \text{Vec}(X)$, where $\text{Vec}(X)$ stands for the Lie algebra of (algebraic) vector fields on X . The construction of this homomorphism goes as follows. For any $x \in X$ let $\mu_x: G \rightarrow X$ be the orbit map $g \mapsto gx$, and denote by

$$d\mu_x: \text{Lie } G := T_e G \rightarrow T_x X$$

its differential in $e \in G$. If $A \in \text{Lie } G$, then the corresponding vector field ξ_A is given by $\xi_A(x) := d\mu_x(A) \in T_x X$. If $L \subseteq \text{Vec}(X)$ denotes the image of $\text{Lie } G$, then one has $T_x Gx = L(x)$ for any $x \in X$. Moreover, the differential $d\mu_x: T_g G \rightarrow T_{gx} Gx$ of the orbit map μ_x is surjective in every $g \in G$. (For a reference one might look at [Kra16, Appendix A.4].)

For a G -variety X we have linear actions of G on the coordinate ring $\mathcal{O}(X)$ and on the vector fields $\text{Vec}(X)$, and these representations are *locally finite and regular*, i.e. every element is contained in a finite dimensional G -invariant subspace and the linear action of G on any finite dimensional G -invariant subspace is regular. Moreover, the homomorphism $\xi: \text{Lie } G \rightarrow \text{Vec}(X)$ is G -equivariant, see [FK18, Sect. 7.3].

1.2. Ind-groups and vector fields. For an affine variety X the group of regular automorphisms $\text{Aut}(X)$ is an *affine ind-group*. We refer to the paper [FK18] for an introduction to ind-varieties and ind-groups and for basic concepts, cf. [Sha81, Kum02]. For an ind-group \mathfrak{G} the tangent space $T_e \mathfrak{G}$ carries a natural structure of a Lie algebra which will be denoted by $\text{Lie } \mathfrak{G}$. In case of $\text{Aut}(X)$ there is a canonical embedding $\xi: \text{Lie } \text{Aut}(X) \hookrightarrow \text{Vec}(X)$ which is an anti-homomorphism of Lie algebras. It is constructed in a similar way as explained above for an algebraic group G , see [FK18, Proposition 7.2.4].

The ind-varieties and ind-groups carry a natural (Zariski-) topology, called the *ind-topology*. All topological notation in this paper will be with respect to the ind-topology.

A subset $Y \subseteq \mathfrak{V}$ of an ind-variety $\mathfrak{V} = \bigcup_k \mathfrak{V}_k$ is called *bounded* if it is contained in \mathfrak{V}_k for some k . It is called *algebraic* if it is bounded and locally closed. If X is a G -variety, then the canonical map $G \rightarrow \text{Aut}(X)$ is a homomorphism of ind-groups, and the image of G is closed and algebraic, see [FK18, Proposition 2.7.1]. In particular, every algebraic subgroup $G \subseteq \text{Aut}(X)$ is closed and thus a linear algebraic group. It follows that $\text{Lie } G \subseteq \text{Lie } \text{Aut}(X)$ is a Lie subalgebra and that $\text{Lie } G$ is canonically embedded into the vector fields $\text{Vec}(X)$.

Convention. By abuse of notation, we will constantly identify, for an algebraic group $G \subseteq \text{Aut}(X)$, the Lie algebra $\text{Lie } G$ with its image $\xi(\text{Lie } G) \subseteq \text{Vec}(X)$.

1.3. Algebraically generated groups. A first result showing a very strong relation between the Lie algebra of an ind-group and the group itself is the following, see [FK18, Proposition 2.2.1(3) and (4)].

Proposition 1.3.1. *Let \mathfrak{G} be an ind-group. Then the connected component \mathfrak{G}° is an algebraic group if and only if $\text{Lie } \mathfrak{G}$ is finite dimensional.*

We will prove a similar statement for so-called *algebraically generated subgroups*. Let $(G_i)_{i \in I}$ be a family of connected algebraic subgroups of $\text{Aut}(X)$ where X is an affine variety. The subgroup $G \subseteq \text{Aut}(X)$ generated by the G_i is called *algebraically generated* ([AFK13]).

Let $L \subseteq \text{Vec}(X)$ be the Lie subalgebra generated by the Lie algebras $\text{Lie } G_i$. We will see in Theorem 2.3.1 that L is invariant under the action of G and of the closure \overline{G} . This implies that $L \subseteq \text{Lie } \overline{G}$ is an ideal.

Question 1. *Do we have $L = \text{Lie } \overline{G}$?*

Our result in this direction is the following.

Theorem A. *Assume that L is finite dimensional. Then G is an algebraic subgroup of $\text{Aut}(X)$ and $\text{Lie } G = L$.*

We will prove a slightly more general result, see Theorems 2.3.1 and 2.5.1. Our example in Section 6 gives an infinite dimensional L where Question 1 has a positive answer, see Remark 6.4.4.

Theorem A above is an important ingredient in the proof of the following result.

Theorem B. *Let $G \subseteq \text{Aut}(X)$ be a solvable subgroup generated by a family of connected algebraic subgroups. If the family is finite, then G is a solvable algebraic subgroup. In general, G is nested, i.e. a filtered*

union of solvable algebraic subgroups, and is of the form $U_G \rtimes T$ where T is a torus and U_G a nested unipotent group.

In particular, if G is generated by a family of unipotent algebraic groups, then $G = U_G$ is a nested unipotent group.

The proof of Theorem B will be given in Section 5.1, see Theorem 5.1.1.

Remarks 1.3.2. (1) If G is an algebraic group and $H_1, \dots, H_m \subseteq G$ a finite set of connected closed subgroups, then the subgroup H generated by the H_i is closed. (In fact, there is a sequence i_1, i_2, \dots, i_N such that the product $P := H_{i_1} H_{i_2} \cdots H_{i_N}$ is dense (and constructible) in \overline{H} . It then follows that $PP = \overline{H}$, hence $H = \overline{H}$.)

As a consequence we see that a subgroup of a nested ind-group \mathfrak{G} (see Definition 4.1.1) generated by finitely many connected algebraic groups is an algebraic group.

Note that this statement does not hold for non-connected groups. E.g. the two matrices $A := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}$ have both order 4, but the product AB has infinite order. Thus the subgroup of $\mathrm{SL}_2(\mathbb{K})$ generated by the finite groups $\langle A \rangle$ and $\langle B \rangle$ is not closed.

(2) If the group $\mathrm{Aut}(X)$ verifies TITS' alternative, then the assumption of solvability of G in Theorem B can be replaced by the weaker assumption that G has no nonabelian free subgroup; see e.g. [AZ21] and the literature therein for a discussion of TITS' alternative in $\mathrm{Aut}(X)$.

1.4. Solvability and triangulation. The DE JONQUIÈRES subgroup $\mathrm{Jonq}(n) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ consists of the automorphisms of the form

$$\varphi = (a_1 x_1 + f_1, a_2 x_2 + f_2(x_1), \dots, a_n x_n + f_n(x_1, \dots, x_{n-1}))$$

where $a_i \in \mathbb{K}^*$ and $f_i \in \mathbb{K}[x_1, \dots, x_{i-1}]$. It is known that the unipotent elements of $\mathrm{Jonq}(n)$ form a solvable subgroup of derived length n which is not nilpotent for $n > 1$ (see Remarks 5.2.1 and 5.3.3). More generally, we have the following result.

Theorem C. *A nested unipotent subgroup $U \subseteq \mathrm{Aut}(X)$ is solvable of derived length $\leq \max\{\dim Ux \mid x \in X\} \leq \dim X$.*

It is known that the derived length of a nilpotent (respectively, a solvable) connected Lie group G acting faithfully on a Hausdorff topological space M is bounded above by $\dim M$ (respectively, by $\dim M + 1$), see

[ET79, Theorem 1.2]. Up to passing to a finite index subgroup, the former estimate works as well in the case of a finitely generated nilpotent group acting faithfully on a quasi-projective variety X defined over a field of characteristic zero [Abb22, Theorem B].

The proof of Theorem C will be given in Section 5.3, see Theorem 5.3.1. It is based on the following important triangulation result, see Theorem 5.2.2.

Theorem D. *If a nested unipotent subgroup $U \subseteq \text{Aut}(\mathbb{A}^n)$ has a dense orbit in \mathbb{A}^n , then U is triangulable, i.e., it is conjugate to a subgroup of the DE JONQUIÈRES subgroup.*

In [Bas84] BASS gave an example of a \mathbb{G}_a -action on \mathbb{A}^3 which is not triangulable, i.e. the image $A \subseteq \text{Aut}(\mathbb{A}^3)$ of \mathbb{G}_a is not conjugate to a subgroup of $\text{Jonq}(3)$. The reason is that the fixed points set $(\mathbb{A}^3)^{\mathbb{G}_a}$ is a hypersurface with an isolated singularity. This is not possible for a triangulable action, because for such an action the fixed point set has the form $X \times \mathbb{A}^1$. With the same idea one can construct non-triangulable \mathbb{G}_a -actions in any dimension, see [Pop87].

The \mathbb{G}_a -action of BASS corresponds to the locally nilpotent vector field $\delta := (xz + y^2)(-2y\partial_x + z\partial_y)$, and the image A of \mathbb{G}_a contains the famous NAGATA automorphism

$$\eta = (x - 2y(xz + y^2) - z(xz + y^2)^2, y + z(xz + y^2), z).$$

Due to the celebrated SHESTAKOV-UMIRBAEV Theorem ([SU03]) this \mathbb{G}_a -subgroup is not contained in the tame subgroup of $\text{Aut}(\mathbb{A}^3)$. However, it becomes tame in \mathbb{A}^4 , see [Smi89], but it is still non-triangulable in \mathbb{A}^4 , see [Fre06, Lemma 3.36]. The question arises if this action is *stably triangulable*, i.e. becomes triangulable in $\mathbb{A}^3 \times \mathbb{A}^d$ for a suitable $d \geq 1$. In this context we have the following negative answer.

Proposition 1.4.1. *Consider a \mathbb{G}_a -action on \mathbb{A}^n and assume that the fixed point set is a hypersurface with an isolated singularity. Then the action is not stably triangulable.*

Proof. (a) Denote by $F \subseteq \mathbb{A}^n$ the fixed point set and by $p \in F$ the isolated singularity. For the quotient morphism $\pi: \mathbb{A}^n \rightarrow \mathbb{A}^n // \mathbb{G}_a$ the fiber $\pi^{-1}(\pi(p))$ has dimension ≥ 1 and thus is not contained in the singularities of F . If we extend the action to $\mathbb{A}^n \times \mathbb{A}^d$ where \mathbb{G}_a acts trivially on \mathbb{A}^d , then $(\mathbb{A}^n \times \mathbb{A}^d) // \mathbb{G}_a = \mathbb{A}^n // \mathbb{G}_a \times \mathbb{A}^d$ and the quotient morphism is equal to $\tilde{\pi} := \pi \times \text{id}$. Moreover, the fixed point set is $\tilde{F} := F \times \mathbb{A}^d$, and the singularities of \tilde{F} are $\tilde{F}_{\text{sing}} := \{p\} \times \mathbb{A}^d$. It follows that $\tilde{\pi}^{-1}(\tilde{\pi}(\tilde{F}_{\text{sing}}))$ is not contained in \tilde{F}_{sing} .

(b) Now assume that the action on $\mathbb{A}^n \times \mathbb{A}^d$ is triangulable, so that the corresponding vector field is equivalent to one of the form

$$\tilde{\delta} = f_1 \partial_{x_1} + f_2 \partial_{x_2} + \cdots + f_m \partial_{x_m}$$

where $m := n + d$ and $f_i \in \mathbb{K}[x_1, \dots, x_{i-1}]$. The fixed point set \tilde{F} is an irreducible hypersurface defined by an invariant (irreducible) function h , and so h must divide all the f_i . Since the singular points of \tilde{F} form a subvariety of dimension d this implies that $f_1 = \dots = f_n = 0$. In fact, if h would depend on less than n variables, then the zero set $\{h = 0\} \subseteq \mathbb{A}^m$ would have the form $X \times \mathbb{A}^{d+1}$ and so the singular set of \tilde{F} would have at least dimension $d + 1$.

Let f_{r+1} be the first non-zero coefficient of $\tilde{\delta}$. Then $r > n$ and x_1, x_2, \dots, x_r are invariants. Since h divides f_{r+1} we see that h only depends on the variables x_1, \dots, x_r . It follows that the linear \mathbb{G}_a -invariant morphism $\varphi: \mathbb{A}^m \rightarrow \mathbb{A}^r$, $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_r)$, maps the fixed point set \tilde{F} to the hypersurface $F_r := \{h = 0\} \subseteq \mathbb{A}^r$, and $\tilde{F} = F_r \times \mathbb{A}^{m-r}$. This shows that $\varphi^{-1}(\varphi(\tilde{F}_{\text{sing}})) = \tilde{F}_{\text{sing}}$. Since the invariant map φ factors through the quotient map $\tilde{\pi}$ this contradicts what we have seen in (a). \square

Remarks 1.4.2.

- (1) The non-triangulable \mathbb{G}_a -actions on \mathbb{A}^n constructed in [Pop87] are not stably triangulable.
- (2) The NAGATA-automorphism η is not contained in a unipotent subgroup $U \subseteq \text{Aut}(\mathbb{A}^3)$ which has a dense orbit in \mathbb{A}^3 , by our Theorem D.
- (3) If we consider the diagonal action on $\mathbb{A}^3 \times \mathbb{A}^1$ where \mathbb{G}_a acts on \mathbb{A}^3 as in Bass' example and on \mathbb{A}^1 by translation, then this new action is triangulable. In fact, we have the following result.

Let U be a unipotent group acting on \mathbb{A}^n . Then the diagonal action on $\mathbb{A}^n \times U$ where U acts by left multiplication on U is triangulable.

Indeed, consider the isomorphism $\Phi: \mathbb{A}^n \times U \xrightarrow{\sim} \mathbb{A}^n \times U$ given by $(x, u) \mapsto (u^{-1}x, u)$. This morphism is U -equivariant with respect to the diagonal action on the left hand side and the action on U by left multiplication (and the trivial action on \mathbb{A}^n) on the right hand side. Since the action on U is triangulable by our Theorem D above the claim follows.

Likewise, given any \mathbb{K} -algebra R and a locally nilpotent derivation ∂ of R , its extension $\bar{\partial}$ to $R[x]$ by letting $\bar{\partial}(x) = 1$ is conjugate to the locally nilpotent R -derivation δ of $R[x]$ defined by $\delta(x) = 1$.

1.5. Locally finite subsets. Let V be a vector space over \mathbb{K} . We denote by $\mathcal{L}(V)$ the algebra of linear endomorphisms of V . An endomorphism $\lambda \in \mathcal{L}(V)$ is called *locally finite* if the linear span $\langle \lambda^j(v) \mid j \in \mathbb{N} \rangle$ is finite dimensional for any $v \in V$. It is called *semisimple* if there is a basis of eigenvectors, and *locally nilpotent* if for any $v \in V$ there is an $m \in \mathbb{N}$ such that $\lambda^m(v) = 0$. Every locally finite endomorphism λ has a uniquely defined *additive JORDAN-decomposition* $\lambda = \lambda_s + \lambda_n$ where λ_s is semisimple, λ_n locally nilpotent, and $\lambda_s \circ \lambda_n = \lambda_n \circ \lambda_s$.

A subset $S \subseteq \mathcal{L}(V)$ is called *locally finite* if every $v \in V$ is contained in a finite dimensional subspace $W \subseteq V$ which is invariant under all elements from S . If $S \subseteq \text{GL}(V)$, then S is locally finite if and only if the group $\langle S \rangle$ generated by S is locally finite.

Note that a locally finite subspace $A \subseteq \mathcal{L}(V)$ is not necessarily finite dimensional. In fact, let (e_1, e_2, \dots) be a basis of $V := \mathbb{K}^\infty$ and define $\lambda_k \in \mathcal{L}(V)$ by $\lambda_k(e_j) := \delta_{kj}e_j$. Then $A := \bigoplus_k \mathbb{K}\lambda_k$ is an infinite dimensional locally finite subspace.

Definition 1.5.1. Let X be an affine variety. A morphism $\varphi: X \rightarrow X$ is called *algebraic* or *locally finite* if the induced endomorphism $\varphi^* \in \mathcal{L}(\mathcal{O}(X))$ is locally finite. It is called *semisimple* if φ^* is semisimple, and *locally nilpotent* if φ^* is locally nilpotent.

A subgroup $H \subseteq \text{Aut}(X)$ is called *locally finite* if the image of H in $\mathcal{L}(\mathcal{O}(X))$ is locally finite.

Note that a subgroup $H \subseteq \text{Aut}(X)$ is locally finite if and only if the closure $\overline{H} \subseteq \text{Aut}(X)$ is an algebraic group. If $g \in \text{Aut}(X)$ is an algebraic element, then the closed commutative subgroup $\overline{\langle g \rangle}$ has the form $\mathbb{G}_m^p \times \mathbb{G}_a^q \times F$ where $p \in \mathbb{Z}_{\geq 0}$, $q \in \{0, 1\}$ and F is a finite cyclic group.

There are many examples of subgroups $G \subseteq \text{Aut}(X)$ that are not locally finite while being generated by algebraic elements. We will discuss such an example in Section 6. We also refer to the interesting discussions in [FK18, 9.4.3–9.4.5] and [PR22] of subgroups $G \subseteq \text{Aut}(X)$ which consist of algebraic elements.

1.6. Locally finite vector fields. Recall that the vector fields $\text{Vec}(X)$ are the derivations $\text{Der}(\mathcal{O}(X))$ of $\mathcal{O}(X)$. Since $\text{Der}(\mathcal{O}(X)) \subseteq \mathcal{L}(\mathcal{O}(X))$ we can talk about *locally finite vector fields* and *locally finite subspaces* of $\text{Vec}(X)$.

Example 1.6.1. If $G \subseteq \text{Aut}(X)$ is an algebraic subgroup, then its Lie algebra $\text{Lie}(G) \subseteq \text{Vec}(X)$ is locally finite. These Lie algebras are called *algebraic*.

In contrast to the general situation we have the following finiteness result.

Lemma 1.6.2. *Let $L \subseteq \text{Vec}(X)$ be a locally finite subspace. Then L is finite dimensional.*

Proof. If $f_1, \dots, f_m \in \mathcal{O}(X)$ is a set of generators, then every vector field δ is determined by the values $\delta(f_1), \dots, \delta(f_m)$. It follows that for any subspace $L \subseteq \text{Vec}(X)$ the linear map $L \rightarrow \mathcal{O}(X)^m$, $\delta \mapsto (\delta(f_1), \dots, \delta(f_m))$ is injective. If L is locally finite, then the image of this map is contained in the finite dimensional subspace $\bigoplus_{i=1}^m Lf_i$, and the claim follows. \square

The fact that a vector field is determined by its values on a finite generating set of $\mathcal{O}(X)$ has the following important consequences.

Lemma 1.6.3. *Let $V \subseteq \mathcal{O}(X)$ be a finite dimensional subspace generating $\mathcal{O}(X)$, and let $\delta \in \text{Vec}(X)$ be a vector field. If V is δ -invariant, then the following holds.*

- (1) δ is locally finite.
- (2) δ is semisimple if and only if $\delta|_V \in \mathfrak{gl}(V)$ is semisimple.
- (3) δ is locally nilpotent if and only if $\delta|_V \in \mathfrak{gl}(V)$ is nilpotent.
- (4) If $\delta = \delta_s + \delta_n$ is the JORDAN decomposition in $\mathcal{L}(\mathcal{O}(X))$, then $\delta_s, \delta_n \in \text{Vec}(X)$.
- (5) V is δ_s - and δ_n -invariant, and $\delta|_V = \delta_s|_V + \delta_n|_V \in \mathfrak{gl}(V)$ is the usual JORDAN decomposition in $\mathfrak{gl}(V)$.

Proof. Denote by $V^{(n)} \subseteq \mathcal{O}(X)$ the linear subspace generated by the n -fold products of elements from V . Then the following is obvious: (a) $V^{(n)}$ is δ -invariant; (b) if $\delta|_V$ is semisimple resp. nilpotent, then so is $\delta|_{V^{(n)}}$. This proves (1)–(3). The claim (4) follows from [FK18, Proposition 7.6.1], and (5) follows from (2) and (3). \square

Question 2. *Let $L_i \subseteq \text{Vec}(X)$ ($i \in I$) be a family of locally finite Lie subalgebras, and denote by $L \subseteq \text{Vec}(X)$ the Lie subalgebra generated by the L_i . Is L locally finite in case L is finite dimensional?*

In this direction we have the following consequence of Theorem A.

Corollary. *Let $L \subseteq \text{Vec}(X)$ be the Lie subalgebra generated by a family of locally finite Lie subalgebras $L_i \subseteq \text{Vec}(X)$, $i \in I$. Assume that each L_i is algebraic, i.e. $L_i = \text{Lie } G_i$ for some algebraic group $G_i \subseteq \text{Aut}(X)$. Then L is locally finite if and only if L is finite dimensional. In this case the subgroup G generated by the G_i is algebraic and $L = \text{Lie } G$. In particular, L is algebraic.*

Proof. If L is locally finite, then it is finite dimensional, by Lemma 1.6.2. If L is finite dimensional, then the claim follows from Theorem A. \square

Remark 1.6.4. For a locally finite vector field $\delta \in \text{Vec}(X)$ there is a uniquely determined minimal algebraic group $H \subseteq \text{Aut}(X)$ such that $\delta \in \text{Lie } H$. Moreover, H is commutative and connected, and H is a torus if δ is semisimple and $H \simeq \mathbb{G}_a$ if δ is locally nilpotent and non-zero, see [FK18, Proposition 7.6.1].

1.7. The adjoint action on $\text{Vec}(X)$. For any vector field δ we have the *adjoint action* $\text{ad } \delta$ on $\text{Vec}(X)$ defined in the usual way:

$$\text{ad } \delta(\eta) := [\delta, \eta] = \delta \circ \eta - \eta \circ \delta.$$

Lemma 1.7.1. *Let $\delta \in \text{Vec}(X)$ be a locally finite vector field with JORDAN decomposition $\delta = \delta_s + \delta_n$. Then $\text{ad } \delta$ acting on $\text{Vec}(X)$ is locally finite, and $\text{ad } \delta = \text{ad } \delta_s + \text{ad } \delta_n$ is the JORDAN decomposition.*

Proof. Let $V \subseteq \mathcal{O}(X)$ be a finite dimensional δ -invariant subspace which generates $\mathcal{O}(X)$. Then we have an inclusion

$$\text{Vec}(X) \hookrightarrow \mathcal{L}(V, \mathcal{O}(X)), \quad \mu \mapsto \mu|_V.$$

Given $\mu \in \text{Vec}(X)$ we choose a finite dimensional δ -invariant subspace W which contains $\mu(V)$. In $\mathcal{L}(\mathcal{O}(X))$ we have

$$(\text{ad } \delta)^m \mu = \sum_{i=0}^m n_i \delta^{m-i} \mu \delta^i$$

with suitable $n_i \in \mathbb{Z}$. Hence $(\text{ad } \delta)^m \mu$ sends V to W for all $m \geq 0$, and so the linear span $\langle (\text{ad } \delta)^m \mu \mid m \geq 0 \rangle \subseteq \text{Vec}(X)$ is finite dimensional. This shows that $\text{ad } \delta$ is locally finite.

If δ is locally nilpotent, then $(\delta|_V)^m = 0$ and $(\delta|_W)^m = 0$ for a suitable $m > 0$. The above formula implies that $(\text{ad } \delta)^{2m-1} \mu = 0$, hence $\text{ad } \delta$ is locally nilpotent.

Next, assume that δ is semisimple. Then we have an eigenspace decomposition $V = \bigoplus_a V_a$ and $\mathcal{O}(X) = \bigoplus_b \mathcal{O}(X)_b$. As a consequence, we get

$$\mathcal{L}(V, \mathcal{O}(X)) = \bigoplus_{a,b} \mathcal{L}(V_a, \mathcal{O}(X)_b)$$

which is an eigenspace decomposition for $\text{ad } \delta$, and so $\text{ad } \delta$ is semisimple.

Finally, $\text{ad } \delta_s$ and $\text{ad } \delta_n$ commute, because $[\delta_s, \delta_n] = 0$, and thus $\text{ad } \delta = \text{ad } \delta_s + \text{ad } \delta_n$ is the JORDAN decomposition. \square

1.8. Toral subalgebras.

Definition 1.8.1. A finite dimensional Lie subalgebra $S \subseteq \text{Vec}(X)$ is called *toral* if it consists of semisimple elements. It follows that S is commutative [Hum72, 8.1, Lemma] and thus locally finite. In particular, we obtain a *weight decomposition* $\mathcal{O}(X) = \bigoplus_{\alpha \in S^*} \mathcal{O}(X)_\alpha$ where

$$\mathcal{O}(X)_\alpha := \{f \in \mathcal{O}(X) \mid \tau f = \alpha(\tau) \cdot f \text{ for } \tau \in S\}.$$

If $T \subseteq \text{Aut}(X)$ is a torus, then $\text{Lie } T \subseteq \text{Vec}(X)$ is a toral subalgebra, but it is not true that every toral subalgebra is of this form. In fact, if $T \subseteq \text{Aut}(X)$ is a two dimensional torus, then the one dimensional tori $S \subseteq T$ are the kernels of certain characters $\chi \in X(T) \simeq \mathbb{Z}^2$ whereas every one dimensional subspace of $\text{Lie } T$ is a toral subalgebra. The best we can get is the following result.

Lemma 1.8.2. *For a toral Lie subalgebra $S \subseteq \text{Vec}(X)$ there exists a uniquely defined smallest torus $T \subseteq \text{Aut}(X)$ such that $\text{Lie } T \supseteq S$. Moreover, if a subspace $M \subseteq \text{Vec}(X)$ is invariant under the adjoint action of S , then M is invariant under T .*

Proof. (a) One easily verifies that the weight decomposition of $\mathcal{O}(X)$ has the following properties, cf. [FK18, Proposition 7.6.1 and its proof]:

- $\mathcal{O}(X)_\alpha \cdot \mathcal{O}(X)_\beta \subseteq \mathcal{O}(X)_{\alpha+\beta}$;
- the set of weights $\Lambda := \{\gamma \mid \mathcal{O}(X)_\gamma \neq 0\} \subseteq S^*$ generates a free subgroup $\mathbb{Z}\Lambda = \bigoplus_{i=1}^k \mathbb{Z}\mu_i$.

This defines a faithful action of the torus $T := (\mathbb{G}_m)^k$ on $\mathcal{O}(X)$ in the following way: For $\alpha = \sum_i n_i \mu_i \in \Lambda$, $f \in \mathcal{O}(X)_\alpha$ and $t = (t_1, \dots, t_k) \in T$ we put

$$t f := t_1^{n_1} \cdots t_k^{n_k} \cdot f.$$

The corresponding action of $c = (c_1, \dots, c_k) \in \text{Lie } T = \mathbb{K}^m$ is then given by the vector field δ_c where

$$\delta_c|_{\mathcal{O}(X)_\alpha} = \text{scalar multiplication with } \sum_i n_i c_i \text{ for } \alpha = \sum_i n_i \mu_i \in \Lambda.$$

It follows that the image of $\text{Lie } T$ contains S and that every torus with this property contains T , see [FK18, proof of Proposition 7.6.1(2)].

(b) With respect to the adjoint action we also have a weight space decomposition $\text{Vec}(X) = \bigoplus_{\gamma \in S^*} \text{Vec}(X)_\gamma$, see Lemma 1.7.1. It has the property that for $\delta \in \text{Vec}(X)_\gamma$ we get $\delta(\mathcal{O}(X)_\alpha) \subseteq \mathcal{O}(X)_{\alpha+\gamma}$. It follows that if $\text{Vec}(X)_\gamma \neq 0$, then $\gamma \in \Lambda$. Hence, the subspaces $\text{Vec}(X)_\gamma$ are weight spaces of the action of T on $\text{Vec}(X)$, and thus every S -invariant subspace of $\text{Vec}(X)$ is invariant under T . \square

1.9. Integration of Lie algebras. The first part of the following result is due to COHEN and DRAISMA, see [CD03, Theorem 1].

Theorem E. *Let $L \subseteq \text{Vec}(X)$ be a Lie subalgebra. Then $L \subseteq \text{Lie } G$ for some algebraic subgroup $G \subseteq \text{Aut}(X)$ if and only if L is locally finite. If L is locally finite and generated by locally nilpotent elements, then L is algebraic, i.e. there is an algebraic group $G \subseteq \text{Aut}(X)$ such that $L = \text{Lie } G$.*

The proof will be given in Section 3.1, see Theorem 3.1.1. A weaker form of Question 2 is the following.

Question 3. *Let $\xi, \eta \in \text{Vec}(X)$ be locally finite vector fields. If the Lie subalgebra $L := \langle \xi, \eta \rangle_{\text{Lie}}$ generated by ξ and η is finite dimensional, does it follow that L is locally finite?*

From Theorem C we get the following result in this direction.

Corollary. *Let $\{\eta_i \mid i \in I\}$ be a family of locally nilpotent vector fields, and denote by $L := \langle \eta_i \mid i \in I \rangle_{\text{Lie}}$ the Lie subalgebra generated by the η_i . If L is finite dimensional, then L is locally finite and algebraic.*

1.10. A free subgroup of $\text{Aut}(\mathbb{A}^2)$ and its closure. In Section 6 we provide an example of a nonabelian free subgroup $F \simeq \mathbb{F}_2$ of $\text{Aut}(\mathbb{A}^2)$ generated by two algebraic elements whose product is not algebraic. In particular, F is not locally finite. We compute its closure $\mathfrak{F} := \overline{F} \subseteq \text{Aut}(\mathbb{A}^2)$ and describe the Lie algebra $\text{Lie } \mathfrak{F}$. It occurs that \mathfrak{F} coincides with the double centralizer of F (Lemma 6.1.1) and is a free product of two abelian nested unipotent ind-subgroups \mathfrak{J} and \mathfrak{J}^- (Theorem 6.4.3(3)). Furthermore, any algebraic subgroup G of \mathfrak{F} is abelian and unipotent and is conjugate to a subgroup of \mathfrak{J} or \mathfrak{J}^- (Theorem 6.4.3(4)).

The construction shows that \mathfrak{F} contains two algebraic subgroups U and V , both isomorphic to \mathbb{G}_a , with the following properties:

$$\mathfrak{F} = \overline{\langle U, V \rangle} \quad \text{and} \quad \text{Lie } \mathfrak{F} = \langle \text{Lie } U, \text{Lie } V \rangle_{\text{Lie}},$$

see Lemma 6.2.1. This is an instance where Question 1 has a positive answer.

Yet another instance is the following. Consider the ind-group $\mathfrak{G} := \text{Aut}_{\mathcal{K}}(\mathcal{K}[x, y])$ where $\mathcal{K} := \mathbb{K}[z]$, and let $\mathfrak{G}^t \subseteq \mathfrak{G}$ be the subgroup consisting of tame automorphisms of $\mathbb{K}[x, y, z]$. Then \mathfrak{G} is connected and closed in $\text{Aut}_{\mathbb{K}}(\mathbb{K}[x, y, z])$, $\mathfrak{G}^t \subsetneq \mathfrak{G}$ is closed and $\text{Lie } \mathfrak{G}^t = \text{Lie } \mathfrak{G}$, see [FK18, Theorem 17.3.1]. The group \mathfrak{G}^t is generated by the linear

and DE JONQUIÈRES transformations which fix z . Hence \mathfrak{G}^t is algebraically generated. By the UMIRBAEV-SHESTAKOV Theorem, the one dimensional unipotent subgroup U of \mathfrak{G} corresponding to the locally nilpotent NAGATA-derivation is not contained in \mathfrak{G}^t . However, since the algebraically generated group $\mathfrak{H} := \langle \mathfrak{G}^t, U \rangle$ is contained in \mathfrak{G} and $\mathrm{Lie} U \subseteq \mathrm{Lie} \mathfrak{G} = \mathrm{Lie} \mathfrak{G}^t$ we have

$$\mathrm{Lie} \overline{\mathfrak{H}} = L_{\mathfrak{H}} := \langle \mathrm{Lie} \mathfrak{G}^t, \mathrm{Lie} U \rangle.$$

1.11. Ind-groups and their Lie algebras. The theory of algebraic group actions on affine varieties can be generalized, to some extent, to ind-groups acting on affine ind-varieties, see [FK18]. If \mathfrak{G} is an ind-group, then a \mathfrak{G} -variety X is a variety with a \mathfrak{G} -action such that the map $\mathfrak{G} \times X \rightarrow X$ is a morphism of ind-varieties. For every ind-group \mathfrak{G} the tangent space $T_e \mathfrak{G}$ at the unit element $e \in \mathfrak{G}$ is well-defined and carries a natural structure of a Lie algebra, denoted by $\mathrm{Lie} \mathfrak{G}$. For every \mathfrak{G} -variety X there is a canonical homomorphism of Lie algebras $\xi: \mathrm{Lie} \mathfrak{G} \rightarrow \mathrm{Vec}(X)$ constructed as in the algebraic case. Again, \mathfrak{G} acts linearly on both, $\mathrm{Lie} \mathfrak{G}$ and $\mathrm{Vec}(X)$, and the homomorphism $\xi: \mathrm{Lie} \mathfrak{G} \rightarrow \mathrm{Vec}(X)$ is \mathfrak{G} -equivariant.

A subset $M \subseteq \mathfrak{X}$ of an ind-variety \mathfrak{X} is called *path-connected* if for any two points $x, y \in M$ there is an irreducible variety Y and a morphism $\varphi: Y \rightarrow \mathfrak{X}$ such that $x, y \in \varphi(Y) \subseteq \mathfrak{X}$, cf. [Ram64, p. 26]. One can always assume that Y is an irreducible curve, because any two points in an irreducible variety are contained in an irreducible curve, cf. [FK18, Section 1.6].

In contrast to the finite-dimensional case, a strict closed subgroup of a path-connected ind-group may have the same Lie algebra, see [FK18, Theorem 17.3.1] and the example in Section 1.10. This disproves [Sha81, Theorems 1 and 2]. However, a homomorphism of a path-connected ind-group to an ind-group is uniquely determined by the induced homomorphism of their Lie algebras, see [FK18, Proposition 7.4.7]. In Section 7 we study surjective homomorphisms of ind-groups and of their Lie algebras, and subgroups of *finite codimension* of ind-groups.

2. ALGEBRAICALLY GENERATED SUBGROUPS

2.1. Orbits of algebraically generated subgroups. Let $\mathfrak{G} = \bigcup_k \mathfrak{G}_k$ be an ind-group. The following concept was introduced in [AFK13].

Definition 2.1.1. A subgroup $G \subseteq \mathfrak{G}$ is called *algebraically generated* if there is a family $(G_i)_{i \in I}$ of connected algebraic subgroups $G_i \subseteq \mathfrak{G}$ generating G as an abstract group: $G = \langle G_i \mid i \in I \rangle$.

It is clear that an algebraically generated subgroup is path-connected, hence connected. The following result can be found in [AFK13, Prop. 1.2 and 1.3]. For the reader's convenience we provide a short argument.

Proposition 2.1.2. *Let $G = \langle G_i \mid i \in I \rangle \subseteq \text{Aut}(X)$ be an algebraically generated subgroup. For every $x \in X$ the G -orbit $Gx \subseteq X$ is open in its closure \overline{Gx} . Moreover, there is a finite sequence $j_1, \dots, j_n \in I$ such that $Gx = G_{j_1} \cdots G_{j_n}x$ for all $x \in X$.*

Proof. (a) Choose a sequence i_1, \dots, i_m such that $\dim \overline{G_{i_1} \cdots G_{i_m}x}$ is maximal. Since $\overline{G_{i_1} \cdots G_{i_m}x}$ is irreducible, it follows that $G_{j_1} \cdots G_{j_\ell}x \subseteq \overline{G_{i_1} \cdots G_{i_m}x}$ for every sequence $j_1, \dots, j_\ell \in I$. Hence $\overline{G_{i_1} \cdots G_{i_m}x} = \overline{Gx}$, and there is an open dense set $U \subseteq \overline{Gx}$ contained in $G_{i_1} \cdots G_{i_m}x$. This implies that $Gx = GU$ is open in \overline{Gx} .

(b) For the second claim we first remark that for every $x \in X$ with $\dim Gx$ maximal there is an open set $U_x \subseteq X$ containing x and a sequence $\mathbf{i}_x = (i_1, \dots, i_m)$, depending on x , such that $\overline{G_{i_1} \cdots G_{i_m}y} = \overline{Gy}$ for all $y \in U_x$. In fact, if $d := \dim Gx$ is maximal and $Gx \subseteq \overline{G_{i_1} \cdots G_{i_m}x}$, then $\dim \overline{G_{i_1} \cdots G_{i_m}y} = d$ for all y in an open neighborhood of x . Now the union $U := \bigcup_x U_x$ is a G -invariant open set, hence covered by finitely many U_{x_j} . By joining the finitely many sequences \mathbf{i}_{x_j} corresponding to the points x_j , we obtain a sequence (j_1, \dots, j_ℓ) such that $\overline{G_{j_1} \cdots G_{j_\ell}y} = \overline{Gy}$ for all $y \in U$. A standard argument implies that $G_{j_\ell}G_{j_{\ell-1}} \cdots G_{j_2}G_{j_1}G_{j_2} \cdots G_{j_\ell}y = Gy$ for all $y \in U$. Since $X \setminus U$ is a closed G -invariant subvariety of smaller dimension, the claim follows by induction. \square

Remark 2.1.3. A similar result holds for the action of a connected ind-group \mathfrak{G} on an affine variety X , see [FK18, Proposition 7.1.2].

2.2. Vector fields associated to subvarieties of $\text{Aut}(X)$. The construction of the embedding $\xi: \text{Lie } G \hookrightarrow \text{Vec}(X)$ for an algebraic subgroup $G \subseteq \text{Aut}(X)$ (see 1.1) has the following generalization, cf. [FK18, Section 3.2]. Consider the orbit map $\mu_x: \text{Aut}(X) \rightarrow X$, $g \mapsto gx$, which is an ind-morphism, and its differential in $g \in \text{Aut}(X)$:

$$(d\mu_x)_g: T_g \text{Aut}(X) \rightarrow T_{gx}X.$$

This defines, for any $v \in T_g \text{Aut}(X)$, a vector field $\xi_v^{(g)} \in \text{Vec}(X)$ by

$$(\xi_v^{(g)})(x) := (d\mu_{g^{-1}x})_g v \in T_x X,$$

and thus a linear map $\xi^{(g)}: T_g \text{Aut}(X) \rightarrow \text{Vec}(X)$.

Lemma 2.2.1 ([FK18, Proposition 3.2.4]). *The map $\xi^{(g)}: T_g \text{Aut}(X) \rightarrow \text{Vec}(X)$ is injective for every $g \in \text{Aut}(X)$.*

Let Z be an affine variety and $\rho: Z \rightarrow \text{Aut}(X)$ a morphism. Composing the differential $d\rho: T_z Z \rightarrow T_{\rho(z)} \text{Aut}(X)$ with $\xi^{(\rho(z))}$ we obtain a linear map $\xi^{(\rho)} = \xi^{(\rho, z)}: T_z Z \rightarrow \text{Vec}(X)$. In the following statement we use the obvious action of $\text{Aut}(X)$ on the vector fields $\text{Vec}(X)$.

Lemma 2.2.2.

- (1) *For $g', g'' \in \text{Aut}(X)$ set $\tilde{\rho}: Z \rightarrow \text{Aut}(X)$, $g \mapsto g'gg''$. Then $\xi^{(\tilde{\rho})} = g' \cdot \xi^{(\rho)}$.*
- (2) *For a closed subvariety $Z \subseteq \text{Aut}(X)$ the map $\xi^{(z)}: T_z Z \rightarrow \text{Vec}(X)$ is injective for any $z \in Z$.*

Proof. (1) Replace the orbit map μ_x by $\tilde{\mu}_x: g \mapsto g'gg''x$, i.e. $\tilde{\mu}_x = g'\mu_{g''x}$. Then $d\tilde{\mu}_x = g'd\mu_{g''x}$. By definition, $\xi^{(g''x)} = \xi^{(x)}$ and so $\xi^{(\tilde{\rho})} = g' \cdot \xi^{(\rho)}$.

(2) This follows from Lemma 2.2.1. \square

Remark 2.2.3. For an algebraic subgroup $G \subseteq \text{Aut}(X)$ we obtain linear maps $\xi^{(g)}: T_g G \rightarrow \text{Vec}(X)$ for any $g \in G$. All these maps have the same image, namely $L := \xi^{(e)}(\text{Lie } G) \subseteq \text{Vec}(X)$.

Proposition 2.2.4. *Let $G_1, \dots, G_m \subseteq \text{Aut}(X)$ be connected algebraic subgroups, and consider the multiplication map*

$$\rho: G_1 \times \dots \times G_m \rightarrow \text{Aut}(X).$$

If $L_i \subseteq \text{Vec}(X)$ denotes the image of $\text{Lie } G_i$, then the image of

$$\xi^{(\rho, \mathbf{g})} = \xi^{(\rho, (g_1, \dots, g_m))}: T_{g_1} G_1 \oplus \dots \oplus T_{g_m} G_m \rightarrow \text{Vec}(X)$$

is given by $\sum_i h_i \cdot L_i$ where $h_1 = e$ and $h_i := g_1 \cdots g_{i-1}$ for $i > 1$.

Proof. The linear map $\xi^{(\rho, \mathbf{g})}$ restricted to the summand $T_{g_i} G_i$ is equal to $\xi^{(\rho_i, g_i)}$ where $\rho_i: G_i \rightarrow \text{Aut}(X)$ is the map $g \mapsto g_1 \cdots g_{i-1}gg_{i+1} \cdots g_m = h_i gg_{i+1} \cdots g_m$. Now Lemma 2.2.2(1) shows that $\xi^{(\rho, \mathbf{g})}(T_{g_i}(G_i)) = h_i \cdot L_i$, and the claim follows. \square

Corollary 2.2.5. *With the notation above consider the “orbit map” in $x \in X$,*

$$\rho_x: G_1 \times \dots \times G_m \rightarrow X, \quad \mathbf{g} = (g_1, \dots, g_m) \mapsto \mathbf{g}x := g_1 \cdots g_m x.$$

Then the image of the differential $(d\rho_x)_{\mathbf{g}}: T_{g_1} G_1 \oplus \dots \oplus T_{g_m} G_m \rightarrow T_{\mathbf{g}x} X$ is given by $\sum_i h_i \cdot L_i(g_i \cdots g_m x)$.

Proof. By definition, $(d\rho_x)_{\mathbf{g}} = \text{ev}_{\mathbf{g}x} \circ \xi^{(\rho, \mathbf{g})}$ where $\text{ev}_y: \text{Vec}(X) \rightarrow T_y X$ is the evaluation map $\xi \mapsto \xi(y)$. Thus, by Proposition 2.2.4, the image of $(d\rho_x)_{\mathbf{g}}$ is given by

$$\text{ev}_{\mathbf{g}x} \left(\sum_i h_i L_i \right) = \sum_i (h_i L_i)(\mathbf{g}x) = \sum_i h_i L_i(g_i \cdots g_m x).$$

□

2.3. Lie algebras of algebraically generated groups. As in 2.1, consider an algebraically generated subgroup $G = \langle G_i \mid i \in I \rangle \subseteq \text{Aut}(X)$. Set $L_i := \text{Lie } G_i \subseteq \text{Vec}(X)$, and let $L := \langle L_i \mid i \in I \rangle_{\text{Lie}} \subseteq \text{Vec}(X)$ be the Lie subalgebra generated by the L_i . If $Y \subseteq X$ is a G -invariant closed subset, then $L_i|_Y \subseteq \text{Vec}(Y)$ for all i and so $L|_Y \subseteq \text{Vec}(Y)$. Choosing $Y := \overline{Gx}$ we get $L(x) \subseteq T_x Gx$ for any $x \in X$.

Theorem 2.3.1. *Let $G = \langle G_i \mid i \in I \rangle \subseteq \text{Aut}(X)$ be an algebraically generated subgroup, and define $L := \langle \text{Lie } G_i \mid i \in I \rangle_{\text{Lie}} \subseteq \text{Vec}(X)$ to be the Lie algebra generated by the $\text{Lie } G_i$. Then*

- (1) *L is stable under the action of G and of $\overline{G} \subseteq \text{Aut}(X)$. In particular, L is an ideal in $\text{Lie } \overline{G}$.*
- (2) *We have $T_x Gx = L(x)$ for all $x \in X$. In particular, any vector field in L is tangent to the G -orbits.*
- (3) *If $Z \subseteq \text{Aut}(X)$ is a closed algebraic subset contained in G , then the image of $T_z Z$ in $\text{Vec}(X)$ is contained in L for any $z \in Z$.*
- (4) *If $H \subseteq \text{Aut}(X)$ is an algebraic subgroup contained in G , then $\text{Lie } H \subseteq L$. In particular, L depends only on G and not on the generating subgroups G_i .*

Proof of Theorem 2.3.1. (1) It suffices to show that L is invariant under all the G_i . Since G_i is connected, we are reduced to prove that L is invariant under the action of $\text{Lie } G_i$, i.e. under the adjoint action of $L_i := \text{Lie } G_i$ (see [FK18, Proposition 6.3.4]) which holds by construction.

(2) We already know that $L(x) \subseteq T_x Gx$. Now we choose a surjective “orbit map”

$$\mu_x: G_{j_1} \times \cdots \times G_{j_n} \rightarrow Gx \subseteq X,$$

see Proposition 2.1.2. It follows that for a suitable $\mathbf{g} = (g_1, \dots, g_n)$ the differential

$$(d\mu_x)_{\mathbf{g}}: T_{g_1} G_{j_1} \oplus \cdots \oplus T_{g_n} G_{j_n} \rightarrow T_{\mathbf{g}x} Gx$$

is also surjective. Thus we get $T_{\mathbf{g}x}Gx = \sum_j h_j L_j(g_j \cdots g_n x)$, by Corollary 2.2.5. Since L is invariant under the action of G , we finally get

$$h_j L_j(g_j \cdots g_n x) \subseteq h_j L(g_j \cdots g_n x) = L(\mathbf{g}x),$$

and the claim follows.

(3) Let $Z \subseteq \text{Aut}(X)$ be a closed algebraic subset contained in G . We first claim that $Z \subseteq G_{i_1} \cdots G_{i_m}$ for a suitable sequence (i_1, \dots, i_m) . In fact, there is a sequence such that the intersection $Z \cap G_{i_1} \cdots G_{i_m}$ is dense in Z , hence contains a dense open set $U \subseteq Z$. Since $\dim(Z \setminus U) < \dim Z$ the claim follows by induction on $\dim Z$.

Applying Proposition 2.2.4 to the multiplication map $\rho: G_{i_1} \times \cdots \times G_{i_m} \rightarrow \text{Aut}(X)$ and using $\xi^{(\rho, \mathbf{g})}$ where $\mathbf{g} = (g_1, \dots, g_m)$ and $g_1 \cdots g_m = z$ it follows that the image of $T_z Z$ in $\text{Vec}(X)$ is contained in $\sum_i h_i L_i \subseteq L$, since L is stable under the action of G , by (1).

(4) This is an immediate consequence of (3). \square

2.4. Ramanujam's alternative. The following result, due to RAMANUJAM, will be used in the proof of our Theorem 2.5.1 below.

Proposition 2.4.1 ([Ram64, Theorem on p. 26]). *Let $G \subseteq \text{Aut}(X)$ be a path-connected subgroup. Then one of the following holds:*

- (i) G is an algebraic subgroup;
- (ii) G contains algebraic subsets of arbitrary large dimension.

Proof. Assume that we are not in case (ii), and let $Y \subseteq G$ be an irreducible algebraic subset of maximal dimension d . We may assume that $e \in Y$. We claim that $G \subseteq \overline{Y} = \overline{G}$. In fact, for any $g \in G$ we can find an irreducible algebraic subset $S \subseteq G$ containing e and g . Then $\overline{S \cdot Y}$ is irreducible and meets G in a dense open set, hence $\overline{S \cdot Y} = \overline{Y}$. Since $g \in S \cdot Y$, the claim follows.

As a consequence, $\overline{G} \subseteq \text{Aut}(X)$ is an algebraic subgroup. Since $Y \subseteq G$ is open and dense in \overline{G} it follows that $\overline{G} = Y^{-1} \cdot Y \subseteq G$, and so (i) holds. \square

2.5. The finite dimensional case. Assume that $G = \langle G_i \mid i \in I \rangle$ is finite dimensional, that is, an algebraic subgroup of $\text{Aut}(X)$, see Proposition 2.4.1. In this case, L is finite dimensional and equal to $\text{Lie } G$, by Theorem 2.3.1(4). The next result shows that the converse is true as well. It also proves Theorem A from Section 1.3 of the introduction.

Theorem 2.5.1. *Let $G = \langle G_i \mid i \in I \rangle \subseteq \text{Aut}(X)$ be an algebraically generated subgroup. Assume that the Lie algebra $L \subseteq \text{Vec}(X)$ generated*

by the $\text{Lie } G_i$ is finite dimensional. Then G is an algebraic subgroup with $\text{Lie } G = L$. In particular, L is locally finite and algebraic.

Proof. Let $Y \subseteq G$ be an irreducible algebraic subset. As in the proof of Theorem 2.3.1(3) there is a sequence (i_1, \dots, i_m) such that $G_{i_1} \cdots G_{i_m}$ contains Y . Thus the image of $T_g Y$ in $\text{Vec}(X)$ is contained in L . Since the map $\xi^{(g)}: T_g \text{Aut}(X) \rightarrow \text{Vec}(X)$ is injective (Lemma 2.2.1) we get $\dim Y \leq \dim L$. Now the claim follows from RAMANUJAM' alternative (Proposition 2.4.1). \square

3. INTEGRATION OF VECTOR FIELDS

3.1. Integration of locally finite Lie algebras. The next result is our Theorem E from the introduction, see Section 1.9.

Theorem 3.1.1. *Let $L \subseteq \text{Vec}(X)$ be a Lie subalgebra.*

- (1) *If L is finite dimensional and generated by locally nilpotent vector fields, then L is algebraic, i.e. there is a connected algebraic subgroup $G \subseteq \text{Aut}(X)$ such that $L = \text{Lie } G$.*
- (2) *If L is locally finite, then there exists a connected algebraic subgroup $G \subseteq \text{Aut}(X)$ such that $L \subseteq \text{Lie } G$.*

Proof of (1). Assume that L is finite dimensional and generated by locally nilpotent vector fields $\eta_1, \eta_2, \dots, \eta_m$. By [FK18, Proposition 7.6.1] there are (uniquely determined) algebraic subgroups $U_i \subseteq \text{Aut}(X)$, $U_i \simeq \mathbb{G}_a$, such that $\text{Lie } U_i = \mathbb{K}\eta_i$. Now Theorem 2.5.1 implies that $G := \langle U_i \mid i = 1, \dots, m \rangle \subseteq \text{Aut}(X)$ is an algebraic group with $\text{Lie } G = L$. \square

Example 3.1.2. Assume that $L \subseteq \text{Vec}(X)$ is a locally finite semisimple Lie subalgebra. Then there is a semisimple group $G \subseteq \text{Aut}(X)$ such that $L = \xi(\text{Lie } G)$. In particular, L is algebraic.

In fact, the semisimple Lie algebra L is generated by the root subspaces with respect to some CARTAN subalgebra (see [Hum72, Proposition 8.4(f)]), hence by nilpotent elements. Lemma 1.6.3(3) shows that these elements are locally nilpotent in $\text{Vec}(X)$, and thus L is generated by locally nilpotent vector fields.

The proof of (2) needs some preparation. Let $L = L_0 \ltimes R$ be the LEVI decomposition where L_0 is semisimple and $R := \text{rad } L$ is the solvable radical (see [Pro07, 10.1.6, pp. 305–306]). The example above shows that $L_0 = \text{Lie } G_0$ for a semisimple subgroup $G_0 \subseteq \text{Aut}(X)$. It remains to find a solvable subgroup $Q \subseteq \text{Aut}(X)$ normalized by G_0 such that $\text{Lie } Q \supseteq R$. Then the product $G := G_0 Q$ has the required property.

Lemma 3.1.3. *Let $M \subseteq \text{Vec}(X)$ be a locally finite solvable Lie subalgebra. Assume that M contains with every element μ its semisimple and locally nilpotent parts μ_s and μ_n . Then the locally nilpotent elements from M form a Lie subalgebra M_n , and $M = S \ltimes M_n$ where S is a maximal toral subalgebra of M .*

Proof. Since M is locally finite we may assume that $M \subseteq \mathfrak{gl}(V)$ where $V \subseteq \mathcal{O}(X)$ is a finite dimensional subspace generating $\mathcal{O}(X)$. Since M is solvable there is a basis of V such that M is contained in the upper triangular matrices $\mathfrak{b}_d \subseteq \mathfrak{gl}_d$ where $d := \dim V$ (LIE's Theorem). Clearly, the nilpotent elements from M belong to the subalgebra $\mathfrak{n}_d \subseteq \mathfrak{b}_d$ of upper triangular matrices with zeros along the diagonal, and so $M_n := M \cap \mathfrak{n}_d$ is the ideal of M consisting of the nilpotent elements. By Lemma 1.6.3 every nilpotent element of M is also locally nilpotent viewed as a vector field on X , and vice versa. If $M = M_n$ then $S = 0$ and $M = S \ltimes M_n$, as desired.

Assume now that M contains a semisimple element, and choose a maximal toral subalgebra $S \subseteq M$. Now consider the corresponding weight space decomposition $M = \bigoplus_{\alpha \in S^*} M_\alpha$ where

$$M_\alpha := \{\mu \in M \mid \text{ad } \tau(\mu) = [\tau, \mu] = \alpha(\tau) \cdot \mu \text{ for } \tau \in S\}.$$

Similarly, we get a decomposition $\mathfrak{gl}_d = \bigoplus_{\alpha \in S^*} (\mathfrak{gl}_d)_\alpha$. Using the identity $[s, a \cdot b] = [s, a] \cdot b + a \cdot [s, b]$ we get $(\mathfrak{gl}_d)_\alpha \cdot (\mathfrak{gl}_d)_\beta \subseteq (\mathfrak{gl}_d)_{\alpha+\beta}$ which implies that M_α consists of nilpotent elements for $\alpha \neq 0$. Hence, $\bigoplus_{\alpha \neq 0} M_\alpha \subseteq \mathfrak{n}_d$ is a solvable subalgebra consisting of nilpotent elements, while $M_0 = \text{Cent}_M(S) \supseteq S$.

Let $\mu \in M_0$ have JORDAN decomposition $\mu = \mu_s + \mu_n$. Since S commutes with μ it also commutes with μ_s and μ_n , and so $\mu_s \in M_0$ and $\mu_n \in M_0 \cap \mathfrak{n}_d$. Since S is a maximal toral subalgebra, we have $\mu_s \in S$, and so $M_0 = S \oplus (M_0 \cap \mathfrak{n}_d)$. As a consequence we see that $M_n = (M_0 \cap \mathfrak{n}_d) \oplus \bigoplus_{\alpha \neq 0} M_\alpha$ is a subalgebra consisting of nilpotent elements, and that $M = S \ltimes M_n$. \square

From the first part of the proof we get the following result.

Corollary 3.1.4. *If a locally finite solvable Lie subalgebra $M \subseteq \text{Vec}(X)$ is generated by locally nilpotent elements, then it consists of locally nilpotent elements.*

3.2. JORDAN-saturated subspaces. The assumption in Lemma 3.1.3 above leads to the following definition. A locally finite subspace $W \subseteq \text{Vec}(X)$ is called *JORDAN-saturated* (shortly *J-saturated*) if it contains with every element η the semisimple and the locally nilpotent parts

η_s and η_n . Clearly, for any algebraic subgroup $G \subseteq \text{Aut}(X)$ the Lie algebra $\text{Lie } G \subseteq \text{Vec}(X)$ is J -saturated.

Lemma 3.2.1. *Let $R \subseteq \text{Vec}(X)$ be a locally finite and solvable Lie subalgebra. Then the smallest J -saturated Lie subalgebra $\tilde{R} \subseteq \text{Vec}(X)$ containing R has the following properties:*

- (1) \tilde{R} is locally finite and solvable.
- (2) If a subspace $V \subseteq \mathcal{O}(X)$ is R -invariant, then it is invariant under \tilde{R} .
- (3) If an automorphism $\varphi \in \text{Aut}(X)$ normalizes R , then it normalizes \tilde{R} .

Notice that any algebraic Lie subalgebra of $\text{Vec}(X)$ which contains R also contains \tilde{R} .

Proof. As before, we choose a finite dimensional R -invariant subspace $V \subseteq \mathcal{O}(X)$ which generates $\mathcal{O}(X)$. Choosing a suitable basis of V we may assume that the image of R in $\mathfrak{gl}(V) = \mathfrak{gl}_d$ is contained in the upper triangular matrices \mathfrak{b}_d .

If $\eta = \eta_s + \eta_n$ is the JORDAN-decomposition of an element $\eta \in R$, then every finite dimensional R -invariant subspace $V \subseteq \mathcal{O}(X)$ is invariant under η_s and η_n , hence invariant under the Lie subalgebra R_1 generated by $R + \mathbb{K}\eta_n$. Thus R_1 is locally finite. The JORDAN-decomposition of η carries over to the image of η in $\mathfrak{gl}(V) = \mathfrak{gl}_d$. Hence the images of η_s and η_n also belong to \mathfrak{b}_d which shows that the image of R_1 is contained in \mathfrak{b}_d .

This procedure of adding the locally nilpotent part and forming the Lie algebra has to end with a locally finite and solvable Lie subalgebra \tilde{R} containing R with the properties (1) and (2).

Finally, assume that $\varphi \in \text{Aut}(X)$ stabilizes R , and let $\eta \in R$. Then $\varphi(\eta) = \varphi(\eta_s) + \varphi(\eta_n)$ is the JORDAN decomposition of $\varphi(\eta) \in R$, hence $\varphi(\eta_s)$ and $\varphi(\eta_n)$ belong to \tilde{R} . Now it follows from the construction that \tilde{R} is invariant under φ . \square

Proof of Theorem 3.1.1(2). Let $L = L_0 \ltimes R$ be the LEVI decomposition where L_0 is semisimple and R is the solvable radical of L . By part (1) and Example 3.1.2 there is a semisimple group $G_0 \subseteq \text{Aut}(X)$ with $\text{Lie } G_0 = L_0$. Clearly, R is G_0 -invariant, as well as \tilde{R} , the J -saturation of R , see Lemma 3.2.1. By Lemma 3.1.3 we have $\tilde{R} = S \ltimes \tilde{R}_n$ where S is toral and \tilde{R}_n is solvable and consists of locally nilpotent elements. Thus, again by part (1), there is a unipotent group $U \subseteq \text{Aut}(X)$ such that $\text{Lie } U = \tilde{R}_n$.

It follows from Lemma 1.8.2 that there is a well-defined minimal torus $T \subseteq \text{Aut}(X)$ such that $\text{Lie } T \supseteq S$, and that T normalizes R_n . By Lemmas 1.8.2 and 3.2.1, T normalizes U and \tilde{R}_n , and so the product $Q := TU = UT \subseteq \text{Aut}(X)$ is a connected solvable subgroup with $\text{Lie } Q \supseteq \tilde{R}$. By construction, Q is the smallest subgroup with this property. It remains to show that G_0 normalizes Q . In fact, gQg^{-1} is a closed subgroup for any $g \in G_0$, and $\text{Lie } gQg^{-1} \supseteq g\tilde{R}g^{-1} = \tilde{R}$. \square

4. NESTED SUBGROUPS

4.1. Nested subgroups and nested Lie algebras. For the following concept see [FK18, Section 9.4] and [KPZ17].

Definition 4.1.1. A subgroup G of an ind-group \mathfrak{G} is called *quasi-nested* if there exists a sequence $G_1 \subseteq G_2 \subseteq \cdots \subseteq \mathfrak{G}$ of bounded subgroups of \mathfrak{G} (cf. Section 1.2) such that $G = \bigcup_i G_i$. It is called *nested* if the G_i are algebraic subgroups. Clearly, a quasi-nested subgroup $G = \bigcup_i G_i$ defines a nested subgroup, namely $\bigcup_i \overline{G_i}$, and every subgroup of a nested group is quasi-nested.

A Lie algebra $L \subseteq \text{Vec}(X)$ is called *nested* if there exists a sequence $L_1 \subseteq L_2 \subseteq \cdots \subseteq L$ of finite dimensional Lie algebras L_i such that $L = \bigcup_i L_i$.

Notice that any two filtrations of a nested subgroup G by algebraic subgroups are compatible. In fact, if $G = \bigcup_j H_j$ is another filtration by algebraic subgroups, then $G_i = \bigcup_j (G_i \cap H_j)$ is an ascending union of closed subgroups, hence $G_i = G_i \cap H_j$ for some j and so $G_i \subseteq H_j$.

This implies that a nested subgroup $G = \bigcup G_i \subseteq \mathfrak{G}$ has a natural Lie algebra defined by $\text{Lie } G = \bigcup_i \text{Lie } G_i \subseteq \text{Lie } \mathfrak{G}$ which is nested and does not depend on the filtration of G by algebraic subgroups.

A nested subgroup $G = \bigcup_i G_i \subseteq \mathfrak{G}$ has a natural structure of an ind-group, and the inclusion $G \hookrightarrow \mathfrak{G}$ is a homomorphism of ind-groups. However, this is not necessarily a closed immersion as the following example shows.

Example 4.1.2. The subgroup $\mu_2 \subseteq \mathbb{G}_m$ of elements of order a power of 2 is nested by the finite subgroups $\simeq \mathbb{Z}/2^n\mathbb{Z}$. But it is not closed.

Lemma 4.1.3. *Assume that the nested subgroup $G \subseteq \mathfrak{G}$ is path-connected. Then G admits an ascending filtration by connected algebraic subgroups.*

Proof. If $G = \bigcup_i G_i$ is path connected, then $G = \bigcup_i G_i^\circ$, see [FK18, Proposition 2.2.1(1)]. \square

Question 4. *Let $G \subseteq \text{Aut}(X)$ be a path-connected nested subgroup. Does it follow that G is closed?*

In order to give a positive answer to this question it suffices to show that for every algebraic subset $A \subseteq \text{Aut}(X)$ the intersection $A \cap G$ is contained in an algebraic group.

The next result is an immediate consequence of Proposition 2.1.2.

Corollary 4.1.4. *Let $G = \bigcup_k G_k \subseteq \text{Aut}(X)$ be a path-connected nested subgroup. Then there is a k_0 such that for each $k \geq k_0$ the groups G and G_k have the same orbits on X .*

4.2. Integration of nested Lie algebras. If $G \subseteq \text{Aut}(X)$ is a nested subgroup, then $\text{Lie } G \subseteq \text{Vec}(X)$ is a nested Lie algebra filtered by locally finite subalgebras. The converse of this is also true.

Proposition 4.2.1. *Let $L = \bigcup_i L_i \subseteq \text{Vec}(X)$ be a nested Lie subalgebra. Then the following hold.*

- (1) *If L is generated by locally nilpotent vector fields, then there exists a nested unipotent subgroup $G = \bigcup_i G_i \subseteq \text{Aut}(X)$ such that $L = \bigcup_i \text{Lie } G_i$.*
- (2) *Assume that all L_i are locally finite. Then there exists a nested subgroup $G \subseteq \text{Aut}(X)$ such that $L \subseteq \text{Lie } G \subseteq \text{Vec}(X)$. Moreover, a closed ind-subgroup $\mathfrak{G} \subseteq \text{Aut}(X)$ contains G if and only if $L \subseteq \text{Lie } \mathfrak{G}$.*

Proof. (1) Any finite collection $S \subseteq L$ of locally nilpotent vector fields generates a Lie subalgebra, say, M_S of some L_j . According to Theorem 3.1.1(1), $M_S = \text{Lie } G_S$ for an algebraic subgroup G_S of $\text{Aut}(X)$. On the other hand, L_i is contained in some $M_i := M_{S_i}$, hence $L = \bigcup_i M_i$. We may suppose that $M_i \subseteq M_{i+1}$, and so $G_i \subseteq G_{i+1}$ where $G_i := G_{S_i}$. The union $G := \bigcup_i G_i$ is a nested subgroup of $\text{Aut } X$ such that $L = \bigcup_i \text{Lie } G_i$.

(2) Let $G_i \subseteq \text{Aut}(X)$ be the minimal algebraic subgroups with $\text{Lie } G_i \supseteq L_i$ (see Theorem 3.1.1(2)). Then $G_i \subseteq G_{i+1}$ for all i and so $G := \bigcup_i G_i$ is a nested subgroup with the required properties.

Now assume that $\text{Lie } \mathfrak{G} \supseteq L$ and define $G'_i := \mathfrak{G} \cap G_i$. Then $\text{Lie } G'_i = \text{Lie } \mathfrak{G} \cap \text{Lie } G_i \supseteq L_i$, hence $G'_i = G_i$ and the claim follows. \square

5. SOLVABILITY AND TRIANGULATION

5.1. Solvable subgroups. This section is devoted to the proof of our Theorem B from the introduction.

Theorem 5.1.1. *Let $G = \langle H_i \mid i \in I \rangle$ be a solvable algebraically generated group. If the family I is finite, then G is a solvable algebraic group with Lie algebra $\text{Lie } G = \langle \text{Lie } H_i \mid i \in I \rangle$. In general, G is nested and is of the form $G = U_G \rtimes T$ where $T \subseteq \text{Aut}(X)$ is a torus and U_G is a nested unipotent group consisting of the unipotent elements of G .*

Corollary 5.1.2. *A solvable subgroup $G \subseteq \text{Aut}(X)$ generated by unipotent elements is quasi-nested and consists of unipotent elements.*

Proof of Corollary. If G is generated by the unipotent elements $(u_i)_{i \in I}$, then the closure $\overline{G} \subseteq \text{Aut}(X)$ contains the subgroup \tilde{G} generated by the unipotent subgroups $\overline{\langle u_i \rangle}$. We will see in the following Lemma 5.1.3(2) that \overline{G} is also solvable. It then follows from the theorem above that $\tilde{G} = U_{\tilde{G}} \rtimes T$ is nested, hence G is quasi-nested and is contained in $U_{\tilde{G}}$. \square

The proof of Theorem 5.1.1 needs some preparation. Given a solvable group G we let $G^{(i)}$ stand for the members of the derived series of G and $d(G)$ for its derived length:

$$G^{(0)} := G \supseteq G^{(1)} := (G, G) \supseteq G^{(2)} := (G^{(1)}, G^{(1)}) \supseteq \cdots \supseteq G^{(d)} = \{e\}.$$

If G is nilpotent, then G_i stands for the members of the lower central series of G and $n(G)$ for its nilpotency class:

$$G_0 := G \supseteq G_1 := (G, G) \supseteq G_2 := (G, G_1) \supseteq \cdots \supseteq G_n = \{e\}.$$

In the following lemma we collect some important facts about closures of solvable and nilpotent groups. Statement (2) can be found in [FP18, Lemma 2.3(3)].

Lemma 5.1.3. *Let $G, H \subseteq \mathfrak{G}$ be subgroups of an ind-group \mathfrak{G} .*

- (1) *We have $(\overline{G}, \overline{H}) \subseteq \overline{(G, H)}$, $\overline{G}^{(i)} \subseteq \overline{G^{(i)}}$ and $\overline{G}_i \subseteq \overline{G_i}$.*
- (2) *If G is solvable, then so is \overline{G} , and $d(\overline{G}) = d(G)$.*
- (3) *If G is nilpotent, then so is \overline{G} , and $n(\overline{G}) = n(G)$.*

Proof. (1) Let $\gamma: \mathfrak{G} \times \mathfrak{G} \rightarrow \mathfrak{G}$ be the commutator map, $(g, h) \mapsto ghg^{-1}h^{-1}$. Then $\gamma(\overline{G} \times \overline{H}) \subseteq \overline{\gamma(G \times H)}$, and so $(\overline{G}, \overline{H}) \subseteq \overline{\langle \gamma(G \times H) \rangle} \subseteq \overline{(G, H)}$. Now the remaining inclusions follow.

(2) By (1) we have $G^{(i)} \subseteq \overline{G}^{(i)} \subseteq \overline{G^{(i)}}$. Hence $\overline{G}^{(d(G))} = \{e\}$ and $\overline{G}^{(i)} \neq \{e\}$ for $i < d(G)$.

(3) By (1) we have $G_i \subseteq \overline{G_i} \subseteq \overline{G_i}$. Hence $\overline{G}_{n(G)} = \{e\}$ and $\overline{G_i} \neq \{e\}$ for $i < n(G)$. \square

Lemma 5.1.4. *Let \mathfrak{G} be an ind-group \mathfrak{G} and \mathfrak{H} be a closed subgroup of \mathfrak{G} . Then the following hold.*

- (1) $[\text{Lie } \mathfrak{H}, \text{Lie } \mathfrak{G}] \subseteq \overline{\text{Lie } (\mathfrak{H}, \mathfrak{G})}$.
- (2) *If \mathfrak{G} is solvable, then so is $\text{Lie } \mathfrak{G}$.*
- (3) *If \mathfrak{G} is nilpotent, then so is $\text{Lie } \mathfrak{G}$.*

Proof. (Cf. [FK18, Section 7.5])

- (1) For a fixed $h \in \mathfrak{H}$ consider the morphism

$$\gamma_h: \mathfrak{G} \rightarrow (\mathfrak{H}, \mathfrak{G}), \quad \gamma_h(g) := hgh^{-1}g^{-1}.$$

The differential $d\gamma_h: \text{Lie } \mathfrak{G} \rightarrow \text{Lie } \overline{(\mathfrak{H}, \mathfrak{G})}$ is given by $d\gamma_h = \text{Ad}(h) - \text{id}$.

Now fix $A \in \text{Lie } \mathfrak{G}$ and consider the morphism

$$\alpha_A: \mathfrak{H} \rightarrow \text{Lie } \mathfrak{G}, \quad \alpha_A(h) := \text{Ad}(h)A - A = d\gamma_h(A).$$

The differential of this map is $-\text{ad}(A)$; it sends $\text{Lie } \mathfrak{H}$ onto $[\text{Lie } \mathfrak{H}, A]$. By the preceding, $d\gamma_h(A) \in \overline{\text{Lie } (\mathfrak{H}, \mathfrak{G})}$ for any $h \in \mathfrak{H}$ and $A \in \text{Lie } \mathfrak{G}$. Hence, $[\text{Lie } \mathfrak{H}, A] \subseteq \overline{\text{Lie } (\mathfrak{H}, \mathfrak{G})}$ for any $A \in \text{Lie } \mathfrak{G}$. This yields (1).

- (2) Now assume that \mathfrak{G} is solvable. From the derived series for \mathfrak{G} we get the normal series

$$\mathfrak{G}^{(0)} = \mathfrak{G} \supseteq \overline{\mathfrak{G}^{(1)}} \supseteq \overline{\mathfrak{G}^{(2)}} \supseteq \dots \supseteq \overline{\mathfrak{G}^{(n)}} = \{e\}$$

of closed subgroups with the property that the factor groups are all commutative (see Lemma 5.1.3(1)). Passing to the series

$$\text{Lie } \mathfrak{G} \supseteq \text{Lie } \overline{\mathfrak{G}^{(1)}} \supseteq \text{Lie } \overline{\mathfrak{G}^{(2)}} \supseteq \dots \supseteq \text{Lie } \overline{\mathfrak{G}^{(n)}} = 0$$

we get from (1)

$$\text{Lie } \overline{\mathfrak{G}^{(i+1)}} = \text{Lie } \overline{(\mathfrak{G}^{(i)}, \mathfrak{G}^{(i)})} = \text{Lie } \overline{(\overline{\mathfrak{G}^{(i)}}, \overline{\mathfrak{G}^{(i)}})} \supseteq [\text{Lie } \overline{\mathfrak{G}^{(i)}}, \text{Lie } \overline{\mathfrak{G}^{(i)}}].$$

Thus the factors $\text{Lie } \overline{\mathfrak{G}^{(i)}} / \text{Lie } \overline{\mathfrak{G}^{(i+1)}}$ are all commutative, which implies that $\text{Lie } \mathfrak{G}$ is solvable.

- (3) Assume now that \mathfrak{G} is nilpotent. From the lower central series for \mathfrak{G} we get the series of closed subgroups

$$\mathfrak{G}_0 = \mathfrak{G} \supseteq \overline{\mathfrak{G}_1} \supseteq \overline{\mathfrak{G}_2} \supseteq \dots \supseteq \overline{\mathfrak{G}_n} = \{e\}$$

and the corresponding series

$$(*) \quad \text{Lie } \mathfrak{G} \supseteq \text{Lie } \overline{\mathfrak{G}_1} \supseteq \text{Lie } \overline{\mathfrak{G}_2} \supseteq \dots \supseteq \text{Lie } \overline{\mathfrak{G}_n} = \{0\}.$$

By virtue of (1) we have

$$\text{Lie } \overline{\mathfrak{G}_{i+1}} = \text{Lie } \overline{(\mathfrak{G}, \mathfrak{G}_i)} = \text{Lie } \overline{(\overline{\mathfrak{G}}, \overline{\mathfrak{G}_i})} \supseteq [\text{Lie } \overline{\mathfrak{G}}, \text{Lie } \overline{\mathfrak{G}_i}].$$

Thus $(*)$ is a central series and so $\text{Lie } \mathfrak{G}$ is nilpotent. \square

Question 5. *Let \mathfrak{G} be a path connected ind-group. Is it true that \mathfrak{G} is solvable (resp. nilpotent) if $\text{Lie } \mathfrak{G}$ is?*

Lemma 5.1.5. *Let L be a finitely generated Lie algebra. If L is solvable, then L is finite dimensional.*

Proof. In the derived series

$$L^{(0)} := L \supseteq L^{(1)} := [L^{(0)}, L^{(0)}] \supseteq L^{(2)} := [L^{(1)}, L^{(1)}] \supseteq \cdots \supseteq L^{(n)} = \{0\}$$

every member $L^{(i)}$ is finitely generated. Therefore $L^{(i)}/L^{(i+1)}$ is a commutative and finitely generated Lie algebra, hence finite dimensional, and the claim follows. \square

Proof of Theorem 5.1.1. (a) We first consider the case where I is finite. The closure \overline{G} is solvable (Lemma 5.1.3(2)) and so $\text{Lie } \overline{G}$ is solvable (Lemma 5.1.4(2)). It follows that $L := \langle \text{Lie } H_i \mid i = 1, \dots, m \rangle_{\text{Lie}}$ is solvable, hence finite dimensional (Lemma 5.1.5). Now the claim follows from Theorem 2.5.1.

(b) In general, it follows from (a) that for any finite subset $F \subseteq I$ the subgroup $G_F := \langle H_i \mid i \in F \rangle$ is a connected solvable algebraic subgroup of \mathfrak{G} . If I is countable we have an ascending filtration $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq I$ by finite subsets, and so $G = \bigcup_j G_{F_j}$ is nested.

(c) For a general I we first remark that the Lie algebra $L \subseteq \text{Vec}(X)$ generated by the $\text{Lie } H_i$ has countable dimension, and so $L = \langle \text{Lie } H_j \mid j \in J \rangle_{\text{Lie}}$ for a countable subset $J \subseteq I$. We claim that G is generated by the H_j , $j \in J$, which implies by (b) that G is nested. In fact, for any H_i there are finitely many H_{j_1}, \dots, H_{j_m} with $j_k \in J$ such that $\text{Lie } H_i \subseteq \langle \text{Lie } H_{j_k} \mid k = 1, \dots, m \rangle_{\text{Lie}}$. Thus $\text{Lie } H_i \subseteq \text{Lie } \tilde{H}_i$ where \tilde{H}_i is the solvable algebraic subgroup generated by the H_{j_k} , and so $H_i \subseteq \tilde{H}_i$ by [FK18, Remark 17.3.3].

(d) If the nested subgroup $G = \bigcup_k G_k$ does not contain semisimple elements, then $G = U_G$ is a nested unipotent group. Otherwise we choose a torus $T \subseteq G$ of maximal dimension. We can assume that T is contained in all G_k , hence $G_k = U_{G_k} \rtimes T$ for all k , and $U_{G_k} \subseteq U_{G_{k+1}}$. Thus $U_G := \bigcup_k U_{G_k}$ is a nested unipotent group containing all unipotent elements of G , and $G = U_G \rtimes T$. \square

5.2. Triangulation in $\text{Aut}(\mathbb{A}^n)$. By definition, an automorphism φ of \mathbb{A}^n belongs to the DE JONQUIÈRES subgroup $\text{Jonq}(n)$ (see Section 1.4) if and only if the comorphism φ^* stabilizes the flag

$$\mathbb{K}[x_1, \dots, x_{n-1}, x_n] \supseteq \mathbb{K}[x_1, \dots, x_{n-1}] \supseteq \cdots \supseteq \mathbb{K}[x_1, x_2] \supseteq \mathbb{K}[x_1].$$

Equivalently, φ stabilizes the coflag

$$\mathcal{F}: \mathbb{A}^n \rightarrow \mathbb{A}^{n-1} \rightarrow \mathbb{A}^{n-2} \rightarrow \cdots \rightarrow \mathbb{A}^2 \rightarrow \mathbb{A}^1$$

which means that φ induce an automorphism on each \mathbb{A}^k such that the projections $\mathbb{A}^k \rightarrow \mathbb{A}^{k-1}$, $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_{n-1})$ are φ -equivariant.

Remark 5.2.1. $\text{Jonq}(n)$ is a connected, nested solvable group of the form $\text{Jonq}(n) = \text{Jonq}(n)_u \rtimes T$ where $T := \mathbb{K}^{*n}$ is the standard maximal torus and $\text{Jonq}(n)_u$ is the subgroup of unipotent elements, cf. [FK18, Section 15.1]. $\text{Jonq}(n)$ has derived length $d(\text{Jonq}(n)) = n + 1$ ([FP18, Lemma 3.2]) and thus $d(\text{Jonq}(n)_u) = n$.

Theorem 5.2.2. *Let $U \subseteq \text{Aut}(\mathbb{A}^n)$ be a nested unipotent subgroup. If U has a dense orbit on \mathbb{A}^n , then U is conjugate to a subgroup of $\text{Jonq}(n)_u$.*

Proof. We construct a U -stable coflag $\mathbb{A}^n \rightarrow \mathbb{A}^{n-1} \rightarrow \cdots \rightarrow \mathbb{A}^1$ where the projections are of the form $\mathbb{A}^k \simeq \mathbb{A}^{k-1} \times \mathbb{A}^1 \xrightarrow{\text{pr}} \mathbb{A}^{k-1}$.

Let $U_x \subseteq U$ be the isotropy group of some point $x \in \mathbb{A}^n$. By Lemma 5.2.3 below we can find an element $a \in \text{Norm}_U U_x \setminus U_x$. Then the subgroup $A := \overline{\langle a \rangle}$ is isomorphic to \mathbb{G}_a and normalizes U_x . The action of A on \mathbb{A}^n is free, and we get an action of U on the quotient $\mathbb{A}^n // A := \text{Spec } \mathcal{O}(\mathbb{A}^n)^A$ such that the quotient map $p: \mathbb{A}^n \rightarrow \mathbb{A}^n // A$ is U -equivariant. It follows that U has a dense orbit in $\mathbb{A}^n // A$ and so $\mathbb{A}^n // A \simeq \mathbb{A}^{n-1}$.

Moreover, the A -action on \mathbb{A}^n admits a local slice, hence a local section of p , see [Fre06, Section 1.4, Principle 1(c)]. Since U acts transitively on \mathbb{A}^n and p is U -equivariant there exist local sections of p in any $z \in \mathbb{A}^n$ and thus $p: \mathbb{A}^n \rightarrow \mathbb{A}^n // A$ is a principal A -bundle. By a theorem of Serre [Ser58, Section 5.1] we have $H^1(X, \mathbb{G}_a) = 0$ for any affine variety X , and hence this bundle is trivial:

$$\begin{array}{ccc} \mathbb{A}^n & \xrightarrow{\simeq} & \mathbb{A}^{n-1} \times \mathbb{A}^1 \\ \downarrow p_i & & \downarrow \text{pr} \\ \mathbb{A}^n // A & \xrightarrow{\simeq} & \mathbb{A}^{n-1} \end{array}$$

Thus, we have constructed a U -equivariant projection $\mathbb{A}^n \rightarrow \mathbb{A}^{n-1}$, and the claim follows by induction. \square

Lemma 5.2.3. *Let $U \subseteq \text{Aut}(X)$ be a nested unipotent subgroup. For any point $x \in X \setminus X^U$ the normalizer $\text{Norm}_U U_x$ strictly contains the isotropy group U_x .*

Proof. The claim of the lemma is well-known for a unipotent algebraic group U . In general, $U = \bigcup_k U_k$ with closed unipotent algebraic subgroups U_k such that $U_k \subseteq U_{k+1}$, and, similarly, $U_x = \bigcup_k (U_k)_x$ with closed unipotent algebraic subgroups $(U_k)_x = U_x \cap U_k$. Now Corollary 4.1.4 implies that we can find a k_0 such that $(U_{k_0})_x$ has the same orbits on X as U_x for all $k \geq k_0$.

Now we use the following general fact. If a group G acts on a space X , and if $x \in X$, then $g \in G$ belongs to the normaliser of the isotropy group G_x if and only if the isotropy group of gx is equal to G_x , i.e. if and only if $gx \in X^{G_x}$.

Now assume that $g \in U_k$ belongs to the normaliser of $(U_k)_x$. Then $gx \in X^{(U_k)_x} = X^{U_x}$, and so g is in the normaliser of U_x . If $g \notin (U_k)_x = U_k \cap U_x$, then $g \notin U_x$, and the claim follows. \square

It is a basic fact in algebraic transformation groups that every affine G -variety X admits a closed G -equivariant embedding $X \hookrightarrow V$ where V is G -module. The corresponding statement for ind-groups does not hold, see [FK18, Proposition 2.6.5].

Question 6. *Let $G \subseteq \text{Aut}(X)$ be a solvable or nilpotent connected subgroup. Does there exist a closed embedding $X \hookrightarrow \mathbb{A}^n$ such that G extends to a subgroup of the DE JONQUÈRES group $\text{Jonq}(n)$? Is this true if G is nested?*

5.3. Nested unipotent subgroups are solvable. We know from Theorem B that a solvable subgroup G of $\text{Aut}(X)$ generated by a family of unipotent algebraic groups is a nested unipotent group. The next result shows that the converse holds as well.

Theorem 5.3.1. *A nested unipotent subgroup $U \subseteq \text{Aut}(X)$ is solvable of derived length $\leq \max\{\dim Ux \mid x \in X\} \leq \dim X$.*

Proof. It suffices to consider the case of a unipotent algebraic subgroup $U \subseteq \text{Aut}(X)$. Then every orbit $O = Ux$ is closed and isomorphic to affine space. Therefore, by Theorem 5.2.2, the image of U in $\text{Aut}(O)$ has derived length $\leq \dim O$. If $m := \max\{\dim Ux \mid x \in X\}$, then the m th member $U^{(m)}$ of the derived series of U acts trivially on every orbit, hence on X , and so $U^{(m)} = \{e\}$. \square

Corollary 5.3.2. *A connected nested subgroup $G \subseteq \text{Aut}(X)$ which is a union of solvable algebraic subgroups is solvable of derived length $\leq \dim X + 1$.*

Proof. Indeed, $G = T \ltimes U$ where T is a torus and U a nested unipotent subgroup (cf. Theorem 5.1.1 or [KPZ17, Theorem 2.11]). \square

Remark 5.3.3. A unipotent algebraic group is nilpotent, but the nilpotency class of unipotent subgroups of $\text{Aut}(X)$ might not be bounded, and thus a nested unipotent subgroup is not necessarily nilpotent.

For example, $\mathcal{J}_n := \text{Jonq}(n)_u$ is a closed nested unipotent subgroup of $\text{Aut}(\mathbb{A}^n)$. For $n \geq 2$ its Lie algebra $\text{Lie } \mathcal{J}_n$ is not nilpotent, and so \mathcal{J}_n is neither, by Lemma 5.1.4(3). Consider, for instance, the case $n = 2$, and define

$$L_d := \langle \partial/\partial x_1, x_1^d \partial/\partial x_2 \rangle \subseteq \text{Lie } \mathcal{J}_2.$$

It is easily seen that the d th member $(L_d)_d$ of the lower central series of L_d does not vanish, and so $(\text{Lie } \mathcal{J}_2)_d \neq 0$ for any $d \geq 1$.

6. AN INTERESTING EXAMPLE

In this section we study the subgroup $F \subseteq \text{Aut}(\mathbb{A}^2)$ generated by $\mathbf{u} := (x + y^2, y)$ and $\mathbf{v} := (x, y + x^2)$. We will show that F is a free group in two generators and we will describe the closure $\mathfrak{F} := \overline{F}$ and its Lie algebra $\text{Lie } \mathfrak{F}$. It turns out that the following holds (see Theorem 6.4.3).

- \mathfrak{F} is a free product of two nested abelian closed unipotent ind-subgroups \mathfrak{J} and $\mathfrak{J}^- \subseteq \text{Aut}(\mathbb{A}^2)$. In particular, \mathfrak{F} is torsion free.
- $\text{Lie } \mathfrak{F}$ is generated, as a Lie algebra, by $\text{Lie } U$ and $\text{Lie } V$ where $U := \langle \mathbf{u} \rangle \simeq \mathbb{G}_a$ and $V := \langle \mathbf{v} \rangle \simeq \mathbb{G}_a$.
- Any algebraic subgroup of \mathfrak{F} is abelian and unipotent and is conjugate to a subgroup of \mathfrak{J} or \mathfrak{J}^- .

6.1. Notation and first results. Let

$$T := \{(ax, by) \mid a, b \in \mathbb{K}^*\} \subseteq \text{GL}_2(\mathbb{K}) \subseteq \text{Aut}(\mathbb{A}^2)$$

denote the standard 2-dimensional torus, and let

$$\text{SAut}(\mathbb{A}^2) := \{\mathbf{g} \in \text{Aut}(X) \mid \text{jac}(\mathbf{g}) = 1\}$$

where the *Jacobian* of $\mathbf{g} = (f, h)$ is defined in the usual way: $\text{jac}(\mathbf{g}) = \det \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix}$. Furthermore, we set

$$\text{Aut}_0(\mathbb{A}^2) := \{\mathbf{g} \in \text{Aut}(\mathbb{A}^2) \mid \mathbf{g}(0) = 0\},$$

$$\mathfrak{A} := \{\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2) \mid d\mathbf{g}_0 = \text{id}\}.$$

Note that $\mathbf{g} \in \text{Aut}(\mathbb{A}^2)$ belongs to \mathfrak{A} if and only if the linear part of $\mathbf{g} = (f, h)$ is equal to (x, y) . \mathfrak{A} is a closed path-connected ind-subgroup of $\text{Aut}(\mathbb{A}^2)$ which contains $\mathbf{u} = (x + y^2, y)$ and $\mathbf{v} = (x, y + x^2)$, hence $\mathfrak{F} \subseteq \mathfrak{A}$.

Let $S := \{(\zeta x, \zeta^2 y) \mid \zeta^3 = 1\} \subseteq \mathrm{SL}_2 \subseteq \mathrm{SAut}(\mathbb{A}^2)$ be the cyclic diagonal subgroup of order 3. The elements of S commute with \mathbf{u} and \mathbf{v} and thus with all elements from F and hence from $\mathfrak{F} = \overline{F}$.

Lemma 6.1.1. *One has $S = \mathrm{Cent}_{\mathrm{Aut}(\mathbb{A}^2)}(F)$.*

Proof. Note that any $\mathbf{g} \in \mathrm{Aut}(\mathbb{A}^2)$ commuting with \mathbf{u} and \mathbf{v} , and therefore with U and V , fixes the origin which is the unique common fixed point of U and V . It also leaves invariant the set of orbits of U and of V , i.e. the pencils of horizontal and vertical lines. Hence, $\mathbf{g} \in T$, and an easy calculation shows that, in fact, $\mathbf{g} \in S$. \square

Considering the action of S on $\mathrm{Aut}(\mathbb{A}^2)$ by conjugation, one sees that $\mathrm{SAut}(\mathbb{A}^2)$ and \mathfrak{A} are stable under this action, hence

$$\mathfrak{F} \subseteq \mathfrak{A}^S := \{\mathbf{g} \in \mathfrak{A} \mid \mathbf{g} \circ s = s \circ \mathbf{g} \text{ for all } s \in S\}.$$

Moreover, S also acts on the Lie algebras $\mathrm{Lie} \mathrm{Aut}(\mathbb{A}^2)$ and $\mathrm{Lie} \mathfrak{A}$, and so

$$(1) \quad \mathrm{Lie} \mathfrak{F} \subseteq \mathrm{Lie} \mathfrak{A}^S \subseteq (\mathrm{Lie} \mathfrak{A})^S.$$

6.2. The Lie algebra of \mathfrak{F} . For the Lie algebras of $\mathrm{Aut}(\mathbb{A}^2)$ and $\mathrm{SAut}(\mathbb{A}^2)$ we have the following description, see [FK18, Lemma 15.7.1]:

- $\mathrm{Lie} \mathrm{Aut}(\mathbb{A}^2) \xrightarrow{\sim} \mathrm{Vec}^c(\mathbb{A}^2) := \{\delta \in \mathrm{Vec}(\mathbb{A}^2) \mid \mathrm{div} \delta \in \mathbb{K}\},$
- $\mathrm{Lie} \mathrm{SAut}(\mathbb{A}^2) \xrightarrow{\sim} \mathrm{Vec}^0(\mathbb{A}^2) := \{\delta \in \mathrm{Vec}(\mathbb{A}^2) \mid \mathrm{div} \delta = 0\},$
- $\mathrm{Lie} \mathfrak{A} \xrightarrow{\sim} \mathrm{Vec}_0^0(\mathbb{A}^2) := \{\delta \in \mathrm{Vec}^0(\mathbb{A}^2) \mid \delta(0) = 0\}$

where the *divergence* of a vector field $\delta = f \frac{\partial}{\partial x} + h \frac{\partial}{\partial y}$ is defined as $\mathrm{div} \delta = \frac{\partial f}{\partial x} + \frac{\partial h}{\partial y}$.

Now consider the following vector fields from $\mathrm{Vec}^0(\mathbb{A}^2)$,

$$\partial_{i,j} := (j+1)x^{i+1}y^j \frac{\partial}{\partial x} - (i+1)x^i y^{j+1} \frac{\partial}{\partial y}$$

where

$$(i, j) \in \Lambda := \{(k, l) \in \mathbb{Z}^2 \mid k, l \geq -1, (k, l) \neq (-1, -1)\}.$$

The torus T acts by conjugation on $\mathrm{Aut}(\mathbb{A}^2)$ and on $\mathrm{Vec}(\mathbb{A}^2)$, and the $\partial_{i,j}$ are eigenvectors of weight (i, j) . Moreover, the action of T on $\mathrm{Vec}^0(\mathbb{A}^2)$ is multiplicity-free, and we have the weight decompositions

$$\mathrm{Vec}^0(\mathbb{A}^2) = \bigoplus_{(i,j) \in \Lambda} \mathbb{K} \partial_{i,j} \quad \text{and} \quad \mathrm{Vec}_0^0(\mathbb{A}^2) = \bigoplus_{(i,j) \in \Lambda, i+j > 0} \mathbb{K} \partial_{i,j}.$$

Note that $\partial_{i,j}$ is fixed by the finite subgroup $S \subseteq T$ if and only if $i - j \equiv 0 \pmod 3$. Setting $\Lambda_0 := \{(k, l) \in \Lambda \mid k - l \equiv 0 \pmod 3, (k, l) \neq (0, 0)\}$ we get

$$(2) \quad \text{Lie } \mathfrak{A}^S \subseteq (\text{Lie } \mathfrak{A})^S = \bigoplus_{(i,j) \in \Lambda_0} \mathbb{K} \partial_{i,j}.$$

The next lemma shows that $\text{Lie } \mathfrak{F} = \text{Lie } \mathfrak{A}^S = (\text{Lie } \mathfrak{A})^S$. In contrast to the situation of algebraic groups, this does not automatically imply that $\mathfrak{F} = \mathfrak{A}^S$. In fact, FURTER-KRAFT give an example of a closed ind-subgroup of a path-connected ind-group which has the same Lie algebra, but is strictly smaller (see [FK18, Theorem 17.3.1]). In order to show that $\mathfrak{F} = \mathfrak{A}^S$ (see Theorem 6.4.3) we therefore need an additional argument.

Lemma 6.2.1. *The Lie algebra $(\text{Lie } \mathfrak{A})^S$ is generated by $\text{Lie } U$ and $\text{Lie } V$, hence $\text{Lie } \mathfrak{F} = \text{Lie } \mathfrak{A}^S = (\text{Lie } \mathfrak{A})^S$.*

Proof. We have $\text{Lie } U = \mathbb{K} \partial_{-1,2}$ and $\text{Lie } V = \mathbb{K} \partial_{2,-1}$. An easy calculation shows that

$$\text{ad } \partial_{-1,2}(\partial_{i,j}) = a \partial_{i-1,j+2} \ (i \geq 0) \text{ and } \text{ad } \partial_{2,-1}(\partial_{i,j}) = b \partial_{i+2,j-1} \ (j \geq 0)$$

with non-zero constants $a, b \in \mathbb{K}$. This implies that the Lie algebra generated by $\partial_{-1,2}$ and $\partial_{2,-1}$ contains all $\partial_{i,j}$ with $(i, j) \in \Lambda_0$, hence $\langle \text{Lie } U, \text{Lie } V \rangle_{\text{Lie}} \supseteq (\text{Lie } \mathfrak{A})^S$, see formula (2). Since $\text{Lie } U, \text{Lie } V \subseteq \text{Lie } \mathfrak{F}$ the claim follows from the inclusions in (1). \square

It is easy to see that the vector field $\partial_{i,j}$ is locally nilpotent if and only if $i = -1$ or $j = -1$. In these cases, there are unique algebraic subgroups $U_{-1,k} \simeq \mathbb{G}_a$ and $U_{k,-1} \simeq \mathbb{G}_a$ with $\text{Lie } U_{-1,k} = \mathbb{K} \partial_{-1,k}$ and $\text{Lie } U_{k,-1} = \mathbb{K} \partial_{k,-1}$. Note that $U_{-1,2} = U = \overline{\langle \mathbf{u} \rangle}$ and $U_{2,-1} = V = \overline{\langle \mathbf{v} \rangle}$. Moreover, the subgroups $U_{-1,k}$ and $U_{-1,l}$ commute as well as the vector fields $\partial_{-1,k}$ and $\partial_{-1,l}$.

6.3. De Jonquières subgroups and amalgamated products. This and the subsequent sections are largely inspired by [AZ13, Section 4]. Consider the DE JONQUIÈRES group

$$\text{Jonq} := \{(ax + h(y), cy + d) \mid a, c \in \mathbb{K}^*, d \in \mathbb{K}, h \in \mathbb{K}[y]\}$$

and its subgroups

$$\begin{aligned} \text{Jonq}_0 &:= \{(ax + yf(y), by) \mid a, b \in \mathbb{K}^*, f \in \mathbb{K}[y]\} = \text{Jonq} \cap \text{Aut}_0(\mathbb{A}^2), \\ \mathfrak{J}_0 &:= \{(x + yf(y), y) \mid f \in \mathbb{K}[y]\} \subseteq \text{Jonq}_0. \end{aligned}$$

\mathfrak{J}_0 is a commutative closed unipotent ind-subgroup, isomorphic to $\mathbb{K}[y]^+$. It admits a filtration by closed unipotent subgroups $\prod_{i=1}^d U_{-1,i} \simeq \mathbb{G}_a^d$, $d \geq 1$. Note that $\text{Jonq}_0 = \mathfrak{J}_0 \rtimes T$.

Lemma 6.3.1. *The group \mathfrak{A}^S is the free product $(\mathfrak{J}_0)^S * (\mathfrak{J}_0^-)^S$ where $\mathfrak{J}_0^- := \tau \circ \mathfrak{J}_0 \circ \tau$ and $\tau := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.*

Proof. (1) The amalgamated product structure $\text{Aut}(\mathbb{A}^2) = \text{Jonq} *_{\mathfrak{B}} \text{Aff}_2$ where $\mathfrak{B} := \text{Jonq} \cap \text{Aff}_2$ implies that every element $\mathbf{g} \in \text{Aut}(\mathbb{A}^2) \setminus \mathfrak{B}$ can be expressed in one of the following 4 forms, called *the type* of \mathbf{g} ,

$$\begin{aligned} \mathbf{g} &= j_1 a_1 j_2 \cdots j_n, & \mathbf{g} &= j_1 a_1 j_2 \cdots j_n a_{n+1}, \\ \mathbf{g} &= a_1 j_1 a_2 \cdots j_n, & \mathbf{g} &= a_1 j_1 a_2 \cdots j_n a_{n+1} \end{aligned}$$

where $j_k \in \text{Jonq} \setminus \mathfrak{B}$ and $a_k \in \text{Aff}_2 \setminus \mathfrak{B}$. The length of the expression, its type and the degrees of the j_k are uniquely determined by \mathbf{g} .

(2) Let now $\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2)$. Using $\text{Jonq} = \text{Jonq}_0 \rtimes \text{Trans}_2$ and $\text{Aff}_2 = \text{GL}_2 \rtimes \text{Trans}_2$ where $\text{Trans}_2 \simeq \mathbb{G}_a^2$ is the group of translations, one can assume that $j_k \in \text{Jonq}_0 \setminus B$ and $a_k \in \text{GL}_2 \setminus B$ where $B \subseteq \text{GL}_2$ is the Borel subgroup of upper triangular matrices. Since $\text{GL}_2 \setminus B = U\tau B = B\tau U$ where $U \subseteq B$ is the unipotent radical, we see that any $\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2)$ has a presentation in one of the 4 forms

$$\begin{aligned} \mathbf{g} &= j_1 \tau j_2 \cdots \tau j_n, & \mathbf{g} &= j_1 \tau j_2 \cdots j_n \tau, \\ \mathbf{g} &= \tau j_1 \tau \cdots \tau j_n, & \mathbf{g} &= \tau j_1 \tau \cdots \tau j_n \tau \end{aligned}$$

where $j_k \in \text{Jonq}_0 \setminus B$. Again, the length, the type and the degrees of the j_k are uniquely determined by \mathbf{g} .

(3) Now we remark that $\text{Jonq}_0 = \mathfrak{J}_0 \rtimes T$ and $T\tau = \tau T$. It follows that we can reach one of the following forms

$$\begin{aligned} \mathbf{g} &= j_1 \tau j_2 \cdots \tau j_n t, & \mathbf{g} &= j_1 \tau j_2 \cdots j_n \tau t, \\ \mathbf{g} &= \tau j_1 \tau \cdots \tau j_n t, & \mathbf{g} &= \tau j_1 \tau \cdots \tau j_n \tau t \end{aligned}$$

where $j_k \in \mathfrak{J}_0$ and $t \in T$. We claim that this form is uniquely determined by \mathbf{g} . Since the type and the length are given by \mathbf{g} , it suffices to compare the similar decompositions

$$\mathbf{g} = j_1 \tau j_2 \cdots j_n t = j'_1 \tau j'_2 \cdots j'_n t' \text{ and } \mathbf{g} = j_1 \tau j_2 \cdots j_n \tau t = j'_1 \tau j'_2 \cdots j'_n \tau t'.$$

It follows that

$$\tau j_2 \cdots j_n t = j_1^{-1} j'_1 \tau j'_2 \cdots j'_n t' \text{ resp. } \tau j_2 \cdots j_n \tau t = j_1^{-1} j'_1 \tau j'_2 \cdots j'_n \tau t',$$

hence $j'_1 = j_1$. Now the claim follows by induction.

(4) Using the equalities $st = ts$ and $s\tau s = \tau$ for $s \in S$ and $t \in T$ we find for an S -invariant $\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2)$ of the form $\mathbf{g} = j_1\tau j_2 \cdots j_nt$ as in (3) above that

$$\mathbf{g} = s\mathbf{g}s^{-1} = \begin{cases} (sj_1s^{-1})\tau(s^{-1}j_2s)\tau(sj_3s^{-1}) \cdots (sj_ns^{-1})t & \text{if } n \text{ is odd,} \\ (sj_1s^{-1})\tau(s^{-1}j_2s)\tau(sj_3s^{-1}) \cdots (s^{-1}j_ns^{-1})t & \text{if } n \text{ is even.} \end{cases}$$

In the first case, we get that each j_k is S -invariant. In the second case, we obtain $sj_ns = j_n$, a contradiction. Looking at the other forms we see that for an S -invariant \mathbf{g} the number of τ 's must be even and each j_k is S -invariant, i.e. $j_k \in (\mathfrak{J}_0)^S \subseteq \mathfrak{A}^S$.

(5) Finally, assume that $\mathbf{g} \in \mathfrak{A}^S$. If \mathbf{g} is given in one of the forms of (3), then $t = \text{id}$, because $d\mathbf{g}_0 = \text{id}$ and $(dj_k)_0 = \text{id}$ for $j_k \in (\mathfrak{J}_0)^S$. Since the number of τ 's is even, we see that such an automorphism \mathbf{g} is a product of the form $\cdots j_k j'_k j_{k+1} j'_{k+1} \cdots$ where $j_k \in (\mathfrak{J}_0)^S$ and $j'_k \in (\mathfrak{J}_0^-)^S = (\tau\mathfrak{J}_0\tau)^S = \tau(\mathfrak{J}_0)^S\tau$. \square

6.4. The fixed points under S . Recall that $\mathfrak{F} \subseteq \text{Aut}(\mathbb{A}^2)$ denotes the closure of the free group F generated by \mathbf{u} and \mathbf{v} .

Proposition 6.4.1. $(\mathfrak{J}_0)^S \subseteq \mathfrak{F}$. In particular, $\mathfrak{F} = \mathfrak{A}^S = (\mathfrak{J}_0)^S * (\mathfrak{J}_0^-)^S$.

The proof needs some preparation. Set

$$\begin{aligned} \varphi &:= \mathbf{u} \circ \mathbf{v} = (x + y^2, y) \circ (x, y + x^2) = \\ &= (x + y^2 + 2x^2y + y^4, y + x^2) \in F = \langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

and consider the powers $\varphi^k = (f_k, h_k)$. It follows that $f_k = x + \text{terms of degree } \geq 2$ and $h_k = y + \text{terms of degree } \geq 2$, and that all coefficients are non-negative integers.

Lemma 6.4.2. *The polynomial f_k contains the term cy^{3k-1} with $c \neq 0$.*

Proof. By definition, we have $f_{k+1} = f_k + h_k^2 + 2f_k^2h_k + h_k^4$. Since all monomials in f_k and h_k have non-negative coefficients, there is no cancellation of terms. Now assume that f_k contains the terms y^2 and y^{3k-1} . Then in the expression of f_{k+1} the summand f_k contains the term y^2 and the summand $f_k^2h_k$ contains $y^2 \cdot y^{3k-1} \cdot y = y^{3k+2}$. Thus the claim follows by induction. \square

Proof of Proposition 6.4.1. We have seen in Section 6.3 that $\mathfrak{J}_0 \simeq \mathbb{K}[y]^+$ admits a filtration by the closed unipotent subgroups $\prod_{i=2}^d U_{-1,i}$, $d \geq 2$. It follows that $\mathfrak{J}_0^S = \{(x + y^2f(y^3), y) \mid f \in \mathbb{K}[y]\} \xrightarrow{\sim} \mathbb{K}[y]^+$, and that \mathfrak{J}_0^S admits a filtration by the closed unipotent subgroups $\prod_{j=1}^k U_{-1,3j-1}$.

Note that this product contains any automorphism of \mathbb{A}^2 of the form $(x + \sum_{j=1}^k c_j y^{3j-1}, y)$. We now show by induction that $U_{-1,3j-1} \subseteq \mathfrak{F}$.

Assume that this holds for $j \leq k$ and consider the product ψ of φ^{k+1} with an element $\mathbf{g} \in \prod_{j=1}^k U_{-1,3j-1} \subseteq \mathfrak{F}$:

$$\begin{aligned} \psi &:= \varphi^{k+1} \circ \mathbf{g} = (f_{k+1}, h_{k+1}) \circ (x + \sum_{j=1}^k c_j y^{3j-1}, y) = \\ &= (f_{k+1}(x + \sum_{j=1}^k c_j y^{3j-1}, y), h_{k+1}(x + \sum_{j=1}^k c_j y^{3j-1}, y)) = (f, h). \end{aligned}$$

Since the lowest degree term of $f_{k+1}(0, y)$ is cy^{3k+2} we can choose the coefficients c_j is such a way that the lowest degree term in $f(0, y)$ is cy^{3k+2} . But then

$$(t^{-3k-2}x, t^{-1}y) \circ (f, h) \circ (t^{3k+2}x, ty) \xrightarrow{t \rightarrow 0} (x + cy^{3k+2}, y) \in U_{-1,3k+2}.$$

Since \mathfrak{F} is closed and stable under the action of the torus T we get $U_{-1,3k+2} \subseteq \mathfrak{F}$, and the claim follows. \square

Summing up we have the following result.

Theorem 6.4.3. *Let $F \subseteq \text{Aut}(\mathbb{A}^2)$ be the subgroup generated by $\mathbf{u} := (x + y^2, y)$ and $\mathbf{v} := (x, y + x^2)$, and denote by $\mathfrak{F} := \overline{F}$ its closure in $\text{Aut}(\mathbb{A}^2)$. Furthermore, define $\mathfrak{A} := \{\mathbf{g} \in \text{Aut}(\mathbb{A}^2) \mid \mathbf{g}(0) = 0, d\mathbf{g}_0 = \text{id}\}$.*

- (1) *F is a free group in the two generators \mathbf{u}, \mathbf{v} , containing non-algebraic elements.*
- (2) *$\mathfrak{F} = \mathfrak{A}^S = \text{Cent}_{\mathfrak{A}}(S)$ where $S := \{(\zeta x, \zeta^2 y) \mid \zeta^3 = 1\} \subseteq T$ is the centralizer of F in $\text{Aut}(\mathbb{A}^2)$.*
- (3) *\mathfrak{F} is a free product $\mathfrak{J} * \mathfrak{J}^-$ where $\mathfrak{J} := \{(x + y^2 f(y^3), y) \mid f \in \mathbb{K}[y]\} = (\mathfrak{J}_0)^S$ and $\mathfrak{J}^- = \{(x, y + x^2 f(x^3)) \mid f \in \mathbb{K}[x]\} = \tau \mathfrak{J} \tau$. In particular, \mathfrak{J} is a commutative nested unipotent closed ind-subgroup of $\text{Aut}(\mathbb{A}^2)$ isomorphic to $\mathbb{K}[y]^+$.*
- (4) *Every algebraic subgroup of \mathfrak{F} is commutative and unipotent and is conjugate to a subgroup of \mathfrak{J} or of \mathfrak{J}^- .*

Proof. (1) The first part is clear. For the second we notice that $\deg(\mathbf{u}\mathbf{v})^k$ tends to infinity for $k \rightarrow \infty$. Hence, $\mathbf{u}\mathbf{v} \in \mathfrak{F}$ is not an algebraic element.

(2) & (3) This is Proposition 6.4.1 and Lemma 6.1.1

(4) This follows from a famous result of SERRE's, see [Ser03, Theorem 8], because an algebraic subgroup of a free product $\mathfrak{J} * \mathfrak{J}^-$ is of bounded length, hence conjugate to a subgroup of one of the factors. \square

Remark 6.4.4. According to Lemma 3.2.1 the subgroup $\mathfrak{F} \subseteq \text{Aut}(X)$ is the closure of the subgroup generated by U and V , and $\text{Lie } \mathfrak{F}$ is the Lie algebra generated by $\text{Lie } U$ and $\text{Lie } V$. Thus our example is a positive instance of our Question 1.

7. HOMOMORPHISMS OF IND-GROUPS AND OF THEIR LIE ALGEBRAS

7.1. A canonical Lie subalgebra of $\text{Lie } \mathfrak{G}$. Let $\mathfrak{G} = \bigcup_d \mathfrak{G}_d$ be an ind-group. For $g \in \mathfrak{G}$ denote by $\lambda_g, \rho_g: \mathfrak{G} \xrightarrow{\sim} \mathfrak{G}$ the left- and the right-multiplications by g , respectively. If $Y \subseteq \mathfrak{G}$ is an algebraic subset and $g \in Y$, then $\rho_{g^{-1}}$ defines an isomorphism $Y \xrightarrow{\sim} Yg^{-1}$ whose differential in g is a linear isomorphism $T_g Y \xrightarrow{\sim} T_e Yg^{-1} \subseteq \text{Lie } \mathfrak{G}$. Define

$$M_Y := \text{span}_{\mathbb{K}} (T_e Yg^{-1} = d\rho_{g^{-1}}(T_g Y) \mid g \in Y) \subseteq \text{Lie } \mathfrak{G}.$$

More generally, if $\varphi: X \rightarrow \mathfrak{G}$ is a morphism where X is an algebraic variety, then we define

$$M_\varphi := \text{span}_{\mathbb{K}} (d\rho_{\varphi(x)^{-1}}(\text{Im } d\varphi_x) \mid x \in X) \subseteq \text{Lie } \mathfrak{G}.$$

Lemma 7.1.1. *Let $Y \subseteq \mathfrak{G}$ be an algebraic subset, and let $\varphi: X \rightarrow \mathfrak{G}$ be a morphism, where X is an algebraic variety. Then the following holds.*

- (a) $\dim M_Y < \infty$ and $\dim M_\varphi < \infty$.
- (b) For $g \in \mathfrak{G}$ we have $M_{Yg} = M_Y$ and $M_{gY} = \text{Ad } g(M_Y)$.
- (c) For $g \in \mathfrak{G}$ we have $M_{\rho_g \circ \varphi} = M_\varphi$ and $M_{\lambda_g \circ \varphi} = \text{Ad } g(M_\varphi)$.
- (d) If $A \subseteq X$ is a closed subset, then $M_{\varphi|_A} \subseteq M_\varphi$.
- (e) If Y is smooth and $U \subseteq Y$ open and dense, then $M_U = M_Y$.
- (f) If $Y = \overline{\varphi(X)}$, then $M_{Y_{\text{reg}}} \subseteq M_\varphi \subseteq M_Y$.

Proof. (a) Consider the closed subset $\tilde{Y} := \overline{\{hg^{-1} \mid h, g \in Y\}}$. Then $\rho_{g^{-1}}(Y) = Yg^{-1} \subseteq \tilde{Y}$ for all $g \in Y$, and so $M_Y \subseteq T_e \tilde{Y}$. For $Y := \overline{\varphi(X)}$ we obviously have $M_\varphi \subseteq M_Y$ and the both claims follow.

(b) For $Z := Yg$, $y \in Y$, $z := yg \in Z$ we have $Zz^{-1} = Yy^{-1}$, hence $M_{Yg} = M_Y$. For $Z := gY$, $y \in Y$, $z := gy$ we get $Zz^{-1} = g(Yy^{-1})g^{-1}$, hence $T_e Zz^{-1} = T_e g(Yy^{-1})g^{-1} = \text{Ad } g(T_e Yy^{-1})$, and so $M_{gY} = \text{Ad } g(M_Y)$.

(c) This is similar to (b).

(d) This is clear.

(e) If V is a finite dimensional vector space and $(U_y)_{y \in Y}$ a family of d -dimensional subspaces parametrized by a variety Y , i.e. given by a morphism of Y to the Grassmannian, then $\langle U_y \mid y \in Y \rangle = \langle U_y \mid y \in Y' \rangle$ for any dense subset $Y' \subseteq Y$.

(f) There is an open dense set $X' \subseteq X_{reg}$ where φ is smooth. This implies that $\text{Im } d\varphi_x = T_{\varphi(x)}Y$ for all $x \in X'$. Since the image $\varphi(X')$ is open and dense in Y_{reg} we obtain from (d) and (e) that $M_{Y_{reg}} = M_{\varphi(X')} \subseteq M_\varphi$. The second inclusion is obvious. \square

Definition 7.1.2. For an ind-group \mathfrak{G} we define

$$L_\mathfrak{G} := \text{span}_\mathbb{K} (M_{Y_{reg}} \mid Y \subseteq \mathfrak{G} \text{ an irreducible closed algebraic subset})$$

where $Y_{reg} \subseteq Y$ is the smooth locus. Given a homomorphism $\Phi: \mathfrak{H} \rightarrow \mathfrak{G}$ of ind-groups, we define $L_\Phi := d\Phi(L_\mathfrak{H}) \subseteq L_\mathfrak{G}$.

The next proposition shows that $L_\mathfrak{G} \subseteq \text{Lie } \mathfrak{G}$ is a canonical Lie sub-algebra.

Proposition 7.1.3.

- (a) $L_\mathfrak{G} \subseteq \text{Lie } \mathfrak{G}$ is stable under the adjoint action of \mathfrak{G} , i.e. $L_\mathfrak{G}$ is an ideal in $\text{Lie } \mathfrak{G}$.
- (b) For every algebraic subgroup $G \subseteq \mathfrak{G}$ we have $\text{Lie } G \subseteq L_\mathfrak{G}$.
- (c) If $\mathfrak{H} \subseteq \mathfrak{G}$ is a closed subgroup then $L_\mathfrak{H} \subseteq L_\mathfrak{G}$.

Proof. (a) For any $h \in \mathfrak{G}$ we have $\text{Ad } h(M_Y) = M_{hYh^{-1}} \subseteq L_\mathfrak{G}$. In fact, for $g \in Y$ one has

$$\text{Ad } h(T_e Y g^{-1}) = T_e(hYg^{-1}h^{-1}) = T_e(hYh^{-1})g_0^{-1} \subseteq M_{hYh^{-1}}$$

where $g_0 = hgh^{-1} \in hYh^{-1}$, and the claim follows.

(b) For an algebraic group $G \subseteq \mathfrak{G}$ we have $d\rho_{g^{-1}}(T_g G) = \text{Lie } G$ (cf. Remark 2.2.3), and so $M_G = \text{Lie } G$.

(c) This is clear. \square

Recall that

$$\text{SAut}(\mathbb{A}^n) := \{\alpha \in \text{Aut}(\mathbb{A}^n) \mid \text{jac}(\alpha) = 1\}$$

is a normal subgroup of $\text{Aut}(\mathbb{A}^n)$ with Lie algebra $\text{Lie SAut}(\mathbb{A}^n)$ being the algebra $\text{Vec}^0(\mathbb{A}^n)$ of polynomial vector fields on \mathbb{A}^n with zero divergence (see Section 6.2).

Corollary 7.1.4. We have $L_{\text{SAut}(\mathbb{A}^n)} = \text{Lie SAut}(\mathbb{A}^n)$ and $L_{\text{Aut}(\mathbb{A}^n)} = \text{Lie Aut}(\mathbb{A}^n)$.

Proof. By [Sha81, Lemma 3], the Lie algebra

$$\text{Lie SAut}(\mathbb{A}^n) = [\text{Lie Aut}(\mathbb{A}^n), \text{Lie Aut}(\mathbb{A}^n)]$$

is simple. It follows from Proposition 7.1.3(a) that $\text{Lie SAut}(\mathbb{A}^n) \subseteq L_{\text{Aut}(\mathbb{A}^n)}$. Now the claim is a consequence of Proposition 7.1.3(b) due to the fact that

$$\text{Lie Aut}(\mathbb{A}^n) = \text{Lie SAut}(\mathbb{A}^n) \oplus \text{Lie } \mathbb{K}^*,$$

where $\mathbb{K}^* \subseteq \text{Aut}(\mathbb{A}^n)$ is realized as the subgroup of the scalar multiplications $\lambda \cdot \text{id}$, $\lambda \in \mathbb{K}^*$, cf. [FK18, Prop. 15.7.2]. \square

7.2. Surjective homomorphisms. Let $\Phi: \mathfrak{H} \rightarrow \mathfrak{G}$ be a homomorphism of ind-groups. If Φ is surjective one expects that the induced homomorphism $d\Phi: \text{Lie } \mathfrak{H} \rightarrow \text{Lie } \mathfrak{G}$ is also surjective.

Proposition 7.2.1. *Let $\Phi: \mathfrak{H} \rightarrow \mathfrak{G}$ be a surjective homomorphism of ind-groups.*

- (a) *L_Φ is stable under the adjoint action of \mathfrak{G} , i.e. $L_\Phi \subseteq \text{Lie } \mathfrak{G}$ is an ideal.*
- (b) *If $H \subseteq \mathfrak{H}$ is an algebraic subgroup, then $L_\Phi \supseteq \text{Lie } \Phi(H)$.*

Proof. (a) For $h \in \mathfrak{H}$ and $g := \Phi(h) \in \mathfrak{G}$ we get from Lemma 7.1.1(c) that $M_{\Phi|_{hX}} = \text{Ad } g(M_{\Phi|_X})$ for any closed algebraic subset $X \subseteq \mathfrak{H}$. Now the assertion follows.

(b) This is clear, since $\Phi(H) \subseteq \mathfrak{G}$ is an algebraic subgroup, and therefore $d\Phi|_H: \text{Lie } H \rightarrow \text{Lie } \Phi(H)$ is surjective. \square

Question 7. *Assume we have a surjective homomorphism $\mathfrak{G} \rightarrow G$ where \mathfrak{G} is a path-connected ind-group and G is an algebraic group. Can one show that \mathfrak{G} contains an algebraic subgroup which is sent surjectively onto G ?*

7.3. Subgroups of finite codimension. Let $\mathfrak{G} = \bigcup_d \mathfrak{G}_d$ be an ind-group.

Definition 7.3.1. We say that a closed subgroup $\mathfrak{H} \subseteq \mathfrak{G}$ is of *finite codimension* if $\mathfrak{G}_d \cdot \mathfrak{H} = \mathfrak{G}$ for some $d \in \mathbb{N}$.

For instance, if \mathfrak{G} is connected and acts on an algebraic variety X , then the stabilizer \mathfrak{G}_x of any point $x \in X$ is of finite codimension. In fact, the orbit $\mathfrak{G}x$ is an irreducible subvariety of X by [FK18, Prop. 7.1.2(1)], and so, $\mathfrak{G}x = \mathfrak{G}_d x$ for some d . The latter implies $\mathfrak{G} = \mathfrak{G}_d \cdot \mathfrak{G}_x$. We do not know if every subgroup \mathfrak{H} of finite codimension of a connected ind-group \mathfrak{G} is the stabilizer of a point of a \mathfrak{G} -variety, i.e. if $\mathfrak{G}/\mathfrak{H}$ has a canonical structure of an algebraic variety.

Proposition 7.3.2. *Let \mathfrak{G} be an ind-group and $\mathfrak{H} \subseteq \mathfrak{G}$ a closed subgroup. Assume that there is a closed irreducible algebraic subset $Z \subseteq \mathfrak{G}$ such that $Z \cdot \mathfrak{H} = \mathfrak{G}$. Then $L_\mathfrak{G} = M_{Z_{\text{reg}}} + L_\mathfrak{H}$. In particular, $L_\mathfrak{H}$ has finite codimension in $L_\mathfrak{G}$.*

Proof. Let $Y \subseteq \mathfrak{G}$ be a closed irreducible algebraic subset. Define $\tilde{Y} := \overline{Z \cdot Y}$. Then $\text{Ad } z(M_{Y_{\text{reg}}}) \subseteq M_{\tilde{Y}_{\text{reg}}}$ for all z in an open dense subset of Z_{reg} . In fact, for all z in an open dense subset of Z we have $zY_{\text{reg}} \cap \tilde{Y}_{\text{reg}} \neq \emptyset$, hence $M_{zY_{\text{reg}}} \subseteq M_{\tilde{Y}_{\text{reg}}}$.

Thus, in Definition 7.1.2 of $L_{\mathfrak{G}}$ we may assume that the irreducible algebraic subsets are of the form $\overline{Z \cdot Y}$.

For any irreducible closed algebraic subset $Y \subseteq \mathfrak{G}$ of the form above we can find an irreducible closed algebraic subset $X \subseteq \mathfrak{H}$ such the $\overline{Z \cdot X} = Y$. Consider the map $\eta: Z \times X \rightarrow Y$, $(z, h) \mapsto z \cdot h$. Then η is smooth on an open dense subset $U \subseteq Z_{\text{reg}} \times X_{\text{reg}}$. In particular, $\eta(U)$ is open and dense in Y_{reg} . Now for every $g = z \cdot h \in \varphi(U)$ where $(z, h) \in U$ we have the following commutative diagram:

$$\begin{array}{ccc} Z \times X & \xrightarrow{\eta} & Y \\ \rho_{z^{-1}} \times \text{Int } z \circ \rho_{h^{-1}} \downarrow \simeq & & \simeq \downarrow \rho_{g^{-1}} \\ Zz^{-1} \times z(Xh^{-1})z^{-1} & \xrightarrow{\eta} & Yg^{-1} \end{array}$$

On the tangent spaces this induces the commutative diagram

$$\begin{array}{ccc} T_z Z \oplus T_h X & \xrightarrow{d\eta} & T_g Y \\ d\rho_{z^{-1}} \times \text{Ad } z \circ d\rho_{h^{-1}} \downarrow \simeq & & \simeq \downarrow d\rho_{g^{-1}} \\ T_e(Zz^{-1}) \oplus T_e(z(Xh^{-1})z^{-1}) & \xrightarrow{d\eta} & T_e(Yg^{-1}) \end{array}$$

By assumption, $d\eta$ is surjective, hence

$$T_e(Yg^{-1}) = T_e(Zz^{-1}) + \text{Ad } z(T_e(Xh^{-1})) \subseteq M_{Z_{\text{reg}}} + M_{zXz^{-1}} \subseteq M_{Z_{\text{reg}}} + L_{\mathfrak{H}}.$$

This holds for every $g \in \varphi(U)$ which is a dense subset of Y_{reg} . Thus, by Lemma 7.1.1(e), we finally get $M_{Y_{\text{reg}}} \subseteq M_{Z_{\text{reg}}} + L_{\mathfrak{H}}$, hence the claim. \square

Corollary 7.3.3. *Let X be an affine variety, and let $\mathfrak{H} \subseteq \text{Aut}(X)$ be a closed ind-subgroup. If \mathfrak{H} has finite codimension, then $\text{Lie } \mathfrak{H} \subseteq \text{Lie Aut}(X)$ has finite codimension, too.*

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DEPARTEMENT MATHEMATIK UND INFORMATIK,
 UNIVERSITÄT BASEL, SPIEGELGASSE 1, CH-4051 BASEL, SWITZERLAND
Email address: `hanspeter.kraft@unibas.ch`

INSTITUT FOURIER, UMR 5582, LABORATOIRE DE MATHÉMATIQUES,
 UNIVERSITÉ GRENOBLE ALPES, CS 40700, 38058 GRENOBLE CEDEX 9, FRANCE
Email address: `mikhail.zaidenberg@univ-grenoble-alpes.fr`