



Algebraically generated groups and their Lie algebras

Hanspeter Kraft, M Zaidenberg

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ALGEBRAICALLY GENERATED GROUPS AND THEIR LIE ALGEBRAS

HANSPETER KRAFT AND MIKHAIL ZAIDENBERG

ABSTRACT. The automorphism group $\text{Aut}(X)$ of an affine variety X is an ind-group. Its Lie algebra is canonically embedded into the Lie algebra $\text{Vec}(X)$ of vector fields on X . We study the relations between closed subgroups \mathfrak{G} of $\text{Aut}(X)$ and the corresponding Lie subalgebras of $\text{Vec}(X)$.

We show that a subgroup $G \subseteq \text{Aut}(X)$ generated by a family of connected algebraic subgroups G_i of $\text{Aut}(X)$ is algebraic if and only if the Lie $G_i \subseteq \text{Vec}(X)$ generate a finite dimensional Lie subalgebra of $\text{Vec}(X)$.

Extending a result by COHEN-DRAISMA [CD03] we prove that a locally finite Lie algebra $L \subseteq \text{Vec}(X)$ generated by locally nilpotent vector fields is algebraic, i.e. $L = \text{Lie } G$ of some algebraic subgroup $G \subseteq \text{Aut}(X)$.

Along the same lines we prove that if a subgroup $G \subseteq \text{Aut}(X)$ generated by finitely many unipotent algebraic groups is solvable, then it is a unipotent algebraic group.

Furthermore, we give an example of a free subgroup $F \subseteq \text{Aut}(\mathbb{A}^2)$ generated by two algebraic elements such that the Zariski closure \bar{F} is a free product of two nested commutative closed unipotent ind-subgroups.

To any ind-group \mathfrak{G} one can associate a canonical ideal $L_{\mathfrak{G}} \subseteq \text{Lie } \mathfrak{G}$ generated by the Lie algebras of algebraic subgroups of \mathfrak{G} , and another one associated to the set of all algebraic subvarieties of \mathfrak{G} . We study functorial properties of these ideals, ind-subgroups of finite codimension in \mathfrak{G} and the corresponding Lie subalgebras of $\text{Lie } \mathfrak{G}$.

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1. INTRODUCTION

1.1. Notation. Our base field \mathbb{K} is algebraically closed and of characteristic zero. \mathbb{A}^n stands for affine n -space, and $\mathbb{G}_a := \mathbb{K}^+$ and $\mathbb{G}_m := \mathbb{K}^*$ denote the additive and the multiplicative group of \mathbb{K} . If G is an algebraic group, then a G -variety X is an affine variety with an action of G such that the corresponding map $G \times X \rightarrow X$ is a morphism.

It is well known that for every G -variety X there is a canonical (anti-) homomorphism of Lie algebras $\xi: \text{Lie } G \rightarrow \text{Vec}(X)$, where $\text{Vec}(X)$

stands for the Lie algebra of (algebraic) vector fields on X . The construction of this homomorphism goes as follows. For any $x \in X$ let $\mu_x: G \rightarrow X$ be the orbit map $g \mapsto gx$, and denote by

$$d\mu_x: \operatorname{Lie} G := T_e G \rightarrow T_x X$$

its differential in $e \in G$. If $A \in \operatorname{Lie} G$, then the corresponding vector field ξ_A is given by $\xi_A(x) := d\mu_x(A) \in T_x X$. If $L \subseteq \operatorname{Vec}(X)$ denotes the image of $\operatorname{Lie} G$, then one has $T_x Gx = L(x)$ for any $x \in X$. Moreover, the differential $d\mu_x: T_g G \rightarrow T_{gx} Gx$ of the orbit map μ_x is surjective in every $g \in G$. (For a reference one might look at [Kra16, Appendix A.4].)

For a G -variety X we have linear actions of G on the coordinate ring $\mathcal{O}(X)$ and on the vector fields $\operatorname{Vec}(X)$, and these representations are *locally finite and regular*, i.e. every element is contained in a finite dimensional G -invariant subspace and the linear action of G on any finite dimensional G -invariant subspace is regular. Moreover, the homomorphism $\xi: \operatorname{Lie} G \rightarrow \operatorname{Vec}(X)$ is G -equivariant, see [FK18, Sect. 7.3].

1.2. Ind-groups and vector fields. For an affine variety X the group of regular automorphisms $\operatorname{Aut}(X)$ is an *affine ind-group*. We refer to the paper [FK18] for an introduction to ind-varieties and ind-groups and for basic concepts, cf. [Sha81, Kum02]. For an ind-group \mathfrak{G} the tangent space $T_e \mathfrak{G}$ carries a natural structure of a Lie algebra which will be denoted by $\operatorname{Lie} \mathfrak{G}$. In case of $\operatorname{Aut}(X)$ there is a canonical embedding $\xi: \operatorname{Lie} \operatorname{Aut}(X) \hookrightarrow \operatorname{Vec}(X)$ which is an anti-homomorphism of Lie algebras. It is constructed in a similar way as explained above for an algebraic group G , see [FK18, Proposition 7.2.4].

The ind-groups and ind-varieties carry a natural (Zariski-) topology, called the *ind-topology*. All topological notation in this paper will be with respect to the ind-topology.

A subset $Y \subseteq \mathfrak{V}$ of an ind-variety $\mathfrak{V} = \bigcup_k \mathfrak{V}_k$ is called *algebraic* if it is locally closed and contained in \mathfrak{V}_k for some k . If X is a G -variety, then the canonical map $G \rightarrow \operatorname{Aut}(X)$ is a homomorphism of ind-groups, and the image of G is closed and algebraic, see [FK18, Proposition 2.7.1]. In particular, every algebraic subgroup $G \subseteq \operatorname{Aut}(X)$ is closed. It follows that $\operatorname{Lie} G \subseteq \operatorname{Lie} \operatorname{Aut}(X)$ is a Lie subalgebra and that $\operatorname{Lie} G$ is canonically embedded into the vector fields $\operatorname{Vec}(X)$.

Convention By abuse of notation, we will constantly identify, for an algebraic group $G \subseteq \operatorname{Aut}(X)$, the Lie algebra $\operatorname{Lie} G$ with its image $\xi(\operatorname{Lie} G) \subseteq \operatorname{Vec}(X)$.

The main objective of these notes is to study the relations between the automorphism group $\operatorname{Aut}(X)$ of an affine variety and its closed

subgroups on the one hand, and the corresponding Lie subalgebras of $\text{Vec}(X)$ on the other. Besides the introduction the paper contains three different sections that we survey below. They are independent and touch three different subjects. Section 2 is devoted to algebraically generated subgroups of $\text{Aut}(X)$ and their Lie algebras. Section 3 studies an example of an ind-subgroup of $\text{Aut}(\mathbb{A}^2)$ which is the closure of a nonabelian free subgroup of $\text{Aut}(\mathbb{A}^2)$, and the final Section 4 is about ind-subgroups of $\text{Aut}(X)$, their Lie algebras and homomorphisms.

1.3. Algebraically generated groups. A first result showing a very strong relation between the Lie algebra of an ind-group and the group itself is the following, see [FK18, Proposition 2.2.1(3) and (4)].

Proposition 1.3.1. *Let \mathfrak{G} be an ind-group. Then the connected component \mathfrak{G}° is an algebraic group if and only if $\text{Lie } \mathfrak{G}$ is finite dimensional.*

We will prove a similar statement for so-called *algebraically generated subgroups*. Let $(G_i)_{i \in I}$ be a family of connected algebraic subgroups of $\text{Aut}(X)$ where X is an affine variety. The subgroup $G \subseteq \text{Aut}(X)$ generated by the G_i is called *algebraically generated* ([AFK13]).

Let $L \subseteq \text{Vec}(X)$ be the Lie subalgebra generated by the Lie algebras $\text{Lie } G_i$. We will see in Theorem 2.3.1 that L is invariant under the action of G and of the closure \overline{G} . This implies that $L \subseteq \text{Lie } \overline{G}$ is an ideal.

Question 1. *Do we have $L = \text{Lie } \overline{G}$?*

Our result in this direction is the following.

Theorem A. *Assume that L is finite dimensional. Then G is an algebraic subgroup of $\text{Aut}(X)$ and $\text{Lie } G = L$.*

We will prove a slightly more general result, see Theorems 2.3.1 and 2.5.1. Our example in Section 3 gives an infinite dimensional L where Question 1 above has a positive answer.

Theorem A above has several interesting application. An important one is the following.

Theorem B. *Let $G \subseteq \text{Aut}(X)$ be a subgroup generated by finitely many unipotent algebraic subgroups. If G is solvable, then G is a unipotent algebraic group.*

The proof of Theorem B will be given in Section 2.7.

We say that $\text{Aut}(X)$ verifies the TITS *alternative* if any subgroup of $\text{Aut}(X)$ is either virtually solvable, or it contains a nonabelian free subgroup. Due to TITS' original theorem, this alternative holds for $\text{GL}(n, \mathbb{K})$, hence for any linear algebraic group. We refer to [AZ21] and

the references therein for a brief survey on the TITS alternative for automorphism groups.

An immediate consequence of Theorem B in this context is the following.

Corollary 1.3.2. *Let $G \subseteq \operatorname{Aut}(X)$ be a subgroup generated by a finite family of unipotent algebraic groups. If the TITS alternative holds for $\operatorname{Aut}(X)$, then either G is a unipotent algebraic group or G contains a nonabelian free subgroup.*

Remark 1.3.3. If G is an algebraic group and $H_1, \dots, H_m \subseteq G$ a finite set of connected closed subgroups, then the subgroup H generated by the H_i is closed. (In fact, there is a sequence i_1, i_2, \dots, i_N such that the product $P := H_{i_1} H_{i_2} \cdots H_{i_N}$ is dense (and constructible) in \overline{H} . It then follows that $PP = \overline{H}$, hence $H = \overline{H}$.)

As a consequence we see that a subgroup of a nested ind-group \mathfrak{G} (see Definition 2.8.1) generated by finitely many connected algebraic groups is an algebraic group.

Note that this statement does not hold for non-connected groups. E.g. the two matrices $A := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}$ have both order 4, but the product AB has infinite order. Thus the subgroup of $\operatorname{SL}_2(\mathbb{K})$ generated by the finite groups $\langle A \rangle$ and $\langle B \rangle$ is not closed.

1.4. Locally finite subsets. Let V be a vector space over \mathbb{K} . We denote by $\mathcal{L}(V)$ the algebra of linear endomorphisms of V . An endomorphism $\lambda \in \mathcal{L}(V)$ is called *locally finite* if the linear span $\langle \lambda^j(v) \mid j \in \mathbb{N} \rangle$ is finite dimensional for any $v \in V$. It is called *semisimple* if there is a basis of eigenvectors, and *locally nilpotent* if for any $v \in V$ there is an $m \in \mathbb{N}$ such that $\lambda^m(v) = 0$. Every locally finite endomorphism λ has a uniquely defined *additive JORDAN-decomposition* $\lambda = \lambda_s + \lambda_n$ where λ_s is semisimple, λ_n locally nilpotent, and $\lambda_s \circ \lambda_n = \lambda_n \circ \lambda_s$.

A subset $S \subseteq \mathcal{L}(V)$ is called *locally finite* if every $v \in V$ is contained in a finite dimensional subspace $W \subseteq V$ which is invariant under all elements from S . If $S \subseteq \operatorname{GL}(V)$, then S is locally finite if and only if the group $\langle S \rangle$ generated by S is locally finite.

Note that a locally finite subspace $A \subseteq \mathcal{L}(V)$ is not necessarily finite dimensional. In fact, let (e_1, e_2, \dots) be a basis of $V := \mathbb{K}^\infty$ and define $\lambda_k \in \mathcal{L}(V)$ by $\lambda_k(e_j) := \delta_{kj} e_j$. Then $A := \bigoplus_k \mathbb{K} \lambda_k$ is an infinite dimensional locally finite subspace.

Definition 1.4.1. Let X be an affine variety. A morphism $\varphi: X \rightarrow X$ is called *algebraic* or *locally finite* if the induced endomorphism $\varphi^* \in$

$\mathcal{L}(\mathcal{O}(X))$ is locally finite. It is called *semisimple* if φ^* is semisimple, and *locally nilpotent* if φ^* is locally nilpotent.

A subgroup $H \subseteq \text{Aut}(X)$ is called *locally finite* if the image of H in $\mathcal{L}(\mathcal{O}(X))$ is locally finite.

Note that a subgroup $H \subseteq \text{Aut}(X)$ is locally finite if and only if the closure $\overline{H} \subseteq \text{Aut}(X)$ is an algebraic group. If $g \in \text{Aut}(X)$ is an algebraic element, then the closed commutative subgroup $\overline{\langle g \rangle}$ has the form $\mathbb{G}_m^p \times \mathbb{G}_a^q \times F$ where $p \in \mathbb{Z}_{\geq 0}$, $q \in \{0, 1\}$ and F is a finite cyclic group.

There are many examples of subgroups $G \subseteq \text{Aut}(X)$ that are not locally finite while being generated by algebraic elements. We will discuss such an example in Section 3. We also refer to the interesting discussions in [FK18, 9.4.3–9.4.5] and [PR22] of subgroups $G \subseteq \text{Aut}(X)$ which consist of algebraic elements.

1.5. Locally finite vector fields. Recall that the vector fields $\text{Vec}(X)$ are the derivations $\text{Der}(\mathcal{O}(X))$ of $\mathcal{O}(X)$. Since $\text{Der}(\mathcal{O}(X)) \subseteq \mathcal{L}(\mathcal{O}(X))$ we can talk about *locally finite vector fields* and *locally finite subspaces* of $\text{Vec}(X)$.

Example 1.5.1. If $G \subseteq \text{Aut}(X)$ is an algebraic subgroup, then its Lie algebra $\text{Lie}(G) \subseteq \text{Vec}(X)$ is locally finite. These Lie algebras are called *algebraic*.

In contrast to the general situation we have the following finiteness result.

Lemma 1.5.2. *Let $L \subseteq \text{Vec}(X)$ be a locally finite subspace. Then L is finite dimensional.*

Proof. If $f_1, \dots, f_m \in \mathcal{O}(X)$ is a set of generators, then every vector field δ is determined by the values $\delta(f_1), \dots, \delta(f_m)$. It follows that for any subspace $L \subseteq \text{Vec}(X)$ the linear map $L \rightarrow \mathcal{O}(X)^m$, $\delta \mapsto (\delta(f_1), \dots, \delta(f_m))$ is injective. If L is locally finite, then the image of this map is contained in the finite dimensional subspace $\bigoplus_{i=1}^m Lf_i$, and the claim follows. \square

The fact that a vector field is determined by its values on a finite generating set of $\mathcal{O}(X)$ has the following important consequences.

Lemma 1.5.3. *Let $V \subseteq \mathcal{O}(X)$ be a finite dimensional subspace generating $\mathcal{O}(X)$, and let $\delta \in \text{Vec}(X)$ be a vector field. If V is δ -invariant, then the following holds.*

- (1) δ is locally finite.

- (2) δ is semisimple if and only if $\delta|_V \in \mathfrak{gl}(V)$ is semisimple.
- (3) δ is locally nilpotent if and only if $\delta|_V \in \mathfrak{gl}(V)$ is nilpotent.
- (4) If $\delta = \delta_s + \delta_n$ is the JORDAN decomposition in $\mathcal{L}(\mathcal{O}(X))$, then $\delta_s, \delta_n \in \text{Vec}(X)$.
- (5) V is δ_s - and δ_n -invariant, and $\delta|_V = \delta_s|_V + \delta_n|_V \in \mathfrak{gl}(V)$ is the usual JORDAN decomposition in $\mathfrak{gl}(V)$.

Proof. Denote by $V^{(n)} \subseteq \mathcal{O}(X)$ the linear subspace generated by the n -fold products of elements from V . Then the following is obvious: (a) $V^{(n)}$ is δ -invariant; (b) if $\delta|_V$ is semisimple resp. nilpotent, then so is $\delta|_{V^{(n)}}$. This proves (1)–(3). The claim (4) follows from [FK18, Proposition 7.6.1], and (5) follows from (2) and (3). \square

Question 2. Let $L_i \subseteq \text{Vec}(X)$ ($i \in I$) be a family of locally finite Lie subalgebras, and denote by $L \subseteq \text{Vec}(X)$ the Lie subalgebra generated by the L_i . Is L locally finite in case L is finite dimensional?

In this direction we have the following consequence of Theorem A.

Corollary. Let $L \subseteq \text{Vec}(X)$ be the Lie subalgebra generated by a family of locally finite Lie subalgebras $L_i \subseteq \text{Vec}(X)$, $i \in I$. Assume that each L_i is algebraic, i.e. $L_i = \text{Lie } G_i$ for some algebraic group $G_i \subseteq \text{Aut}(X)$. Then L is locally finite if and only if L is finite dimensional. In this case the subgroup G generated by the G_i is algebraic and $L = \text{Lie } G$. In particular, L is algebraic.

Proof. If L is locally finite, then it is finite dimensional, by Lemma 1.5.2. If L is finite dimensional, then the claim follows from Theorem A. \square

Remark 1.5.4. For a locally finite vector field $\delta \in \text{Vec}(X)$ there is a uniquely determined minimal algebraic group $H \subseteq \text{Aut}(X)$ such that $\delta \in \text{Lie } H$. Moreover, H is commutative and connected, and H is a torus if δ is semisimple and $H \simeq \mathbb{G}_a$ if δ is locally nilpotent and non-zero, see [FK18, Proposition 7.6.1].

1.6. The adjoint action on $\text{Vec}(X)$. For any vector field δ we have the *adjoint action* $\text{ad } \delta$ on $\text{Vec}(X)$ defined in the usual way:

$$\text{ad } \delta(\eta) := [\delta, \eta] = \delta \circ \eta - \eta \circ \delta.$$

Lemma 1.6.1. Let $\delta \in \text{Vec}(X)$ be a locally finite vector field with JORDAN decomposition $\delta = \delta_s + \delta_n$. Then $\text{ad } \delta$ acting on $\text{Vec}(X)$ is locally finite, and $\text{ad } \delta = \text{ad } \delta_s + \text{ad } \delta_n$ is the JORDAN decomposition.

Proof. Let $V \subseteq \mathcal{O}(X)$ be a finite dimensional δ -invariant subspace which generates $\mathcal{O}(X)$. Then we have an inclusion

$$\text{Vec}(X) \hookrightarrow \mathcal{L}(V, \mathcal{O}(X)), \quad \mu \mapsto \mu|_V.$$

Given $\mu \in \text{Vec}(X)$ we choose a finite dimensional δ -invariant subspace W which contains $\mu(V)$. In $\mathcal{L}(\mathcal{O}(X))$ we have

$$(\text{ad } \delta)^m \mu = \sum_{i=0}^m n_i \delta^{m-i} \mu \delta^i$$

with suitable $n_i \in \mathbb{Z}$. Hence $(\text{ad } \delta)^m \mu$ sends V to W for all $m \geq 0$, and so the linear span $\langle (\text{ad } \delta)^m \mu \mid m \geq 0 \rangle \subseteq \text{Vec}(X)$ is finite dimensional. This shows that $\text{ad } \delta$ is locally finite.

If δ is locally nilpotent, then $(\delta|_V)^m = 0$ and $(\delta|_W)^m = 0$ for a suitable $m > 0$. The above formula implies that $(\text{ad } \delta)^{2m-1} \mu = 0$, hence $\text{ad } \delta$ is locally nilpotent.

Next, assume that δ is semisimple. Then we have an eigenspace decomposition $V = \bigoplus_a V_a$ and $\mathcal{O}(X) = \bigoplus_b \mathcal{O}(X)_b$. As a consequence, we get

$$\mathcal{L}(V, \mathcal{O}(X)) = \bigoplus_{a,b} \mathcal{L}(V_a, \mathcal{O}(X)_b)$$

which is an eigenspace decomposition for $\text{ad } \delta$, and so $\text{ad } \delta$ is semisimple.

Finally, $\text{ad } \delta_s$ and $\text{ad } \delta_n$ commute, because $[\delta_s, \delta_n] = 0$, and thus $\text{ad } \delta = \text{ad } \delta_s + \text{ad } \delta_n$ is the JORDAN decomposition. \square

1.7. Toral subalgebras.

Definition 1.7.1. A finite dimensional Lie subalgebra $S \subseteq \text{Vec}(X)$ is called *toral* if it consists of semisimple elements. It follows that S is commutative [Hum72, 8.1, Lemma] and thus locally finite. In particular, we obtain a *weight decomposition* $\mathcal{O}(X) = \bigoplus_{\alpha \in S^*} \mathcal{O}(X)_\alpha$ where

$$\mathcal{O}(X)_\alpha := \{f \in \mathcal{O}(X) \mid \tau f = \alpha(\tau) \cdot f \text{ for } \tau \in S\}.$$

If $T \subseteq \text{Aut}(X)$ is a torus, then $\text{Lie } T \subseteq \text{Vec}(X)$ is a toral subalgebra, but it is not true that every toral subalgebra is of this form. In fact, if $T \subseteq \text{Aut}(X)$ is a two dimensional torus, then the one dimensional tori $S \subseteq T$ are the kernels of certain characters $\chi \in X(T) \simeq \mathbb{Z}^2$ whereas every one dimensional subspace of $\text{Lie } T$ is a toral subalgebra. The best we can get is the following result.

Lemma 1.7.2. *For a toral Lie subalgebra $S \subseteq \text{Vec}(X)$ there exists a uniquely defined smallest torus $T \subseteq \text{Aut}(X)$ such that $\text{Lie } T \supseteq S$. Moreover, if a subspace $M \subseteq \text{Vec}(X)$ is invariant under the adjoint action of S , then M is invariant under T .*

Proof. (a) One easily verifies that the weight decomposition of $\mathcal{O}(X)$ has the following properties, cf. [FK18, Proposition 7.6.1 and its proof]:

- $\mathcal{O}(X)_\alpha \cdot \mathcal{O}(X)_\beta \subseteq \mathcal{O}(X)_{\alpha+\beta}$;

- the set of weights $\Lambda := \{\gamma \mid \mathcal{O}(X)_\gamma \neq 0\} \subseteq S^*$ generates a free subgroup $\mathbb{Z}\Lambda = \bigoplus_{i=1}^k \mathbb{Z}\mu_i$.

This defines a faithful action of the torus $T := (\mathbb{G}_m)^k$ on $\mathcal{O}(X)$ in the following way: For $\alpha = \sum_i n_i \mu_i \in \Lambda$, $f \in \mathcal{O}(X)_\alpha$ and $t = (t_1, \dots, t_k) \in T$ we put

$$t f := t_1^{n_1} \cdots t_k^{n_k} \cdot f.$$

The corresponding action of $c = (c_1, \dots, c_k) \in \text{Lie } T = \mathbb{K}^m$ is then given by the vector field δ_c where

$$\delta_c|_{\mathcal{O}(X)_\alpha} = \text{scalar multiplication with } \sum_i n_i c_i \text{ for } \alpha = \sum_i n_i \mu_i \in \Lambda.$$

It follows that the image of $\text{Lie } T$ contains S and that every torus with this property contains T , see [FK18, proof of Proposition 7.6.1(2)].

(b) With respect to the adjoint action we also have a weight space decomposition $\text{Vec}(X) = \bigoplus_{\gamma \in S^*} \text{Vec}(X)_\gamma$, see Lemma 1.6.1. It has the property that for $\delta \in \text{Vec}(X)_\gamma$ we get $\delta(\mathcal{O}(X)_\alpha) \subseteq \mathcal{O}(X)_{\alpha+\gamma}$. It follows that if $\text{Vec}(X)_\gamma \neq 0$, then $\gamma \in \Lambda$. Hence, the subspaces $\text{Vec}(X)_\gamma$ are weight spaces of the action of T on $\text{Vec}(X)$, and thus every S -invariant subspace of $\text{Vec}(X)$ is invariant under T . \square

1.8. Integration of Lie algebras. The first part of the following result is due to COHEN and DRAISMA, see [CD03, Theorem 1].

Theorem C. *Let $L \subseteq \text{Vec}(X)$ be a Lie subalgebra. Then $L \subseteq \text{Lie } G$ for some algebraic subgroup $G \subseteq \text{Aut}(X)$ if and only if L is locally finite. If L is locally finite and generated by locally nilpotent elements, then L is algebraic, i.e. there is an algebraic group $G \subseteq \text{Aut}(X)$ such that $L = \text{Lie } G$.*

The proof will be given in Section 2.6, see Theorem 2.6.1. A weaker form of Question 2 is the following.

Question 3. *Let $\xi, \eta \in \text{Vec}(X)$ be locally finite vector fields. If the Lie subalgebra $L := \langle \xi, \eta \rangle_{\text{Lie}}$ generated by ξ and η is finite dimensional, does it follow that L is locally finite?*

From Theorem C we get the following result in this direction.

Corollary. *Let $\{\eta_i \mid i \in I\}$ be a family of locally nilpotent vector fields, and denote by $L := \langle \eta_i \mid i \in I \rangle_{\text{Lie}}$ the Lie subalgebra generated by the η_i . If L is finite dimensional, then L is locally finite and algebraic.*

1.9. A free subgroup of $\text{Aut}(\mathbb{A}^2)$ and its closure. In Section 3 we provide an example of a nonabelian free subgroup $F \simeq \mathbb{F}_2$ of $\text{Aut}(\mathbb{A}^2)$ generated by two algebraic elements whose product is not algebraic. In particular, F is not locally finite. We compute its closure $\mathfrak{F} := \overline{F} \subseteq \text{Aut}(\mathbb{A}^2)$ and describe the Lie algebra $\text{Lie } \mathfrak{F}$. It occurs that \mathfrak{F} coincides with the double centralizer of F (Lemma 3.1.1) and is a free product of two abelian nested unipotent ind-subgroups \mathfrak{J} and \mathfrak{J}^- (Theorem 3.4.3(3)). Furthermore, any algebraic subgroup G of \mathfrak{F} is abelian and unipotent and is conjugate to a subgroup of \mathfrak{J} or \mathfrak{J}^- (Theorem 3.4.3(4)).

The construction shows that \mathfrak{F} contains two algebraic subgroups U and V , both isomorphic to \mathbb{G}_a , with the following properties:

$$\mathfrak{F} = \overline{\langle U, V \rangle} \quad \text{and} \quad \text{Lie } \mathfrak{F} = \langle \text{Lie } U, \text{Lie } V \rangle_{\text{Lie}},$$

see Lemma 3.2.1. This is an instance where Question 1 has a positive answer.

Yet another instance is the following. Consider the ind-group $\mathfrak{G} := \text{Aut}_{\mathcal{K}}(\mathcal{K}[x, y])$ where $\mathcal{K} := \mathbb{K}[z]$, and let $\mathfrak{G}^t \subseteq \mathfrak{G}$ be the subgroup consisting of tame automorphisms of $\mathbb{K}[x, y, z]$. Then \mathfrak{G} is connected and closed in $\text{Aut}_{\mathbb{K}}(\mathbb{K}[x, y, z])$, $\mathfrak{G}^t \subsetneq \mathfrak{G}$ is closed and $\text{Lie } \mathfrak{G}^t = \text{Lie } \mathfrak{G}$, see [FK18, Theorem 17.3.1]. The group \mathfrak{G}^t is generated by the linear and DE JONQUIÈRES transformations which fix z . Hence \mathfrak{G}^t is algebraically generated. By the UMIRBAEV-SHESTAKOV Theorem, the one dimensional unipotent subgroup U of \mathfrak{G} corresponding to the locally nilpotent NAGATA-derivation is not contained in \mathfrak{G}^t . However, since the algebraically generated group $\mathfrak{H} := \langle \mathfrak{G}^t, U \rangle$ is contained in \mathfrak{G} and $\text{Lie } U \subseteq \text{Lie } \mathfrak{G} = \text{Lie } \mathfrak{G}^t$ we have

$$\text{Lie } \overline{\mathfrak{H}} = L_{\mathfrak{H}} := \langle \text{Lie } \mathfrak{G}^t, \text{Lie } U \rangle.$$

1.10. Ind-groups and their Lie algebras. The theory of algebraic group actions on affine varieties can be generalized, to some extent, to ind-groups acting on affine ind-varieties, see [FK18]. If \mathfrak{G} is an ind-group, then a \mathfrak{G} -variety X is a variety with a \mathfrak{G} -action such that the map $\mathfrak{G} \times X \rightarrow X$ is a morphism of ind-varieties. For every ind-group \mathfrak{G} the tangent space $T_e \mathfrak{G}$ at the unit element $e \in \mathfrak{G}$ is well-defined and carries a natural structure of a Lie algebra, denoted by $\text{Lie } \mathfrak{G}$. For every \mathfrak{G} -variety X there is a canonical homomorphism of Lie algebras $\xi: \text{Lie } \mathfrak{G} \rightarrow \text{Vec}(X)$ constructed as in the algebraic case. Again, \mathfrak{G} acts linearly on both, $\text{Lie } \mathfrak{G}$ and $\text{Vec}(X)$, and the homomorphism $\xi: \text{Lie } \mathfrak{G} \rightarrow \text{Vec}(X)$ is \mathfrak{G} -equivariant.

A subset $M \subseteq \mathfrak{X}$ of an ind-variety \mathfrak{X} is called *path-connected* if for any two points $x, y \in M$ there is an irreducible variety Y and a morphism $\varphi: Y \rightarrow \mathfrak{X}$ such that $x, y \in \varphi(Y) \subseteq \mathfrak{X}$, cf. [Ram64, p. 26]. One can always assume that Y is an irreducible curve, because any two points in an irreducible variety are contained in an irreducible curve, cf. [FK18, Section 1.6].

In contrast to the finite-dimensional case, a strict closed subgroup of a path-connected ind-group may have the same Lie algebra, see [FK18, Theorem 17.3.1]; cf. the example in Section 1.9. This disproves [Sha81, Theorems 1 and 2]. However, a homomorphism of a path-connected ind-group to an ind-group is uniquely determined by the induced homomorphism of their Lie algebras, see [FK18, Proposition 7.4.7]. In Section 4 we study surjective homomorphisms of ind-groups and of their Lie algebras, and subgroups of *finite codimension* of ind-groups.

2. ALGEBRAICALLY GENERATED SUBGROUPS

2.1. Orbits of algebraically generated subgroups. Let $\mathfrak{G} = \bigcup_k \mathfrak{G}_k$ be an ind-group. The following concept was introduced in [AFK13].

Definition 2.1.1. A subgroup $G \subseteq \mathfrak{G}$ is called *algebraically generated* if there is a family $(G_i)_{i \in I}$ of connected algebraic subgroups $G_i \subseteq \mathfrak{G}$ generating G as an abstract group: $G = \langle G_i \mid i \in I \rangle$.

It is clear that an algebraically generated subgroup is path-connected, hence connected. The following result can be found in [AFK13, Prop. 1.2 and 1.3]. For the reader's convenience we provide a short argument.

Proposition 2.1.2. *Let $G = \langle G_i \mid i \in I \rangle \subseteq \text{Aut}(X)$ be an algebraically generated subgroup. For every $x \in X$ the G -orbit $Gx \subseteq X$ is open in its closure \overline{Gx} . Moreover, there is a finite sequence $j_1, \dots, j_n \in I$ such that $Gx = G_{j_1} \cdots G_{j_n}x$ for all $x \in X$.*

Proof. (a) Choose a sequence i_1, \dots, i_m such that $\dim \overline{G_{i_1} \cdots G_{i_m}x}$ is maximal. Since $\overline{G_{i_1} \cdots G_{i_m}x}$ is irreducible, it follows that $G_{j_1} \cdots G_{j_\ell}x \subseteq \overline{G_{i_1} \cdots G_{i_m}x}$ for every sequence $j_1, \dots, j_\ell \in I$. Hence $\overline{G_{i_1} \cdots G_{i_m}x} = \overline{Gx}$, and there is an open dense set $U \subseteq \overline{Gx}$ contained in $G_{i_1} \cdots G_{i_m}x$. This implies that $Gx = GU$ is open in \overline{Gx} .

(b) For the second claim we first remark that for every $x \in X$ with $\dim Gx$ maximal there is an open set $U_x \subseteq X$ containing x and a sequence $\mathbf{i}_x = (i_1, \dots, i_m)$, depending on x , such that $\overline{G_{i_1} \cdots G_{i_m}y} = \overline{Gy}$ for all $y \in U_x$. In fact, if $d := \dim Gx$ is maximal and $Gx \subseteq \overline{G_{i_1} \cdots G_{i_m}x}$, then $\dim \overline{G_{i_1} \cdots G_{i_m}y} = d$ for all y in an open neighborhood of x . Now the union $U := \bigcup_x U_x$ is a G -invariant open set, hence

covered by finitely many U_{x_j} . By joining the finitely many sequences i_{x_j} corresponding to the points x_j , we obtain a sequence (j_1, \dots, j_ℓ) such that $\overline{G_{j_1} \cdots G_{j_\ell} y} = \overline{Gy}$ for all $y \in U$. A standard argument implies that $G_{j_\ell} G_{j_{\ell-1}} \cdots G_{j_2} G_{j_1} G_{j_2} \cdots G_{j_\ell} y = Gy$ for all $y \in U$. Since $X \setminus U$ is a closed G -invariant subvariety of smaller dimension, the claim follows by induction. \square

Remark 2.1.3. A similar result holds for the action of a connected ind-group \mathfrak{G} on an affine variety X , see [FK18, Proposition 7.1.2].

2.2. Vector fields associated to subvarieties of $\text{Aut}(X)$. The construction of the embedding $\xi: \text{Lie } G \hookrightarrow \text{Vec}(X)$ for an algebraic subgroup $G \subseteq \text{Aut}(X)$ (see 1.1) has the following generalization, cf. [FK18, Section 3.2]. Consider the orbit map $\mu_x: \text{Aut}(X) \rightarrow X$, $g \mapsto gx$, which is an ind-morphism, and its differential in $g \in \text{Aut}(X)$:

$$(d\mu_x)_g: T_g \text{Aut}(X) \rightarrow T_{gx} X.$$

This defines, for any $v \in T_g \text{Aut}(X)$, a vector field $\xi_v^{(g)} \in \text{Vec}(X)$ by

$$(\xi_v^{(g)})(x) := (d\mu_{g^{-1}x})_g v \in T_x X,$$

and thus a linear map $\xi^{(g)}: T_g \text{Aut}(X) \rightarrow \text{Vec}(X)$.

Lemma 2.2.1 ([FK18, Proposition 3.2.4]). *The map $\xi^{(g)}: T_g \text{Aut}(X) \rightarrow \text{Vec}(X)$ is injective for every $g \in \text{Aut}(X)$.*

Let Z be an affine variety and $\rho: Z \rightarrow \text{Aut}(X)$ a morphism. Composing the differential $d\rho: T_z Z \rightarrow T_{\rho(z)} \text{Aut}(X)$ with $\xi^{(\rho(z))}$ we obtain a linear map $\xi^{(\rho)} = \xi^{(\rho, z)}: T_z Z \rightarrow \text{Vec}(X)$. In the following statement we use the obvious action of $\text{Aut}(X)$ on the vector fields $\text{Vec}(X)$.

Lemma 2.2.2.

- (1) *For $g', g'' \in \text{Aut}(X)$ set $\tilde{\rho}: Z \rightarrow \text{Aut}(X)$, $g \mapsto g'gg''$. Then $\xi^{(\tilde{\rho})} = g' \cdot \xi^{(\rho)}$.*
- (2) *For a closed subvariety $Z \subseteq \text{Aut}(X)$ the map $\xi^{(z)}: T_z Z \rightarrow \text{Vec}(X)$ is injective for any $z \in Z$.*

Proof. (1) Replace the orbit map μ_x by $\tilde{\mu}_x: g \mapsto g'gg''x$, i.e. $\tilde{\mu}_x = g'\mu_{g''x}$. Then $d\tilde{\mu}_x = g'd\mu_{g''x}$. By definition, $\xi^{(g'x)} = \xi^{(x)}$ and so $\xi^{(\tilde{\rho})} = g' \cdot \xi^{(\rho)}$.

(2) This follows from Lemma 2.2.1. \square

Remark 2.2.3. For an algebraic subgroup $G \subseteq \text{Aut}(X)$ we obtain linear maps $\xi^{(g)}: T_g G \rightarrow \text{Vec}(X)$ for any $g \in G$. All these maps have the same image, namely $L := \xi^{(e)}(\text{Lie } G) \subseteq \text{Vec}(X)$.

The following result is crucial and will be used at several places along the paper.

Proposition 2.2.4. *Let $G_1, \dots, G_m \subseteq \text{Aut}(X)$ be connected algebraic subgroups, and consider the multiplication map*

$$\rho: G_1 \times \cdots \times G_m \rightarrow \text{Aut}(X).$$

If $L_i \subseteq \text{Vec}(X)$ denotes the image of $\text{Lie } G_i$, then the image of

$$\xi^{(\rho, \mathbf{g})} = \xi^{(\rho, (g_1, \dots, g_m))}: T_{g_1} G_1 \oplus \cdots \oplus T_{g_m} G_m \rightarrow \text{Vec}(X)$$

is given by $\sum_i h_i \cdot L_i$ where $h_1 = e$ and $h_i := g_1 \cdots g_{i-1}$ for $i > 1$.

Proof. The linear map $\xi^{(\rho, \mathbf{g})}$ restricted to the summand $T_{g_i} G_i$ is equal to $\xi^{(\rho_i, g_i)}$ where $\rho_i: G_i \rightarrow \text{Aut}(X)$ is the map $g \mapsto g_1 \cdots g_{i-1} g g_{i+1} \cdots g_m = h_i g g_{i+1} \cdots g_m$. Now Lemma 2.2.2(1) shows that $\xi^{(\rho, \mathbf{g})}(T_{g_i}(G_i)) = h_i \cdot L_i$, and the claim follows. \square

Corollary 2.2.5. *With the notation above consider the “orbit map” in $x \in X$,*

$$\rho_x: G_1 \times \cdots \times G_m \rightarrow X, \quad \mathbf{g} = (g_1, \dots, g_m) \mapsto \mathbf{g}x := g_1 \cdots g_m x.$$

Then the image of the differential $(d\rho_x)_{\mathbf{g}}: T_{g_1} G_1 \oplus \cdots \oplus T_{g_m} G_m \rightarrow T_{\mathbf{g}x} X$ is given by $\sum_i h_i \cdot L_i(g_i \cdots g_m x)$.

Proof. By definition, $(d\rho_x)_{\mathbf{g}} = \text{ev}_{\mathbf{g}x} \circ \xi^{(\rho, \mathbf{g})}$ where $\text{ev}_y: \text{Vec}(X) \rightarrow T_y X$ is the evaluation map $\xi \mapsto \xi(y)$. Thus, by Proposition 2.2.4, the image of $(d\rho_x)_{\mathbf{g}}$ is given by

$$\text{ev}_{\mathbf{g}x} \left(\sum_i h_i L_i \right) = \sum_i (h_i L_i)(\mathbf{g}x) = \sum_i h_i L_i(g_i \cdots g_m x).$$

\square

2.3. Lie algebras of algebraically generated groups. As in 2.1, consider an algebraically generated subgroup $G = \langle G_i \mid i \in I \rangle \subseteq \text{Aut}(X)$. Set $L_i := \text{Lie } G_i \subseteq \text{Vec}(X)$, and let $L := \langle L_i \mid i \in I \rangle_{\text{Lie}} \subseteq \text{Vec}(X)$ be the Lie subalgebra generated by the L_i . If $Y \subseteq X$ is a G -invariant closed subset, then $L_i|_Y \subseteq \text{Vec}(Y)$ for all i and so $L|_Y \subseteq \text{Vec}(Y)$. Choosing $Y := \overline{Gx}$ we get $L(x) \subseteq T_x Gx$ for any $x \in X$.

Theorem 2.3.1. *Let $G = \langle G_i \mid i \in I \rangle \subseteq \text{Aut}(X)$ be an algebraically generated subgroup, and define $L := \langle \text{Lie } G_i \mid i \in I \rangle_{\text{Lie}} \subseteq \text{Vec}(X)$ to be the Lie algebra generated by the $\text{Lie } G_i$. Then*

- (1) *L is stable under the action of G and of $\overline{G} \subseteq \text{Aut}(X)$. In particular, L is an ideal in $\text{Lie } \overline{G}$.*

- (2) We have $T_x Gx = L(x)$ for all $x \in X$. In particular, any vector field in L is tangent to the G -orbits.
- (3) If $Z \subseteq \text{Aut}(X)$ is a closed algebraic subset contained in G , then the image of $T_z Z$ in $\text{Vec}(X)$ is contained in L for any $z \in Z$.
- (4) If $H \subseteq \text{Aut}(X)$ is an algebraic subgroup contained in G , then $\text{Lie } H \subseteq L$. In particular, L depends only on G and not on the generating subgroups G_i .

Proof of Theorem 2.3.1. (1) It suffices to show that L is invariant under all the G_i . Since G_i is connected, we are reduced to prove that L is invariant under the action of $\text{Lie } G_i$, i.e. under the adjoint action of $L_i := \text{Lie } G_i$ (see [FK18, Proposition 6.3.4]) which holds by construction.

(2) We already know that $L(x) \subseteq T_x Gx$. Now we choose a surjective “orbit map”

$$\mu_x: G_{j_1} \times \cdots \times G_{j_n} \rightarrow Gx \subseteq X,$$

see Proposition 2.1.2. It follows that for a suitable $\mathbf{g} = (g_1, \dots, g_n)$ the differential

$$(d\mu_x)_{\mathbf{g}}: T_{g_1} G_{j_1} \oplus \cdots \oplus T_{g_n} G_{j_n} \rightarrow T_{\mathbf{g}x} Gx$$

is also surjective. Thus we get $T_{\mathbf{g}x} Gx = \sum_j h_j L_j(g_j \cdots g_n x)$, by Corollary 2.2.5. Since L is invariant under the action of G , we finally get

$$h_j L_j(g_j \cdots g_n x) \subseteq h_j L(g_j \cdots g_n x) = L(\mathbf{g}x),$$

and the claim follows.

(3) Let $Z \subseteq \text{Aut}(X)$ be a closed algebraic subset contained in G . We first claim that $Z \subseteq G_{i_1} \cdots G_{i_m}$ for a suitable sequence (i_1, \dots, i_m) . In fact, there is a sequence such that the intersection $Z \cap G_{i_1} \cdots G_{i_m}$ is dense in Z , hence contains a dense open set $U \subseteq Z$. Since $\dim(Z \setminus U) < \dim Z$ the claim follows by induction on $\dim Z$.

Applying Proposition 2.2.4 to the multiplication map $\rho: G_{i_1} \times \cdots \times G_{i_m} \rightarrow \text{Aut}(X)$ and using $\xi^{(\rho, \mathbf{g})}$ where $\mathbf{g} = (g_1, \dots, g_m)$ and $g_1 \cdots g_m = z$ it follows that the image of $T_z Z$ in $\text{Vec}(X)$ is contained in $\sum_i h_i L_i \subseteq L$, since L is stable under the action of G , by (1).

(4) This is an immediate consequence of (3). □

2.4. Ramanujam’s alternative. The following result, due to RAMANUJAM, will be used in the proof of our Theorem 2.5.1 below.

Proposition 2.4.1 ([Ram64, Theorem on p. 26]). *Let $G \subseteq \text{Aut}(X)$ be a path-connected subgroup. Then one of the following holds:*

- (i) G is an algebraic subgroup;

(ii) G contains algebraic subsets of arbitrary large dimension.

Proof. Assume that we are not in case (ii), and let $Y \subseteq G$ be an irreducible algebraic subset of maximal dimension d . We may assume that $e \in Y$. We claim that $G \subseteq \overline{Y} = \overline{G}$. In fact, for any $g \in G$ we can find an irreducible algebraic subset $S \subseteq G$ containing e and g . Then $\overline{S \cdot Y}$ is irreducible and meets G in a dense open set, hence $\overline{S \cdot Y} = \overline{Y}$. Since $g \in S \cdot Y$, the claim follows.

As a consequence, $\overline{G} \subseteq \text{Aut}(X)$ is an algebraic subgroup. Since $Y \subseteq G$ is open and dense in \overline{G} it follows that $\overline{G} = Y^{-1} \cdot Y \subseteq G$, and so (i) holds. \square

2.5. The finite dimensional case. Assume that $G = \langle G_i \mid i \in I \rangle$ is finite dimensional, that is, an algebraic subgroup of $\text{Aut}(X)$, see Proposition 2.4.1. In this case, L is finite dimensional and equal to $\text{Lie } G$, by Theorem 2.3.1(4). The next result shows that the converse is true as well. It also proves Theorem A from Section 1.3 of the introduction.

Theorem 2.5.1. *Let $G = \langle G_i \mid i \in I \rangle \subseteq \text{Aut}(X)$ be an algebraically generated subgroup. Assume that the Lie algebra $L \subseteq \text{Vec}(X)$ generated by the $\text{Lie } G_i$ is finite dimensional. Then G is an algebraic subgroup with $\text{Lie } G = L$. In particular, L is locally finite and algebraic.*

Proof. Let $Y \subseteq G$ be an irreducible algebraic subset. As in the proof of Theorem 2.3.1(3) there is a sequence (i_1, \dots, i_m) such that $G_{i_1} \cdots G_{i_m}$ contains Y . Thus the image of $T_g Y$ in $\text{Vec}(X)$ is contained in L . Since the map $\xi^{(g)}: T_g \text{Aut}(X) \rightarrow \text{Vec}(X)$ is injective (Lemma 2.2.1) we get $\dim Y \leq \dim L$. Now the claim follows from RAMANUJAM' alternative (Proposition 2.4.1). \square

2.6. Integration of locally finite Lie algebras. The next result is our Theorem C from the introduction, see Section 1.8.

Theorem 2.6.1. *Let $L \subseteq \text{Vec}(X)$ be a Lie subalgebra.*

- (1) *If L is finite dimensional and generated by locally nilpotent vector fields, then L is algebraic, i.e. there is a connected algebraic subgroup $G \subseteq \text{Aut}(X)$ such that $L = \text{Lie } G$.*
- (2) *If L is locally finite, then there exists a connected algebraic subgroup $G \subseteq \text{Aut}(X)$ such that $L \subseteq \text{Lie } G$.*

Proof of (1). Assume that L is finite dimensional and generated by locally nilpotent vector fields $\eta_1, \eta_2, \dots, \eta_m$. By [FK18, Proposition 7.6.1] there are (uniquely determined) algebraic subgroups $U_i \subseteq \text{Aut}(X)$, $U_i \simeq \mathbb{G}_a$, such that $\text{Lie } U_i = \mathbb{K}\eta_i$. Now Theorem 2.5.1 implies that

$G := \langle U_i \mid i = 1, \dots, m \rangle \subseteq \text{Aut}(X)$ is an algebraic group with $\text{Lie } G = L$. \square

Example 2.6.2. Assume that $L \subseteq \text{Vec}(X)$ is a locally finite semisimple Lie subalgebra. Then there is a semisimple group $G \subseteq \text{Aut}(X)$ such that $L = \xi(\text{Lie } G)$. In particular, L is algebraic.

In fact, the semisimple Lie algebra L is generated by the root subspaces with respect to some CARTAN subalgebra (see [Hum72, Proposition 8.4(f)]), hence by nilpotent elements. Lemma 1.5.3(3) shows that these elements are locally nilpotent in $\text{Vec}(X)$, and thus L is generated by locally nilpotent vector fields.

The proof of (2) needs some preparation. Let $L = L_0 \ltimes R$ be the LEVI decomposition where L_0 is semisimple and $R := \text{rad } L$ is the solvable radical (see [Pro07, 10.1.6, pp. 305–306]). The example above shows that $L_0 = \text{Lie } G_0$ for a semisimple subgroup $G_0 \subseteq \text{Aut}(X)$. It remains to find a solvable subgroup $Q \subseteq \text{Aut}(X)$ normalized by G_0 such that $\text{Lie } Q \supseteq R$. Then the product $G := G_0 Q$ has the required property.

Lemma 2.6.3. *Let $M \subseteq \text{Vec}(X)$ be a locally finite solvable Lie subalgebra. Assume that M contains with every element μ its semisimple and locally nilpotent parts μ_s and μ_n . Then the locally nilpotent elements from M form a Lie subalgebra M_n , and $M = S \ltimes M_n$ where S is a maximal toral subalgebra of M .*

Proof. Since M is locally finite we may assume that $M \subseteq \mathfrak{gl}(V)$ where $V \subseteq \mathcal{O}(X)$ is a finite dimensional subspace generating $\mathcal{O}(X)$. Since M is solvable there is a basis of V such that M is contained in the upper triangular matrices $\mathfrak{b}_d \subseteq \mathfrak{gl}_d$ where $d := \dim V$ (LIE's Theorem). Clearly, the nilpotent elements from M belong to the subalgebra $\mathfrak{n}_d \subseteq \mathfrak{b}_d$ of upper triangular matrices with zeros along the diagonal, and so $M_n := M \cap \mathfrak{n}_d$ is the ideal of M consisting of the nilpotent elements. By Lemma 1.5.3 every nilpotent element of M is also locally nilpotent viewed as a vector field on X , and vice versa. If $M = M_n$ then $S = 0$ and $M = S \ltimes M_n$, as desired.

Assume now that M contains a semisimple element, and choose a maximal toral subalgebra $S \subseteq M$. Now consider the corresponding weight space decomposition $M = \bigoplus_{\alpha \in S^*} M_\alpha$ where

$$M_\alpha := \{\mu \in M \mid \text{ad } \tau(\mu) = [\tau, \mu] = \alpha(\tau) \cdot \mu \text{ for } \tau \in S\}.$$

Similarly, we get a decomposition $\mathfrak{gl}_d = \bigoplus_{\alpha \in S^*} (\mathfrak{gl}_d)_\alpha$. Using the identity $[s, a \cdot b] = [s, a] \cdot b + a \cdot [s, b]$ we get $(\mathfrak{gl}_d)_\alpha \cdot (\mathfrak{gl}_d)_\beta \subseteq (\mathfrak{gl}_d)_{\alpha+\beta}$ which implies that M_α consists of nilpotent elements for $\alpha \neq 0$. Hence, $\bigoplus_{\alpha \neq 0} M_\alpha \subseteq \mathfrak{n}_d$

is a solvable subalgebra consisting of nilpotent elements, while $M_0 = \text{Cent}_M(S) \supseteq S$.

Let $\mu \in M_0$ have JORDAN decomposition $\mu = \mu_s + \mu_n$. Since S commutes with μ it also commutes with μ_s and μ_n , and so $\mu_s \in M_0$ and $\mu_n \in M_0 \cap \mathfrak{n}_d$. Since S is a maximal toral subalgebra, we have $\mu_s \in S$, and so $M_0 = S \oplus (M_0 \cap \mathfrak{n}_d)$. As a consequence we see that $M_n = (M_0 \cap \mathfrak{n}_d) \oplus \bigoplus_{\alpha \neq 0} M_\alpha$ is a subalgebra consisting of nilpotent elements, and that $M = S \ltimes M_n$. \square

The assumption in the lemma above leads to the following definition. A locally finite subspace $W \subseteq \text{Vec}(X)$ is called *JORDAN-saturated* (shortly *J-saturated*) if it contains with every element η the semisimple and the locally nilpotent parts η_s and η_n . Clearly, for any algebraic subgroup $G \subseteq \text{Aut}(X)$ the Lie algebra $\text{Lie } G \subseteq \text{Vec}(X)$ is *J-saturated*.

Lemma 2.6.4. *Let $R \subseteq \text{Vec}(X)$ be a locally finite and solvable Lie subalgebra. Then the smallest J-saturated Lie subalgebra $\tilde{R} \subseteq \text{Vec}(X)$ containing R has the following properties:*

- (1) \tilde{R} is locally finite and solvable.
- (2) If a subspace $V \subseteq \mathcal{O}(X)$ is R -invariant, then it is invariant under \tilde{R} .
- (3) If an automorphism $\varphi \in \text{Aut}(X)$ normalizes R , then it normalizes \tilde{R} .

Notice that any algebraic Lie subalgebra of $\text{Vec}(X)$ which contains R also contains \tilde{R} . Hence \tilde{R} is the smallest algebraic Lie subalgebra which contains R .

Proof. As before, we choose a finite dimensional R -invariant subspace $V \subseteq \mathcal{O}(X)$ which generates $\mathcal{O}(X)$. Choosing a suitable basis of V we may assume that the image of R in $\mathfrak{gl}(V) = \mathfrak{gl}_d$ is contained in the upper triangular matrices \mathfrak{b}_d .

If $\eta = \eta_s + \eta_n$ is the JORDAN-decomposition of an element $\eta \in R$, then every finite dimensional R -invariant subspace $V \subseteq \mathcal{O}(X)$ is invariant under η_s and η_n , hence invariant under the Lie subalgebra R_1 generated by $R + \mathbb{K}\eta_n$. Thus R_1 is locally finite. The JORDAN-decomposition of η carries over to the image of η in $\mathfrak{gl}(V) = \mathfrak{gl}_d$. Hence the images of η_s and η_n also belong to \mathfrak{b}_d which shows that the image of R_1 is contained in \mathfrak{b}_d .

This procedure of adding the locally nilpotent part and forming the Lie algebra has to end with a locally finite and solvable Lie subalgebra \tilde{R} containing R with the properties (1) and (2).

Finally, assume that $\varphi \in \text{Aut}(X)$ stabilizes R , and let $\eta \in R$. Then $\varphi(\eta) = \varphi(\eta_s) + \varphi(\eta_n)$ is the JORDAN decomposition of $\varphi(\eta) \in R$, hence

$\varphi(\eta_s)$ and $\varphi(\eta_n)$ belong to \tilde{R} . Now it follows from the construction that \tilde{R} is invariant under φ . \square

Proof of Theorem 2.6.1(2). Let $L = L_0 \ltimes R$ be the LEVI decomposition where L_0 is semisimple and R is the solvable radical of L . By part (1) and Example 2.6.2 there is a semisimple group $G_0 \subseteq \text{Aut}(X)$ with $\text{Lie } G_0 = L_0$. Clearly, R is G_0 -invariant, as well as \tilde{R} , the J -saturation of R , see Lemma 2.6.4. By Lemma 2.6.3 we have $\tilde{R} = S \ltimes \tilde{R}_n$ where S is toral and \tilde{R}_n is solvable and consists of locally nilpotent elements. Thus, again by part (1), there is a unipotent group $U \subseteq \text{Aut}(X)$ such that $\text{Lie } U = \tilde{R}_n$.

It follows from Lemma 1.7.2 that there is a well-defined minimal torus $T \subseteq \text{Aut}(X)$ such that $\text{Lie } T \supseteq S$, and that T normalizes R_n . By Lemmas 1.7.2 and 2.6.4, T normalizes U and \tilde{R}_n , and so the product $Q := TU = UT \subseteq \text{Aut}(X)$ is a connected solvable subgroup with $\text{Lie } Q \supseteq \tilde{R}$. By construction, Q is the smallest subgroup with this property. It remains to show that G_0 normalizes Q . In fact, gQg^{-1} is a closed subgroup for any $g \in G_0$, and $\text{Lie } gQg^{-1} \supseteq g\tilde{R}g^{-1} = \tilde{R}$. \square

2.7. Solvable subgroups. This section is devoted to the proof of our Theorem B.

Theorem 2.7.1. *Let $G \subseteq \text{Aut}(X)$ be a subgroup generated by a finite collection of unipotent algebraic subgroups. If G is solvable, then G is a unipotent algebraic group.*

The proof needs some preparation.

Lemma 2.7.2. *Let $G \subseteq \mathfrak{G}$ be a subgroup of an ind-group \mathfrak{G} .*

- (1) *We have $(\overline{G}, \overline{G}) \subseteq \overline{(G, G)}$.*
- (2) *If G is solvable, then so is \overline{G} .*

Proof. (1) Let $\gamma: \mathfrak{G} \times \mathfrak{G} \rightarrow \mathfrak{G}$ be the commutator map, $(g, h) \mapsto ghg^{-1}h^{-1}$. Then $\gamma(\overline{G} \times \overline{G}) \subseteq \overline{\gamma(G \times G)}$, and so $(\overline{G}, \overline{G}) \subseteq \langle \gamma(G \times G) \rangle \subseteq \overline{(G, G)}$.

(2) From the derived series

$$G^{(0)} := G \supseteq G^{(1)} := (G, G) \supseteq G^{(2)} := (G^{(1)}, G^{(1)}) \supseteq \dots \supseteq G^{(n)} = \{e\}$$

we find the series of closed subgroups of \mathfrak{G} :

$$\overline{G} \supseteq \overline{G^{(1)}} = \overline{(G, G)} \supseteq \overline{G^{(2)}} = \overline{(G^{(1)}, G^{(1)})} \supseteq \dots \supseteq \overline{G^{(n)}} = \{e\}.$$

It follows from (1) that the factor groups $\overline{G^{(i)}}/\overline{G^{(i+1)}}$ are commutative, and the claim follows. \square

Lemma 2.7.3. *Let \mathfrak{G} be an ind-group. Then $\text{Lie } \overline{(\mathfrak{G}, \mathfrak{G})} \supseteq [\text{Lie } \mathfrak{G}, \text{Lie } \mathfrak{G}]$. In particular, if \mathfrak{G} is solvable, then so is $\text{Lie } \mathfrak{G}$.*

Proof. (Cf. [FK18, Section 7.5])

(a) For a fixed $h \in \mathfrak{G}$ consider the morphism $\gamma_h: \mathfrak{G} \rightarrow \mathfrak{G}$, $\gamma_h(g) := hgh^{-1}g^{-1}$. Then the differential $d\gamma_h: \text{Lie } \mathfrak{G} \rightarrow \text{Lie } \mathfrak{G}$ is given by $d\gamma_h := \text{Ad}(h) - \text{id}$. Moreover, the image of $d\gamma_h$ is contained in $\text{Lie } \overline{(\mathfrak{G}, \mathfrak{G})}$.

Now fix $A \in \text{Lie } \mathfrak{G}$ and consider the morphism $\alpha_A: \mathfrak{G} \rightarrow \text{Lie } \mathfrak{G}$, $\alpha_A(h) := \text{Ad}(h)A - A$. The differential of this map is $-\text{ad}(A)$. By (a), the image of α_A and hence of its differential $d\alpha_A$ belongs to $\text{Lie } \overline{(\mathfrak{G}, \mathfrak{G})}$ which implies that $\text{Lie } \overline{(\mathfrak{G}, \mathfrak{G})}$ contains $[\text{Lie } \mathfrak{G}, \text{Lie } \mathfrak{G}]$.

(b) Now assume that \mathfrak{G} is solvable. From the derived series for \mathfrak{G} we get the series

$$\mathfrak{G}^{(0)} = \mathfrak{G} \supseteq \overline{\mathfrak{G}^{(1)}} \supseteq \overline{\mathfrak{G}^{(2)}} \supseteq \dots \supseteq \overline{\mathfrak{G}^{(n)}} = \{e\}.$$

of closed subgroups with the property that the factor groups are all commutative (see Lemma 2.7.2 above and its proof). From (a) we get

$$\text{Lie } \overline{\mathfrak{G}^{(i+1)}} = \text{Lie } \overline{(\mathfrak{G}^{(i)}, \mathfrak{G}^{(i)})} = \text{Lie } \overline{(\overline{\mathfrak{G}^{(i)}}, \overline{\mathfrak{G}^{(i)}})} \supseteq [\text{Lie } \overline{\mathfrak{G}^{(i)}}, \text{Lie } \overline{\mathfrak{G}^{(i)}}].$$

Thus the Lie algebra $\overline{\mathfrak{G}^{(i)}} / \text{Lie } \overline{\mathfrak{G}^{(i+1)}}$ is commutative which implies that $\text{Lie } \mathfrak{G}$ is solvable. \square

Lemma 2.7.4. *Let L be a finitely generated Lie algebra. If L is solvable, then L is finite dimensional.*

Proof. In the derived series

$$L^{(0)} := L \supseteq L^{(1)} := [L^{(0)}, L^{(0)}] \supseteq L^{(2)} := [L^{(1)}, L^{(1)}] \supseteq \dots \supseteq L^{(n)} = \{0\}$$

every member $L^{(i)}$ is finitely generated. Therefore $L^{(i)} / L^{(i+1)}$ is a commutative and finitely generated Lie algebra, hence finite dimensional, and the claim follows. \square

Proof of Theorem 2.7.1. We can assume that G is generated by one dimensional unipotent subgroups $U_i \subseteq \text{Aut}(X)$, $i = 1, \dots, m$. Then the image of $\text{Lie } U_i$ in $\text{Vec}(X)$ is of the form $\mathbb{K}\eta_i$ with a locally nilpotent vector field η_i . Denote by $L \subseteq \text{Vec}(X)$ the Lie algebra generated by the η_i , $i = 1, \dots, m$. Since the $U_i \subseteq G \subseteq \overline{G}$ we see that $L \subseteq \text{Lie } \overline{G}$.

By Lemma 2.7.2 the closure \overline{G} is solvable, hence $\text{Lie } \overline{G}$ is solvable, by Lemma 2.7.3. As a consequence, L is solvable, hence finite dimensional by Lemma 2.7.4. Now Theorem 2.6.1(1) implies that G is a connected algebraic group with $\text{Lie } G = L$. In particular, L is locally finite. Since L is generated by locally nilpotent elements, it consists of locally nilpotent elements (cf. first part of the proof of Lemma 2.6.3). Thus G is a unipotent algebraic group. \square

2.8. Nested Lie algebras. For the following concept see [FK18, Section 9.4] and [KPZ17].

Definition 2.8.1. A subgroup G of an ind-group \mathfrak{G} is called *nested* if there exists a sequence $G_1 \subseteq G_2 \subseteq \cdots \subseteq \mathfrak{G}$ of algebraic subgroups of \mathfrak{G} such that $G = \bigcup_i G_i$.

A Lie algebra $L \subseteq \text{Vec}(X)$ is called *nested* if there exists a sequence $L_1 \subseteq L_2 \subseteq \cdots \subseteq L$ of finite dimensional Lie algebras L_i such that $L = \bigcup_i L_i$.

Note that any two filtrations of G by algebraic subgroups are compatible. In fact, if $G = \bigcup_j H_j$ is another filtration by algebraic subgroups, then $G_i = \bigcup_j (G_i \cap H_j)$ is an ascending union of closed subgroups, hence $G_i = G_i \cap H_j$ for some j and so $G_i \subseteq H_j$.

This implies that a nested subgroup $G = \bigcup G_i \subseteq \mathfrak{G}$ has a natural Lie algebra defined by $\text{Lie } G = \bigcup_i \text{Lie } G_i$ which is nested and does not depend on the filtration of G by algebraic subgroups.

A nested subgroup $G = \bigcup_i G_i \subseteq \mathfrak{G}$ has a natural structure of an ind-group, and the inclusion $G \hookrightarrow \mathfrak{G}$ is a homomorphism of ind-groups. However, this is not necessarily a closed immersion as the following example shows.

Example 2.8.2. The subgroup $\mu_2 \subseteq \mathbb{G}_m$ of elements of order a power of 2 is nested by the finite subgroups $\simeq \mathbb{Z}/2^n\mathbb{Z}$. But it is not closed.

Lemma 2.8.3. *Assume that the nested subgroup $G \subseteq \mathfrak{G}$ is path-connected. Then G admits an ascending filtration by connected algebraic subgroups.*

Proof. If $G = \bigcup_i G_i$ is path connected, then $G = \bigcup_i G_i^\circ$. □

Question 4. *Let $G \subseteq \text{Aut}(X)$ be a path-connected nested subgroup. Does it follow that G is closed?*

In order to give a positive answer to this question one has to show that for every algebraic subset $A \subseteq \text{Aut}(X)$ the intersection $A \cap G$ is contained in an algebraic group.

Clearly, if $G \subseteq \text{Aut}(X)$ is a nested subgroup, then $\text{Lie } G \subseteq \text{Vec}(X)$ is a nested Lie algebra filtered by locally finite subalgebras. The converse of this is also true.

Proposition 2.8.4. *Let $L = \bigcup_i L_i \subseteq \text{Vec}(X)$ be a nested Lie subalgebra where all L_i are locally finite. Then there exists a nested subgroup $G \subseteq \text{Aut}(X)$ such that $\text{Lie } G \subseteq \text{Vec}(X)$ contains L . Moreover, every closed ind-subgroup $\mathfrak{G} \subseteq \text{Aut}(X)$ such that $\text{Lie } \mathfrak{G} \supseteq L$ contains G .*

Proof. Let $G_i \subseteq \text{Aut}(X)$ be the minimal algebraic subgroups with $\text{Lie } G_i \supseteq L_i$ (Theorem 2.6.1). Then $G_i \subseteq G_{i+1}$ for all i and so $G := \bigcup_i G_i$ is a nested subgroup with the required properties.

Assume that $\text{Lie } \mathfrak{G} \supseteq L$ and define $G'_i := \mathfrak{G} \cap G_i$. Then $\text{Lie } G'_i = \text{Lie } \mathfrak{G} \cap \text{Lie } G_i \supseteq L_i$, hence $G'_i = G_i$ and the claim follows. \square

3. AN INTERESTING EXAMPLE

In this section we study the subgroup $F \subseteq \text{Aut}(\mathbb{A}^2)$ generated by $\mathbf{u} := (x + y^2, y)$ and $\mathbf{v} := (x, y + x^2)$. We will show that F is a free group in two generators and we will describe the closure $\mathfrak{F} := \overline{F}$ and its Lie algebra $\text{Lie } \mathfrak{F}$. It turns out that the following holds (see Theorem 3.4.3).

- \mathfrak{F} is a free product of two nested abelian closed unipotent ind-subgroups \mathfrak{J} and $\mathfrak{J}^- \subseteq \text{Aut}(\mathbb{A}^2)$. In particular, \mathfrak{F} is torsion free.
- $\text{Lie } \mathfrak{F}$ is generated, as a Lie algebra, by $\text{Lie } U$ and $\text{Lie } V$ where $U := \langle \mathbf{u} \rangle \simeq \mathbb{G}_a$ and $V := \langle \mathbf{v} \rangle \simeq \mathbb{G}_a$.
- Any algebraic subgroup of \mathfrak{F} is abelian and unipotent and is conjugate to a subgroup of \mathfrak{J} or \mathfrak{J}^- .

3.1. Notation and first results. Let

$$T := \{(ax, by) \mid a, b \in \mathbb{K}^*\} \subseteq \text{GL}_2(\mathbb{K}) \subseteq \text{Aut}(\mathbb{A}^2)$$

denote the standard 2-dimensional torus, and let

$$\text{SAut}(\mathbb{A}^2) := \{\mathbf{g} \in \text{Aut}(X) \mid \text{jac}(\mathbf{g}) = 1\}$$

where the *Jacobian* of $\mathbf{g} = (f, h)$ is defined in the usual way: $\text{jac}(\mathbf{g}) = \det \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix}$. Furthermore, we set

$$\text{Aut}_0(\mathbb{A}^2) := \{\mathbf{g} \in \text{Aut}(\mathbb{A}^2) \mid \mathbf{g}(0) = 0\},$$

$$\mathfrak{A} := \{\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2) \mid d\mathbf{g}_0 = \text{id}\}.$$

Note that $\mathbf{g} \in \text{Aut}(\mathbb{A}^2)$ belongs to \mathfrak{A} if and only if the linear part of $\mathbf{g} = (f, h)$ is equal to (x, y) . \mathfrak{A} is a closed path-connected ind-subgroup of $\text{Aut}(\mathbb{A}^2)$ which contains $\mathbf{u} = (x + y^2, y)$ and $\mathbf{v} = (x, y + x^2)$, hence $\mathfrak{F} \subseteq \mathfrak{A}$.

Let $S := \{(\zeta x, \zeta^2 y) \mid \zeta^3 = 1\} \subseteq \text{SL}_2 \subseteq \text{SAut}(\mathbb{A}^2)$ be the cyclic diagonal subgroup of order 3. The elements of S commute with \mathbf{u} and \mathbf{v} and thus with all elements from F and hence from $\mathfrak{F} = \overline{F}$.

Lemma 3.1.1. *One has $S = \text{Cent}_{\text{Aut}(\mathbb{A}^2)}(F)$.*

Proof. Note that any $\mathbf{g} \in \text{Aut}(\mathbb{A}^2)$ commuting with \mathbf{u} and \mathbf{v} , and therefore with U and V , fixes the origin which is the unique common fixed point of U and V . It also leaves invariant the set of orbits of U and of V , i.e. the pencils of horizontal and vertical lines. Hence, $\mathbf{g} \in T$, and an easy calculation shows that, in fact, $\mathbf{g} \in S$. \square

Considering the action of S on $\text{Aut}(\mathbb{A}^2)$ by conjugation, one sees that $\text{SAut}(\mathbb{A}^2)$ and \mathfrak{A} are stable under this action, hence

$$\mathfrak{F} \subseteq \mathfrak{A}^S := \{\mathbf{g} \in \mathfrak{A} \mid \mathbf{g} \circ s = s \circ \mathbf{g} \text{ for all } s \in S\}.$$

Moreover, S also acts on the Lie algebras $\text{Lie Aut}(\mathbb{A}^2)$ and $\text{Lie } \mathfrak{A}$, and so

$$(1) \quad \text{Lie } \mathfrak{F} \subseteq \text{Lie } \mathfrak{A}^S \subseteq (\text{Lie } \mathfrak{A})^S.$$

3.2. The Lie algebra of \mathfrak{F} . For the Lie algebras of $\text{Aut}(\mathbb{A}^2)$ and $\text{SAut}(\mathbb{A}^2)$ we have the following description, see [FK18, Lemma 15.7.1]:

- $\text{Lie Aut}(\mathbb{A}^2) \xrightarrow{\sim} \text{Vec}^c(\mathbb{A}^2) := \{\delta \in \text{Vec}(\mathbb{A}^2) \mid \text{div } \delta \in \mathbb{K}\},$
- $\text{Lie SAut}(\mathbb{A}^2) \xrightarrow{\sim} \text{Vec}^0(\mathbb{A}^2) := \{\delta \in \text{Vec}(\mathbb{A}^2) \mid \text{div } \delta = 0\},$
- $\text{Lie } \mathfrak{A} \xrightarrow{\sim} \text{Vec}_0^0(\mathbb{A}^2) := \{\delta \in \text{Vec}^0(\mathbb{A}^2) \mid \delta(0) = 0\}$

where the *divergence* of a vector field $\delta = f \frac{\partial}{\partial x} + h \frac{\partial}{\partial y}$ is defined as $\text{div } \delta = \frac{\partial f}{\partial x} + \frac{\partial h}{\partial y}$.

Now consider the following vector fields from $\text{Vec}^0(\mathbb{A}^2)$,

$$\partial_{i,j} := (j+1)x^{i+1}y^j \frac{\partial}{\partial x} - (i+1)x^i y^{j+1} \frac{\partial}{\partial y}$$

where

$$(i, j) \in \Lambda := \{(k, l) \in \mathbb{Z}^2 \mid k, l \geq -1, (k, l) \neq (-1, -1)\}.$$

The torus T acts by conjugation on $\text{Aut}(\mathbb{A}^2)$ and on $\text{Vec}(\mathbb{A}^2)$, and the $\partial_{i,j}$ are eigenvectors of weight (i, j) . Moreover, the action of T on $\text{Vec}^0(\mathbb{A}^2)$ is multiplicity-free, and we have the weight decompositions

$$\text{Vec}^0(\mathbb{A}^2) = \bigoplus_{(i,j) \in \Lambda} \mathbb{K} \partial_{i,j} \quad \text{and} \quad \text{Vec}_0^0(\mathbb{A}^2) = \bigoplus_{(i,j) \in \Lambda, i+j > 0} \mathbb{K} \partial_{i,j}.$$

Note that $\partial_{i,j}$ is fixed by the finite subgroup $S \subseteq T$ if and only if $i - j \equiv 0 \pmod{3}$. Setting $\Lambda_0 := \{(k, l) \in \Lambda \mid k - l \equiv 0 \pmod{3}, (k, l) \neq (0, 0)\}$ we get

$$(2) \quad \text{Lie } \mathfrak{A}^S \subseteq (\text{Lie } \mathfrak{A})^S = \bigoplus_{(i,j) \in \Lambda_0} \mathbb{K} \partial_{i,j}.$$

The next lemma shows that $\text{Lie } \mathfrak{F} = \text{Lie } \mathfrak{A}^S = (\text{Lie } \mathfrak{A})^S$. In contrast to the situation of algebraic groups, this does not automatically imply

that $\mathfrak{F} = \mathfrak{A}^S$. In fact, FURTER-KRAFT give an example of a closed ind-subgroup of a path-connected ind-group which has the same Lie algebra, but is strictly smaller (see [FK18, Theorem 17.3.1]). In order to show that $\mathfrak{F} = \mathfrak{A}^S$ (see Theorem 3.4.3) we therefore need an additional argument.

Lemma 3.2.1. *The Lie algebra $(\text{Lie } \mathfrak{A})^S$ is generated by $\text{Lie } U$ and $\text{Lie } V$, hence $\text{Lie } \mathfrak{F} = \text{Lie } \mathfrak{A}^S = (\text{Lie } \mathfrak{A})^S$.*

Proof. We have $\text{Lie } U = \mathbb{K}\partial_{-1,2}$ and $\text{Lie } V = \mathbb{K}\partial_{2,-1}$. An easy calculation shows that

$$\text{ad } \partial_{-1,2}(\partial_{i,j}) = a \partial_{i-1,j+2} \ (i \geq 0) \text{ and } \text{ad } \partial_{2,-1}(\partial_{i,j}) = b \partial_{i+2,j-1} \ (j \geq 0)$$

with non-zero constants $a, b \in \mathbb{K}$. This implies that the Lie algebra generated by $\partial_{-1,2}$ and $\partial_{2,-1}$ contains all $\partial_{i,j}$ with $(i, j) \in \Lambda_0$, hence $\langle \text{Lie } U, \text{Lie } V \rangle_{\text{Lie}} \supseteq (\text{Lie } \mathfrak{A})^S$, see formula (2). Since $\text{Lie } U, \text{Lie } V \subseteq \text{Lie } \mathfrak{F}$ the claim follows from the inclusions in (1). \square

It is easy to see that the vector field $\partial_{i,j}$ is locally nilpotent if and only if $i = -1$ or $j = -1$. In these cases, there are unique algebraic subgroups $U_{-1,k} \simeq \mathbb{G}_a$ and $U_{k,-1} \simeq \mathbb{G}_a$ with $\text{Lie } U_{-1,k} = \mathbb{K}\partial_{-1,k}$ and $\text{Lie } U_{k,-1} = \mathbb{K}\partial_{k,-1}$. Note that $U_{-1,2} = U = \overline{\langle \mathbf{u} \rangle}$ and $U_{2,-1} = V = \overline{\langle \mathbf{v} \rangle}$. Moreover, the subgroups $U_{-1,k}$ and $U_{-1,l}$ commute as well as the vector fields $\partial_{-1,k}$ and $\partial_{-1,l}$.

3.3. De Jonquière subgroups and amalgamated products. This and the subsequent sections are largely inspired by [AZ13, Section 4]. Consider the DE JONQUIÈRE group

$$\text{Jonq} := \{(ax + h(y), cy + d) \mid a, c \in \mathbb{K}^*, d \in \mathbb{K}, h \in \mathbb{K}[y]\}$$

and its subgroups

$$\text{Jonq}_0 := \{(ax + yf(y), by) \mid a, b \in \mathbb{K}^*, f \in \mathbb{K}[y]\} = \text{Jonq} \cap \text{Aut}_0(\mathbb{A}^2),$$

$$\mathfrak{J}_0 := \{(x + yf(y), y) \mid f \in \mathbb{K}[y]\} \subseteq \text{Jonq}_0.$$

\mathfrak{J}_0 is a commutative closed unipotent ind-subgroup, isomorphic to $\mathbb{K}[y]^+$. It admits a filtration by closed unipotent subgroups $\prod_{i=1}^d U_{-1,i} \simeq \mathbb{G}_a^d$, $d \geq 1$. Note that $\text{Jonq}_0 = \mathfrak{J}_0 \rtimes T$.

Lemma 3.3.1. *The group \mathfrak{A}^S is the free product $(\mathfrak{J}_0)^S * (\mathfrak{J}_0^-)^S$ where $\mathfrak{J}_0^- := \tau \circ \mathfrak{J}_0 \circ \tau$ and $\tau := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.*

Proof. (1) The amalgamated product structure $\text{Aut}(\mathbb{A}^2) = \text{Jonq} *_{\mathfrak{B}} \text{Aff}_2$ where $\mathfrak{B} := \text{Jonq} \cap \text{Aff}_2$ implies that every element $\mathbf{g} \in \text{Aut}(\mathbb{A}^2) \setminus \mathfrak{B}$

can be expressed in one of the following 4 forms, called *the type* of \mathbf{g} ,

$$\begin{aligned} \mathbf{g} &= j_1 a_1 j_2 \cdots j_n, & \mathbf{g} &= j_1 a_1 j_2 \cdots j_n a_{n+1}, \\ \mathbf{g} &= a_1 j_1 a_2 \cdots j_n, & \mathbf{g} &= a_1 j_1 a_2 \cdots j_n a_{n+1} \end{aligned}$$

where $j_k \in \text{Jonq} \setminus \mathfrak{B}$ and $a_k \in \text{Aff}_2 \setminus \mathfrak{B}$. The length of the expression, its type and the degrees of the j_k are uniquely determined by \mathbf{g} .

(2) Let now $\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2)$. Using $\text{Jonq} = \text{Jonq}_0 \rtimes \text{Trans}_2$ and $\text{Aff}_2 = \text{GL}_2 \rtimes \text{Trans}_2$ where $\text{Trans}_2 \simeq \mathbb{G}_a^2$ is the group of translations, one can assume that $j_k \in \text{Jonq}_0 \setminus B$ and $a_k \in \text{GL}_2 \setminus B$ where $B \subseteq \text{GL}_2$ is the Borel subgroup of upper triangular matrices. Since $\text{GL}_2 \setminus B = U\tau B = B\tau U$ where $U \subseteq B$ is the unipotent radical, we see that any $\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2)$ has a presentation in one of the 4 forms

$$\begin{aligned} \mathbf{g} &= j_1 \tau j_2 \cdots \tau j_n, & \mathbf{g} &= j_1 \tau j_2 \cdots j_n \tau, \\ \mathbf{g} &= \tau j_1 \tau \cdots \tau j_n, & \mathbf{g} &= \tau j_1 \tau \cdots \tau j_n \tau \end{aligned}$$

where $j_k \in \text{Jonq}_0 \setminus B$. Again, the length, the type and the degrees of the j_k are uniquely determined by \mathbf{g} .

(3) Now we remark that $\text{Jonq}_0 = \mathfrak{J}_0 \rtimes T$ and $T\tau = \tau T$. It follows that we can reach one of the following forms

$$\begin{aligned} \mathbf{g} &= j_1 \tau j_2 \cdots \tau j_n t, & \mathbf{g} &= j_1 \tau j_2 \cdots j_n \tau t, \\ \mathbf{g} &= \tau j_1 \tau \cdots \tau j_n t, & \mathbf{g} &= \tau j_1 \tau \cdots \tau j_n \tau t \end{aligned}$$

where $j_k \in \mathfrak{J}_0$ and $t \in T$. We claim that this form is uniquely determined by \mathbf{g} . Since the type and the length are given by \mathbf{g} , it suffices to compare the similar decompositions

$$\mathbf{g} = j_1 \tau j_2 \cdots j_n t = j_1' \tau j_2' \cdots j_n' t' \text{ and } \mathbf{g} = j_1 \tau j_2 \cdots j_n \tau t = j_1' \tau j_2' \cdots j_n' \tau t'.$$

It follows that

$$\tau j_2 \cdots j_n t = j_1^{-1} j_1' \tau j_2' \cdots j_n' t' \text{ resp. } \tau j_2 \cdots j_n \tau t = j_1^{-1} j_1' \tau j_2' \cdots j_n' \tau t',$$

hence $j_1' = j_1$. Now the claim follows by induction.

(4) Using the equalities $st = ts$ and $s\tau s = \tau$ for $s \in S$ and $t \in T$ we find for an S -invariant $\mathbf{g} \in \text{Aut}_0(\mathbb{A}^2)$ of the form $\mathbf{g} = j_1 \tau j_2 \cdots j_n t$ as in (3) above that

$$\mathbf{g} = s\mathbf{g}s^{-1} = \begin{cases} (sj_1 s^{-1})\tau(s^{-1}j_2 s)\tau(sj_3 s^{-1}) \cdots (sj_n s^{-1})t & \text{if } n \text{ is odd,} \\ (sj_1 s^{-1})\tau(s^{-1}j_2 s)\tau(sj_3 s^{-1}) \cdots (s^{-1}j_n s^{-1})t & \text{if } n \text{ is even.} \end{cases}$$

In the first case, we get that each j_k is S -invariant. In the second case, we obtain $sj_n s = j_n$, a contradiction. Looking at the other forms we see that for an S -invariant \mathbf{g} the number of τ 's must be even and each j_k is S -invariant, i.e. $j_k \in (\mathfrak{J}_0)^S \subseteq \mathfrak{A}^S$.

(5) Finally, assume that $\mathbf{g} \in \mathfrak{A}^S$. If \mathbf{g} is given in one of the forms of (3), then $t = \text{id}$, because $d\mathbf{g}_0 = \text{id}$ and $(dj_k)_0 = \text{id}$ for $j_k \in (\mathfrak{J}_0)^S$. Since the number of τ 's is even, we see that such an automorphism \mathbf{g} is a product of the form $\cdots j_k j'_k j_{k+1} j'_{k+1} \cdots$ where $j_k \in (\mathfrak{J}_0)^S$ and $j'_k \in (\mathfrak{J}_0^-)^S = (\tau \mathfrak{J}_0 \tau)^S = \tau (\mathfrak{J}_0)^S \tau$. \square

3.4. The fixed points under S . Recall that $\mathfrak{F} \subseteq \text{Aut}(\mathbb{A}^2)$ denotes the closure of the free group F generated by \mathbf{u} and \mathbf{v} .

Proposition 3.4.1. $(\mathfrak{J}_0)^S \subseteq \mathfrak{F}$. In particular, $\mathfrak{F} = \mathfrak{A}^S = (\mathfrak{J}_0)^S * (\mathfrak{J}_0^-)^S$.

The proof needs some preparation. Set

$$\begin{aligned} \varphi &:= \mathbf{u} \circ \mathbf{v} = (x + y^2, y) \circ (x, y + x^2) = \\ &= (x + y^2 + 2x^2y + y^4, y + x^2) \in F = \langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

and consider the powers $\varphi^k = (f_k, h_k)$. It follows that $f_k = x +$ terms of degree ≥ 2 and $h_k = y +$ terms of degree ≥ 2 , and that all coefficients are non-negative integers.

Lemma 3.4.2. *The polynomial f_k contains the term cy^{3k-1} with $c \neq 0$.*

Proof. By definition, we have $f_{k+1} = f_k + h_k^2 + 2f_k^2h_k + h_k^4$. Since all monomials in f_k and h_k have non-negative coefficients, there is no cancellation of terms. Now assume that f_k contains the terms y^2 and y^{3k-1} . Then in the expression of f_{k+1} the summand f_k contains the term y^2 and the summand $f_k^2h_k$ contains $y^2 \cdot y^{3k-1} \cdot y = y^{3k+2}$. Thus the claim follows by induction. \square

Proof of Proposition 3.4.1. We have seen in Section 3.3 that $\mathfrak{J}_0 \simeq \mathbb{K}[y]^+$ admits a filtration by the closed unipotent subgroups $\prod_{i=2}^d U_{-1,i}$, $d \geq 2$. It follows that $\mathfrak{J}_0^S = \{(x + y^2 f(y^3), y) \mid f \in \mathbb{K}[y]\} \xrightarrow{\sim} \mathbb{K}[y]^+$, and that \mathfrak{J}_0^S admits a filtration by the closed unipotent subgroups $\prod_{j=1}^k U_{-1,3j-1}$. Note that this product contains any automorphism of \mathbb{A}^2 of the form $(x + \sum_{j=1}^k c_j y^{3j-1}, y)$. We now show by induction that $U_{-1,3j-1} \subseteq \mathfrak{F}$.

Assume that this holds for $j \leq k$ and consider the product ψ of φ^{k+1} with an element $\mathbf{g} \in \prod_{j=1}^k U_{-1,3j-1} \subseteq \mathfrak{F}$:

$$\begin{aligned} \psi &:= \varphi^{k+1} \circ \mathbf{g} = (f_{k+1}, h_{k+1}) \circ (x + \sum_{j=1}^k c_j y^{3j-1}, y) = \\ &= (f_{k+1}(x + \sum_{j=1}^k c_j y^{3j-1}, y), h_{k+1}(x + \sum_{j=1}^k c_j y^{3j-1}, y)) = (f, h). \end{aligned}$$

Since the lowest degree term of $f_{k+1}(0, y)$ is cy^{3k+2} we can choose the coefficients c_j is such a way that the lowest degree term in $f(0, y)$ is cy^{3k+2} . But then

$$(t^{-3k-2}x, t^{-1}y) \circ (f, h) \circ (t^{3k+2}x, ty) \xrightarrow{t \rightarrow 0} (x + cy^{3k+2}, y) \in U_{-1, 3k+2}.$$

Since \mathfrak{F} is closed and stable under the action of the torus T we get $U_{-1, 3k+2} \subseteq \mathfrak{F}$, and the claim follows. \square

Summing up we have the following result.

Theorem 3.4.3. *Let $F \subseteq \text{Aut}(\mathbb{A}^2)$ be the subgroup generated by $\mathbf{u} := (x + y^2, y)$ and $\mathbf{v} := (x, y + x^2)$, and denote by $\mathfrak{F} := \overline{F}$ its closure in $\text{Aut}(\mathbb{A}^2)$. Furthermore, define $\mathfrak{A} := \{\mathbf{g} \in \text{Aut}(\mathbb{A}^2) \mid \mathbf{g}(0) = 0, d\mathbf{g}_0 = \text{id}\}$.*

- (1) *F is a free group in the two generators \mathbf{u}, \mathbf{v} , containing non-algebraic elements.*
- (2) *$\mathfrak{F} = \mathfrak{A}^S = \text{Cent}_{\mathfrak{A}}(S)$ where $S := \{(\zeta x, \zeta^2 y) \mid \zeta^3 = 1\} \subseteq T$ is the centralizer of F in $\text{Aut}(\mathbb{A}^2)$.*
- (3) *\mathfrak{F} is a free product $\mathfrak{J} * \mathfrak{J}^-$ where $\mathfrak{J} := \{(x + y^2 f(y^3), y) \mid f \in \mathbb{K}[y]\} = (\mathfrak{J}_0)^S$ and $\mathfrak{J}^- = \{(x, y + x^2 f(x^3)) \mid f \in \mathbb{K}[x]\} = \tau \mathfrak{J} \tau$. In particular, \mathfrak{J} is a commutative nested unipotent closed ind-subgroup of $\text{Aut}(\mathbb{A}^2)$ isomorphic to $\mathbb{K}[y]^+$.*
- (4) *Every algebraic subgroup of \mathfrak{F} is commutative and unipotent and is conjugate to a subgroup of \mathfrak{J} or of \mathfrak{J}^- .*

Proof. (1) The first part is clear. For the second we notice that $\deg(\mathbf{u}\mathbf{v})^k$ tends to infinity for $k \rightarrow \infty$. Hence, $\mathbf{u}\mathbf{v} \in \mathfrak{F}$ is not an algebraic element.

(2) & (3) This is Proposition 3.4.1 and Lemma 3.1.1

(4) This follows from a famous result of SERRE's, see [Ser03, Theorem 8], because an algebraic subgroup of a free product $\mathfrak{J} * \mathfrak{J}^-$ is of bounded length, hence conjugate to a subgroup of one of the factors. \square

Remark 3.4.4. According to Lemma 3.2.1 the subgroup $\mathfrak{F} \subseteq \text{Aut}(X)$ is the closure of the subgroup generated by U and V , and $\text{Lie } \mathfrak{F}$ is the Lie algebra generated by $\text{Lie } U$ and $\text{Lie } V$. Thus our example is a positive instance of our Question 1.

4. HOMOMORPHISMS OF IND-GROUPS AND OF THEIR LIE ALGEBRAS

4.1. A canonical Lie subalgebra of $\text{Lie } \mathfrak{G}$. Let $\mathfrak{G} = \bigcup_d \mathfrak{G}_d$ be an ind-group. For $g \in \mathfrak{G}$ denote by $\lambda_g, \rho_g: \mathfrak{G} \xrightarrow{\sim} \mathfrak{G}$ the left- and the right-multiplications by g , respectively. If $Y \subseteq \mathfrak{G}$ is an algebraic subset and

$g \in Y$, then $\rho_{g^{-1}}$ defines an isomorphism $Y \xrightarrow{\sim} Yg^{-1}$ whose differential in g is a linear isomorphism $T_g Y \xrightarrow{\sim} T_e Yg^{-1} \subseteq \text{Lie } \mathfrak{G}$. Define

$$M_Y := \text{span}_{\mathbb{K}} (T_e Yg^{-1} = d\rho_{g^{-1}}(T_g Y) \mid g \in Y) \subseteq \text{Lie } \mathfrak{G}.$$

More generally, if $\varphi: X \rightarrow \mathfrak{G}$ is a morphism where X is an algebraic variety, then we define

$$M_\varphi := \text{span}_{\mathbb{K}} (d\rho_{\varphi(x)^{-1}}(\text{Im } d\varphi_x) \mid x \in X) \subseteq \text{Lie } \mathfrak{G}.$$

Lemma 4.1.1. *Let $Y \subseteq \mathfrak{G}$ be an algebraic subset, and let $\varphi: X \rightarrow \mathfrak{G}$ be a morphism, where X is an algebraic variety. Then the following holds.*

- (a) $\dim M_Y < \infty$ and $\dim M_\varphi < \infty$.
- (b) For $g \in \mathfrak{G}$ we have $M_{Yg} = M_Y$ and $M_{gY} = \text{Ad } g(M_Y)$.
- (c) For $g \in \mathfrak{G}$ we have $M_{\rho_g \circ \varphi} = M_\varphi$ and $M_{\lambda_g \circ \varphi} = \text{Ad } g(M_\varphi)$.
- (d) If $A \subseteq X$ is a closed subset, then $M_{\varphi|_A} \subseteq M_\varphi$.
- (e) If Y is smooth and $U \subseteq Y$ open and dense, then $M_U = M_Y$.
- (f) If $Y = \overline{\varphi(X)}$, then $M_{Y_{\text{reg}}} \subseteq M_\varphi \subseteq M_Y$.

Proof. (a) Consider the closed subset $\tilde{Y} := \overline{\{hg^{-1} \mid h, g \in Y\}}$. Then $\rho_{g^{-1}}(Y) = Yg^{-1} \subseteq \tilde{Y}$ for all $g \in Y$, and so $M_Y \subseteq T_e \tilde{Y}$. For $Y := \overline{\varphi(X)}$ we obviously have $M_\varphi \subseteq M_Y$ and the both claims follow.

(b) For $Z := Yg$, $y \in Y$, $z := yg \in Z$ we have $Zz^{-1} = Yy^{-1}$, hence $M_{Yg} = M_Y$. For $Z := gY$, $y \in Y$, $z := gy$ we get $Zz^{-1} = g(Yy^{-1})g^{-1}$, hence $T_e Zz^{-1} = T_e g(Yy^{-1})g^{-1} = \text{Ad } g(T_e Yy^{-1})$, and so $M_{gY} = \text{Ad } g(M_Y)$.

(c) This is similar to (b).

(d) This is clear.

(e) If V is a finite dimensional vector space and $(U_y)_{y \in Y}$ a family of d -dimensional subspaces parametrized by a variety Y , i.e. given by a morphism of Y to the Grassmannian, then $\langle U_y \mid y \in Y \rangle = \langle U_y \mid y \in Y' \rangle$ for any dense subset $Y' \subseteq Y$.

(f) There is an open dense set $X' \subseteq X_{\text{reg}}$ where φ is smooth. This implies that $\text{Im } d\varphi_x = T_{\varphi(x)} Y$ for all $x \in X'$. Since the image $\varphi(X')$ is open and dense in Y_{reg} we obtain from (d) and (e) that $M_{Y_{\text{reg}}} = M_{\varphi(X')} \subseteq M_\varphi$. The second inclusion is obvious. \square

Definition 4.1.2. For an ind-group \mathfrak{G} we define

$$L_{\mathfrak{G}} := \text{span}_{\mathbb{K}} (M_{Y_{\text{reg}}} \mid Y \subseteq \mathfrak{G} \text{ an irreducible closed algebraic subset})$$

where $Y_{\text{reg}} \subseteq Y$ is the smooth locus. Given a homomorphism $\Phi: \mathfrak{H} \rightarrow \mathfrak{G}$ of ind-groups, we define $L_\Phi := d\Phi(L_{\mathfrak{H}}) \subseteq L_{\mathfrak{G}}$.

The next proposition shows that $L_{\mathfrak{G}} \subseteq \text{Lie } \mathfrak{G}$ is a canonical Lie subalgebra.

Proposition 4.1.3.

- (a) $L_{\mathfrak{G}} \subseteq \text{Lie } \mathfrak{G}$ is stable under the adjoint action of \mathfrak{G} , i.e. $L_{\mathfrak{G}}$ is an ideal in $\text{Lie } \mathfrak{G}$.
- (b) For every algebraic subgroup $G \subseteq \mathfrak{G}$ we have $\text{Lie } G \subseteq L_{\mathfrak{G}}$.
- (c) If $\mathfrak{H} \subseteq \mathfrak{G}$ is a closed subgroup then $L_{\mathfrak{H}} \subseteq L_{\mathfrak{G}}$.

Proof. (a) We have $\text{Ad } h(M_Y) = M_{hYh^{-1}}$. In fact,

$$\text{Ad } h(T_e Y g^{-1}) = T_e(hYg^{-1}h^{-1}) = T_e(hYh^{-1})g_0^{-1}$$

where $g_0 = hgh^{-1}$, and the claim follows.

(b) For an algebraic group $G \subseteq \mathfrak{G}$ we have $d\rho_{g^{-1}}(T_g G) = \text{Lie } G$ (cf. Remark 2.2.3), and so $M_G = \text{Lie } G$.

(c) This is clear. □

Recall that

$$\text{SAut}(\mathbb{A}^n) := \{\alpha \in \text{Aut}(\mathbb{A}^n) \mid \text{jac}(\alpha) = 1\}$$

is a normal subgroup of $\text{Aut}(\mathbb{A}^n)$ with Lie algebra $\text{Lie SAut}(\mathbb{A}^n)$ being the algebra $\text{Vec}^0(\mathbb{A}^n)$ of polynomial vector fields on \mathbb{A}^n with zero divergence (see Section 3.2).

Corollary 4.1.4. *We have $L_{\text{SAut}(\mathbb{A}^n)} = \text{Lie SAut}(\mathbb{A}^n)$ and $L_{\text{Aut}(\mathbb{A}^n)} = \text{Lie Aut}(\mathbb{A}^n)$.*

Proof. By [Sha81, Lemma 3], the Lie algebra

$$\text{Lie SAut}(\mathbb{A}^n) = [\text{Lie Aut}(\mathbb{A}^n), \text{Lie Aut}(\mathbb{A}^n)]$$

is simple. It follows from Proposition 4.1.3(a) that $\text{Lie SAut}(\mathbb{A}^n) \subseteq L_{\text{Aut}(\mathbb{A}^n)}$. Now the claim is a consequence of Proposition 4.1.3(b) due to the fact that

$$\text{Lie Aut}(\mathbb{A}^n) = \text{Lie SAut}(\mathbb{A}^n) \oplus \text{Lie } \mathbb{K}^*,$$

where $\mathbb{K}^* \subseteq \text{Aut}(\mathbb{A}^n)$ is realized as the subgroup of the scalar multiplications $\lambda \cdot \text{id}$, $\lambda \in \mathbb{K}^*$, cf. [FK18, Prop. 15.7.2]. □

4.2. Surjective homomorphisms. Let $\Phi: \mathfrak{H} \rightarrow \mathfrak{G}$ be a homomorphism of ind-groups. If Φ is surjective one expects that the induced homomorphism $d\Phi: \text{Lie } \mathfrak{H} \rightarrow \text{Lie } \mathfrak{G}$ is also surjective.

Proposition 4.2.1. *Let $\Phi: \mathfrak{H} \rightarrow \mathfrak{G}$ be a surjective homomorphism of ind-groups.*

- (a) M_Φ is stable under the adjoint action of \mathfrak{G} , i.e. $M_\Phi \subseteq \text{Lie } \mathfrak{G}$ is an ideal.
- (b) If $H \subseteq \mathfrak{H}$ is an algebraic subgroup, then $M_\Phi \supseteq \text{Lie } \Phi(H)$.

Proof. (a) For $h \in \mathfrak{H}$ and $g := \Phi(h) \in \mathfrak{G}$ we get from Lemma 4.1.1(c) that $M_{\Phi|_{hX}} = \text{Ad } g(M_{\Phi|_X})$ for any closed algebraic subset $X \subseteq \mathfrak{H}$.

(b) This is clear, since $\Phi(H) \subseteq \mathfrak{G}$ is an algebraic subgroup, and therefore $d\Phi|_H: \text{Lie } H \rightarrow \text{Lie } \Phi(H)$ is surjective. \square

Question 5. Assume we have a surjective homomorphism $\mathfrak{G} \rightarrow G$ where \mathfrak{G} is a path-connected ind-group and G is an algebraic group. Can one show that \mathfrak{G} contains an algebraic subgroup which is sent surjectively onto G ?

4.3. Subgroups of finite codimension. Let $\mathfrak{G} = \bigcup_d \mathfrak{G}_d$ be an ind-group.

Definition 4.3.1. We say that a closed subgroup $\mathfrak{H} \subseteq \mathfrak{G}$ is of *finite codimension* if $\mathfrak{G}_d \cdot \mathfrak{H} = \mathfrak{G}$ for some $d \in \mathbb{N}$.

For instance, if \mathfrak{G} is connected and acts on an algebraic variety X , then the stabilizer \mathfrak{G}_x of any point $x \in X$ is of finite codimension. In fact, the orbit $\mathfrak{G}x$ is an irreducible subvariety of X by [FK18, Prop. 7.1.2(1)], and so, $\mathfrak{G}x = \mathfrak{G}_d x$ for some d . The latter implies $\mathfrak{G} = \mathfrak{G}_d \cdot \mathfrak{G}_x$. We do not know if every subgroup \mathfrak{H} of finite codimension of a connected ind-group \mathfrak{G} is the stabilizer of a point of a \mathfrak{G} -variety, i.e. if $\mathfrak{G}/\mathfrak{H}$ has a canonical structure of an algebraic variety.

Proposition 4.3.2. Let \mathfrak{G} be an ind-group and $\mathfrak{H} \subseteq \mathfrak{G}$ a closed subgroup. Assume that there is a closed irreducible algebraic subset $Z \subseteq \mathfrak{G}$ such that $Z \cdot \mathfrak{H} = \mathfrak{G}$. Then $L_\mathfrak{G} = M_{Z_{\text{reg}}} + L_\mathfrak{H}$. In particular, $L_\mathfrak{H}$ has finite codimension in $L_\mathfrak{G}$.

Proof. Let $Y \subseteq \mathfrak{G}$ be a closed irreducible algebraic subset. Define $\tilde{Y} := \overline{Z \cdot Y}$. Then $\text{Ad } z(M_{Y_{\text{reg}}}) \subseteq M_{\tilde{Y}_{\text{reg}}}$ for all z in an open dense subset of Z_{reg} . In fact, for all z in an open dense subset of Z we have $zY_{\text{reg}} \cap \tilde{Y}_{\text{reg}} \neq \emptyset$, hence $M_{zY_{\text{reg}}} \subseteq M_{\tilde{Y}_{\text{reg}}}$.

Thus, in Definition 4.1.2 of $L_\mathfrak{G}$ we may assume that the irreducible algebraic subsets are of the form $\overline{Z \cdot Y}$.

For any irreducible closed algebraic subset $Y \subseteq \mathfrak{G}$ of the form above we can find an irreducible closed algebraic subset $X \subseteq \mathfrak{H}$ such the $\overline{Z \cdot X} = Y$. Consider the map $\eta: Z \times X \rightarrow Y$, $(z, h) \mapsto z \cdot h$. Then η is smooth on an open dense subset $U \subseteq Z_{\text{reg}} \times X_{\text{reg}}$. In particular,

$\eta(U)$ is open and dense in Y_{reg} . Now for every $g = z \cdot h \in \varphi(U)$ where $(z, h) \in U$ we have the following commutative diagram:

$$\begin{array}{ccc} Z \times X & \xrightarrow{\quad \eta \quad} & Y \\ \rho_{z^{-1}} \times \text{Int } z \circ \rho_{h^{-1}} \downarrow \simeq & & \simeq \downarrow \rho_{g^{-1}} \\ Zz^{-1} \times z(Xh^{-1})z^{-1} & \xrightarrow[\eta]{} & Yg^{-1} \end{array}$$

On the tangent spaces this induces the commutative diagram

$$\begin{array}{ccc} T_z Z \oplus T_h X & \xrightarrow[\quad d\eta \quad]{} & T_g Y \\ d\rho_{z^{-1}} \times \text{Ad } z \circ d\rho_{h^{-1}} \downarrow \simeq & & \simeq \downarrow d\rho_{g^{-1}} \\ T_e(Zz^{-1}) \oplus T_e(z(Xh^{-1})z^{-1}) & \xrightarrow[\quad d\eta \quad]{} & T_e(Yg^{-1}) \end{array}$$

By assumption, $d\eta$ is surjective, hence

$$T_e(Yg^{-1}) = T_e(Zz^{-1}) + \text{Ad } z(T_e(Xh^{-1})) \subseteq M_{Z_{reg}} + M_{zXz^{-1}} \subseteq M_{Z_{reg}} + L_{\mathfrak{H}}.$$

This holds for every $g \in \varphi(U)$ which is a dense subset of Y_{reg} . Thus, by Lemma 4.1.1(e), we finally get $M_{Y_{reg}} \subseteq M_{Z_{reg}} + L_{\mathfrak{H}}$, hence the claim. \square

Corollary 4.3.3. *Let X be an affine variety, and let $\mathfrak{H} \subseteq \text{Aut}(X)$ be a closed ind-subgroup. If \mathfrak{H} has finite codimension, then $\text{Lie } \mathfrak{H} \subseteq \text{Lie Aut}(X)$ has finite codimension, too.*

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MATHEMATISCHES INSTITUT, UNIVERSITÄT BASEL,
RHEINSPRUNG 21, CH-4051 BASEL, SWITZERLAND
Email address: hanspeter.kraft@unibas.ch

INSTITUT FOURIER, UMR 5582, LABORATOIRE DE MATHÉMATIQUES,
UNIVERSITÉ GRENOBLE ALPES, CS 40700, 38058 GRENOBLE CEDEX 9, FRANCE
Email address: mikhail.zaidenberg@univ-grenoble-alpes.fr