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# The neighbour sum distinguishing edge-weighting with local constraints

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## Abstract

A  $k$ -edge-weighting of  $G$  is a mapping  $\omega : E(G) \rightarrow \{1, \dots, k\}$ . The edge-weighting naturally induces a vertex colouring  $\sigma_\omega : V(G) \rightarrow \mathbb{N}$  given by  $\sigma_\omega(v) = \sum_{u \in N_G(v)} \omega(vu)$  for every  $v \in V(G)$ . The edge-weighting  $\omega$  is neighbour sum distinguishing if it yields a proper vertex colouring  $\sigma_\omega$ , *i.e.*,  $\sigma_\omega(u) \neq \sigma_\omega(v)$  for every edge  $uv$  of  $G$ .

We investigate a neighbour sum distinguishing edge-weighting with local constraints, namely, we assume that the set of edges incident to a vertex of large degree is not monochromatic. The graph is nice if it has no components isomorphic to  $K_2$ . We prove that every nice graph with maximum degree at most 5 admits a neighbour sum distinguishing  $(\Delta(G) + 2)$ -edge-weighting such that all the vertices of degree at least 2 are incident with at least two edges of different weights. Furthermore, we prove that every nice graph admits a neighbour sum distinguishing 7-edge-weighting such that all the vertices of degree at least 6 are incident with at least two edges of different weights. Finally, we show that nice bipartite graphs admit a neighbour sum distinguishing 6-edge-weighting such that all the vertices of degree at least 2 are incident with at least two edges of different weights.

# 1 Introduction

Let  $G$  be a graph and  $k \in \mathbb{N}$ . A *neighbour sum distinguishing  $k$ -edge-weighting* is a mapping  $\omega : E(G) \rightarrow \{1, \dots, k\}$  such that the induced vertex colouring  $\sigma_\omega : V(G) \rightarrow \mathbb{N}$  where  $\sigma_\omega(v) = \sum_{u \in N_G(v)} \omega(vu)$  is proper, *i.e.*  $\sigma_\omega(u) \neq \sigma_\omega(v)$  for every edge  $uv$  of  $G$ . Observe that  $G$  always admits such a neighbour sum distinguishing edge-weighting, unless it includes  $K_2$  as a component. Thus, we call  $G$  *nice* whenever it has no such component. We say that the edge-weighting  $\omega$  *distinguishes* vertices  $v, w \in V(G)$  if  $\sigma_\omega(v) \neq \sigma_\omega(w)$ .

In 2004 Karoński et al. [8] posed the conjecture, called the 1-2-3 Conjecture, that asks whether every nice graph admits a 3-edge-weighting that is neighbour sum distinguishing. The 1-2-3 Conjecture inspired a lot of studies on the original conjecture and variants of it. For more information on that topic, we refer the reader to the survey by Seamone [16]. The best result towards the 1-2-3 Conjecture is due to Kalkowski et al. [7], who proved that every nice graph admits a neighbour sum distinguishing 5-edge-weighting. The conjecture cannot be pushed further down, since there are graphs that require 3 weights, as an example, see cycles or complete graphs. It was proved by Dudek and Wajc [4] that deciding whether there is a neighbour sum distinguishing 2-edge-weighting for a given graph  $G$  is NP-complete in general, while Thomassen, Wu and Zhang [14] showed that the same problem is polynomial in the family of bipartite graphs. Recently Przybyło [12] proved that every  $d$ -regular graph ( $d \geq 2$ ) admits a neighbour sum distinguishing 4-edge-weighting and that the 1-2-3 Conjecture is true for  $d$ -regular graphs with  $d \geq 10^8$ .

In the version of the neighbour sum distinguishing edge-weighting, introduced by Karoński et al. [8], the edges incident with a vertex may have the same weight. On the other hand, Flandrin et al. [5] introduced the version of the edge-weighting, called a neighbour sum distinguishing  $k$ -edge colouring, which distinguishes vertices and in which incident edges must have different weights. A  *$k$ -edge-colouring* of  $G$  is a mapping  $\omega : E(G) \rightarrow \{1, \dots, k\}$  such that  $\omega(e_1) \neq \omega(e_2)$  for every two adjacent edges  $e_1, e_2 \in E(G)$ . If the  $k$ -edge colouring  $\omega$  satisfies  $\sigma_\omega(v) \neq \sigma_\omega(u)$  for every edge  $uv \in E(G)$ , then we call such a colouring a *neighbour sum distinguishing  $k$ -edge colouring*. The smallest value  $k$  for which  $G$  admits a neighbour sum distinguishing  $k$ -edge colouring is denoted by  $\chi'_\Sigma(G)$ . Wang and Yan [15] proved that  $\chi'_\Sigma(G) \leq \lceil (10\Delta(G) + 2)/3 \rceil$  when  $\Delta(G) \geq 18$ . It is known that  $\chi'_\Sigma(G) \leq 2\Delta(G) + \text{col}(G) - 1$  [10] and  $\chi'_\Sigma(G) \leq \Delta(G) + 3\text{col}(G) - 4$  [13], where  $\text{col}(G)$  denote the *colouring number* of  $G$ , *i.e.* the smallest integer  $k$  such that  $G$  has a vertex ordering in which each vertex is preceded by fewer than  $k$  of its neighbours. Recently, Przybyło [11] proved that  $\chi'_\Sigma(G) \leq \Delta + O(\sqrt{\Delta})$ , where  $\Delta = \Delta(G)$ .

In this paper, we consider an edge-weighting which allows a vertex to be incident with edges having the same weight, in a limited way. We require that a vertex of large degree is incident with at least two edges of different weights. Such a version is, on the one hand, stronger than the classical edge-weighting, while, on the other hand, it is weaker than the edge-colouring. Our paper is organized as follows. In Sections 2 and 3, we consider nice graphs with degree at most 4 and at most 5, respectively. We prove that every nice graph  $G$  with degree at most 5 admits a neighbour sum distinguishing  $(\Delta(G)+2)$ -edge-weighting such that all the vertices of degree at least 2 are incident with at least two edges of different weights. In Section 4, we prove that every nice graph admits a neighbour sum distinguishing 7-edge-weighting such that all the vertices of degree at least 6 are incident with at least two edges of different weights. In Section 5, we show that the result from Section 4 can be improved for bipartite graphs: we prove that every nice bipartite graph admits a neighbour sum distinguishing 6-edge-weighting such that all the vertices of degree at least 2 are incident with at least two edges of different weights. Furthermore, we show that every connected bipartite graph on at least three vertices having a vertex partition  $(V_1, V_2)$  such that  $|V_1|$  is even admits a neighbour sum distinguishing 4-edge-weighting such that every vertex of degree at least 2 is incident with at least two edges of different weights.

Another variant of the distinguishing edge colouring, called a neighbour sum distinguishing relaxed edge colouring, was introduced in [3]. A  $d$ -relaxed  $k$ -edge colouring is a mapping  $\omega : E(G) \rightarrow \{1, \dots, k\}$  such that each monochromatic set of edges induces a subgraph with maximum degree at most  $d$ . If a  $d$ -relaxed  $k$ -edge colouring  $\omega$  satisfies  $\sigma_\omega(v) \neq \sigma_\omega(u)$  for every edge  $uv \in E(G)$ , then it is called a *neighbour sum distinguishing  $d$ -relaxed  $k$ -edge colouring*. By  $\chi_\Sigma^d(G)$ , we denote the smallest value  $k$  for which  $G$  admits a neighbour sum distinguishing  $d$ -relaxed  $k$ -edge colouring. Hence,  $\chi_\Sigma^\Delta(G) = \chi_\Sigma'(G)$ . Observe that if  $G$  admits a neighbour sum distinguishing  $k$ -edge-weighting such that every vertex of degree at least 2 (or at least 6 for graphs with maximum degree at least 6) is incident with at least two edges of different weights, then  $\chi_\Sigma^{\Delta-1}(G) \leq k$ . In [3] it was proved that every nice subcubic graph with no component isomorphic to  $C_5$  admits a neighbour sum distinguishing 2-relaxed 4-edge colouring such that every vertex of degree two is incident with edges coloured differently. We will need this result to prove the theorem for graphs with maximum degree 4. This result can be equivalently rewritten in the following way:

**Theorem 1.** [3] *If  $G$  is a nice subcubic graph with no component isomorphic to  $C_5$ , then it admits a neighbour sum distinguishing 4-edge-weighting such that every vertex of degree at least 2 is incident with at least two edges of different weights.*

The following theorem by Alon [1] will be frequently used in arguments to prove results for graphs with maximum degree at most 5.

**Theorem 2** (Combinatorial Nullstellensatz [1]). *Let  $\mathbb{F}$  be an arbitrary field, and let  $P = P(x_1, \dots, x_n)$  be a polynomial in  $\mathbb{F}[x_1, \dots, x_n]$ . Suppose the degree  $\deg(P)$  of  $P$  equals  $\sum_{i=1}^n k_i$ , where each  $k_i$  is a nonnegative integer, and suppose the coefficient of  $x_1^{k_1} \cdots x_n^{k_n}$  in  $P$  is nonzero. Then if  $S_1, \dots, S_n$  are subsets of  $\mathbb{F}$  with  $|S_i| > k_i$ , there are  $s_1 \in S_1, \dots, s_n \in S_n$  such that  $P(s_1, \dots, s_n) \neq 0$ .*

## 2 Graphs with maximum degree at most 4

**Theorem 3.** *Every nice graph  $G$  with  $\Delta(G) \leq 4$  admits a neighbour sum distinguishing 6-edge-weighting such that all the vertices of degree at least 2 are incident with at least two edges of different weights.*

*Proof.* We proceed by induction on the number of edges. It is easy to see that the theorem is true for graphs with two and three edges. Assume that the theorem is true for graphs with at most  $m - 1$  edges. Let  $G$  be a graph with  $m$  edges. We may assume that  $G$  is connected, since otherwise, by induction, every component has a 6-edge-weighting that satisfies the theorem. Furthermore, we may assume that  $\Delta(G) = 4$ , because, by Theorem 1, the theorem is true for all nice subcubic graphs except  $C_5$ , and  $C_5$  admits a neighbour sum distinguishing 5-edge-weighting such that all the vertices are incident with edges of different weights. Let  $u$  be a vertex of degree 4 in  $G$ .

**Case 1. There is an edge in the subgraph induced by  $N(u)$**

Let  $N(u) = \{v, w, u_1, u_2\}$  and  $vw \in E(G)$ . Let  $G'$  be obtained from  $G$  by removing the two edges  $uv$  and  $uw$ .  $G'$  has at most two components. Each component of  $G'$  with at least two edges admits an edge-weighting that satisfies the theorem. If  $d_{G'}(v) \geq 2$  or  $d_{G'}(w) \geq 2$ , then every component has at least two edges, otherwise one component is isomorphic to  $K_2$ . Let  $\omega$  be an edge-weighting of components of  $G'$  with at least two edges that satisfies the theorem, and additionally we extend the edge-weighting  $\omega$  on the component isomorphic to  $K_2$  (if such exists), which we label with an arbitrary weight.

To obtain our final edge-weighting, we just need to label the two edges  $uv$  and  $uw$  while making sure that all the vertices of  $\{u, v, w\}$  are distinguished with their neighbours and the vertices  $v$  and  $w$  are incident with two edges of different weights. Note that the vertex  $u$  already has two incident edges of distinct weights, because  $d_{G'}(u) = 2$ . If  $d_{G'}(v) \geq 2$  and  $d_{G'}(w) \geq 2$ , then  $v$  and  $w$  also have two incident

edges of distinct weights, otherwise we have to choose a weight on  $uv$  and  $uw$  that is different from  $\omega(vw)$ .

First, we consider how many weights we have to forbid for the edges  $uv$  and  $uw$  such that we obtain an edge-weighting that distinguishes all adjacent vertices except the pairs  $(v, w), (u, u_1), (u, u_2)$  and such that all vertices of degree at least 2 are incident with two edges of different weights. The vertex  $v$  must be distinguished from its neighbours in  $G' - w$ . If  $v$  has two neighbours in  $G' - w$ , then there are potentially two forbidden weights for  $vu$ . Thus, four possible weights remain for  $uv$ . If  $v$  has exactly one neighbour in  $G' - w$ , then there is at most one forbidden weight for  $uv$ . If  $w$  is the only neighbour of  $v$  in  $G'$ , then the weight of  $uv$  must be different from the weight of  $vw$  and hence there is at most one forbidden weight for  $uv$ . Thus, summarizing, there are at most two forbidden weights for  $uv$ . Similarly, we can observe that for  $uw$  there are at most two forbidden weights. Let  $S_1$  be the set of weights that are not forbidden for  $uv$  and  $S_2$  be the set of weights that are not forbidden for  $uw$ , so  $|S_1| \geq 4$  and  $|S_2| \geq 4$ . Observe that if we choose for  $uv$  a weight from  $S_1$  and for  $uw$  a weight from  $S_2$ , then we obtain an edge-weighting of  $G$  in which all pairs of adjacent vertices, except  $(v, w), (u, u_1), (u, u_2)$ , are distinguished, and all vertices of degree at least 2 are incident with two edges of different weights. Let  $x_1 \in S_1$  and  $x_2 \in S_2$  be the weights attributed to  $uv$  and  $uw$ , respectively. To obtain an edge-weighting that satisfies the theorem for the weights  $x_1$  and  $x_2$ , we must have the following:

- $x_1 + x_2 + \sigma_\omega(u) \neq \sigma_\omega(u_i)$ , because  $u$  must be distinguished from  $u_i$  for  $i \in \{1, 2\}$ ;
- $x_2 + \sigma_\omega(u) \neq \sigma_\omega(v)$ , because we have to distinguish  $u$  and  $v$ ;
- $x_1 + \sigma_\omega(u) \neq \sigma_\omega(w)$ , because we have to distinguish  $u$  and  $w$ ;
- $x_1 + \sigma_\omega(v) \neq x_2 + \sigma_\omega(w)$ , because we have to distinguish  $v$  and  $w$ .

To prove that there are weights  $x_1$  and  $x_2$  that satisfy all the above conditions, we construct the polynomial:

$$\begin{aligned}
 P(x_1, x_2) = & (x_1 + x_2 + \sigma_\omega(u) - \sigma_\omega(u_1)) \\
 & (x_1 + x_2 + \sigma_\omega(u) - \sigma_\omega(u_2)) \\
 & (x_2 + \sigma_\omega(u) - \sigma_\omega(v)) \\
 & (x_1 + \sigma_\omega(u) - \sigma_\omega(w)) \\
 & (x_1 - x_2 + \sigma_\omega(v) - \sigma_\omega(w)).
 \end{aligned}$$

If there exist  $x_1$  and  $x_2$  such that  $P(x_1, x_2) \neq 0$  and  $x_i \in S_i$  ( $i \in \{1, 2\}$ ), then the  $x_i$  satisfy all the conditions. By labeling  $uv, uw$  with  $x_1, x_2$ , we can extend the edge-weighting  $\omega$  to an edge-weighting that satisfies the theorem. We apply Theorem 2

to prove that  $x_1$  and  $x_2$  exist. First, we claim that the coefficient of the monomial  $x_1^3x_2^2$  is non-zero. Observe that this coefficient in  $P$  is the same as in the following polynomial:

$$P_1(x_1, x_2) = (x_1 + x_2)^2(x_1 - x_2)x_1x_2.$$

The coefficient of the monomial  $x_1^3x_2^2$  is 1. Since  $|S_1| > 3$  and  $|S_2| > 2$ , Theorem 2 implies that there are  $x_1 \in S_1$  and  $x_2 \in S_2$  such that  $P(x_1, x_2) \neq 0$  and equivalently there is the desired edge-weighting of  $G$ .

**Case 2.  $N(u)$  is an independent set.**

Let  $N(u) = \{u_1, u_2, u_3, u_4\}$  and  $G' = G - u$ . Each component of  $G'$  with at least two edges admits an edge-weighting that satisfies the theorem. Let  $\omega$  be an edge-weighting of components of  $G'$  with at least two edges that satisfies the theorem, and additionally we extend the edge-weighting  $\omega$  to the components isomorphic to  $K_2$  (if such exist), which we label with arbitrary weights.

To obtain a final edge-weighting, we just need to label the edges  $uu_i$  for  $i \in \{1, 2, 3, 4\}$ . We choose a weight for  $uu_i$  in such a way that ensures that  $u_i$  is distinguished with its neighbours in  $G'$  and if  $u_i$  has exactly one neighbour in  $G'$ , then the weight of  $uu_i$  is different from the weight of the edge incident with  $u_i$  in  $G'$ . Furthermore, after labeling the four edges  $uu_1, uu_2, uu_3, uu_4$ , the vertex  $u$  must be distinguished from its neighbours and these edges cannot be monochromatic.

First, we consider how many weights we have to forbid for edges  $uu_i$  such that we obtain an edge-weighting in which the pairs of adjacent vertices of  $G'$  are still distinguished and all vertices of  $G'$  are incident with two edges of distinct weights. Since  $u_i$  must be distinguished from its neighbours in  $G'$ , we have at most three forbidden weights for  $uu_i$ . If  $u_i$  has exactly one neighbour in  $G'$ , then in order to distinguish  $u_i$  from its neighbour there is at most one forbidden weight and the weight of  $uu_i$  must be different from the weight of the edge incident with  $u_i$  in  $G'$ , so together we have at most two forbidden weights. Thus,  $uu_i$  has at most three forbidden weights. Let  $S_i$  be a set of weights that are not forbidden for  $uu_i$ , thus  $|S_i| \geq 3$  for  $i \in \{1, 2, 3, 4\}$ . After labeling the edge  $uu_i$  with weight  $x_i \in S_i$  for  $i \in \{1, 2, 3, 4\}$  we obtain an edge-weighting that distinguishes all vertices of  $G'$  and every vertex of  $G'$  is incident with at least two edges of different weights. Let  $x_i \in S_i$  be weights attributed to  $uu_i$  for  $i \in \{1, 2, 3, 4\}$ . To obtain an edge-weighting that satisfies the theorem for  $x_i$ , it must additionally hold:

- $x_1 + x_2 + x_3 + x_4 - x_i \neq \sigma_\omega(u_i)$ , because we have to distinguish  $u$  and  $u_i$  for  $i \in \{1, 2, 3, 4\}$ ;

- $x_i \neq x_j$  for some  $i, j \in \{1, 2, 3, 4\}$ , because  $u$  must be incident with at least two edges of different weights.

We consider the polynomial

$$\begin{aligned}
P(x_1, x_2, x_3, x_4) = & (x_2 + x_3 + x_4 - \sigma_\omega(u_1)) \\
& (x_1 + x_3 + x_4 - \sigma_\omega(u_2)) \\
& (x_1 + x_2 + x_4 - \sigma_\omega(u_3)) \\
& (x_1 + x_2 + x_3 - \sigma_\omega(u_4)) \\
& (x_3 - x_4).
\end{aligned}$$

If there exist  $x_1, x_2, x_3, x_4$  such that  $P(x_1, x_2, x_3, x_4) \neq 0$  and  $x_i \in S_i$  ( $i \in \{1, 2, 3, 4\}$ ), then the  $x_i$  satisfy all the conditions. By labeling  $uu_i$  with  $x_i$ , we can extend the edge-weighting  $\omega$  to an edge-weighting that satisfies the theorem. To prove that there are such  $x_i$  we again apply Theorem 2. We consider the coefficient of the monomial  $x_1^2 x_2 x_3^2$ . Observe that this coefficient in  $P$  is the same as in the following polynomial:

$$P_1(x_1, x_2, x_3, x_4) = (x_2 + x_3 + x_4)(x_1 + x_3 + x_4)(x_1 + x_2 + x_4)(x_1 + x_2 + x_3)(x_3 - x_4).$$

The coefficient of the monomial  $x_1^2 x_2 x_3^2$  is non-zero. Since  $|S_1| > 2$ ,  $|S_2| > 1$  and  $|S_3| \geq 2$ , Theorem 2 implies that there are  $x_i \in S_i$  such that  $P(x_1, x_2, x_3, x_4) \neq 0$  and so an edge-weighting of  $G$  that satisfies the theorem exists. □

### 3 Graphs with maximum degree at most 5

**Theorem 4.** *Every nice graph  $G$  with  $\Delta(G) \leq 5$  admits a neighbour sum distinguishing 7-edge-weighting such that all the vertices of degree at least 2 are incident with at least two edges of different weights.*

*Proof.* We proceed by induction on the number of edges. It is easy to see that the theorem is true for graphs with two, three and four edges. Assume that the theorem is true for graphs with at most  $m - 1$  edges. Let  $G$  be a graph with  $m$  edges. We may assume that  $G$  is connected, since otherwise, by induction, every component admits a 7-edge-weighting that satisfies the theorem. Furthermore, by Theorem 3, we may assume that  $\Delta(G) = 5$  since, otherwise, the result holds. Let  $u$  be a vertex of degree 5 in  $G$ .

**Case 1. There is an edge in the subgraph induced by  $N(u)$**

Let  $N(u) = \{v, w, u_1, u_2, u_3\}$  and  $vw \in E(G)$ .

First, we consider the case where  $d_G(v) \leq 3$  or  $d_G(w) \leq 3$ , say without loss of generality  $d_G(v) \leq 3$ . Let  $G'$  be the graph obtained by removing from  $G$  the two edges  $uv$  and  $uw$ . Each component of  $G'$  with at least two edges admits an edge-weighting that satisfies the theorem. Let  $\omega$  be an edge-weighting of components of  $G'$  with at least two edges that satisfies the theorem, and additionally we extend the edge-weighting  $\omega$  on the components isomorphic to  $K_2$ , which we label with arbitrary weights.

To obtain our desired edge-weighting, we need to label the two edges  $uv$  and  $uw$ , making sure that the vertices  $u, v$  and  $w$  are distinguished from their neighbours and the vertices  $v$  and  $w$  are incident with two edges of distinct weights. Note that the vertex  $u$  already verifies this property, since  $d_{G'}(u) = 3$ .

First, we consider how many weights we have to forbid for edges  $uv$  and  $uw$  for us to obtain an edge-weighting that distinguishes all adjacent vertices except the pairs  $(v, w), (u, u_1), (u, u_2), (u, u_3)$  and in which all vertices of degree at least 2 are incident with two edges of different weights. The vertex  $v$  must be distinguished from its neighbour in  $G' - w$ . If  $v$  has one neighbour in  $G' - w$ , then there is potentially one forbidden weight for  $vu$ , such that  $v$  is just incident with two edges labeled differently. If  $v$  has no neighbour in  $G' - w$ , then the weight of  $uv$  must be different from the weight of  $vw$ , so again there is one forbidden weight for  $uv$ . Thus, there are 6 possible weights for  $uv$ . To distinguish  $w$  from its neighbours in  $G' - w$  there are at most three forbidden weights. If  $w$  is the only neighbour of  $v$  in  $G'$ , then the weight of  $uw$  must be different from the weight of  $vw$  and hence there is at most one forbidden weight for  $uw$ . In conclusion, there are at least 4 possible weights for  $uw$ . Let  $S_1$  be the set of weights that are not forbidden for  $uv$  and  $S_2$  be the set of weights that are not forbidden for  $uw$ , so  $|S_1| \geq 6$  and  $|S_2| \geq 4$ . To prove that we can choose weights from  $S_1$  and  $S_2$  such that we result in an edge-weighting that satisfies the conditions of the theorem, we use Theorem 2. Let  $x_1 \in S_1$  and  $x_2 \in S_2$  be weights attributed to  $uv$  and  $uw$ , respectively. To obtain the final edge-weighting, the weights  $x_1$  and  $x_2$  must additionally verify:

- $x_1 + x_2 + \sigma_\omega(u) \neq \sigma_\omega(u_i)$ , because  $u$  must be distinguished from  $u_i$ , for  $i = 1, 2, 3$ ;
- $x_2 + \sigma_\omega(u) \neq \sigma_\omega(v)$ , because we have to distinguish  $u$  and  $v$ ;
- $x_1 + \sigma_\omega(u) \neq \sigma_\omega(w)$ , because we have to distinguish  $u$  and  $w$ ;
- $x_1 + \sigma_\omega(v) \neq x_2 + \sigma_\omega(w)$ , because we have to distinguish  $v$  and  $w$ .

We construct a polynomial

$$\begin{aligned}
P(x_1, x_2) = & \prod_{i=1,2,3} (x_1 + x_2 + \sigma_\omega(u) - \sigma_\omega(u_i)) \\
& (x_2 + \sigma_\omega(u) - \sigma_\omega(v)) \\
& (x_1 + \sigma_\omega(u) - \sigma_\omega(w)) \\
& (x_1 - x_2 + \sigma_\omega(v) - \sigma_\omega(w)).
\end{aligned}$$

We consider the coefficient of the monomial  $x_1^5 x_2$ . Observe that this coefficient in  $P$  is the same as in the following polynomial:

$$P_1(x_1, x_2) = (x_1 + x_2)^3 (x_1 - x_2) x_1 x_2.$$

The coefficient of the monomial  $x_1^5 x_2$  is 1. Since  $|S_1| > 5$  and  $|S_2| > 1$ , Theorem 2 implies that there are  $x_1 \in S_1$  and  $x_2 \in S_2$  such that  $P(x_1, x_2) \neq 0$  and equivalently we can construct the desired edge-weighting of  $G$ .

Consider now the case when  $d_G(v) \geq 4$  and  $d_G(w) \geq 4$ . Let  $G'$  be obtained from  $G$  by removing the three edges  $uv, uw$  and  $vw$ . Each component of  $G'$  has at least two edges, so it admits an edge-weighting that satisfies the theorem. Let  $\omega$  be an edge-weighting of components of  $G'$  that satisfies the theorem. Observe that in  $G'$  the vertices  $u, v, w$  are just incident with at least two edges of different weights, since  $d_{G'}(u) = 3, d_{G'}(v) \geq 2$  and  $d_{G'}(w) \geq 2$ .

Let  $x_1, x_2$  and  $x_3$  be weights attributed to  $uv, uw$  and  $vw$ , respectively. To obtain an edge-weighting of  $G$  that satisfies the theorem,  $x_1$  and  $x_2$  must verify:

- $x_1 + x_2 + \sigma_\omega(u) \neq \sigma_\omega(u_i)$ , because  $u$  must be distinguished from  $u_i$  for  $i \in \{1, 2, 3\}$ ;
- $x_1 + x_3 + \sigma_\omega(v) \neq \sigma_\omega(v_i)$ , where  $i \in \{1, 2\}$  if  $v$  has two neighbours  $v_1, v_2$  in  $G'$  and  $i \in \{1, 2, 3\}$  if  $v$  has three neighbours  $v_1, v_2, v_3$  in  $G'$ , because  $v$  must be distinguished from its neighbours in  $G'$ ;
- $x_2 + x_3 + \sigma_\omega(w) \neq \sigma_\omega(w_i)$ , where  $i \in \{1, 2\}$  if  $w$  has two neighbours  $w_1, w_2$  in  $G'$  and  $i \in \{1, 2, 3\}$  if  $w$  has three neighbours  $w_1, w_2, w_3$  in  $G'$ , because  $w$  must be distinguished from its neighbours in  $G'$ ;
- $x_1 + \sigma_\omega(v) \neq x_2 + \sigma_\omega(w)$ , because  $v$  must be distinguished from  $w$ ;
- $x_1 + \sigma_\omega(u) \neq x_3 + \sigma_\omega(w)$ , because  $u$  must be distinguished from  $w$ ;
- $x_2 + \sigma_\omega(u) \neq x_3 + \sigma_\omega(v)$ , because  $u$  must be distinguished from  $v$ .

We construct a polynomial

$$\begin{aligned}
P(x_1, x_2, x_3) = & \prod_{i=1,2,3} (x_1 + x_2 + \sigma_\omega(u) - \sigma_\omega(u_i)) \\
& \prod_{i=1,2,3} (x_1 + x_3 + \sigma_\omega(v) - \sigma_\omega(v_i)) \\
& \prod_{i=1,2,3} (x_2 + x_3 + \sigma_\omega(w) - \sigma_\omega(w_i)) \\
& (x_1 + \sigma_\omega(v) - x_2 - \sigma_\omega(w)) \\
& (x_1 + \sigma_\omega(u) - x_3 - \sigma_\omega(w)) \\
& (x_2 + \sigma_\omega(u) - x_3 - \sigma_\omega(v)).
\end{aligned}$$

If there are  $x_i \in \{1, \dots, 7\}$  ( $i \in \{1, 2, 3\}$ ) such that  $P(x_1, x_2, x_3) \neq 0$ , then by labeling  $uv, uw, vw$  with  $x_1, x_2, x_3$  we can extend the edge-weighting  $\omega$  of  $G'$  to an edge-weighting of  $G$  that satisfies the theorem whenever  $d_G(v) = d_G(w) = 5$ . If  $d_G(v) = 4$  or  $d_G(w) = 4$ , then the polynomial  $R$ , which we should construct for proving that the weights  $x_1, x_2, x_3$  exist, is a factor of  $P(x_1, x_2, x_3)$ . However, if  $P(x_1, x_2, x_3) \neq 0$ , then also for the factor  $R$  we have  $R(x_1, x_2, x_3) \neq 0$ . So it is enough to consider the polynomial  $P$ .

To prove that there are  $x_i \in \{1, \dots, 7\}$  ( $i \in \{1, 2, 3\}$ ) such that  $P(x_1, x_2, x_3) \neq 0$  we apply Theorem 2. Consider the coefficient of the monomial  $x_1^5 x_2^4 x_3^3$ . Observe that this coefficient in  $P$  is the same as in the following polynomial:

$$P(x_1, x_2, x_3) = (x_1 + x_2)^3 (x_1 + x_3)^3 (x_2 + x_3)^3 (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

The coefficient of the monomial  $x_1^5 x_2^4 x_3^3$  is 2. Theorem 2 implies that there are  $x_i \in \{1, \dots, 7\}$  such that  $P(x_1, x_2, x_3) \neq 0$  and equivalently there is the desired edge-weighting of  $G$ .

**Case 2.  $N(u)$  is an independent set.**

This part of the proof is very similar to **Case 2** of the proof of Theorem 3. Let  $N(u) = \{u_1, u_2, u_3, u_4, u_5\}$ . Let  $G' = G - u$ . Each component of  $G'$  with at least two edges has an edge-weighting that satisfies the theorem. Let  $\omega$  be an edge-weighting of components of  $G'$  with at least two edges that satisfies the theorem, and additionally we extend the edge-weighting  $\omega$  to the components isomorphic to  $K_2$ , which we label with an arbitrary weight.

First, we consider how many weights we have to forbid for edges  $uu_i$  such that we result in an edge-weighting in which the pairs of adjacent vertices of  $G'$  are still

distinguished and all vertices of  $G'$  are incident with two edges of distinct weights. Since the vertex  $u_i$  must be distinguished from its neighbours in  $G'$  and  $d_{G'}(u_i) \leq 4$ , we have at most four forbidden weights for  $uu_i$ . If  $u_i$  has exactly one neighbour in  $G'$ , then in order to distinguish  $u_i$  from its neighbour there is at most one forbidden weight and the weight of  $uu_i$  must be different from the weight of the edge incident with  $u_i$  in  $G'$ , so together we have at most two forbidden weights. Let  $S_i$  be the set of weights that are not forbidden for  $uu_i$ , thus  $|S_i| \geq 3$  for  $i \in \{1, 2, 3, 4, 5\}$ . After labeling the edge  $uu_i$  with weight  $x_i \in S_i$  for  $i \in \{1, 2, 3, 4, 5\}$ , we obtain an edge-weighting that distinguishes all vertices of  $G'$  and every vertex of  $G'$  is incident with at least two edges of different weights. Let  $x_i \in S_i$  be weights attributed to  $uu_i$  for  $i \in \{1, 2, 3, 4, 5\}$ . To obtain an edge-weighting that satisfies the theorem, the weights  $x_i$  must additionally verify:

- $x_1 + x_2 + x_3 + x_4 + x_5 - x_i \neq \sigma_\omega(u_i)$ , because we have to distinguish  $u$  and  $u_i$  for  $i \in \{1, 2, 3, 4, 5\}$ ;
- $x_i \neq x_j$  for some  $i, j \in \{1, 2, 3, 4, 5\}$ , because  $u$  must be adjacent to at least two edges of different weights.

We construct a polynomial

$$\begin{aligned}
P(x_1, x_2, x_3, x_4, x_5) = & (x_2 + x_3 + x_4 + x_5 - \sigma_\omega(u_1)) \\
& (x_1 + x_3 + x_4 + x_5 - \sigma_\omega(u_2)) \\
& (x_1 + x_2 + x_4 + x_5 - \sigma_\omega(u_3)) \\
& (x_1 + x_2 + x_3 + x_5 - \sigma_\omega(u_4)) \\
& (x_1 + x_2 + x_3 + x_4 - \sigma_\omega(u_5)) \\
& (x_3 - x_4).
\end{aligned}$$

If there are  $x_i \in S_i$  ( $i \in \{1, 2, 3, 4, 5\}$ ) such that  $P(x_1, x_2, x_3, x_4, x_5) \neq 0$ , then, by labeling  $uu_i$  with  $x_i$ , we extend the edge-weighting  $\omega$  to an edge-weighting that satisfies the theorem. We again apply Theorem 2 to prove that there are such  $x_i$ 's. We consider the coefficient of the monomial  $x_1^2 x_2 x_3^2$ . Observe that this coefficient in  $P$  is the same as in the following polynomial:

$$\begin{aligned}
P_1(x_1, x_2, x_3, x_4, x_5) = & (x_2 + x_3 + x_4 + x_5)(x_1 + x_3 + x_4 + x_5)(x_1 + x_2 + x_4 + x_5) \\
& (x_1 + x_2 + x_3 + x_5)(x_1 + x_2 + x_3 + x_4)(x_3 - x_4).
\end{aligned}$$

The coefficient of the monomial  $x_1^2 x_2 x_3^2$  is non-zero. Since  $|S_1| > 2$ ,  $|S_2| > 2$  and  $|S_3| > 2$ , Theorem 2 implies that there are  $x_i \in S_i$  such that  $P(x_1, x_2, x_3, x_4, x_5) \neq 0$  and so we can construct the desired edge-weighting of  $G$ .  $\square$

## 4 Graphs with maximum degree at least 6

In this section, we prove that every nice graph admits a neighbour sum distinguishing 7-edge-weighting such that every vertex of degree at least 6 is incident with at least two edges of different weights. Our approach is based on the algorithm given in [8] for proving that every nice graph admits a neighbour sum distinguishing 5-edge weighting. It is worth mentioning that modifications of that algorithm allowed getting new results for the neighbour sum distinguishing edge-weighting and its variants. For example, Bensmail [2] proved that every 5-regular graph admits a neighbour sum distinguishing 4-edge-weighting and Gao et al. [6] proved that the 1-2-3 Conjecture is true if we allow the vertices with the same incident sum to induce a forest.

We prove the following theorem:

**Theorem 5.** *Every nice graph  $G$  admits a neighbour sum distinguishing 7-edge-weighting of  $G$  such that all the vertices of degree at least 6 are incident with at least two edges of different weights.*

*Rough ideas of the proof of Theorem 5*

We give an algorithm which constructs a vertex assignment  $w$  and a 7-edge-weighting  $\omega$ . The vertex-assignment  $w$  will be almost the vertex colouring  $\sigma_\omega$ , namely  $\sigma_\omega(u) = w(u)$  or  $\sigma_\omega(u) = w(u) + 3$  for  $u \in V(G)$ . The 7-edge-weighting  $\omega$  will satisfy the conditions of Theorem 5. The algorithm processes the vertices one after another, following a special ordering. First, we define that ordering and prove that every nice graph, except stars, admits such an ordering of vertices. Then, we give the algorithm and prove that every step of the algorithm is always executable. Finally, we prove that the 7-edge-weighting  $\omega$  given by the algorithm is neighbour sum distinguishing and that all vertices of degree at least 6 are incident with at least two edges of different weights.

Before we define the ordering of vertices (in Lemma 8) we need the following notations.

Let  $(v_1, v_2, \dots, v_n)$  be an ordering of vertices of  $G$ . We say that  $v_j$  follows  $v_i$  in the ordering if  $i < j$ . A predecessor (resp. successor) of  $v_i$  is every neighbour of  $v_i$  in  $\{v_1, \dots, v_{i-1}\}$  (resp. in  $\{v_{i+1}, \dots, v_n\}$ ) for  $i \in \{1, \dots, n\}$ . Let us define a partial ordering induced by a given vertex ordering  $(v_1, v_2, \dots, v_n)$  in the following way

$v_j \prec v_i \Leftrightarrow$  there is a path  $v_j v_{k_1} v_{k_2} \dots v_{k_d} v_i$  in  $G$  such that  $j < k_1 < k_2 < \dots < k_d < i$ .

**Remark 6.** *Two different vertex orderings of the graph  $G$  may induce the same partial ordering.*

**Remark 7.** If  $y \prec x$ , then  $x$  has a predecessor and  $y$  has a successor.

An *inversion* of the ordering  $(v_1, v_2, \dots, v_n)$  is the ordering  $(v_n, v_{n-1}, \dots, v_1)$ .

**Lemma 8.** Let  $G$  be a connected graph on  $n$  vertices and  $G \neq K_{1,n-1}$ . There is a vertex ordering  $(v_1, v_2, \dots, v_n)$  of  $G$  such that

- (i)  $d(v_1) \geq 2$  and  $d(v_2) \geq 2$ ;
- (ii)  $v_i$  has a predecessor for  $i \in \{2, \dots, n\}$ ;
- (iii) if  $v_i$  has no successor, then, in  $N_G(v_i)$ , there is at most one vertex having a successor in  $\{v_{i+1}, \dots, v_n\}$  for  $i \in \{1, \dots, n\}$ .

**Remark 9.** The condition (ii) can be equivalently replaced by the following one:  $v_1 \prec v_i$  for  $i \in \{2, \dots, n\}$ .

*Proof of Lemma 8.* It is easy to see that if  $G$  is a connected graph and  $G$  is not a star, then there is an ordering that satisfies the conditions (i) and (ii). On the contrary, suppose that there is no ordering that satisfies (i), (ii), and (iii). For an ordering  $\mathbf{v} = (v_1, v_2, \dots, v_n)$  by  $B(\mathbf{v})$  we denote the set of vertices which have no successor.

Let  $\mathbf{v} = (v_1, v_2, \dots, v_n)$  be an ordering that satisfies (i) and (ii) with minimum  $|B(\mathbf{v})|$ .

Let  $\prec$  be the partial ordering induced by  $\mathbf{v}$  and  $v$  be the first vertex in  $\mathbf{v}$  for which (iii) fails, so  $v \in B(\mathbf{v})$ . Let  $\mathbf{v}'$  be an ordering of  $V(G)$  which induces the same partial ordering  $\prec$  as  $\mathbf{v}$ , but in which the index of  $v$  is minimum and let  $v = v_i$  in  $\mathbf{v}'$ . Observe that every vertex has the same predecessors and successors in both orderings, so  $|B(\mathbf{v}')| = |B(\mathbf{v})|$  and the vertex  $v$  still makes (iii) fails in the ordering  $\mathbf{v}'$ . Furthermore, the choice of  $\mathbf{v}'$  implies that for any  $x \in \{v_1, \dots, v_{i-1}\}$  we have  $x \prec v_i$ . Let  $j$  be the largest integer smaller than  $i$  such that  $v_j$  is a predecessor of  $v_i$  and  $v_j$  has a successor in  $\{v_{i+1}, \dots, v_n\}$ .

**Case 1.**  $j > 3$

Let  $\mathbf{w} = (v_j, v_{k_1}, v_{k_2}, \dots, v_{k_\ell})$  be a subordering of  $\mathbf{v}'$  containing  $v_j$  and all vertices  $x$  such that  $v_j \prec x \prec v_i$ . Let  $\mathbf{w}'$  be the inverse of  $\mathbf{w}$ . We reorder the vertices of  $\mathbf{v}'$  in the following way:  $\mathbf{v}'' = (v_1, \dots, v_{j-1}, \dots, v_i, \mathbf{w}', v_{i+1}, \dots, v_n)$ . Let  $\prec'$  be the partial ordering induced by  $\mathbf{v}''$ . Since  $v_j$  was the last predecessor of  $v_i$  having a successor in  $\{v_{i+1}, \dots, v_n\}$ ,  $v_i$  still has predecessor in  $\mathbf{v}''$  and now  $v_i$  has a successor. Furthermore, for any  $x \in \mathbf{w}' \setminus \{v_j\}$  we have  $v_i \prec' x \prec' v_j$  and hence every vertex of  $\mathbf{w}' \setminus \{v_j\}$  has a predecessor and a successor. Also  $v_j$  has a predecessor and a successor in  $\mathbf{v}''$ . Thus  $\mathbf{v}''$  satisfies the conditions (i) and (ii) and  $|B(\mathbf{v}'')| < |B(\mathbf{v})|$ , a contradiction.

**Case 2.**  $j \leq 2$

Since  $v_i$  has at least two predecessors having a successor in  $\{v_{i+1}, \dots, v_n\}$ ,  $j = 2$  and  $v_1, v_2$  have successors in  $\{v_{i+1}, \dots, v_n\}$ . Furthermore,  $v_1, v_2$  are the only predecessor of  $v_i$  having successors in  $\{v_{i+1}, \dots, v_n\}$ . If  $i = 3$ , then we reorder the vertices of  $\mathbf{v}'$  in the following way:  $\mathbf{v}'' = (v_1, v_3, v_2, v_4, \dots, v_n)$ . In  $\mathbf{v}''$  the vertex  $v_3$  has a successor, so  $|B(\mathbf{v}'')| < |B(\mathbf{v})|$  and  $\mathbf{v}''$  satisfies the conditions (i) and (ii), a contradiction. Suppose that  $i > 3$ . The condition (ii) implies that  $v_3$  is adjacent to  $v_2$  or  $v_1$ . If  $v_3v_2 \in E(G)$ , then we reorder  $\mathbf{v}'$  in the following way:  $\mathbf{v}'' = (v_2, v_3, v_4, \dots, v_i, v_1, v_{i+1}, \dots, v_n)$ , otherwise, we reorder  $\mathbf{v}'$  in the following way:  $\mathbf{v}'' = (v_1, v_3, v_4, \dots, v_i, v_2, v_{i+1}, \dots, v_n)$ . In both cases,  $\mathbf{v}''$  satisfies the conditions (i) and (ii) and  $|B(\mathbf{v}'')| < |B(\mathbf{v})|$ , a contradiction.  $\square$

**ALGORITHM**

Let  $G$  be an  $n$ -vertex connected graph and  $G \neq K_{1,n-1}$ . Let  $\mathbf{v} = (v_1, v_2, \dots, v_n)$  be a vertex ordering that satisfies the conditions (i)–(iii) of Lemma 8. Let

$$V' = \{v_i \in \{v_1, v_2, \dots, v_n\} : v_i \text{ has a successor}\},$$

$$V'' = \{v_i \in \{v_1, v_2, \dots, v_n\} : v_i \text{ has no successor}\}.$$

We start by assigning the provisional weight 4 to every edge, then we process the  $v'_i$ s one after another, following the ordering  $\mathbf{v}$ . Whenever we treat a new vertex  $v_i$ , we modify the weights of the edges incident with  $v_i$  under some restrictions and, at the end of the step, we define  $w(v_i)$  as the sum of the weights of the edges incident with  $v_i$  at the end of the step  $i$ . The weights of edges must be in  $\{1, \dots, 7\}$ .

In the  $i$ -th step of ALGORITHM we treat vertex  $v_i$ , however, we merge the first and the second steps of ALGORITHM, the vertices  $v_1$  and  $v_2$  are treated together. Then, we consider the remaining vertices according to the ordering  $\mathbf{v}$ . We assume  $\omega_1(e) = 4$  for any  $e \in E(G)$ . Let  $\omega_2$  be the edge-weighting after the second step of ALGORITHM,  $\omega_i$  be the edge-weighting after treating the vertex  $v_i$  (i.e. after  $i$ -th step of ALGORITHM), and finally  $\omega_n = \omega$ .

**Step 1,2**

We have  $\sigma_{\omega_1}(v_1) = 4d_G(v_1)$  and  $\sigma_{\omega_1}(v_2) = 4d_G(v_2)$ . Observe that  $4d \in \{0, 2, 4\} \pmod{6}$  for every integer  $d$ . Let  $e_1$  be the edge between  $v_1$  and its first successor distinct from  $v_2$ , let  $e_2$  be the edge between  $v_2$  and its first successor. In Table 1 we give the new weights of edges  $v_1v_2, e_1, e_2$ .

We then put  $w(v_1) := \sigma_{\omega_2}(v_1), w(v_2) := \sigma_{\omega_2}(v_2)$ .

Observe that after the first and the second steps of ALGORITHM, the vertex assignment  $w$  and the edge-weighting  $\omega_2$  have the following properties.

$(4d(v_1), 4d(v_2))$ (mod 6)	(0,0)	(0,2)	(2,0)	(0,4)	(4,0)	(2,2)	(2,4)	(4,2)	(4,4)
$\omega_2(v_1, v_2)$	7	7	7	5	5	5	6	6	2
$\omega_2(e_1)$	1	1	1	3	1	3	2	4	4
$\omega_2(e_2)$	2	1	1	1	3	1	4	2	3

Table 1: Step 1,2 of ALGORITHM.

- Observation 10.**
- $\sigma_{\omega_2}(v_1), \sigma_{\omega_2}(v_1) \in \{0, 1, 2\} \pmod{6}$ ,
  - $w(v_1) \neq w(v_2)$ , namely  $w(v_1) \not\equiv w(v_2) \pmod{6}$ ,
  - the weight of the first successor of  $v_i$  is at most 4 for  $i \in \{1, 2\}$ .

**Step  $i$ ,**  $i \in \{3, \dots, n\}$

Let  $v_k$  be the first successor of  $v_i$ . For an edge  $e$  the weight  $w(e)$  can only be modified if either  $e = v_j v_i$  with  $j < i$  or  $e = v_i v_k$ . The weight of every edge must be in  $\{1, \dots, 7\}$ . Furthermore, the modification of weights has to result in an edge-weighting  $\omega_i$  that satisfies the following properties:

- (1)  $\omega_i(v_i v_k) \leq 4$ .
- (2) If  $v_i \in V'$ , then  $\sigma_{\omega_i}(v_i) \in \{0, 1, 2\} \pmod{6}$ .
- (3) For  $j < i$ ,  $v_j \in N(v_i)$ 
  - (i) if  $v_i \in V'$ , then  $\sigma_{\omega_i}(v_i) \neq w(v_j)$ .
  - (ii) if  $v_i \in V''$ , then
    - $\sigma_{\omega_i}(v_i) \neq \sigma_{\omega_i}(v_j)$ , when  $v_j$  has no successor that follows  $v_i$ ,
    - $\sigma_{\omega_i}(v_i) \notin \{w(v_j), w(v_j) + 3\}$ , when  $v_j$  has a successor that follows  $v_i$ .
- (4) For  $j < i$ ,  $v_j \in N(v_i)$ ,  $\sigma_{\omega_i}(v_j) \in \{w(v_j), w(v_j) + 3\}$ .
- (5) If  $d(v_i) \geq 6$ , then edges  $\{v_j v_i : j < i, v_j \in N(v_i)\}$  are not monochromatic or the weight of edges  $\{v_j v_i : j < i, v_j \in N(v_i)\}$  is not in  $\{\omega_i(v_i v_k), \omega_i(v_i v_k) + 3\}$ .

When we obtain an edge-weighting that satisfies properties (1)–(5), we assign  $w(v_i) := \sigma_{\omega_i}(v_i)$ .

We will often use the following property of the vertex-assignment  $w$  given by ALGORITHM:

**Observation 11.** *If  $u \in V'$ , then  $w(u) \in \{0, 1, 2\} \pmod{6}$ .*

**Lemma 12.** *Every step  $i$  ( $i \in \{3, \dots, n\}$ ) of ALGORITHM is executable.*

*Proof.* Let us consider the  $i$ -th step of ALGORITHM. We prove that we can modify the weights of edges between  $v_i$  and its predecessors, and between  $v_i$  and its first successor (if it exists), in such a way that we obtain an edge-weighting that satisfies the properties (1)–(5). We consider two cases, whether  $v_i$  has a successor or not, each leading to several subcases.

**Case 1**  $v_i \in V'$ , *i.e.*,  $v_i$  has a successor

Let  $v_{j_1}, v_{j_2}, \dots, v_{j_d}$  be the predecessors of  $v_i$  and  $v_{j_\ell} v_i = e_\ell$  for  $\ell \in \{1, \dots, d\}$ . Let  $e'$  be the edge that joins  $v_i$  with its first successor. Recall that  $\omega_{i-1}(e') = 4$ ,  $\omega_{i-1}(e_\ell) = 4$  if  $v_i$  is not the first successor of  $v_{j_\ell}$ , and  $\omega_{i-1}(e_\ell) \leq 4$  if  $v_i$  is the first successor of  $v_{j_\ell}$ . We put the lower possible weights on every  $e_\ell$  for  $\ell \in \{1, \dots, d\}$ , *i.e.* we provisionally modify weights in the following way:  $\omega'_{i-1}(e_\ell) := \omega_{i-1}(e_\ell) - 3$  if  $\sigma_{\omega_{i-1}}(v_{j_\ell}) = w(v_{j_\ell}) + 3$  and  $\omega'_{i-1}(e) := \omega_{i-1}(e)$ , otherwise. Observe that such a modification results in weights that belong to  $\{1, \dots, 4\}$ , since  $\omega_{i-1}(e_\ell) < 4$  only if  $v_i$  is the first successor of  $v_{j_\ell}$  and then  $\sigma_{\omega_{i-1}}(v_{j_\ell}) = w(v_{j_\ell})$ , otherwise  $\omega_{i-1}(e_\ell) = 4$ . To simplify the notations, we state  $\omega_{i-1} := \omega'_{i-1}$ . After such a modification we have  $\sigma_{\omega_{i-1}}(v_{j_\ell}) = w(v_{j_\ell})$  for  $\ell \in \{1, \dots, d\}$ .

We will modify edges by adding 3 to  $e_\ell$  for some  $\ell \in \{1, \dots, d\}$  or subtracting 1, 2 or 3 from the weight of  $e'$ . As we observe above, by adding 3 to  $e_\ell$ , the weight of  $e_\ell$  is still in  $\{1, \dots, 7\}$ . If we subtract 1, 2, or 3 from the weight of  $e'$ , then the weight of  $e'$  is in  $\{1, 2, 3\}$ , since  $\omega_{i-1}(e') = 4$ . Furthermore, observe that adding 3 to some  $e_\ell$  or subtracting 1, 2 or 3 from the weight of  $e'$  maintains the properties (1) and (4). We show now that, by such a modification of weights, we are able to result in the edge-weighting that also satisfies the properties (2),(3), and (5).

By the modification weights of edges  $e_1, \dots, e_d, e'$ , we see that  $\sigma_{\omega_i}(v_i)$  can take any value in the interval  $[\alpha - 3, \alpha - 2, \dots, \alpha + 3d]$ , where  $\alpha = \sigma_{\omega_{i-1}}(v_i)$ .

**Subcase 1.1**  $v_i$  has at least three predecessors.

To satisfy the property (2), we have to choose weights for edges such that  $\sigma_{\omega_i}(v_i) \in \{0, 1, 2\} \pmod{6}$ , in the interval there are at least  $d + 3$  integers that are congruent

to 0, 1 or 2 (mod 6). The property (3) can block at most  $d$  values and hence 3 values remain open for  $\sigma_{\omega_i}(v_i)$ . Let  $\beta_i \in [\alpha - 3, \alpha - 2, \dots, \alpha + 3d]$  ( $i \in \{1, 2, 3\}$ ) be the values open for  $\sigma_{\omega_i}(v_i)$ , *i.e.*  $\beta_i \in \{0, 1, 2\} \pmod{6}$  and  $\beta_i \neq \sigma_{\omega_i}(v_{j_\ell})$  for all  $\ell \in \{1, \dots, d\}$ . Let us denote  $\beta_i = \alpha + 3p_i - r_i$ , where  $p_i \in \{0, \dots, d\}$  and  $r_i \in \{0, 1, 2, 3\}$  (*i.e.*  $p_i$  denotes the number of edges to which we have to add 3,  $r_i$  denotes the value which we have to subtract from the weight of  $e'$ ). Now we have to guarantee the property (5).

Suppose that there is  $i$  such that  $p_i \in \{1, \dots, d - 1\}$ . We choose exactly  $p_i$  edges from the set  $\{e_1, \dots, e_d\}$  and add 3 to their weights, next we subtract  $r_i$  from the weight of  $e'$ . Since we chose  $p_i$  edges from the set of  $d$  edges and  $0 < p_i < d$ , we can do this in such a way that the property (5) holds.

Suppose that  $p_i = 0$  or  $p_i = d$  for all  $i \in \{1, 2, 3\}$ . If edges  $\{e_\ell : \ell \in \{1, \dots, d\}\}$  are not monochromatic, then every  $\beta_i$  is good for  $\sigma_{\omega_i}(v_i)$ . Thus, we relabel only the edge  $e'$  with  $\omega_i(e') := \omega_{i-1}(e') - r_1$ , whenever  $p_1 = 0$  or  $\omega_i(e_\ell) := \omega_{i-1}(e_\ell) + 3$  for  $\ell \in \{1, \dots, d\}$  and  $\omega_i(e') := \omega_{i-1}(e') - r_1$ , otherwise.

Then assume that  $p_i = 0$  or  $p_i = d$  and edges  $\{e_\ell : \ell \in \{1, \dots, d\}\}$  are monochromatic. Thus  $\beta_i \in \{\alpha - 3, \alpha - 2, \alpha - 1, \alpha + 3d - 2, \alpha + 3d - 1, \alpha + 3d\}$  for all  $i \in \{1, 2, 3\}$ . There are at least two indexes  $i$ , say  $i = 1$  and  $i = 2$ , such that  $\beta_1, \beta_2 \in \{\alpha - 3, \alpha - 2, \alpha - 1\}$  or  $\beta_1, \beta_2 \in \{\alpha + 3d - 2, \alpha + 3d - 1, \alpha + 3d\}$  and so we have two choices for the weight of  $e'$ . We can see that one of them results in an edge-weighting  $\omega_i$  such that the weight of edges  $\{e_\ell : \ell \in \{1, \dots, d\}\}$  is not in  $\{\omega_i(e'), \omega_i(e') + 3\}$  and hence the property (5) holds.

**Subcase 1.2  $v_i$  has two predecessors.**

	$\alpha - 3$	$\alpha - 2$	$\alpha - 1$	$\alpha$	$\alpha + 1$	$\alpha + 2$	$\alpha + 3$	$\alpha + 4$	$\alpha + 5$	$\alpha + 6$
1	0	1	2	3	4	5	0	1	2	3
2	1	2	3	4	5	0	1	2	3	2
3	2	3	4	5	0	1	2	3	4	5
4	3	4	5	0	1	2	3	4	5	0
5	4	5	0	1	2	3	4	5	0	1
6	5	0	1	2	3	4	5	0	1	2

Table 2: Subcase 1.2, all possible values (mod 6) in the interval.

Thus, in the interval  $[\alpha - 3, \alpha - 2, \dots, \alpha + 3d] = [\alpha - 3, \alpha - 2, \dots, \alpha + 6]$ , there are at least 4 integers that are congruent to 0, 1 or 2 (mod 6). The property (3) can block at most two values and hence two values remain open for  $\sigma_{\omega_i}(v_i)$ . Let

$\beta_i \in [\alpha - 3, \alpha - 2, \dots, \alpha + 6]$  ( $i \in \{1, 2\}$ ) be the values open for  $\sigma_{\omega_i}(v_i)$ . Similarly as above, let  $\beta_i = \alpha + 3p_i - r_i$ , where  $p_i \in \{0, 1, 2\}$  and  $r_i \in \{0, 1, 2, 3\}$  for  $i \in \{1, 2\}$ .

Suppose that either  $p_1 = 1$  or  $p_2 = 1$ , say  $p_1 = 1$ . Then we add 3 to either  $e_1$  or  $e_2$  to obtain the edge-weighting such that  $\omega_i(e_1) \neq \omega_i(e_2)$  and put  $\omega_i(e') := \omega_{i-1}(e') - r_1$ .

Thus, we may assume that  $p_i \in \{0, 2\}$  and so  $\beta_i \in \{\alpha - 3, \alpha - 2, \alpha - 1, \alpha + 4, \alpha + 5, \alpha + 6\}$  for all  $i \in \{1, 2\}$ . If the edges  $e_1$  and  $e_2$  have different weights, then every  $\beta_i$  is good for  $\sigma_{\omega_i}(v_i)$ . Thus, we recolour only the edge  $e'$  with  $\omega_i(e') := \omega_{i-1}(e') - r_1$ , whenever  $p_1 = 0$  or  $\omega_i(e_1) := \omega_{i-1}(e_1) + 3, \omega_i(e_2) := \omega_{i-1}(e_2) + 3$  and  $\omega_i(e') := \omega_{i-1}(e') - r_1$ , otherwise.

Assume then that  $p_i \in \{0, 2\}$  and that  $e_1$  and  $e_2$  have the same weight. If we have either  $\beta_1, \beta_2 \in \{\alpha - 3, \alpha - 2, \alpha - 1\}$  or  $\beta_1, \beta_2 \in \{\alpha + 4, \alpha + 5, \alpha + 6\}$ , then we have two choices for the weight of  $e'$ . We can see that one of them gives an edge-weighting  $\omega_i$  such that the weight of the edges  $\{e_1, e_2\}$  is not in  $\{\omega_i(e'), \omega_i(e') + 3\}$  and hence the property (5) holds.

We claim that we always have either  $\beta_1, \beta_2 \in \{\alpha - 3, \alpha - 2, \alpha - 1\}$  or  $\beta_1, \beta_2 \in \{\alpha + 4, \alpha + 5, \alpha + 6\}$ . Let us consider the integers  $\{\alpha, \alpha + 1, \alpha + 2, \alpha + 3\}$ , we can see that there is at least one value congruent to 0, 1 or 2 (mod 6) (see Table 2). We may assume that all values congruent to 0, 1 or 2 (mod 6) are blocked by the property (3), otherwise we are in the case considered above. Thus, we are not in the case described in lines 3 or 4 of Table 2. If there is exactly one value congruent to 0, 1 or 2 (mod 6) in  $\{\alpha, \alpha + 1, \alpha + 2, \alpha + 3\}$  (it is blocked by the property (3)), then there are five values congruent to 0, 1 or 2 (mod 6) in  $\{\alpha - 3, \alpha - 2, \alpha - 1, \alpha + 4, \alpha + 5, \alpha + 6\}$  (see Table 2, lines 1 and 6), at least four are not blocked by the property (2), and hence two of them are in either  $\{\alpha - 3, \alpha - 2, \alpha - 1\}$  or  $\{\alpha + 4, \alpha + 5, \alpha + 6\}$ . If there are two values congruent to 0, 1 or 2 (mod 6) in  $\{\alpha, \alpha + 1, \alpha + 2, \alpha + 3\}$ , then, there are three values congruent to 0, 1 or 2 (mod 6) in  $\{\alpha - 3, \alpha - 2, \alpha - 1, \alpha + 4, \alpha + 5, \alpha + 6\}$  (see Table 2, lines 2 and 5) and none of them are blocked by the property (3) and hence two of them are in either  $\{\alpha - 3, \alpha - 2, \alpha - 1\}$  or  $\{\alpha + 4, \alpha + 5, \alpha + 6\}$ .

### Subcase 1.3 $v_i$ has one predecessor.

Suppose first that  $\alpha \notin \{0, 1, 2\} \pmod{6}$ . Then, there are at least four values congruent to 0, 1 or 2 (mod 6) in the interval  $[\alpha - 3, \alpha - 2, \dots, \alpha + 3d] = [\alpha - 3, \alpha - 2, \dots, \alpha + 3]$  (see Table 3). One of them can be blocked by the property (3), so three values remain open for  $\sigma_{\omega_i}(v_i)$ . Let  $\beta_i$  ( $i \in \{1, 2, 3\}$ ) be the values open for  $\sigma_{\omega_i}(v_i)$ . Thus at least two of them are in either  $\{\alpha - 3, \alpha - 2, \alpha - 1\}$  or  $\{\alpha + 1, \alpha + 2, \alpha + 3\}$ , and so we have two choices for the weight of  $e'$ . We can see that one of them gives an edge-weighting  $\omega_i$  such that  $\omega_i(e_1) \notin \{\omega_i(e'), \omega_i(e') + 3\}$ , which guarantee that the property (5) holds.

	$\alpha - 3$	$\alpha - 2$	$\alpha - 1$	$\alpha$	$\alpha + 1$	$\alpha + 2$	$\alpha + 3$
1	0	1	2	3	4	5	0
2	1	2	3	4	5	0	1
3	2	3	4	5	0	1	2
4	3	4	5	0	1	2	3
5	4	5	0	1	2	3	4
6	5	0	1	2	3	4	5

Table 3: Subcase 1.3, all possible values (mod 6) in the interval.

Finally, suppose that  $\alpha \in \{0, 1, 2\} \pmod{6}$ . Assume that there is  $\beta_i$  such that  $\beta_i = \alpha$  (*i.e.*  $\alpha$  is not blocked by the property (3) for  $\sigma_{\omega_i}(v_i)$ ). Recall that  $\omega_{i-1}(e_1) \leq 4$  and  $\omega_{i-1}(e') = 4$ . If  $\omega_{i-1}(e_1) \neq 4$ , then we assign  $\omega_i(e) := \omega_{i-1}(e)$  for every  $e \in E(G)$ . If  $\omega_{i-1}(e_1) = 4$ , then we recolour edges  $\omega_i(e_1) := 7$  and  $\omega_i(e') := 1$ . Suppose that  $\alpha$  is blocked by the property (3). If  $\alpha \equiv 0 \pmod{6}$ , then there is a value congruent to 1 and there is a value congruent to 2 (mod 6) in  $\{\alpha + 1, \alpha + 2, \alpha + 3\}$  (see Table 2, line 4) and hence one of them gives an edge-weighting  $\omega_i$  such that  $\omega_i(e_1) \notin \{\omega_i(e'), \omega_i(e') + 3\}$ . If  $\alpha \equiv 2 \pmod{6}$ , then there is a value congruent to 0 and there is a value congruent to 1 (mod 6) in  $\{\alpha - 3, \alpha - 2, \alpha - 1\}$  (see Table 2, line 6), so similarly as above we are done. If  $\alpha \equiv 1 \pmod{6}$ , then  $\beta_1 = \alpha - 1$  and  $\beta_2 = \alpha + 1$  (see Table 2, line 5). If  $\omega_{i-1}(e_1) \neq 3$ , then we assign  $\omega_i(e') := 3$  and so  $\sigma_{\omega_i}(v_i) = \alpha - 1$ . Otherwise, we modify the weights of two edges  $\omega_i(e_1) := 6, \omega_i(e') := 2$  and then  $\sigma_{\omega_i}(v_i) = \alpha + 1$ .

**Case 2**  $v_i \in V''$ , *i.e.*,  $v_i$  has no successor

Let  $v_{j_1}, v_{j_2}, \dots, v_{j_d}$  be the neighbours of  $v_i$  and  $v_{j_\ell} v_i = e_\ell$  for  $\ell \in \{1, \dots, d\}$ . Let  $v_{j_1}$  be a vertex that has a successor in  $\{v_{i+1}, \dots, v_n\}$  if such one exists. Recall that by our choice of the ordering of vertices  $\mathbf{v}$ , there is at most one such a vertex (Lemma 8 (iii)). To guarantee the property (3), we choose the weight of the edges incident with  $v_i$  in such a way that  $\sigma_{\omega_i}(v_i) \neq \sigma_{\omega_i}(v_{j_\ell})$  for  $\ell \in \{2, \dots, d\}$  and  $\sigma_{\omega_i}(v_i) \notin \{w(v_{j_1}), w(v_{j_1}) + 3\}$  even if  $v_{j_1}$  has no successor in  $\{v_{i+1}, \dots, v_n\}$ .

Similarly as in Case 1, we put the lower possible weights on every  $e_\ell$  for  $\ell \in \{1, \dots, d\}$ , we provisionally modify the weights of edges in the following way:  $\omega'_{i-1}(e_\ell) := \omega_{i-1}(e_\ell) - 3$  if  $\sigma_{\omega_{i-1}}(v_{j_\ell}) = w(v_{j_\ell}) + 3$ , and  $\omega'_{i-1}(e) := \omega_{i-1}(e)$  otherwise. Similarly as in Case 1, we can see that after such a modification, the weight of  $e_\ell$  is in  $\{1, 2, 3, 4\}$ . To simplify notations, we state  $\omega_{i-1} = \omega'_{i-1}$ . Observe that  $\sigma_{\omega_{i-1}}(v_{j_\ell}) = w(v_{j_\ell})$  for  $\ell \in \{1, \dots, d\}$  and  $\sigma_{\omega_{i-1}}(v_{j_\ell}) \in \{0, 1, 2\} \pmod{6}$  for  $\ell \in \{1, \dots, d\}$  (every  $v_{j_\ell}$  belongs

to  $V'$ ).

We will modify weights by adding 3 to  $\omega_{i-1}(e_\ell)$  for some  $\ell \in \{1, \dots, d\}$ . We can see that after adding 3 to the weight of  $e_\ell$ , the weight is still in  $\{1, \dots, 7\}$ . Furthermore, adding 3 to some  $e_\ell$  maintains the property (4). Since  $v_i$  has no successor, the properties (1) and (2) hold. We prove that we can add 3 to some edges in such a way that the properties (3) and (5) will be satisfied. Let  $\sigma_{\omega_{i-1}}(v_i) = \alpha$ .

**Subcase 2.1**  $d(v_i) \geq 3$

Observe that if  $\alpha \in \{0, 1, 2\} \pmod{6}$ , then  $\alpha + 3 \notin \{0, 1, 2\} \pmod{6}$ . Thus, we consider two cases.

**Subcase 2.1.1**  $\alpha \in \{0, 1, 2\} \pmod{6}$

If  $\alpha \neq \sigma_{\omega_{i-1}}(v_{j_1})$  and  $\alpha \neq \sigma_{\omega_{i-1}}(v_{j_\ell})$  for  $\ell \in \{2, \dots, d\}$ , then we assign  $\omega_i(e) := \omega_{i-1}(e)$  for all  $e \in E(G)$ . Recall that  $w(v_{j_1}) = \sigma_{\omega_{i-1}}(v_{j_1})$ , so  $\sigma_{\omega_i}(v_i) \neq w(v_{j_1})$ . We also have  $\sigma_{\omega_i}(v_i) \neq w(v_{j_1}) + 3$ , since  $\sigma_{\omega_i}(v_i) \in \{0, 1, 2\} \pmod{6}$  and  $w(v_{j_1}) + 3 \notin \{0, 1, 2\} \pmod{6}$ . Thus,  $\omega_i$  satisfies (3). If the edges incident with  $v_i$  are not monochromatic or  $d(v_i) \leq 5$ , then we are done. Otherwise, we relabel the edge  $\omega_i(e_1) := \omega_{i-1}(e_1) + 3$ . Thus,  $\sigma_{\omega_i}(v_i) = \alpha + 3$ . Our assumption  $\alpha \in \{0, 1, 2\} \pmod{6}$  implies that  $\alpha + 3 \notin \{0, 1, 2\} \pmod{6}$  and consequently  $\sigma_{\omega_i}(v_i) \neq \sigma_{\omega_i}(v_{j_\ell})$  for  $\ell \in \{2, \dots, d\}$ . Furthermore,  $\alpha + 3 = \sigma_{\omega_i}(v_i) \neq w(v_{j_1})$ , since  $\alpha + 3 \notin \{0, 1, 2\} \pmod{6}$  and  $w(v_{j_1}) \in \{0, 1, 2\} \pmod{6}$ . We also have  $\sigma_{\omega_i}(v_i) \neq w(v_{j_1}) + 3$ , since  $w(v_{j_1}) + 3 = \sigma_{\omega_{i-1}}(v_{j_1}) + 3 \neq \alpha + 3 = \sigma_{\omega_i}(v_i)$ . Thus, we have a weighting  $\omega_i$  that satisfies the properties (1)–(5).

Assume now that  $\alpha = \sigma_{\omega_{i-1}}(v_{j_1})$  or there is  $\ell \in \{2, \dots, d\}$  such that  $\alpha = \sigma_{\omega_{i-1}}(v_{j_\ell})$ .

Suppose first that  $\alpha = \sigma_{\omega_{i-1}}(v_{j_1})$ . Assume that there are at least two vertices  $v_{j_a}, v_{j_b} \in \{v_{j_2}, \dots, v_{j_d}\}$  such that  $\sigma_{\omega_{i-1}}(v_{j_a}) \neq \alpha + 6, \sigma_{\omega_{i-1}}(v_{j_b}) \neq \alpha + 6$ . We assign  $\omega_i(e_1) := \omega_{i-1}(e_1) + 3, \omega_i(e_a) := \omega_{i-1}(e_a) + 3, \omega_i(e_b) := \omega_{i-1}(e_b) + 3$ . Thus,  $\sigma_{\omega_i}(v_i) = \alpha + 9$ . We show that the property (3) holds. Since  $\alpha + 9 \notin \{0, 1, 2\} \pmod{6}$ , we have  $\sigma_{\omega_{i-1}}(v_{j_\ell}) \neq \alpha + 9$  for  $\ell \in \{2, \dots, d\}$  and so  $\sigma_{\omega_i}(v_i) \neq \sigma_{\omega_i}(u)$  for  $u \in \{v_{j_2}, \dots, v_{j_d}\} \setminus \{v_{j_a}, v_{j_b}\}$ . Our assumptions  $\sigma_{\omega_{i-1}}(v_{j_a}) \neq \alpha + 6, \sigma_{\omega_{i-1}}(v_{j_b}) \neq \alpha + 6$  imply that  $\sigma_{\omega_i}(v_{j_a}) \neq \alpha + 9 = \sigma_{\omega_i}(v_i), \sigma_{\omega_i}(v_{j_b}) \neq \alpha + 9 = \sigma_{\omega_i}(v_i)$ . Now consider  $v_{j_1}$ . Since  $w(v_{j_1}) = \alpha$ , we have  $\sigma_{\omega_i}(v_i) \neq w(v_{j_1})$  and  $\sigma_{\omega_i}(v_i) \neq w(v_{j_1}) + 3$ . Thus, the edge-weighting  $\omega_i$  verifies property (3). If  $d(v_i) \leq 5$  or edges incident with  $v_i$  are not monochromatic, then we are done. Otherwise, if there is another vertex  $v_{j_c} \in \{v_{j_2}, \dots, v_{j_d}\}$  such that  $\sigma_{\omega_{i-1}}(v_{j_c}) \neq \alpha + 6$ , then we can relabel edges in the following way:  $\omega_i(e_1) := \omega_{i-1}(e_1) + 3, \omega_i(e_a) := \omega_{i-1}(e_a) + 3, \omega_i(e_c) := \omega_{i-1}(e_c) + 3$ . Thus, suppose that this is not the case: in  $\{v_{j_2}, \dots, v_{j_d}\}$ , there are at most two vertices with colour other than  $\alpha + 6$ . Then, we add 3 to the weight of  $e_1$  and edges incident with vertices with colours other than  $\alpha + 6$ . Next, from the remaining edges, we

choose one edge if we have two vertices with colours other than  $\alpha + 6$ , two edges if we have one vertex with colour other than  $\alpha + 6$  and three edges if we have no vertices with colour other than  $\alpha + 6$ , and add 3 to their weights. Thus, we obtain  $\sigma_{\omega_i}(v_i) = \alpha + 12$ . Since  $d(v_i) \geq 6$ , we can choose edges for the relabeling in such a way that the edges incident with  $v_i$  are not monochromatic. Observe that the only neighbours of  $v_i$  that have in  $\omega_i$  the same colour as in  $\omega_{i-1}$  are those with colour  $\alpha + 6$ . Those vertices are distinguished with  $v_i$  in  $\omega_i$ . Now, the remaining neighbours of  $v_i$  have colours that are not in  $\{0, 1, 2\} \pmod{6}$ . Thus, they are also distinguished from  $v_i$  in  $\omega_i$ , since  $\alpha + 12 \in \{0, 1, 2\} \pmod{6}$ . So the edge-weighting  $\omega_i$  satisfies the properties (1)–(5).

Finally, assume that  $\alpha \neq \sigma_{\omega_{i-1}}(v_{j_1})$  and there is  $\ell \in \{2, \dots, d\}$  such that  $\alpha = \sigma_{\omega_{i-1}}(v_{j_\ell})$ . If the edges  $\{e_2, \dots, e_d\}$  are not monochromatic, or the weight of  $\{e_2, \dots, e_d\}$  is different from  $\omega_{i-1}(e_1) + 3$ , or  $d(v_i) \leq 5$ , then we assign  $\omega_i(e_1) := \omega_{i-1}(e_1) + 3$ . Since  $\sigma_{\omega_i}(v_i) = \alpha + 3 \notin \{0, 1, 2\} \pmod{6}$ ,  $v_i$  is distinguished from every vertex in  $\{v_{j_2}, \dots, v_{j_d}\}$ . Our assumption  $\alpha \neq \sigma_{\omega_{i-1}}(v_{j_1})$  implies  $\sigma_{\omega_i}(v_i) \neq w(v_{j_1}) + 3$ . Furthermore,  $\sigma_{\omega_i}(v_i) \neq w(v_{j_1})$  since  $\sigma_{\omega_i}(v_i) \notin \{0, 1, 2\} \pmod{6}$  and  $w(v_{j_1}) \in \{0, 1, 2\} \pmod{6}$ . Thus, the edge-weighting  $\omega_i$  verifies properties (1)–(5). Thus, we may assume that  $d(v_i) \geq 6$  and  $\omega_{i-1}(e_2) = \dots = \omega_{i-1}(e_d) = \omega_{i-1}(e_1) + 3$ .

If there is a  $v_{j_a} \in \{v_{j_2}, \dots, v_{j_d}\}$  with colour other than  $\alpha$ , then we assign  $\omega_i(e_a) := \omega_{i-1}(e_a) + 3$ . The edge-weighting  $\omega_i$  verifies properties (1)–(5) (recall that  $\alpha + 3 \neq \sigma_{\omega_i}(v_{j_\ell})$ , since  $\sigma_{\omega_i}(v_{j_\ell}) \in \{0, 1, 2\} \pmod{6}$  for  $\ell \in \{e_2, \dots, e_d\} \setminus \{e_a\}$  and similarly as above we can observe that  $\sigma_{\omega_i}(v_i) \notin \{w(v_{j_1}), w(v_{j_1}) + 3\}$ ).

Suppose that all vertices  $\{v_{j_2}, \dots, v_{j_d}\}$  are coloured with  $\alpha$ . If  $\alpha + 6 \neq w(v_{j_1})$ , then we add 3 to the weights of two edges from  $\{e_2, \dots, e_d\}$ . Since we can choose which edges to relabel, we can maintain the property (5). Since  $\sigma_{\omega_i}(v_i) = \alpha + 6$ ,  $\sigma_{\omega_i}(u) = \alpha$  or  $\alpha + 3$  for  $u \in \{v_{j_2}, \dots, v_{j_d}\}$  and  $\sigma_{\omega_i}(v_i) \notin \{w(v_{j_1}), w(v_{j_1}) + 3\}$ ,  $\omega_i$  verifies properties (1)–(5). If  $\alpha + 6 = w(v_{j_1})$ , then we add 3 to the weights of four edges. Again, we can choose which edges to relabel, since  $d(v_i) \geq 6$ . Hence, we are able to maintain property (5). Similarly as above, we can check that  $\omega_i$  also verifies property (3), so we are done.

**Subcase 2.1.2**  $\alpha + 3 \in \{0, 1, 2\} \pmod{6}$

Since  $\alpha + 3 \in \{0, 1, 2\} \pmod{6}$ , we have  $\alpha \notin \{0, 1, 2\} \pmod{6}$  and, in  $\{v_{j_1}, \dots, v_{j_d}\}$ , there is no vertex with colour  $\alpha$  or  $\alpha + 6$ .

First, we consider the case when  $\sigma_{\omega_{i-1}}(v_{j_1}) = \alpha - 3$ .

If, in  $\{v_{j_2}, \dots, v_{j_d}\}$ , there is a vertex  $v_{j_a}$  with a colour other than  $\alpha + 3$ , then we add 3 to the weights of  $e_a$  and  $e_1$ . If the edges incident with  $v_i$  are not monochromatic or  $d(v_i) \leq 5$ , then we are done. Thus, suppose that  $d(v_i) \geq 6$  and all these edges

have the same weight. If there is another vertex  $v_{j_b}$ ,  $b \neq a$ , with a colour other than  $\alpha + 3$ , then we add 3 to the weights of  $e_b$  and  $e_1$ . In the resulting edge-weighting, the edges incident with  $v_i$  are not monochromatic. If  $v_{j_a}$  is the only vertex with a colour other than  $\alpha + 3$ , *i.e.* all vertices in  $\{v_{j_2}, \dots, v_{j_d}\} \setminus \{v_{j_a}\}$  have the colour  $\alpha + 3$ , then we choose one edge in  $\{e_2, \dots, e_d\} \setminus \{e_a\}$ , say  $e_b$ , and assign  $\omega_i(e_1) := \omega_{i-1}(e_1) + 3, \omega_i(e_a) := \omega_{i-1}(e_a) + 3, \omega_i(e_b) := \omega_{i-1}(e_b) + 3$ . Since we have a choice, we can maintain property (5). Now, we have  $\sigma_{\omega_i}(v_i) = \alpha + 9, \sigma_{\omega_i}(v_b) = \alpha + 6, \sigma_{\omega_i}(u) = \alpha + 3$  for  $u \in \{v_{j_2}, \dots, v_{j_d}\} \setminus \{v_{j_a}, v_{j_b}\}$ , so  $\omega_i$  distinguishes  $v_i$  and vertices from  $\{v_{j_1}, \dots, v_{j_d}\} \setminus \{v_{j_b}\}$ . Furthermore, we have  $\sigma_{\omega_i}(v_a) = \sigma_{\omega_{i-1}}(v_{j_a}) + 3$ . As observed before,  $\sigma_{\omega_{i-1}}(v_{j_a}) \neq \alpha + 6$ , which implies that  $\sigma_{\omega_i}(v_a) \neq \sigma_{\omega_i}(v_i)$ . For  $v_{j_1}$  we have  $w(v_{j_1}) = \sigma_{\omega_{i-1}}(v_{j_1}) = \alpha - 3$ , so  $\sigma_{\omega_i}(v_i) \notin \{w(v_{j_1}), w(v_{j_1}) + 3\}$ . Thus, property (3) holds.

If all vertices in  $\{v_{j_2}, \dots, v_{j_d}\}$  have colour  $\alpha + 3$ , then we choose three edges from  $\{e_2, \dots, e_d\}$  for the relabeling, and since we can choose freely, we can construct an edge-weighting  $\omega_i$  satisfying property (5). Since  $\sigma_{\omega_i}(v_i) = \alpha + 9$  and  $\sigma_{\omega_i}(u) = \alpha + 3$  or  $\alpha + 6$  for  $u \in \{v_{j_2}, \dots, v_{j_d}\}$ ,  $\omega_i$  distinguishes  $v_i$  and vertices from  $\{v_{j_1}, \dots, v_{j_d}\}$ . Similarly as above, we can see that  $\sigma_{\omega_i}(v_i) \notin \{w(v_{j_1}), w(v_{j_1}) + 3\}$ , so we are done.

Suppose that  $\sigma_{\omega_{i-1}}(v_{j_1}) = \alpha + 3$ . If the edges incident to  $v_i$  are not monochromatic or  $d(v_i) \leq 5$ , then the edge-weighting  $\omega_i := \omega_{i-1}$  satisfies (1)–(5). Thus, we may assume that all edges have the same weight and  $d(v_i) \geq 6$ .

If, in  $\{v_{j_2}, \dots, v_{j_d}\}$ , there are three vertices  $v_{j_a}, v_{j_b}, v_{j_c}$  with colour other than  $\alpha + 9$ , then we add 3 to the weights of  $e_a, e_b, e_c$  and  $e_1$ . Thus, the edges incident with  $v_i$  are not monochromatic (the property (5) holds) and  $\sigma_{\omega_i}(v_i) = \alpha + 12$ . Since the colour of  $v_{j_a}, v_{j_b}, v_{j_c}$  is not equal to  $\alpha + 9$  in  $\omega_{i-1}$ , the colour of  $v_{j_a}, v_{j_b}, v_{j_c}$  is not equal to  $\alpha + 12$  in  $\omega_i$ . Thus,  $\omega_i$  distinguishes  $v_i$  from  $v_{j_a}, v_{j_b}, v_{j_c}$ . Since  $\alpha + 12 \notin \{0, 1, 2\} \pmod{6}$ , no vertex in  $\{v_{j_2}, \dots, v_{j_d}\} \setminus \{v_{j_a}, v_{j_b}, v_{j_c}\}$  has colour  $\alpha + 12$ . Furthermore,  $w(v_{j_1}) = \sigma_{\omega_{i-1}}(v_{j_1}) = \alpha + 3$ , so  $\sigma_{\omega_i}(v_i) \notin \{w(v_{j_1}), w(v_{j_1}) + 3\}$ , so the the resulting edge weighting satisfies properties (1)–(5).

Assume that, in  $\{v_{j_2}, \dots, v_{j_d}\}$ , there are at most two vertices with colour other than  $\alpha + 9$ . Then, we add 3 to the weights of  $e_1$  and edges incident with vertices having colour different from  $\alpha + 9$ . Next, from the remaining edges, we choose two edges if we have two vertices with colours other than  $\alpha + 9$ , three edges if we have one vertex with colour other than  $\alpha + 9$  and four edges if we have no vertex with colour other than  $\alpha + 9$ , and add 3 to their weights. Since  $d(v_i) \geq 6$ , we can choose the edges for relabeling in such a way that the edges incident with  $v_i$  are not monochromatic. We obtain  $\sigma_{\omega_i}(v_i) = \alpha + 15$ . The vertices that had colour  $\alpha + 9$  in  $\omega_{i-1}$  have colour either  $\alpha + 9$  or  $\alpha + 12$  in  $\omega_i$ , so  $\omega_i$  distinguishes  $v_i$  and these vertices. Consider the

vertices that had a colour different from  $\alpha + 9$  in  $\omega_{i-1}$ . We added 3 to the edges incident with these vertices. In  $\omega_{i-1}$ , the colours of these vertices were in  $\{0, 1, 2\} \pmod{6}$ , so now these vertices have colours that are not in  $\{0, 1, 2\} \pmod{6}$ , but  $\sigma_{\omega_i}(v_i) = \alpha + 15 \in \{0, 1, 2\} \pmod{6}$ . Thus,  $\omega_i$  distinguishes also these vertices. Furthermore,  $w(v_{j_1}) = \sigma_{\omega_{i-1}}(v_{j_1}) = \alpha + 3$ , so  $\sigma_{\omega_i}(v_i) \notin \{w(v_{j_1}), w(v_{j_1}) + 3\}$ , and so we are done.

Finally, suppose that  $\sigma_{\omega_{i-1}}(v_{j_1}) \notin \{\alpha - 3, \alpha + 3\}$ . Since  $\alpha \notin \{0, 1, 2\} \pmod{6}$ , there is no vertex with colour  $\alpha$  in  $\{v_{j_1}, \dots, v_{j_d}\}$ . If the edges incident with  $v_i$  are not monochromatic or  $d(v_i) \leq 5$ , then the edge-weighting satisfies properties (1)–(5). Thus, we may assume that all edges have the same weight and  $d(v_i) \geq 6$ .

If, in  $\{v_{j_2}, \dots, v_{j_d}\}$ , there is a vertex  $v_{j_a}$  with colour other than  $\alpha + 3$ , then we add 3 to weights of  $e_a$  and  $e_1$ . Thus,  $\sigma_{\omega_i}(v_i) = \alpha + 6$  and so  $\sigma_{\omega_i}(v_{j_a}) = \sigma_{\omega_{i-1}}(v_{j_a}) + 3 \neq \sigma_{\omega_i}(v_i)$ ,  $\sigma_{\omega_i}(v_i) \neq \sigma_{\omega_i}(u)$  for  $u \in \{v_{j_2}, \dots, v_{j_d}\} \setminus \{v_{j_a}\}$ , because  $\sigma_{\omega_i}(v_i) \notin \{0, 1, 2\} \pmod{6}$  and  $\sigma_{\omega_i}(u) \in \{0, 1, 2\} \pmod{6}$ . Consider  $v_{j_1}$ : we have  $\sigma_{\omega_i}(v_i) \neq w(v_{j_1})$ , since  $w(v_{j_1}) = \sigma_{\omega_{i-1}}(v_{j_1}) \in \{0, 1, 2\} \pmod{6}$ , and  $\sigma_{\omega_i}(v_i) \neq w(v_{j_1}) + 3$ , since  $\sigma_{\omega_{i-1}}(v_{j_1}) \neq \alpha + 3$  by our assumption. Thus, the resulting edge-weighting satisfies properties (1)–(5).

Thus, we may assume that  $\sigma_{\omega_{i-1}}(v_{j_\ell}) = \alpha + 3$  for  $\ell \in \{2, \dots, d\}$ . If  $\sigma_{\omega_{i-1}}(v_{j_1}) \neq \alpha + 9$ , then we choose three edges from  $\{e_2, \dots, e_d\}$  and add 3 to their weights. Since  $d(v_i) \geq 6$ , we can choose edges in such a way that we maintain property (5). Now, we have  $\sigma_{\omega_i}(v_i) = \alpha + 9$ , so  $\omega_i$  distinguishes  $v_i$  from  $v_{j_\ell}$  for  $\ell \in \{2, \dots, d\}$ . By our assumption,  $\sigma_{\omega_i}(v_i) \neq w(v_{j_1}) = \sigma_{\omega_{i-1}}(v_{j_1})$ . Since  $w(v_{j_1}) + 3 \notin \{0, 1, 2\} \pmod{6}$  and  $\sigma_{\omega_i}(v_i) \in \{0, 1, 2\} \pmod{6}$ ,  $\sigma_{\omega_i}(v_i) \neq w(v_{j_1}) + 3$ , so we are done. If  $\sigma_{\omega_{i-1}}(v_{j_1}) = \alpha + 9$ , then we choose five edges from  $\{e_1, \dots, e_d\}$  and add 3 to their weights. Since  $d(v_i) \geq 6$ , we can choose edges in such a way that we maintain property (5). Now  $\sigma_{\omega_i}(v_i) = \alpha + 15$  and  $\sigma_{\omega_i}(u) = \alpha + 3$  or  $\alpha + 6$  for  $u \in \{v_{j_2}, \dots, v_{j_d}\}$ . Thus,  $v_i$  is distinguished from  $\{v_{j_2}, \dots, v_{j_d}\}$  by  $\omega_i$ . By our assumption,  $w(v_{j_1}) = \alpha + 9$  and so the property (3ii) also holds.

### Subcase 2.2 $d(v_i) = 2$

Thus,  $v_i$  has two neighbours  $v_{j_1}, v_{j_2}$ , and  $v_{j_1}$  may have a successor that follows  $v_i$ . Since  $d(v_i) = 2$ , the property (5) holds. If  $\alpha \neq \sigma_{\omega_{i-1}}(v_{j_2})$  and  $\alpha \notin \{w(v_{j_1}), w(v_{j_1}) + 3\}$ , then the edge-weighting  $\omega_i := \omega_{i-1}$  satisfies properties (1)–(5). Thus, we may assume that either  $\alpha = \sigma_{\omega_{i-1}}(v_{j_2})$  or  $\alpha \in \{w(v_{j_1}), w(v_{j_1}) + 3\}$ .

Suppose that  $\alpha \in \{w(v_{j_1}), w(v_{j_1}) + 3\}$ . First, assume that  $w(v_{j_1}) = \alpha$  (*i.e.*  $\sigma_{\omega_{i-1}}(v_{j_1}) = \alpha$ ). Thus, we must have  $\alpha \in \{0, 1, 2\} \pmod{6}$  and hence  $\alpha + 3 \notin \{0, 1, 2\} \pmod{6}$  which implies  $\sigma_{\omega_{i-1}}(v_{j_2}) \neq \alpha + 3$ . We assign  $\omega_i(e_1) := \omega_{i-1}(e_1) + 3$  and  $\omega_i(e_2) := \omega_{i-1}(e_2) + 3$ , so we are done, since now  $\sigma_{\omega_i}(v_i) = \alpha + 6 \neq \sigma_{\omega_i}(v_{j_2})$  and

$\sigma_{\omega_i}(v_i) \notin \{w(v_{j_1}), w(v_{j_1}) + 3\}$ . Suppose that  $w(v_{j_1}) + 3 = \alpha$ . If  $\sigma_{\omega_{i-1}}(v_{j_2}) = \alpha + 3$ , then we assign  $\omega_i(e_2) := \omega_{i-1}(e_2) + 3$ , otherwise, we assign  $\omega_i(e_1) := \omega_{i-1}(e_1) + 3$  and  $\omega_i(e_2) := \omega_{i-1}(e_2) + 3$ . We can check that in both cases property (3) holds.

Thus, we may assume that  $\alpha \notin \{w(v_{j_1}), w(v_{j_1}) + 3\}$  and  $\sigma_{\omega_{i-1}}(v_{j_2}) = \alpha$ . The assumption  $\sigma_{\omega_{i-1}}(v_{j_2}) = \alpha$  implies that  $\alpha \in \{0, 1, 2\} \pmod{6}$  and hence  $\alpha + 3 \notin \{0, 1, 2\} \pmod{6}$ . In this case, we assign  $\omega_i(e_1) := \omega_{i-1}(e_1) + 3$ . Thus  $\alpha + 3 = \sigma_{\omega_i}(v_i) \neq \sigma_{\omega_i}(v_{j_2}) = \alpha$ . Furthermore,  $\sigma_{\omega_i}(v_i) \neq w(v_{j_1})$ , since  $\sigma_{\omega_i}(v_i) \notin \{0, 1, 2\} \pmod{6}$  and  $w(v_{j_1}) \in \{0, 1, 2\} \pmod{6}$ . Also  $\sigma_{\omega_i}(v_i) \neq w(v_{j_1}) + 3$ , since by our assumption  $\alpha \neq w(v_{j_1})$ .

**Subcase 2.3**  $d(v_i) = 1$

Thus  $N(v_i) = \{v_{j_1}\}$ ,  $v_{j_1}$  may have a successor that follows  $v_i$ , and  $\sigma_{\omega_{i-1}}(v_i) = \omega_{i-1}(v_{j_1}v_i)$ . Since  $G \neq K_2$ , we have  $\sigma_{\omega_{i-1}}(v_i) < \sigma_{\omega_{i-1}}(v_{j_1})$  and so  $\sigma_{\omega_{i-1}}(v_i) \notin \{w(v_{j_1}), w(v_{j_1}) + 3\}$ . Thus, the edge-weighting  $\omega_i := \omega_{i-1}$  verifies properties (1)–(5). □

**Lemma 13.** *Let  $\omega$  be the edge-weighting given by ALGORITHM. Then  $\omega$  is a neighbour sum distinguishing 7-edge-weighting.*

*Proof.* It is obvious that  $\omega$  is a 7-edge-weighting, since the weight of every edge is in  $\{1, \dots, 7\}$ . We show that  $\omega$  is neighbour sum distinguishing. Let  $\mathbf{v}, V', V''$  and  $\omega_i$  be defined the same as in ALGORITHM. Let  $w$  be the vertex-assignment determined by ALGORITHM. First, observe the following property of every vertex:

**Claim 14.** (i) *If  $u \in V'$ , then  $\sigma_\omega(u) \in \{w(u), w(u) + 3\}$ .*

(ii) *If  $u \in V''$ , then  $\sigma_\omega(u) = w(u)$ .*

*Proof.* Let  $u = v_i$ .

Suppose that  $i = 1$  or  $2$ . Since  $v_1$  and  $v_2$  have successors,  $v_1, v_2 \in V'$ . The values  $w(v_1)$  and  $w(v_2)$  were assigned at the end of steps 1 and 2, by  $w(v_1) = \sigma_{\omega_2}(v_1), w(v_2) = \sigma_{\omega_2}(v_2)$ . Observe that the weight of  $v_1v_2$  will not change in steps  $\{3, \dots, n\}$ . ALGORITHM has to respect property (4), so the weights of the remaining edges incident with either  $v_1$  or  $v_2$  can be modified only in such a way that  $\sigma_{\omega_k}(v_1) \in \{w(v_1), w(v_1) + 3\}$  and  $\sigma_{\omega_k}(v_2) \in \{w(v_2), w(v_2) + 3\}$  for  $k \in \{3, \dots, n\}$ . Thus, finally,  $\sigma_\omega(v_1) \in \{w(v_1), w(v_1) + 3\}$  and  $\sigma_\omega(v_2) \in \{w(v_2), w(v_2) + 3\}$ .

Suppose that  $i \geq 3$ . Assume first that  $v_i \in V'$ . Let  $v_{j_1}, v_{j_2}, \dots, v_{j_d}$  be the predecessors of  $v_i$ . Observe that the weight which we assigned to  $v_{j_\ell}v_i$  in the  $i$ -th step of ALGORITHM will not change in the next steps (*i.e.*  $\omega_i(v_{j_\ell}v_i) = \omega(v_{j_\ell}v_i)$ ) and

$w(v_i) = \sigma_{\omega_i}(v_i)$ . ALGORITHM has to verify property (4), so the weights of edges incident with the successors of  $v_i$  can be modified only in such a way that  $\sigma_{\omega_k}(v_i) \in \{w(v_i), w(v_i) + 3\}$  for  $k \in \{i + 1, \dots, n\}$ . Thus, finally,  $\sigma_\omega(v_i) \in \{w(v_i), w(v_i) + 3\}$ .

Assume now that  $v_i \in V''$ . Let  $v_{j_1}, v_{j_2}, \dots, v_{j_d}$  be the neighbours of  $v_i$ . The weight which we assigned to  $v_{j_\ell}v_i$  in the  $i$ -th step of ALGORITHM will not change in the next steps (*i.e.*  $\omega_i(v_{j_\ell}v_i) = \omega(v_{j_\ell}v_i)$ ). Thus  $\sigma_{\omega_i}(v_i) = \sigma_\omega(v_i)$ , which implies that  $w(v_i) = \sigma_\omega(v_i)$ .  $\square$

Let  $uw \in E(G)$ , we show that  $\sigma_\omega(u) \neq \sigma_\omega(w)$ . Suppose that  $uw = v_1v_2$ . Steps 1 and 2 imply that  $\{w(v_1), w(v_1) + 3\} \cap \{w(v_2), w(v_2) + 3\} = \emptyset$  and so  $\sigma_\omega(v_1) \neq \sigma_\omega(v_2)$  by Claim 14. Suppose that  $u = v_j, w = v_i$  and  $j < i$  ( $i \neq 2$ ). We have ( $v_i \in V'$  or  $v_i \in V''$ ) and  $v_j \in V'$ , since  $v_j$  has a successor. Suppose that  $v_i \in V'$ . By property (3i),  $w(v_i) \neq w(v_j)$ , since  $w(v_i) = \sigma_{\omega_i}(v_i)$ . As we noticed in Observation 11,  $w(v_i), w(v_j) \in \{0, 1, 2\} \pmod{6}$ , thus  $w(v_i) \neq w(v_j) + 3$  and  $w(v_j) \neq w(v_i) + 3$ , and so  $\{w(v_i), w(v_i) + 3\} \cap \{w(v_j), w(v_j) + 3\} = \emptyset$ . Thus, Claim 14 implies that  $\sigma_\omega(v_i) \neq \sigma_\omega(v_j)$ . Suppose now that  $v_i \in V''$ . Since  $v_i$  has no successor, the weights of edges incident with  $v_i$  will not change in steps  $\{i + 1, \dots, n\}$ , so  $\sigma_{\omega_i}(v_i) = \sigma_\omega(v_i)$ . If  $v_j$  has no successor that follows  $v_i$ , then the weights of edges incident with  $v_j$  will also not change in steps  $\{i + 1, \dots, n\}$  and so  $\sigma_{\omega_i}(v_j) = \sigma_\omega(v_j)$ . By property (3ii), we have  $\sigma_{\omega_i}(v_i) \neq \sigma_{\omega_i}(v_j)$ , thus  $\sigma_\omega(v_i) \neq \sigma_\omega(v_j)$  if  $v_j$  has no successor that follows  $v_i$ . If  $v_j$  has a successor that follows  $v_i$ , then property (3ii) implies that  $\sigma_{\omega_i}(v_i) \notin \{w(v_j), w(v_j) + 3\}$ . As we observed,  $\sigma_{\omega_i}(v_i) = \sigma_\omega(v_i)$  and then  $\sigma_\omega(v_i) \neq \sigma_\omega(v_j)$  by Claim 14.  $\square$

**Lemma 15.** *Let  $\omega$  be the edge-weighting given by ALGORITHM. Then, for every vertex  $u$  of degree at least 6, there are edges  $e'$  and  $e''$  incident with  $u$  verifying  $\omega(e') \neq \omega(e'')$ .*

*Proof.* Let  $\mathbf{v}, V', V''$  and  $\omega_i$  be defined the same as in ALGORITHM. Let  $w$  be the vertex-assignment determined by ALGORITHM. First, we prove that the lemma is true for  $v_1$  and  $v_2$ . Let  $v_i$  be the first successor of  $v_1$  different from  $v_2$ . Let  $v_j$  be the first successor of  $v_2$ . Let  $e_1 = v_1v_i, e_2 = v_2v_j$ . Steps 1 and 2 of ALGORITHM imply that  $\omega_2(v_1v_2) \notin \{\omega_2(e_1), \omega_2(e_1) + 3\}, \omega_2(v_1v_2) \notin \{\omega_2(e_2), \omega_2(e_2) + 3\}$ . Observe that the weight of  $v_1v_2$  will not change in steps  $\{3, \dots, n\}$ , *i.e.*  $\omega(v_1v_2) = \omega_2(v_1v_2)$ . When  $v_i$  ( $v_j$ ) is treated, then  $\sigma_{\omega_i}(v_1) = \sigma_{\omega_2}(v_1) = w(v_1)$  ( $\sigma_{\omega_j}(v_2) = \sigma_{\omega_2}(v_2) = w(v_2)$ ), because  $v_i$  ( $v_j$ ) is the first successor. Thus, the weight of  $e_1$  ( $e_2$ ) can be modified only by adding 3, because the property (4) must hold. So  $\omega(e_1) \in \{\omega_2(e_1), \omega_2(e_1) + 3\}$  ( $\omega(e_2) \in \{\omega_2(e_2), \omega_2(e_2) + 3\}$ ). Thus, the argument that  $\omega_2(v_1v_2) \notin \{\omega_2(e_1), \omega_2(e_1) + 3\}$

$3\}, \omega_2(v_1v_2) \notin \{\omega_2(e_2), \omega_2(e_2) + 3\}$  implies that  $\omega(v_1v_2) \neq \omega(e_1)$  and  $\omega(v_1v_2) \neq \omega(e_2)$ , so we are done.

Suppose that  $u \in V'$  and  $u \notin \{v_1, v_2\}$ . Assume that  $u = v_i$  and  $v_k$  is the first successor of  $v_i$ . By property (5) of ALGORITHM, the edges  $\{v_jv_i : j < i, v_j \in N(v_i)\}$  are not monochromatic or the weight of the edges  $\{v_jv_i : j < i, v_j \in N(v_i)\}$  is not in  $\{\omega_i(v_iv_k), \omega_i(v_iv_k) + 3\}$ . If the edges  $\{v_jv_i : j < i, v_j \in N(v_i)\}$  are not monochromatic, then there are two edges  $v_{j'}v_i$  and  $v_{j''}v_i$  such that  $\omega(v_{j'}v_i) \neq \omega(v_{j''}v_i)$  and we are done. Otherwise, observe that the weight of  $v_iv_k$  can be modified only if  $v_k$  is being treated. When  $v_k$  is being treated, then  $\sigma_{\omega_{k-1}}(v_i) = \sigma_{\omega_i}(v_i) = w(v_i)$ , because  $v_k$  is the first successor of  $v_i$ . Since ALGORITHM restricts property (4), the weight of  $v_iv_k$  can be modified only by adding 3. Thus,  $\omega(v_iv_k)$  is different from the weight of the edges incident with the predecessors of  $v_i$ .

Suppose that  $u \in V''$ . Since  $v_i$  has no successor and property (5) of ALGORITHM must hold, the edges  $\{v_jv_i : j < i, v_j \in N(v_i)\}$  are not monochromatic. Thus, there are two edges  $v_{j'}v_i$  and  $v_{j''}v_i$  such that  $\omega(v_{j'}v_i) \neq \omega(v_{j''}v_i)$  and we are done.  $\square$

*Proof of Theorem 5.* We may assume that  $G$  is connected, since otherwise the theorem holds by induction on each component. The theorem is obviously true if  $G = K_{1,n-1}$ . Thus, we assume that  $G \neq K_{1,n-1}$ . By Lemma 8, there is an ordering  $\mathbf{v} = (v_1, v_2, \dots, v_n)$  of vertices of  $G$  that satisfies the conditions (i)–(iii). Thus, we can apply ALGORITHM on  $G$ . Let  $\omega$  be the the edge-weighting  $\omega$  given by ALGORITHM. By Lemmas 13 and 15,  $\omega$  is a neighbour sum distinguishing 7-edge-weighting and all the vertices of degree at least 6 are incident with at least two edges of different weights, which proves the theorem.  $\square$

## 5 Bipartite graphs

In this section, we show that every nice bipartite graph has a 6-edge-weighting which distinguishes adjacent vertices and in which every vertex of degree at least 2 is incident with at least two edges of different weights. In order to prove this result, we apply a result obtained by Karoński et al. in [8]. They considered edge-weightings with elements of a group and proved the following theorem:

**Theorem 16.** [8] *Let  $\Gamma$  be a finite abelian group of odd order and let  $G$  be a non-trivial  $|\Gamma|$ -colourable graph. Then, there is an edge-weighting of  $G$  with elements of  $\Gamma$  such that the resulting vertex colouring is proper.*

Theorem 16 implies that if  $k$  is odd and  $G$  is non-trivially  $k$ -vertex colourable, then  $G$  admits a neighbour sum distinguishing  $k$ -edge-weighting. Furthermore, the

proof of Theorem 16 implies that if  $U_1, \dots, U_k$ ,  $|U_i| > 0$ ,  $1 \leq i \leq k$  are colour classes of  $G$ , then there is a neighbour sum distinguishing  $k$ -edge-weighting  $\omega$  such that  $\sigma_\omega(v_i) = i \pmod{k}$  for every  $v_i \in U_i$  ( $1 \leq i \leq k$ ). For convenience, we restate the part of the proof of Theorem 16 for bipartite graphs.

**Theorem 17.** [8] *Let  $G$  be a connected bipartite graph on at least three vertices with the vertex partition  $(V_1, V_2)$ . Then,  $G$  admits a neighbour sum distinguishing 3-edge-weighting. Moreover, there is a neighbour sum distinguishing 3-edge-weighting  $\omega$  of  $G$  such that  $\sigma_\omega(v_1) \neq \sigma_\omega(v_2) \pmod{3}$  for every  $v_1 \in V_1$  and  $v_2 \in V_2$ .*

*Proof.* Let  $x \in V(G)$  and  $d(x) \geq 2$ . Without loss of generality, assume that  $x \in V_1$ . Let  $e_1 = xv'_2, e_2 = xv''_2$ . We start with the weight 3 on all edges, so  $\sum_{e \in E(G)} \omega(e) = 0 \pmod{3}$ . We now try to modify the weights of edges, maintaining the sum of edge weights congruent to 0  $\pmod{3}$ , until all vertices of  $V_1 \setminus \{x\}$  have colours congruent to 1  $\pmod{3}$ . To do that, for each vertex  $v$  of  $V_1 \setminus \{x\}$ , we consider a path from  $v$  to  $x$  and add alternately 1 and 2 to the values of the edges along this path. After such an operation, the colour of  $v$  is 1  $\pmod{3}$ , the colour of  $x$  is changed, and all the colours of the other vertices are unchanged. Now, the only vertex of  $V_1$  which may have a colour different from 1  $\pmod{3}$  is  $x$ , and all the vertices of  $V_2$  still have a colour congruent to 0  $\pmod{3}$ . If the colour of  $x$  is not congruent to 0  $\pmod{3}$ , we are done; if not, we can finish by relabeling edge  $e_1$  on  $c_1$  and  $e_2$  on  $c_2$ , where  $c_1, c_2 \in \{1, 2, 3\}$  and  $c_1 = \omega(e_1) + 2, c_2 = \omega(e_2) + 2 \pmod{3}$ . Finally, we obtain the desired edge-weighting  $\omega$ , because

- either  $\sigma_\omega(v_1) = 1 \pmod{3}$  for  $v_1 \in V_1$  and  $\sigma_\omega(v_2) = 0$  for  $v_2 \in V_2 \setminus \{v'_2, v''_2\}$ ,  $\sigma_\omega(v'_2) \in \{0, 2\} \pmod{3}$  and  $\sigma_\omega(v''_2) \in \{0, 2\} \pmod{3}$ ,
- or  $\sigma_\omega(v_1) = 1 \pmod{3}$  for  $v_1 \in V_1 \setminus \{x\}$ ,  $\sigma_\omega(x) = 2 \pmod{3}$  and  $\sigma_\omega(v_2) = 0$  for  $v_2 \in V_2$ .

□

We can apply Theorem 17 for our version of the neighbour sum distinguishing edge-weighting. To prove the main result of this section, we need also the following lemma:

**Lemma 18.** *If  $G$  is bipartite, then there is a 2-edge-weighting of  $G$  such that every vertex of degree at least 2 is incident with two edges with different weights.*

*Proof.* We proceed by induction on the number of vertices. The lemma is true for bipartite graphs with one or two vertices. Assume that the lemma is true for every

bipartite graph with less than  $n$  vertices. Let  $G$  be a bipartite graph with  $n \geq 3$ . If  $G$  is not connected, then by induction there is a 2-edge-weighting of every component of  $G$  such that every vertex of degree at least 2 is incident with at least two edges of different weights, so we are done. Assume that  $G$  is connected and  $v$  is a vertex of minimum degree. Let  $G' = G - v$  and  $\omega$  be a 2-edge weighting of  $G'$  such that every vertex of degree at least 2 is incident with at least two edges labeled differently. We extend  $\omega$  to all the edges of  $G$ .

First, assume that  $d_G(v) = 1$ , then there are two possibilities. Let  $u$  be the neighbour of  $v$ . If  $d_{G'}(u) \geq 2$ , then, by induction hypothesis,  $u$  is already incident with two edges labeled differently, hence we can label the edge  $uv$  with any weight. Otherwise, we label the edge  $uv$  with the weight not used by the edge incident with  $u$  in  $G'$ .

Now, assume that  $d_G(v) = 2$  and let  $N(v) = \{u, w\}$ . Suppose first that  $v$  has a neighbour of degree at least 2 in  $G'$ , say  $d_{G'}(u) \geq 2$ . In this case, the edge  $uv$  can be labeled with either 1 or 2, because the vertex  $u$  is already incident with two edges labeled differently in  $G'$ . So, we first label the edge  $vw$  in such a way that  $w$  is incident with two edges of different weights, and next we label  $vu$  with the weight different from  $\omega(vw)$ .

Thus, we may assume that  $d_{G'}(u) = 1$  and  $d_{G'}(w) = 1$ . Observe that if  $\omega(uu_1) \neq \omega(wu_1)$  (where  $u_1, w_1$  is the neighbour in  $G'$  of  $u, w$ , respectively), then we can extend the colouring on all the edges of  $G$ . In such a case we label  $vu$  with the weight  $\omega(wu_1)$  and  $vw$  with the weight  $\omega(uu_1)$ .

Thus, we may assume that  $\omega(uu_1) = \omega(wu_1)$ , say without loss of generality  $\omega(uu_1) = \omega(wu_1) = 1$ . We relabel some edges of  $G'$ . If  $u_1$  is incident with at least two edges labeled with 1, then we relabel the edge  $uu_1$  with 2. In the new weighting of  $G'$ , every vertex of degree at least 2 is incident with at least two edges labeled differently and there are two neighbours of  $v$  having incident edges labeled differently, so as observed above we can extend the weighting to the desired edge-weighting of  $G$ . Suppose that  $uu_1$  is the only edge incident with  $u_1$  labeled with 1, the remaining edges having weight 2. Let  $u_2 \in N(u_1) \setminus \{u\}$ . If  $u_2$  is incident with at least two edges labeled with 2, then we relabel the edge  $u_1u_2$  with 1 and the edge  $uu_1$  with 2. We obtain an edge-weighting of  $G'$  in which every vertex of degree at least 2 is incident with at least two edges labeled differently and there are two neighbours of  $v$  having incident edges labeled differently, so we are done. Otherwise, we repeat this relabeling process. Suppose that, after  $k$  steps, we obtain a relabeled path  $P = u_0, u_1, u_2, \dots, u_k$  ( $u = u_0$ ). Let  $u_{k+1} \in N(u_k) \setminus \{u_{k-1}\}$ . Since every vertex  $u_i$  ( $i \in \{1, \dots, k-1\}$ ) is incident with exactly one edge labeled with

$\omega(u_{i-1}u_i)$  and there is no odd cycle in  $G$ , we have  $u_{k+1} \notin V(P) \setminus \{u_0\}$  and  $u_{k+1} \neq w$ . Furthermore,  $u_{k+1} \neq u$  because  $d_{G'}(u) = 1$ . Thus, the relabeling process eventually ends, and we obtain an alternating path  $P = u_0, u_1, u_2, \dots, u_t$  ( $u_0 = u$ ). Every vertex  $u_i$  ( $i \in \{1, \dots, t\}$ ) of  $P$  has degree at least 2 in  $G'$  and is incident with exactly one edge labeled with  $\omega(u_{i-1}u_i)$ , while  $u_t$  is incident with at least two edges labeled with  $\omega(u_{t-1}u_t)$ . We can swap the label of the edges of  $P$ , keeping an alternating path and obtaining a 2-edge-weighting of  $G'$  in which every vertex of degree at least 2 is incident with at least two edges labeled differently and where two neighbours of  $v$  have incident edges labeled differently, so we can extend the weighting on the edges incident with  $v$  in such a way that we obtain the desired edge-weighting.

Finally, assume that  $d_G(v) > 2$ . Since  $v$  is a vertex of minimum degree, each neighbour of  $v$  has degree at least 2 in  $G'$ . Thus, every neighbour of  $v$  is incident with two edges labeled differently in  $G'$ . Hence, we can label every edge incident with  $v$  with either colour 1 or 2, ensuring that the edges adjacent with  $v$  are not monochromatic.  $\square$

**Theorem 19.** *Let  $G$  be a nice bipartite graph. Then, there is a neighbour sum distinguishing 6-edge-weighting such that every vertex of degree at least 2 is incident with at least two edges with different weights.*

*Proof.* Let  $(V_1, V_2)$  be the vertex partition of  $G$ . By Theorem 17, there is a neighbour sum distinguishing 3-edge-weighting  $\omega$  of  $G$  such that  $\sigma_\omega(v_1) \neq \sigma_\omega(v_2) \pmod{3}$  for every  $v_1 \in V_1$  and  $v_2 \in V_2$ . Let  $E_i = \{e \in E(G) : \omega(e) = i\}$  for  $i \in \{1, 2, 3\}$ . By Lemma 18, every subgraph induced by  $E_i$  can be labeled with two weights in such a way that every vertex of degree at least 2 is incident with at least two edges labeled differently. Thus, we relabel  $E_1$  with weights 1 and 4 in such a way that every vertex of degree at least 2 is incident with at least two edges labeled differently, and similarly we relabel the edges of  $E_2$  with 2 and 5, and the edges of  $E_3$  with 3 and 6. Let us denote by  $\omega'$  the resultant edge-weighting. Observe that  $\sigma_\omega(v) = \sigma_{\omega'}(v) \pmod{3}$ . Thus,  $\omega'$  is neighbour sum distinguishing, so  $\omega'$  is the desired edge-weighting.  $\square$

The following theorem was proved in [9]:

**Theorem 20.** [9] *Let  $G$  be a connected bipartite graph on at least three vertices with vertex partition  $(V_1, V_2)$ . If  $|V_1|$  is even, then, there is a neighbour sum distinguishing 2-edge-weighting  $\omega$  of  $G$  such that  $\sigma_\omega(v_1) \neq \sigma_\omega(v_2) \pmod{2}$  for every  $v_1 \in V_1$  and  $v_2 \in V_2$ .*

Thus, we can apply Theorem 20 and, similarly as Theorem 19, we can prove the following result:

**Theorem 21.** *Let  $G$  be a connected bipartite graph on at least three vertices with vertex partition  $(V_1, V_2)$  and  $|V_1|$  be even. Then,  $G$  admits a neighbour sum distinguishing 4-edge-weighting such that every vertex of degree at least 2 is incident with at least two edges of different weights.*

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