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# BILINEAR OPTIMAL CONTROL FOR A FRACTIONAL DIFFUSIVE EQUATION

CYRILLE KENNE, GISÈLE MOPHOU, AND MAHAMADI WARMA

ABSTRACT. We consider a bilinear optimal control for an evolution equation involving the fractional Laplace operator of order  $0 < s < 1$ . We first give some existence and uniqueness results for the considered evolution equation. Next, we establish some weak maximum principle results allowing us to obtain more regularity of our state equation. Then, we consider an optimal control problem which consists to bring the state of the system at final time to a desired state. We show that this optimal control problem has a solution and we derive the first and second order optimality conditions. Finally, under additional assumptions on the initial datum and the given target, we prove that local uniqueness of optimal solutions can be achieved.

## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ) be a bounded domain with boundary  $\partial\Omega$  and  $\omega \subset \Omega$  an open set. Given  $T > 0$ ,  $\alpha > 0$  and  $\rho^d \in L^\infty(\Omega)$ , we are interested to the optimal control problem: Find

$$\inf_{v \in \mathcal{U}} J(v) := \frac{1}{2} \|\rho(\cdot, T) - \rho^d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|v\|_{L^2(\omega \times (0, T))}^2, \quad (1.1)$$

subject to the constraints that  $\rho$  solves the space fractional diffusion equation

$$\begin{cases} \rho_t + (-\Delta)^s \rho &= v \rho \chi_\omega & \text{in } Q := \Omega \times (0, T), \\ \rho &= 0 & \text{in } \Sigma := (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ \rho(\cdot, 0) &= \rho^0 & \text{in } \Omega, \end{cases} \quad (1.2)$$

and the set of admissible controls is given by

$$\mathcal{U} := \{v \in L^\infty(\omega \times (0, T)) : m \leq v \leq M, m, M \in \mathbb{R}, M > m\}. \quad (1.3)$$

In (1.2),  $(-\Delta)^s$  denotes the fractional Laplace operator of order  $0 < s < 1$ ,  $\rho^0 \in L^\infty(\Omega)$  and  $\chi_\omega$  is the characteristic function of  $\omega$ .

Bilinear systems are used to describe many processes in biology, ecology and engineering. These systems which are nonlinear due to the product between the input and the state variable are gained in interest to many researchers. We refer to Bruni et al. [8] for instance. Optimal control of such systems has been widely investigated. In the case of control depending only on time, Bradley et al. [6] proved the existence and uniqueness of a bilinear optimal control. Actually, the control which acts as a multiplier of a velocity term is a positive uniformly bounded function of time. The uniqueness of the optimal control were achieved for a time  $T$  sufficiently small. Addou et al. [1] studied a bilinear optimal control of a system governed by a fourth-order parabolic operator. The bilinearity appeared in the form of the scalar product of the vector of controls and the gradient of the state. The authors proved under suitable hypotheses the existence of an optimal control that they characterized with an optimality system. Then, assuming that the initial state is small enough, they obtained the uniqueness of the

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optimal control. In [16], an optimal bilinear control of an abstract Schrödinger equation was considered. The existence of an optimal control depending only on time is proved and the first order optimality system is derived. The paper [33] considered a regional quadratic control problem for distributed bilinear systems. They proved that an optimal control exists and gave an optimality system that characterizes the control. Considering a control depending on time and space, [21] studied an optimal control of linear heat equation with convective boundary condition in which the heat transfer is took as the control. They proved the existence and uniqueness of the optimal control and the solution of the optimality system. The results were achieved by means of compactness and maximum principle results. In [13] they investigated a constrained regional control problem of a bilinear plate equation. They proved using some compactness results the existence of an optimal control that they characterized with an optimality system. The cases of time or space dependent control were also discussed. Recently, [5] studied an optimal control problem subject to the Fokker-Planck equation. They proved the existence of optimal controls and derived the first and second order optimality conditions. We refer to [4, 7, 22, 27, 28, 30, 34, 35, 36] for more literature on bilinear optimal control problems involving PDEs of integer order.

Actually, since bilinear controlled PDEs are nonlinear, the most challenging issue in controlling such models is not only to find appropriate compactness results to obtain the existence of an optimal control, but also to derive necessary and sufficient optimality conditions and the uniqueness of optimal solutions. This can be achieved by improving the regularity of solutions to the optimality systems.

To be the best of our knowledge this is the first work on control problems associated to fractional bilinear PDEs. We have obtained the following specific results:

- Corollary 3.4 and Theorem 3.6 show existence, uniqueness and regularity of solutions of (1.2). The regularity is obtained by proving some results of maximum principle that are interesting in their own independently of the application given here.
- Theorem 4.1 gives the existence of solutions to the control problem (1.2)-(1.1).
- The first order necessary optimality conditions are given in Theorem 5.8.
- The second order necessary and sufficient conditions are contained in Theorems 5.16 and 5.18.
- Finally, in Theorem 5.20 we prove the local uniqueness of optimal solutions under additional assumptions on the initial datum  $\rho^0$  and the target  $\rho^d$ .

The rest of the paper is organized as follows. In Section 2, we give some definitions and properties of the fractional Laplacian and some known results. In Section 3, we first prove the existence and uniqueness of weak solutions to the bilinear fractional diffusive system. Then, under some assumptions on the data, we establish appropriate maximum principle results. We prove in Section 4 that there exists at least one optimal control solution of (1.2)-(1.1). In Section 5 we derive the first and second order optimality conditions and systems. Under smallness assumptions on the initial datum and the given target, we show that the optimal solutions are locally unique.

## 2. PRELIMINARIES

For the sake of completeness, we give some well-known results that are used throughout the paper. We start by introducing the fractional Laplace operator. Given  $0 < s < 1$ , we let

$$\mathcal{L}_s^1(\mathbb{R}^N) := \left\{ w : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable and } \int_{\mathbb{R}^N} \frac{|w(x)|}{(1+|x|)^{N+2s}} dx < \infty \right\}.$$

For  $w \in \mathcal{L}_s^1(\mathbb{R}^N)$  and  $\varepsilon > 0$ , we set

$$(-\Delta)_\varepsilon^s w(x) := C_{N,s} \int_{\{y \in \mathbb{R}^N : |x-y| > \varepsilon\}} \frac{w(x) - w(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where  $C_{N,s}$  is a normalization constant given by  $C_{N,s} := \frac{s2^{2s}\Gamma(\frac{2s+N}{2})}{\pi^{\frac{N}{2}}\Gamma(1-s)}$ . The fractional Laplacian  $(-\Delta)^s$  is defined by the following singular integral:

$$(-\Delta)^s w(x) := C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{w(x) - w(y)}{|x - y|^{N+2s}} dy = \lim_{\varepsilon \downarrow 0} (-\Delta)_\varepsilon^s w(x), \quad x \in \mathbb{R}^N, \quad (2.1)$$

provided that the limit exists for a.e.  $x \in \mathbb{R}^N$ . We refer to [12, 20] for equivalent definitions of  $(-\Delta)^s$ .

Next, we introduce the function spaces needed to investigate our problem. Let  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ) be an arbitrary open set and  $0 < s < 1$ . We define the fractional order Sobolev space

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\}$$

and we endow it with the norm given by

$$\|u\|_{H^s(\Omega)} = \left( \int_{\Omega} |u|^2 dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}.$$

We set

$$H_0^s(\Omega) := \left\{ w \in H^s(\mathbb{R}^N) : w = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}.$$

Then,  $H_0^s(\Omega)$  endowed with the norm

$$\|w\|_{H_0^s(\Omega)} := \left( \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(w(x) - w(y))^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}, \quad (2.2)$$

is a Hilbert space (see e.g. [29, Lemma 7]). We let  $H^{-s}(\Omega) := (H_0^s(\Omega))^*$  be the dual space of  $H_0^s(\Omega)$  with respect to the pivot space  $L^2(\Omega)$ , so that we have the following continuous embeddings (see e.g. [3]):

$$H_0^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-s}(\Omega). \quad (2.3)$$

From now on, for any  $\rho, \psi \in H_0^s(\Omega)$ , we set

$$\mathcal{F}(\rho, \psi) := \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\rho(x) - \rho(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} dx dy. \quad (2.4)$$

Hence, the norm on  $H_0^s(\Omega)$  given by (2.2) becomes  $\|w\|_{H_0^s(\Omega)} = (\mathcal{F}(w, w))^{1/2}$ .

We let the operator  $(-\Delta)_D^s$  on  $L^2(\Omega)$  be given by

$$D((-\Delta)_D^s) := \{u \in H_0^s(\Omega) : (-\Delta)^s u \in L^2(\Omega)\}, \quad (-\Delta)_D^s u := (-\Delta)^s u \text{ in } \Omega. \quad (2.5)$$

Then,  $(-\Delta)_D^s$  is the realization in  $L^2(\Omega)$  of  $(-\Delta)^s$  with the zero Dirichlet exterior condition.

The following result is well-known (see e.g. [11, 14]).

**Proposition 2.1.** *Let  $(-\Delta)_D^s$  be the operator defined in (2.5). Then,  $(-\Delta)_D^s$  can be also viewed as a bounded operator from  $H_0^s(\Omega)$  into  $H^{-s}(\Omega)$  given by*

$$\langle (-\Delta)_D^s u, v \rangle_{H^{-s}(\Omega), H_0^s(\Omega)} = \mathcal{F}(u, v), \quad u, v \in H_0^s(\Omega). \quad (2.6)$$

We also need the following compactness result.

**Theorem 2.2.** [23, Theorem 5.1, Page 58] *Let  $B_0, B, B_1$  be three Banach spaces such that we have the continuous embeddings  $B_0 \hookrightarrow B \hookrightarrow B_1$ , with  $B_i$  being reflexive,  $i = 0, 1$ . Assume that the embedding  $B_0 \hookrightarrow B$  is also compact and set*

$$\mathcal{W} := \{ \rho \in L^2((0, T); B_0) : \rho_t \in L^2((0, T); B_1) \},$$

*with  $T < \infty$ . Then,  $\mathcal{W}$  is compactly embedded in  $L^2((0, T); B)$ .*

Let us recall the following result given in [24, Page 37].

**Theorem 2.3.** *Let  $(F, \|\cdot\|_F)$  be a Hilbert space. Let  $\Phi$  be a subspace of  $F$  endowed with a pre-Hilbert scalar product  $((\cdot, \cdot))$ , with associated norm  $\|\cdot\|$ . Moreover, let  $\mathcal{E} : F \times \Phi \rightarrow \mathbb{C}$  be a sesquilinear form. Assume that the following hypotheses hold:*

(a) *The embedding  $\Phi \hookrightarrow F$  is continuous, i.e., there is a constant  $C_1 > 0$  such that*

$$\|\varphi\|_F \leq C_1 \|\varphi\| \text{ for all } \varphi \in \Phi.$$

(b) *For all  $\varphi \in \Phi$ , the mapping  $u \mapsto \mathcal{E}(u, \varphi)$  is continuous on  $F$ .*

(c) *There is a constant  $C_2 > 0$  such that*

$$\mathcal{E}(\varphi, \varphi) \geq C_2 \|\varphi\|^2 \text{ for all } \varphi \in \Phi.$$

*If  $\varphi \mapsto L(\varphi)$  is a continuous linear functional on  $\Phi$ , then there exists  $u \in F$  verifying  $\mathcal{E}(u, \varphi) = L(\varphi)$  for all  $\varphi \in \Phi$ .*

Next, let  $\mathbb{X}$  be a Banach space with dual  $\mathbb{X}^*$ . We set

$$W(0, T; \mathbb{X}) := \{\psi \in L^2((0, T); \mathbb{X}) : \psi_t \in L^2((0, T); \mathbb{X}^*)\}. \quad (2.7)$$

Then  $W(0, T; \mathbb{X})$  endowed with the norm given by

$$\|\psi\|_{W(0, T; \mathbb{X})}^2 = \|\psi\|_{L^2(0, T; \mathbb{X})}^2 + \|\psi_t\|_{L^2(0, T; \mathbb{X}^*)}^2, \quad (2.8)$$

is a Hilbert space. Moreover, if  $\mathbb{Y}$  is a Hilbert space that can be identified with its dual  $\mathbb{Y}^*$  and we have the continuous embeddings  $\mathbb{X} \hookrightarrow \mathbb{Y} = \mathbb{Y}^* \hookrightarrow \mathbb{X}^*$ , then using [25, Theorem 1.1, page 102], we have the continuous embedding

$$W(0, T; \mathbb{X}) \hookrightarrow C([0, T]; \mathbb{Y}). \quad (2.9)$$

### 3. EXISTENCE RESULTS AND MAXIMUM PRINCIPLE

From now on, we simply denote by  $\|\cdot\|_\infty$  the  $L^\infty$ -norm in  $L^\infty(\omega_T)$  and  $\Omega \subset \mathbb{R}^N$  is an arbitrary bounded domain. In addition, for  $u, w \in H_0^s(\Omega)$ , we let  $\mathcal{F}(u, w)$  denote the bilinear form given in (2.4). To simplify the notations, we set

$$\mathbb{V} := H_0^s(\Omega) \text{ and } \mathbb{V}^* := H^{-s}(\Omega), \quad (3.1)$$

and we let  $\langle \cdot, \cdot \rangle_{\mathbb{V}^*, \mathbb{V}}$  denote the duality mapping between  $\mathbb{V}^*$  and  $\mathbb{V}$ .

**3.1. Existence results.** For  $r > 0$  a real number, we consider the system

$$\begin{cases} z_t + (-\Delta)^s z + rz &= vz\chi_\omega + e^{-rt}f & \text{in } Q, \\ z &= 0 & \text{in } \Sigma, \\ z(\cdot, 0) &= \rho^0 & \text{in } \Omega. \end{cases} \quad (3.2)$$

**Definition 3.1.** Let  $f \in L^2((0, T); \mathbb{V}^*)$ ,  $v \in L^\infty(\omega_T)$  and  $\rho^0 \in L^2(\Omega)$ . We say that  $z \in L^2((0, T); \mathbb{V})$  is a weak solution of (3.2), if the equality

$$\begin{aligned} - \int_0^T \langle \phi_t, z \rangle_{\mathbb{V}^*, \mathbb{V}} dt &+ \int_0^T \mathcal{F}(z, \phi) dt + r \int_Q z \phi dx dt - \int_{\omega_T} vz \phi dx dt \\ &= \int_0^T e^{-rt} \langle f, \phi \rangle_{\mathbb{V}^*, \mathbb{V}} dt + \int_\Omega \rho^0 \phi(0) dx, \end{aligned}$$

holds, for every  $\phi \in H(Q)$ , where

$$H(Q) := \{\varphi \in W(0, T; \mathbb{V}) \text{ and } \varphi(\cdot, T) = 0 \text{ a.e. in } \Omega\}. \quad (3.3)$$

We have the following existence result.

**Theorem 3.2.** *Let  $f \in L^2((0, T); \mathbb{V}^*)$ ,  $v \in L^\infty(\omega_T)$ ,  $r = \|v\|_\infty$  and  $\rho^0 \in L^2(\Omega)$ . Then, there exists a unique weak solution  $z \in W(0, T; \mathbb{V})$  of (3.2). In addition, there is a constant  $C = C(N, s, \Omega) > 0$  such that*

$$\sup_{\tau \in [0, T]} \|z(\cdot, \tau)\|_{L^2(\Omega)}^2 \leq \|f\|_{L^2((0, T); \mathbb{V}^*)}^2 + \|\rho^0\|_{L^2(\Omega)}^2, \quad (3.4a)$$

$$\|z\|_{L^2((0, T); \mathbb{V})}^2 \leq \|f\|_{L^2((0, T); \mathbb{V}^*)}^2 + \|\rho^0\|_{L^2(\Omega)}^2, \quad (3.4b)$$

$$\|z\|_{W(0, T; \mathbb{V})} \leq (C\|v\|_\infty + 3) (\|f\|_{L^2((0, T); \mathbb{V}^*)} + \|\rho^0\|_{L^2(\Omega)}). \quad (3.4c)$$

*Proof.* We proceed in four steps.

**Step 1.** We prove existence by using Theorem 2.3. Recall that the norm on  $L^2((0, T); \mathbb{V})$  is given by

$$\|z\|_{L^2((0, T); \mathbb{V})}^2 = \int_0^T \|z(\cdot, t)\|_{\mathbb{V}}^2 dt.$$

We consider the norm defined on  $H(Q)$  by  $\|z\|_{H(Q)}^2 := \|z\|_{L^2((0, T); \mathbb{V})}^2 + \|z(\cdot, 0)\|_{L^2(\Omega)}^2$ . It is clear that we have the continuous embedding  $H(Q) \hookrightarrow L^2((0, T); \mathbb{V})$ .

Now, let  $\varphi \in H(Q)$  and consider the bilinear form  $\mathcal{E}(\cdot, \cdot) : L^2((0, T); \mathbb{V}) \times H(Q) \rightarrow \mathbb{R}$  given by

$$\mathcal{E}(z, \varphi) := - \int_0^T \langle \phi_t, z \rangle_{\mathbb{V}^*, \mathbb{V}} dt + \int_0^T \mathcal{F}(z, \varphi) dt + r \int_Q z \varphi dx dt - \int_{\omega_T} v z \varphi dx dt. \quad (3.5)$$

Using Cauchy-Schwarz's inequality, we get that

$$|\mathcal{E}(z, \varphi)| \leq \left( \|\varphi_t\|_{L^2((0, T); \mathbb{V}^*)} + (r + \|v\|_\infty) \|\varphi\|_{L^2(Q)} + \|\varphi\|_{L^2((0, T); \mathbb{V})} \right) \|z\|_{L^2((0, T); \mathbb{V})}.$$

Consequently, for every fixed  $\varphi \in H(Q)$ , the functional  $z \mapsto \mathcal{E}(z, \varphi)$  is continuous on  $L^2((0, T); \mathbb{V})$ .

Next, since  $r = \|v\|_\infty$ , we have that for every  $\varphi \in H(Q)$ ,

$$\begin{aligned} \mathcal{E}(\varphi, \varphi) &= - \int_0^T \langle \varphi_t, \varphi \rangle_{\mathbb{V}^*, \mathbb{V}} dt + \int_0^T \mathcal{F}(\varphi, \varphi) dt + r \int_Q \varphi^2 dx dt - \int_{\omega_T} v \varphi^2 dx dt \\ &\geq \frac{1}{2} \|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 + \int_0^T \mathcal{F}(\varphi, \varphi) dt + (r - \|v\|_\infty) \int_Q \varphi^2 dx dt \\ &\geq \frac{1}{2} \|\varphi\|_{H(Q)}^2. \end{aligned}$$

Hence,  $\mathcal{E}$  is coercive on  $H(Q)$ .

Finally, let us consider the functional  $L : H(Q) \rightarrow \mathbb{R}$  defined by

$$L(\varphi) := \int_0^T e^{-rt} \langle f, \phi \rangle_{\mathbb{V}^*, \mathbb{V}} dt + \int_\Omega \rho^0 \varphi(x, 0) dx.$$

Using Cauchy-Schwarz's inequality, there is a constant  $C > 0$  such that

$$|L(\varphi)| \leq C (\|\rho^0\|_{L^2(\Omega)} + \|f\|_{L^2((0, T); \mathbb{V}^*)}) \|\varphi\|_{H(Q)}.$$

Therefore,  $L$  is continuous and linear on  $H(Q)$ . It follows from Theorem 2.3 that there exists  $z \in L^2((0, T); \mathbb{V})$  such that  $\mathcal{E}(z, \varphi) = L(\varphi)$ ,  $\forall \varphi \in H(Q)$ . We have shown that the system (3.2) has a solution  $z \in L^2((0, T); \mathbb{V})$  in the sense of Definition 3.1.

**Step 2.** We show that  $z_t \in L^2((0, T); \mathbb{V}^*)$ . Notice that (3.2) can be rewritten as the abstract Cauchy problem

$$\begin{cases} z_t + (-\Delta)_D^s z + rz &= vz\chi_\omega + e^{-rt} f & \text{in } Q, \\ z(\cdot, 0) &= \rho^0 & \text{in } \Omega, \end{cases} \quad (3.6)$$

where  $(-\Delta)_D^s$  is defined in (2.5). Since  $z \in L^2((0, T); \mathbb{V})$ , it follows from Proposition 2.1 that  $(-\Delta)_D^s z(\cdot, t) \in \mathbb{V}^*$ . Since  $v \in L^\infty(\omega_T)$ , we have  $zv\chi_\omega \in L^2(\omega_T) \hookrightarrow \mathbb{V}^*$  and we can deduce that  $z_t(\cdot, t) = -(-\Delta)_D^s z(\cdot, t) - rz(\cdot, t) + (vz)(\cdot, t)\chi_\omega + e^{-rt}f(\cdot, t) \in \mathbb{V}^*$ .

If we take the duality map between (3.6) and  $\phi \in L^2((0, T); \mathbb{V})$ , and use Proposition 2.1, we obtain

$$\begin{aligned} \langle z_t(t), \phi(t) \rangle_{\mathbb{V}^*, \mathbb{V}} + \mathcal{F}(z(t), \phi(t)) + r \int_{\Omega} z(t) \phi(t) dx &= \int_{\omega} v(t) z(t) \phi(t) dx \\ &+ e^{-rt} \langle f(t), \phi(t) \rangle_{\mathbb{V}^*, \mathbb{V}}. \end{aligned}$$

This implies that there is a constant  $C > 0$  such that

$$|\langle z_t(t), \phi(t) \rangle_{\mathbb{V}^*, \mathbb{V}}| \leq [(C\|v\|_\infty + 1)\|z(t)\|_{\mathbb{V}} + \|f(t)\|_{\mathbb{V}^*}] \|\phi(t)\|_{\mathbb{V}}. \quad (3.7)$$

Integrating (3.7) over  $(0, T)$ , we get that there are two constants  $C = C(N, s, \Omega) > 0$  and  $C_1 := (C\|v\|_\infty + 1)$  such that

$$\int_0^T |\langle z_t(t), \phi(t) \rangle_{\mathbb{V}^*, \mathbb{V}}| dt \leq [C_1 \|z\|_{L^2((0, T); \mathbb{V})} + \|f\|_{L^2((0, T); \mathbb{V}^*)}] \|\phi\|_{L^2((0, T); \mathbb{V})}. \quad (3.8)$$

Using (3.4b) we get from (3.8) that

$$\|z_t\|_{L^2((0, T); \mathbb{V}^*)} \leq (C\|v\|_\infty + 2) (\|f\|_{L^2((0, T); \mathbb{V}^*)} + \|\rho^0\|_{L^2(\Omega)}). \quad (3.9)$$

Thus,  $z_t \in L^2((0, T); \mathbb{V}^*)$  and we have shown that  $z \in W(0, T; \mathbb{V})$ .

**Step 3.** We show (3.4a), (3.4b) and (3.4c). If we take the duality map between (3.6) and  $z \in W(0, T; \mathbb{V})$ , use Proposition 2.1 and Young's inequality, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|z(t)\|_{L^2(\Omega)}^2 + \mathcal{F}(z(t), z(t)) + r \|z(t)\|_{L^2(\Omega)}^2 &= \\ e^{-rt} \langle f(t), z(t) \rangle_{\mathbb{V}^*, \mathbb{V}} + \int_{\omega} v(t) z^2(t) dx &\leq \frac{1}{2} \|f(t)\|_{\mathbb{V}^*}^2 + \frac{1}{2} \|z(t)\|_{\mathbb{V}}^2 + \|v\|_\infty \|z(t)\|_{L^2(\Omega)}^2. \end{aligned}$$

Hence,

$$\frac{d}{dt} \|z(t)\|_{L^2(\Omega)}^2 + \|z(t)\|_{\mathbb{V}}^2 \leq \|f(t)\|_{\mathbb{V}^*}^2, \quad (3.10)$$

because  $r = \|v\|_\infty$ . Integrating (3.10) over  $(0, \tau)$ , with  $\tau \in [0, T]$ , we get that

$$\|z(\tau)\|_{L^2(\Omega)}^2 + \int_0^\tau \|z(t)\|_{\mathbb{V}}^2 dt \leq \|f\|_{L^2((0, T); \mathbb{V}^*)}^2 + \|\rho^0\|_{L^2(\Omega)}^2,$$

from which we deduce (3.4a) and (3.4b). Combining (3.4b)-(3.9) and recalling (2.8), we get (3.4c).

**Step 4.** We prove uniqueness. Assume that there exist  $z_1$  and  $z_2$  solutions to (3.2) with the same right hand side  $f$ ,  $v$  and initial datum  $\rho^0$ . Set  $\tilde{z} := z_1 - z_2$ . Then,  $\tilde{z}$  satisfies

$$\begin{cases} \tilde{z}_t + (-\Delta)^s \tilde{z} + r\tilde{z} = v\tilde{z}\chi_\omega & \text{in } Q, \\ \tilde{z} = 0 & \text{in } \Sigma, \\ \tilde{z}(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \quad (3.11)$$

From Step 2, we have that  $\tilde{z}$  has the regularity to be taken as a test function in (3.11). So, if we take the duality map of (3.11) with  $\tilde{z}$ , use Proposition 2.1 and integrate over  $(0, T)$  we obtain

$$\frac{1}{2} \|\tilde{z}(\cdot, T)\|_{L^2(\Omega)}^2 + \int_0^T \mathcal{F}(\tilde{z}, \tilde{z}) dt + r \|\tilde{z}\|_{L^2(Q)}^2 = \int_{\omega_T} v \tilde{z}^2 dx dt \leq \|v\|_\infty \|\tilde{z}\|_{L^2(Q)}^2.$$

Since  $r = \|v\|_\infty$ , we can deduce that  $\frac{1}{2} \|\tilde{z}(\cdot, T)\|_{L^2(\Omega)}^2 + \|\tilde{z}\|_{L^2((0, T); \mathbb{V})}^2 \leq 0$ . Hence,  $\tilde{z} = 0$  in  $\mathbb{R}^N \times [0, T]$ . Thus,  $z_1 = z_2$  in  $\mathbb{R}^N \times [0, T]$  and we have shown uniqueness.  $\square$

Next, we consider the system

$$\begin{cases} \rho_t + (-\Delta)^s \rho = f + v\rho\chi_\omega & \text{in } Q, \\ \rho = 0 & \text{in } \Sigma, \\ \rho(\cdot, 0) = \rho^0 & \text{in } \Omega. \end{cases} \quad (3.12)$$

We have the following result that can be viewed as a corollary of Theorem 3.2.

**Corollary 3.3.** *Let  $f \in L^2((0, T); \mathbb{V}^*)$ ,  $v \in L^\infty(\omega_T)$  and  $\rho^0 \in L^2(\Omega)$ . Then, there exists a unique weak solution  $\rho \in W(0, T; \mathbb{V})$  of (3.12). In addition, there is a constant  $C = C(N, s, \Omega) > 0$  such that*

$$\sup_{\tau \in [0, T]} \|\rho(\cdot, \tau)\|_{L^2(\Omega)}^2 \leq e^{2\|v\|_\infty T} \left[ \|f\|_{L^2((0, T); \mathbb{V}^*)}^2 + \|\rho^0\|_{L^2(\Omega)}^2 \right], \quad (3.13a)$$

$$\|\rho\|_{L^2((0, T); \mathbb{V})}^2 \leq e^{2\|v\|_\infty T} \left[ \|f\|_{L^2((0, T); \mathbb{V}^*)}^2 + \|\rho^0\|_{L^2(\Omega)}^2 \right], \quad (3.13b)$$

$$\|\rho\|_{W(0, T; \mathbb{V})} \leq \left[ (1 + C\|v\|_\infty) e^{\|v\|_\infty T} + 1 \right] \left[ \|f\|_{L^2((0, T); \mathbb{V}^*)} + \|\rho^0\|_{L^2(\Omega)} \right]. \quad (3.13c)$$

*Proof.* Since  $\rho := e^{\|v\|_\infty t} z$  is a weak solution of (3.12) if and only if  $z$  is a weak solution of (3.2), we have from Theorem 3.2 that there exists a unique solution  $\rho \in W(0, T; \mathbb{V})$  of (3.12).

Next, letting  $z = e^{-\|v\|_\infty t} \rho$  in (3.4a) and (3.4b) we respectively deduce that (3.13a) and (3.13b) hold true.

Finally, we show (3.13c). Let  $\phi \in L^2((0, T); \mathbb{V})$ . If we take the duality map between (3.12) and  $\phi(t)$ , and use Proposition 2.1, we obtain for a.e.  $t \in (0, T)$ ,

$$\langle \rho_t(t), \phi(t) \rangle_{\mathbb{V}^*, \mathbb{V}} + \mathcal{F}(\rho(t), \phi(t)) = \int_\omega v(t) \rho(t) \phi(t) \, dx + \langle f(t), \phi(t) \rangle_{\mathbb{V}^*, \mathbb{V}}.$$

Using the Cauchy-Schwarz inequality and integrating over  $(0, T)$  we can deduce that

$$\|\rho_t\|_{L^2((0, T); \mathbb{V}^*)} \leq (1 + C\|v\|_\infty) e^{\|v\|_\infty T} \left[ \|f\|_{L^2((0, T); \mathbb{V}^*)} + \|\rho^0\|_{L^2(\Omega)} \right], \quad (3.14)$$

where we have also used (3.13b). Adding (3.13b) to (3.14) we get (3.13c).  $\square$

**Corollary 3.4.** *Let  $v \in L^\infty(\omega_T)$  and  $\rho^0 \in L^2(\Omega)$ . Then, there exists a unique solution  $\rho \in W(0, T; \mathbb{V})$  of (1.2). In addition, there is a constant  $C = C(N, s, \Omega) > 0$  such that*

$$\sup_{\tau \in [0, T]} \|\rho(\cdot, \tau)\|_{L^2(\Omega)} \leq C \|\rho\|_{W(0, T; \mathbb{V})} \leq (1 + C\|v\|_\infty) e^{\|v\|_\infty T} \|\rho^0\|_{L^2(\Omega)}. \quad (3.15)$$

*Proof.* It suffice to apply Corollary 3.3 with  $f \equiv 0$ .  $\square$

**3.2. Maximum principle.** We give some useful results of maximum principle.

**Lemma 3.5.** *Let  $\rho^0 \in L^2(\Omega)$  be such that  $\rho^0 \geq 0$  a.e. in  $\Omega$  and  $v \in L^\infty(\omega_T)$ . Then, the weak solution  $\rho$  of (1.2) satisfies  $\rho \geq 0$  a.e. in  $\mathbb{R}^N \times [0, T]$ .*

*Proof.* We write  $\rho = \rho^+ - \rho^-$ , where  $\rho^+ := \max(\rho, 0)$  and  $\rho^- := \max(0, -\rho)$ . It is sufficient to show that  $\rho^- = 0$  a.e. in  $\mathbb{R}^N \times [0, T]$ . Notice that

$$\rho^- = 0 \text{ in } \Sigma \text{ and } \rho^-(\cdot, 0) = \max(0, -\rho^0) = 0 \text{ a.e. in } \Omega.$$

Moreover, we have that  $\rho^- \in W(0, T; \mathbb{V})$  (see e.g. [32]). We set

$$\begin{cases} \mathcal{O}^- := \{x \in \mathbb{R}^N : \rho(x, t) \leq 0 \text{ for a.e. } t \in (0, T)\}, \\ \mathcal{O}^+ := \{x \in \mathbb{R}^N : \rho(x, t) > 0 \text{ for a.e. } t \in (0, T)\}. \end{cases} \quad (3.16)$$



If we take the duality map between (1.2) and  $\psi \in \mathbb{V}$ , and use Proposition 2.1, we get for a.e.  $t \in [0, T]$ ,

$$\langle \rho_t(t), \psi \rangle_{\mathbb{V}^*, \mathbb{V}} + \mathcal{F}(\rho(t), \psi) = \int_{\omega} v(t) \rho(t) \psi dx. \quad (3.17)$$

Taking  $\psi = \rho^-(\cdot, t)$  in (3.17) and noticing that  $\rho_t^- = \rho_t \chi_{\mathcal{O}^-}$  and  $\rho^+ \rho^- = 0$ , we get

$$\int_{\mathcal{O}^-} \frac{d}{dt} |\rho^-(t)|^2 dx - \mathcal{F}(\rho(t), \rho^-(t)) = \int_{\mathcal{O}^-} v(t) |\rho^-(t)|^2 dx. \quad (3.18)$$

Observe also that

$$\mathcal{F}(\rho(t), \rho^-(t)) = \mathcal{F}(\rho^+(t), \rho^-(t)) - \mathcal{F}(\rho^-(t), \rho^-(t)). \quad (3.19)$$

It is well-known (see e.g. [11, 14, 32]) that  $\mathcal{F}(\rho^+(t), \rho^-(t)) \leq 0$ . Thus, it follows from (3.19) that  $\mathcal{F}(\rho(t), \rho^-(t)) \leq 0$ . Since  $-\mathcal{F}(\rho(t), \rho^-(t)) \geq 0$ , it follows from (3.18) that

$$\frac{1}{2} \frac{d}{dt} \|\rho^-(t)\|_{L^2(\mathcal{O}^-)}^2 \leq \int_{\mathcal{O}^-} v(t) (\rho^-(t))^2 dx \leq \|v\|_{\infty} \|\rho^-(t)\|_{L^2(\mathcal{O}^-)}^2. \quad (3.20)$$

Using Gronwall's Lemma we can deduce that

$$\|\rho^-(\cdot, t)\|_{L^2(\mathcal{O}^-)}^2 \leq e^{2t\|v\|_{\infty}} \|\rho^-(\cdot, 0)\|_{L^2(\mathcal{O}^-)}^2 = 0,$$

where we have also used that  $\rho^-(\cdot, 0) = 0$  a.e. in  $\mathcal{O}^-$ . We have shown that  $\rho^- = 0$  a.e. in  $\mathcal{O}^- \times [0, T]$ . Thus,  $\rho^- = 0$  a.e. in  $\mathbb{R}^N \times [0, T]$ . Consequently,  $\rho \geq 0$  a.e. in  $\mathbb{R}^N \times [0, T]$ . The proof is finished.  $\square$

We have the following maximum principle.

**Theorem 3.6.** *Let  $\rho^0 \in L^\infty(\Omega)$  and  $v \in L^\infty(\omega_T)$ . Then, the unique weak solution  $\rho$  of (1.2) belongs to  $W(0, T; \mathbb{V}) \cap L^\infty(\mathbb{R}^N \times (0, T))$  and*

$$\|\rho\|_{L^\infty(\mathbb{R}^N \times (0, T))} \leq e^{\|v\|_{\infty} T} \|\rho^0\|_{L^\infty(\Omega)}. \quad (3.21)$$

*Proof.* We set  $z := e^{-\|v\|_{\infty} t} \rho$ , where  $\rho$  is the solution of (1.2). Then,  $z \in W(0, T; \mathbb{V})$ . We claim that

$$z \leq \|\rho^0\|_{L^\infty(\Omega)} \quad \text{a.e. in } \mathbb{R}^N \times [0, T]. \quad (3.22)$$

We set  $w := \|\rho^0\|_{L^\infty(\Omega)} - z$ . Then,  $w(x, 0) = \|\rho^0\|_{L^\infty(\Omega)} - \rho^0(x) \geq 0$  for a.e.  $x \in \Omega$ . Moreover,  $w$  satisfies

$$\begin{cases} w_t + (-\Delta)^s w + \|v\|_{\infty} w &= v w \chi_{\omega} + (\|v\|_{\infty} - v \chi_{\omega}) \|\rho^0\|_{L^\infty(\Omega)} & \text{in } Q, \\ w &= \|\rho^0\|_{L^\infty(\Omega)} & \text{in } \Sigma, \\ w(\cdot, 0) &= \|\rho^0\|_{L^\infty(\Omega)} - \rho^0 & \text{in } \Omega. \end{cases} \quad (3.23)$$

To obtain our result, it is sufficient to show that  $w^- = 0$  a.e. in  $\mathbb{R}^N \times [0, T]$ . Since  $\|\rho^0\|_{L^\infty(\Omega)} \geq 0$ , we have that  $w^- = 0$  in  $\Sigma$ . It is also clear that  $w^- \in W(0, T; \mathbb{V})$ . Let  $\mathcal{O}^-$  and  $\mathcal{O}^+$  be as in (3.16) with  $\rho$  replaced by  $w$ . If we take the duality map between (3.23) and  $w^-$ , use the same argument as in the proof of Lemma 3.5, we get that

$$\begin{aligned} & -\frac{1}{2} \int_{\mathcal{O}^-} \frac{d}{dt} |w^-(t)|^2 dx + \mathcal{F}(w(t), w^-(t)) - \|v\|_{\infty} \int_{\mathcal{O}^-} (w^-(t))^2 dx \\ &= - \int_{\mathcal{O}^-} v(t) (w^-(t))^2 dx + \|\rho^0\|_{L^\infty(\Omega)} \int_{\mathcal{O}^-} (\|v\|_{\infty} - v(t)) w^-(t) dx. \end{aligned}$$

We have shown that

$$\begin{aligned} & \frac{1}{2} \int_{\mathcal{O}^-} \frac{d}{dt} |w^-(t)|^2 dx - \mathcal{F}(w(t), w^-(t)) + \|v\|_{\infty} \int_{\mathcal{O}^-} (w^-(t))^2 dx \\ &= \int_{\mathcal{O}^-} v(t) (w^-(t))^2 dx - \|\rho^0\|_{L^\infty(\Omega)} \int_{\mathcal{O}^-} (\|v\|_{\infty} - v(t)) w^-(t) dx. \end{aligned} \quad (3.24)$$

As in the proof of Lemma 3.5 we have that  $\mathcal{F}(w(t), w^-(t)) \leq 0$ . Since

$$\|\rho^0\|_{L^\infty(\Omega)} \int_{\Omega} (\|v\|_{\infty} - v(t)) w^-(t) \, dx \geq 0$$

and

$$-\mathcal{F}(w(t), w^-(x, t)) + \|v\|_{\infty} \int_{\mathcal{O}^-} (w^-(t))^2 \, dx \geq 0,$$

we can deduce from (3.24) that

$$\frac{1}{2} \frac{d}{dt} \|w^-(\cdot, t)\|_{L^2(\mathcal{O}^-)}^2 \leq \|v\|_{\infty} \|w^-(\cdot, t)\|_{L^2(\mathcal{O}^-)}^2. \quad (3.25)$$

It follows from Gronwall's Lemma again that

$$\|w^-(\cdot, t)\|_{L^2(\mathcal{O}^-)}^2 \leq e^{2t\|v\|_{\infty}} \|w^-(\cdot, 0)\|_{L^2(\mathcal{O}^-)}^2 = 0,$$

where we have also used that  $w^-(\cdot, 0) = 0$  a.e. in  $\mathcal{O}^-$ . Hence,  $w^- = 0$  a.e. in  $\mathcal{O}^- \times [0, T]$ . We can deduce that  $w \geq 0$  a.e. in  $\mathbb{R}^N \times [0, T]$ . We have shown the claim.

Replacing  $z$  by  $e^{-\|v\|_{\infty} t} \rho$  in (3.22), we obtain (3.21). The proof is finished.  $\square$

Now, let  $\rho$  be the solution of (1.2) and consider the system

$$\begin{cases} -q_t + (-\Delta)^s q &= vq\chi_{\omega} & \text{in } Q, \\ q &= 0 & \text{in } \Sigma, \\ q(\cdot, T) &= \rho(\cdot, T) - \rho^d & \text{in } \Omega, \end{cases} \quad (3.26)$$

which can be viewed as the dual system associated with (1.2).

**Corollary 3.7.** *Let  $\rho^0, \rho^d \in L^\infty(\Omega)$  and  $v \in L^\infty(\omega_T)$ . Then, (3.26) has a unique weak solution  $q$  that belongs to  $W(0, T; \mathbb{V}) \cap L^\infty(\mathbb{R}^N \times (0, T))$ . In addition, there is a constant  $C = C(N, s, \Omega) > 0$  such that*

$$\|q\|_{W(0, T; \mathbb{V})} \leq (2 + C\|v\|_{\infty}) e^{\|v\|_{\infty} T} (\|\rho^0\|_{L^2(\Omega)} + \|\rho^d\|_{L^2(\Omega)}) \quad (3.27)$$

and

$$\|q\|_{L^\infty(\mathbb{R}^N \times (0, T))} \leq e^{\|v\|_{\infty} T} (\|\rho^0\|_{L^\infty(\Omega)} + \|\rho^d\|_{L^\infty(\Omega)}). \quad (3.28)$$

*Proof.* Making the change of variable  $t \mapsto T - t$ , we have that  $\varphi(x, t) := q(x, T - t)$  satisfies

$$\begin{cases} \varphi_t + (-\Delta)^s \varphi &= \tilde{v}\varphi\chi_{\omega} & \text{in } Q, \\ \varphi &= 0 & \text{in } \Sigma, \\ \varphi(\cdot, 0) &= \rho(\tilde{v}; \cdot, 0) - \rho^d & \text{in } \Omega, \end{cases} \quad (3.29)$$

where  $\tilde{v}(x, t) = v(x, T - t)$ . Observing that  $\rho = \rho(\tilde{v})$  is a solution of (1.2) with  $\rho^0 \in L^\infty(\Omega)$  and  $v = \tilde{v}$ , we have from Theorem 3.6 that  $\rho \in W(0, T; \mathbb{V}) \cap L^\infty(\mathbb{R}^N \times [0, T])$ . Hence,  $\rho(\tilde{v}; \cdot, 0) - \rho^d = \rho^0 - \rho^d \in L^\infty(\Omega)$  and it follows from Corollary 3.4 and Theorem 3.6 that there exists a unique  $\varphi \in W(0, T; \mathbb{V}) \cap L^\infty(\mathbb{R}^N \times [0, T])$  solution of (3.29). Thanks to (3.21) and (3.15), we have that

$$\|q\|_{L^\infty(\mathbb{R}^N \times (0, T))} = \|\varphi\|_{L^\infty(\mathbb{R}^N \times (0, T))} \leq e^{\|\tilde{v}\|_{\infty} T} \|\rho^0 - \rho^d\|_{L^\infty(\Omega)}$$

and there is a constant  $C = C(N, s, \Omega) > 0$  such that

$$\|q\|_{W(0, T; \mathbb{V})} = \|\varphi\|_{W(0, T; \mathbb{V})} \leq (2 + C\|\tilde{v}\|_{\infty}) e^{\|\tilde{v}\|_{\infty} T} (\|\rho^0\|_{L^2(\Omega)} + \|\rho^d\|_{L^2(\Omega)}).$$

Hence,  $q$  satisfies (3.27) and (3.28). The proof is finished.  $\square$

## 4. EXISTENCE OF OPTIMAL SOLUTIONS

We are concerned with the optimal control problem (1.2)-(1.1). In view of Theorem 3.6 and (2.9) the cost function  $J$  is well defined. We define the control-to-state mapping

$$G : L^\infty(\omega_T) \rightarrow W(0, T; \mathbb{V}) \cap L^\infty(\mathbb{R}^N \times (0, T)), \quad v \mapsto G(v) = \rho \quad (4.1)$$

which associates to each  $v \in L^\infty(\omega_T)$  the unique weak solution  $\rho$  of (1.2). Then, the optimal control problem (1.2)-(1.1) can be rewritten as

$$\inf_{v \in \mathcal{U}} J(v) = \frac{1}{2} \|G(v)(\cdot, T) - \rho^d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|v\|_{L^2(\omega_T)}^2. \quad (4.2)$$

**Theorem 4.1.** *Let  $\alpha > 0$ ,  $v \in \mathcal{U}$  and  $\rho^0, \rho^d \in L^\infty(\Omega)$ . Then, there exists a solution  $u \in \mathcal{U}$  of (4.2), and hence, of (1.2)-(1.1).*

*Proof.* Let  $v^n \in \mathcal{U}$  be a minimizing sequence such that

$$\lim_{n \rightarrow \infty} J(v^n) = \inf_{v \in \mathcal{U}} J(v).$$

Since  $\rho^n := G(v^n)$  is the state associated to the control  $v^n$ , there is a constant  $C > 0$  independent of  $n$  such that

$$\|\rho^n(\cdot, T)\|_{L^2(\Omega)} \leq C \quad \text{and} \quad \|v^n\|_{L^2(\omega_T)} \leq C. \quad (4.3)$$

Moreover,  $\rho^n$  satisfies

$$\begin{cases} (\rho^n)_t + (-\Delta)^s \rho^n &= v^n \rho^n \chi_\omega & \text{in } Q, \\ \rho^n &= 0 & \text{in } \Sigma, \\ \rho^n(\cdot, 0) &= \rho^0 & \text{in } \Omega. \end{cases} \quad (4.4)$$

It follows from Corollary 3.4 that (4.4) has a unique solution  $\rho^n \in W(0, T; \mathbb{V})$  satisfying

$$- \int_0^T \langle \phi_t, \rho^n \rangle_{\mathbb{V}^*, \mathbb{V}} dt + \int_0^T \mathcal{F}(\rho^n, \phi) dt = \int_{\omega_T} v^n \rho^n \phi \, dx \, dt + \int_\Omega \rho^0 \phi(0) \, dx \quad (4.5)$$

for every  $\phi \in H(Q)$ . Thanks to (3.15) and  $\|v^n\|_\infty \leq \max\{|m|, |M|\}$ , we have that there is a constant  $C > 0$  independent of  $n$  such that

$$\|\rho^n\|_{W(0, T; \mathbb{V})} \leq C e^{CT} \|\rho^0\|_{L^2(\Omega)}. \quad (4.6)$$

We can deduce that

$$\|v^n \rho^n\|_{L^2(\omega_T)} \leq C \|\rho^n\|_{L^2(Q)} \leq C \|\rho^n\|_{W(0, T; \mathbb{V})} \leq C e^{CT} \|\rho^0\|_{L^2(\Omega)}. \quad (4.7)$$

From (4.3)-(4.7), there exist  $\eta \in L^2(\Omega)$ ,  $u \in L^2(\omega_T)$ ,  $\beta \in L^2(\omega_T)$  and  $\rho \in W(0, T; \mathbb{V})$  such that (up to a subsequence if necessary), as  $n \rightarrow \infty$ , we have that

$$v^n \rightharpoonup u \text{ weakly in } L^2(\omega_T), \quad (4.8)$$

$$\rho^n(\cdot, T) \rightharpoonup \eta \text{ weakly in } L^2(\Omega), \quad (4.9)$$

$$\rho^n \rightharpoonup \rho \text{ weakly in } W(0, T; \mathbb{V}), \quad (4.10)$$

$$v^n \rho^n \rightharpoonup \beta \text{ weakly in } L^2(\omega_T). \quad (4.11)$$

Since  $\mathcal{U}$  is a closed convex subset of  $L^2(\omega_T)$ , we have that  $\mathcal{U}$  is weakly closed and so

$$u \in \mathcal{U}. \quad (4.12)$$

It follows from Theorem 2.2 that the embedding  $W(0, T; \mathbb{V}) \hookrightarrow L^2(Q)$  is compact. Hence, from (4.10) we have that, as  $n \rightarrow \infty$ ,

$$\rho^n \rightarrow \rho \text{ strongly in } L^2(Q). \quad (4.13)$$

Taking (4.8) and (4.13) into account and using the weak-strong convergence, we get that, as  $n \rightarrow \infty$ ,

$$v^n \rho^n \rightharpoonup u \rho \text{ weakly in } L^1(\omega_T), \quad (4.14)$$

which in view of the continuous embedding  $L^2(\omega_T) \hookrightarrow L^1(\omega_T)$ , (4.11) and the uniqueness of the weak limit, imply that  $\beta = u\rho$ . We have shown that, as  $n \rightarrow \infty$ ,

$$v^n \rho^n \rightharpoonup u \rho \text{ weakly in } L^2(\omega_T). \quad (4.15)$$

Passing to the limit, as  $n \rightarrow \infty$ , in (4.5), while using (4.10) and (4.15), we get

$$-\int_0^T \langle \phi_t, \rho \rangle_{\mathbb{V}^*, \mathbb{V}} dt + \int_0^T \mathcal{F}(\rho, \phi) dt = \int_{\omega_T} u \rho \phi \, dx \, dt + \int_{\Omega} \rho^0 \phi(0) dx \quad (4.16)$$

for every  $\phi \in H(Q)$ . Thus,  $\rho \in W(0, T; \mathbb{V})$  is the unique solution of (1.2) with  $v = u$ .

Now, let  $\phi \in W(0, T; \mathbb{V})$ . If we take the duality map between (4.4) and  $\phi$ , use Proposition 2.1 and integrate over  $(0, T)$ , we obtain

$$\begin{aligned} -\int_0^T \langle \phi_t, \rho^n \rangle_{\mathbb{V}^*, \mathbb{V}} dt + \int_0^T \mathcal{F}(\rho^n, \phi) dt &= \int_{\omega_T} v^n \rho^n \phi \, dx \, dt - \int_{\Omega} \rho^n(T) \phi(T) \, dx \\ &\quad + \int_{\Omega} \rho^0 \phi(0) dx. \end{aligned} \quad (4.17)$$

Passing to the limit in (4.17) while using (4.9), (4.10) and (4.15), we obtain

$$\int_{\Omega} \eta \phi(T) \, dx - \int_0^T \langle \phi_t, \rho \rangle_{\mathbb{V}^*, \mathbb{V}} dx \, dt + \int_0^T \mathcal{F}(\rho, \phi) dt = \int_{\omega_T} u \rho \phi \, dx \, dt + \int_{\Omega} \rho^0 \phi(0) \, dx.$$

Using again Proposition 2.1 we can deduce that

$$\begin{aligned} \int_0^T \langle \rho_t + (-\Delta)^s \rho, \phi \rangle_{\mathbb{V}^*, \mathbb{V}} dt &= \int_{\omega_T} u \rho \phi \, dx \, dt - \int_{\Omega} \rho^0 \phi(0) \, dx \\ &\quad + \int_{\Omega} \rho^0 \phi(0) \, dx - \int_{\Omega} \phi(T) (\eta - \rho(T)) dx, \end{aligned} \quad (4.18)$$

for all  $\phi \in W(0, T; \mathbb{V})$ . Since  $\rho$  is a solution of (1.2), we get from (4.18) that

$$\int_{\Omega} \phi(T) (\eta - \rho(T)) \, dx = 0, \quad \forall \phi \in W(0, T; \mathbb{V}).$$

Hence, we can deduce that

$$\eta = \rho(\cdot, T) \quad \text{a.e. in } \Omega. \quad (4.19)$$

Combining (4.9)-(4.19) we obtain that, as  $n \rightarrow \infty$ ,

$$\rho^n(\cdot, T) \rightharpoonup \rho(\cdot, T) \text{ weakly in } L^2(\Omega). \quad (4.20)$$

Using (4.20), (4.8), (4.12) and the lower semi-continuity of  $J$ , we can deduce that  $J(u) \leq \liminf_{n \rightarrow \infty} J(v^n) = \inf_{v \in \mathcal{U}} J(v)$ . This completes the proof.  $\square$

**Remark 4.2.** In Theorem 4.1 we only proved the existence of optimal solutions. Uniqueness will necessitate additional assumptions. This will be done in Section 5.3

## 5. OPTIMALITY CONDITIONS

In this section, we give the first and second order optimality conditions for the problem (4.2), and hence, for (1.2)-(1.1).

**5.1. First order necessary optimality conditions.** The aim of this section is to derive the first order necessary optimality conditions and to characterize the optimal control. But before going further, we need some regularity results for the control-to-state operator. Let us define the mapping

$$\begin{cases} \mathcal{G} : W(0, T; \mathbb{V}) \times L^\infty(\omega_T) \rightarrow L^2((0, T); \mathbb{V}^*) \times L^2(\Omega), \\ \mathcal{G}(\rho, v) := (\rho_t + (-\Delta)^s \rho - v\rho\chi_\omega, \rho(0) - \rho^0). \end{cases} \quad (5.1)$$

Then, the state equation (1.2) can be viewed as  $\mathcal{G}(\rho, v) = (0, 0)$ .

**Lemma 5.1.** *The mapping  $\mathcal{G}$  defined in (5.1) is of class  $\mathcal{C}^\infty$ .*

*Proof.* We write the first component  $\mathcal{G}_1$  of  $\mathcal{G}$  as

$$\mathcal{G}_1(\rho, v)(\phi) = \mathcal{G}_{11}(\rho, v)(\phi) + \mathcal{G}_{12}(\rho, v)(\phi), \quad \forall \phi \in L^2((0, T); \mathbb{V})$$

where

$$\mathcal{G}_{11}(\rho, v)(\phi) := \int_0^T \langle \rho_t, \phi \rangle_{\mathbb{V}^*, \mathbb{V}} dt + \int_0^T \mathcal{F}(\rho, \phi) dt$$

and

$$\mathcal{G}_{12}(\rho, v)(\phi) := - \int_{\omega_T} v \rho \phi \, dx \, dt.$$

It is clear that  $\mathcal{G}_{11}$  is linear and continuous from  $W(0, T; \mathbb{V})$  to  $L^2((0, T); \mathbb{V}^*)$  and  $\mathcal{G}_{12}$  is bilinear and continuous from  $W(0, T; \mathbb{V}) \times L^\infty(\omega_T)$  to  $L^2((0, T); \mathbb{V}^*)$ . Thus, they are of class  $\mathcal{C}^\infty$ . The second component of  $\mathcal{G}$  is clearly of class  $\mathcal{C}^\infty$ .  $\square$

**Lemma 5.2.** *The mapping  $G : L^\infty(\omega_T) \rightarrow W(0, T; \mathbb{V})$ ,  $u \mapsto \rho$  is of class  $\mathcal{C}^\infty$ .*

*Proof.* Let  $v \in L^\infty(\omega_T)$ . It follows from Lemma 5.1 that  $\mathcal{G}$  defined in (5.1) is of class  $\mathcal{C}^\infty$ . Moreover,

$$\partial_\rho \mathcal{G}(\rho, v)\varphi = \left( \varphi_t + (-\Delta)^s \varphi - v\varphi\chi_\omega, \varphi(0) \right).$$

For  $\varphi^0 \in L^2(\Omega)$  and  $f \in L^2((0, T); \mathbb{V}^*)$ , Corollary 3.3 shows that (3.12) with  $\rho = \varphi$  has a unique solution  $\varphi$  in  $W(0, T; \mathbb{V})$  which depends continuously on  $\varphi^0$  and  $f$ . Hence,  $\partial_\rho \mathcal{G}(\rho, v)$  defines an isomorphism from  $W(0, T; \mathbb{V})$  to  $L^2((0, T); \mathbb{V}^*) \times L^2(\Omega)$ . Using the Implicit Function Theorem, we can deduce that  $\mathcal{G}(\rho, v) = (0, 0)$  has a unique solution  $\rho = G(v)$  for any  $v$  in  $B_\varepsilon^\infty(u)$ , where  $B_\varepsilon^\infty(u)$  denotes the open ball in  $L^\infty(Q)$  of radius  $\varepsilon$  centered at  $u$ , solution of (1.1)-(1.3). Moreover, the operator  $G : v \mapsto \rho$  is itself of class  $\mathcal{C}^\infty$ . The proof is finished.  $\square$

**Remark 5.3.** Actually, we do not only have a unique solution  $\rho = G(v)$  for any  $v$  in a suitable neighborhood of  $u$ , but we have a unique solution  $\rho = G(v)$  of the state equation (1.2) for any given  $v \in L^\infty(\omega_T)$  (see Corollary 3.4). But this does not show that the optimal control found in Theorem 4.1 is unique.

The following result shows the Lipschitz continuity of  $G$ . The proof follows using similar arguments as in [17, 18, 19].

**Proposition 5.4.** *Let  $v \in L^\infty(\omega_T)$  and  $\rho^0 \in L^\infty(\Omega)$ . Then, the mapping  $v \mapsto G(v)$  is a Lipschitz continuous function from  $L^2(\omega_T)$  into  $W(0, T; \mathbb{V})$ . More precisely, there is a constant  $C_1 = C_1(N, s, \Omega) > 0$  such that for all  $v_1, v_2 \in L^\infty(\omega_T)$ ,*

$$\|G(v_1) - G(v_2)\|_{W(0, T; \mathbb{V})} \leq C \|v_1 - v_2\|_{L^2(\omega_T)} \quad (5.2)$$

where  $C = [(2 + C_1 \|v_1\|_\infty) e^{\|v_1\|_\infty T} + 1] e^{\|v_2\|_\infty T} \|\rho^0\|_{L^\infty(\Omega)}$ .

*Proof.* Let  $v_1, v_2 \in L^\infty(\omega_T)$ . Set  $z := \rho(v_1) - \rho(v_2)$ , where  $\rho(v_1), \rho(v_2)$  are solutions of (1.2) with  $v = v_1$  and  $v = v_2$ , respectively. We have that  $z$  satisfies

$$\begin{cases} z_t + (-\Delta)^s z &= (v_1 z + (v_1 - v_2)\rho(v_2))\chi_\omega & \text{in } Q, \\ z &= 0 & \text{in } \Sigma, \\ z(\cdot, 0) &= 0 & \text{in } \Omega. \end{cases} \quad (5.3)$$

Since  $v_1, v_2 \in L^\infty(\omega_T)$  and  $(v_1 - v_2)\rho(v_2)\chi_\omega \in L^2(Q)$ , using Corollary 3.3, we have that there is a unique  $z \in W(0, T; \mathbb{V})$  solution to (5.3). Thanks to (3.13c), it follows that

$$\|z\|_{W(0, T; \mathbb{V})} \leq \left[ (2 + C\|v_1\|_\infty) e^{\|v_1\|_\infty T} + 1 \right] \|(v_1 - v_2)\rho(v_2)\|_{L^2(\omega_T)}. \quad (5.4)$$

Since  $\rho^0 \in L^\infty(\Omega)$ , we know that  $\rho = \rho(v_2)$  solution of (1.2) satisfies (3.21), that is,

$$\|\rho(v_2)\|_{L^\infty(\mathbb{R}^N \times (0, T))} \leq e^{\|v_2\|_\infty T} \|\rho^0\|_{L^\infty(\Omega)}. \quad (5.5)$$

Using the Cauchy-Schwarz inequality and (5.5) in (5.4), we can deduce (5.2).  $\square$

**Lemma 5.5.** *Let  $G : L^\infty(\omega_T) \rightarrow W(0, T; \mathbb{V}) \cap L^\infty(\mathbb{R}^N \times (0, T))$ ,  $v \mapsto \rho$  be the control-to-state operator, where  $\rho$  is the weak solution of (1.2). Then, the directional derivative of  $G$  in the direction  $w \in L^\infty(\omega_T)$  is given by  $G'(v)w = y$ , where  $y \in W(0, T; \mathbb{V})$  is the unique weak solution of*

$$\begin{cases} y_t + (-\Delta)^s y &= (vy + w\rho)\chi_\omega & \text{in } Q, \\ y &= 0 & \text{in } \Sigma, \\ y(\cdot, 0) &= 0 & \text{in } \Omega. \end{cases} \quad (5.6)$$

Moreover, for every  $v \in L^\infty(\omega_T)$ , the linear mapping  $w \mapsto G'(v)w$  can be extended to a linear continuous mapping from  $L^2(\omega_T) \rightarrow W(0, T; \mathbb{V})$ . In addition, there is a constant  $C = C(N, s, \Omega) > 0$  such that

$$\|y\|_{W(0, T; \mathbb{V})} \leq \left[ (2 + C\|v\|_\infty) e^{\|v\|_\infty T} + 1 \right] e^{\|v_2\|_\infty T} \|\rho^0\|_{L^\infty(\Omega)} \|w\|_{L^2(\omega_T)}. \quad (5.7)$$

*Proof.* Since  $\rho \in W(0, T; \mathbb{V})$  and  $w \in L^\infty(\omega_T)$ , we have that  $w\rho\chi_\omega \in L^2(Q)$ . Since  $v \in L^\infty(\omega_T)$ , it follows from Corollary 3.3 that there exists a unique  $y \in W(0, T; \mathbb{V})$  solution of (5.6). Thanks to (3.13c) and the fact that  $\rho$  satisfies (3.21), we can deduce (5.7). Now, let  $\lambda > 0$  and set  $y_\lambda := \frac{\rho(v + \lambda w) - \rho(v)}{\lambda}$ . Then,  $y_\lambda$  is a solution of

$$\begin{cases} (y_\lambda)_t + (-\Delta)^s y_\lambda &= (vy_\lambda + w\rho(v + \lambda w))\chi_\omega & \text{in } Q, \\ y_\lambda &= 0 & \text{in } \Sigma, \\ y_\lambda(\cdot, 0) &= 0 & \text{in } \Omega. \end{cases}$$

Define  $p_\lambda := e^{-rt}(y_\lambda - y)$ , with  $r > 0$ . Then  $p_\lambda$  is a weak solution of

$$\begin{cases} (p_\lambda)_t + (-\Delta)^s p_\lambda + rp_\lambda &= [vp_\lambda + e^{-rt}w(\rho(v + \lambda w) - \rho(v))]\chi_\omega & \text{in } Q, \\ p_\lambda &= 0 & \text{in } \Sigma, \\ p_\lambda(\cdot, 0) &= 0 & \text{in } \Omega. \end{cases} \quad (5.8)$$

It follows from Corollary 3.3 that there exists a unique  $p_\lambda \in W(0, T; \mathbb{V})$  solution to (5.8). Thanks to (3.13c), we have that there is a constant  $C > 0$  such that

$$\|p_\lambda\|_{W(0, T; \mathbb{V})} \leq \left[ (2 + C\|v\|_\infty) e^{\|v\|_\infty T} + 1 \right] \|w\|_\infty \|\rho(v + \lambda w) - \rho(v)\|_{L^2(\omega_T)}.$$

Taking the limit as  $\lambda \downarrow 0$  of the latter inequality and using Proposition 5.4, we obtain that  $p_\lambda \rightarrow 0$  strongly in  $W(0, T; \mathbb{V})$ . We can deduce that  $y_\lambda \rightarrow y$  strongly in  $W(0, T; \mathbb{V})$  as  $\lambda \downarrow 0$ . This proves (5.6).

For the extension, it is sufficient to prove that for any  $w \in L^2(\omega_T)$ , the system (5.6) has a unique solution  $y \in W(0, T; \mathbb{V})$ . This follows directly from Corollary 3.3, with  $f := w\rho \in L^2(Q)$ . Note that  $f \in L^2(Q)$  because  $\rho \in L^\infty(\mathbb{R}^N \times (0, T))$ .  $\square$

We have the following result.

**Proposition 5.6 (Fréchet differentiability of  $J$ ).** *Let  $\rho$  be the solution of (1.2). Under the hypothesis of Lemma 5.5, the functional  $J : L^\infty(\omega_T) \rightarrow \mathbb{R}$  defined in (4.2) is continuously Fréchet differentiable and for every  $v, w \in L^\infty(\omega_T)$ ,*

$$J'(v)w = \int_{\Omega} y(x, T)(\rho(x, T) - \rho^d(x)) dx + \alpha \int_{\omega_T} vw dx dt, \quad (5.9)$$

where  $y \in W(0, T; \mathbb{V})$  is the unique weak solution of (5.6).

*Proof.* Observing that  $\rho(v) = G(v)$  is a solution of (1.2), we have that  $J$  is continuously Fréchet differentiable, since by Lemma 5.2,  $G$  has this property.

Let  $v, w \in L^\infty(\omega_T)$  and  $y \in W(0, T; \mathbb{V})$  the unique solution of (5.6). After some straightforward calculations and using Lemma 5.5 we get

$$\lim_{\lambda \downarrow 0} \frac{J(v + \lambda w) - J(v)}{\lambda} = \int_{\Omega} y(x, T)(\rho(x, T) - \rho^d(x)) dx + \alpha \int_{\omega_T} vw dx dt.$$

We have shown (5.9). The proof is finished.  $\square$

Since the functional  $J$  is non-convex, in general, we cannot expect a unique solution to the minimization problem (4.2). We introduce the following notion of local solutions.

**Definition 5.7.** We say that  $u \in \mathcal{U}$  is an  $L^\infty$ -local solution of (4.2) if there exists  $\varepsilon > 0$  such that  $J(u) \leq J(v)$  for every  $v \in \mathcal{U} \cap B_\varepsilon^\infty(u)$ .

The following result is crucial for the rest of the paper.

**Theorem 5.8 (First order necessary optimality conditions).** *Let  $\alpha > 0$ ,  $v \in \mathcal{U}$  and  $\rho^0, \rho^d \in L^\infty(\Omega)$ . Let  $u \in \mathcal{U}$  be an  $L^\infty$ -local minimum for (4.2). Then,*

$$J'(u)(v - u) \geq 0 \quad \text{for every } v \in \mathcal{U}. \quad (5.10)$$

Moreover, there is a unique  $\rho \in W(0, T; \mathbb{V})$  and a unique  $q \in W(0, T; \mathbb{V})$  such that  $\rho$  satisfies (1.2) with  $v = u$ ,  $q$  satisfies (3.26) with  $v = u$  and

$$\int_{\omega_T} (\alpha u + \rho q)(v - u) dx dt \geq 0 \quad \forall v \in \mathcal{U}. \quad (5.11)$$

The condition (5.11) is equivalent to the following: for a.e.  $(x, t) \in \omega_T$ , we have that

$$\begin{cases} u(x, t) = m & \text{if } \alpha u(x, t) + \rho(u(x, t))q(x, t) > 0 \\ u(x, t) \in [m, M] & \text{if } \alpha u(x, t) + \rho(u(x, t))q(x, t) = 0 \\ u(x, t) = M & \text{if } \alpha u(x, t) + \rho(u(x, t))q(x, t) < 0. \end{cases} \quad (5.12)$$

**Remark 5.9.** Note that (5.11) or equivalently (5.12) can be rewritten as

$$u = \min \left( \max \left( m, -\frac{q}{\alpha} \rho \right), M \right) \text{ a.e. in } \omega_T. \quad (5.13)$$

**Proof of Theorem 5.8.** Let  $v \in \mathcal{U}$  be arbitrary. Since  $\mathcal{U}$  is convex, we have that  $u + \lambda(v - u) \in \mathcal{U}$  for all  $\lambda \in (0, 1]$ . But  $u$  is an  $L^\infty$ -local minimum, so  $J(u + \lambda(v - u)) \geq J(u)$ . Hence,  $\frac{J(u + \lambda(v - u)) - J(u)}{\lambda} \geq 0$  for all  $\lambda \in (0, 1]$ . Letting  $\lambda \downarrow 0$  in the latter inequality, we obtain (5.10). We have already shown in the

proof of Theorem 4.1 that the state  $\rho$  satisfies (1.2). We use Proposition 5.6 and (5.10), with  $w = v - u$ , to obtain

$$\begin{aligned} \lim_{\lambda \downarrow 0} \frac{J(u + \lambda(v - u)) - J(u)}{\lambda} &= \int_{\Omega} y(x, T)(\rho(x, T) - \rho^d(x)) dx \\ &+ \alpha \int_{\omega_T} u(v - u) dx dt \geq 0 \quad \forall v \in \mathcal{U}, \end{aligned} \quad (5.14)$$

where  $y \in W(0, T; \mathbb{V})$  is the unique solution of (5.6). We let  $q$  be the adjoint state associated with (1.1)-(1.2). Since  $q$  satisfies (3.26) with  $v = u$ , we have from Corollary 3.7 that (3.26) has a unique weak solution  $q \in W(0, T; \mathbb{V}) \cap L^\infty(\mathbb{R}^N \times (0, T))$ . Taking the duality map of (5.6) and  $q$ , use Proposition 2.1 and integrate over  $(0, T)$ , we get

$$\int_{\Omega} y(x, T)(\rho(x, T) - \rho^d(x)) dx = \int_{\omega_T} w \rho q dx dt. \quad (5.15)$$

Combining (5.14)-(5.15) we obtain (5.11). The equivalent between (5.11) and (5.12) is proved as in the classical case contained in [31, Chapter 4].  $\square$

**Remark 5.10.** Using the change of variable  $t \mapsto T - t$  and Lemma 5.2, one can show that the mapping  $u \mapsto q$ , solution of (3.26) with  $v = u$ , is also of class  $\mathcal{C}^\infty$ .

**Proposition 5.11.** *Let  $u \in L^\infty(\omega_T)$  and  $\rho^0, \rho^d \in L^\infty(\Omega)$ . Then, the mapping  $u \mapsto q(u)$ , where  $q$  is the adjoint state solution to (3.26) with  $v = u$  is a Lipschitz continuous function from  $L^2(\omega_T)$  into  $W(0, T; \mathbb{V})$ . More precisely, there is a constant  $C_1 = C_1(N, s, \Omega) > 0$  such that for all  $u_1, u_2 \in L^\infty(\omega_T)$ ,*

$$\|q(u_1) - q(u_2)\|_{W(0, T; \mathbb{V})} \leq C \|u_1 - u_2\|_{L^2(\omega_T)}, \quad (5.16)$$

where  $C = e^{\|u_2\|_\infty T} [(2 + C_1 \|u_1\|_\infty) e^{\|u_1\|_\infty T} + 1] (\|\rho^0\|_{L^\infty(\Omega)} + \|\rho^d\|_{L^\infty(\Omega)})$ .

*Proof.* We proceed as in the proof of Proposition 5.4. Let  $u_1, u_2 \in L^\infty(\omega_T)$ . Set  $z := q(u_1) - q(u_2)$ , where  $q(u_1), q(u_2) \in W(0, T; \mathbb{V})$  are solutions of (3.26) with  $v = u_1$  and  $v = u_2$ , respectively. We have that  $z$  satisfies

$$\begin{cases} z_t + (-\Delta)^s z &= (u_1 z + (u_1 - u_2)q(u_2))\chi_\omega & \text{in } Q, \\ z &= 0 & \text{in } \Sigma, \\ z(\cdot, 0) &= 0 & \text{in } \Omega. \end{cases} \quad (5.17)$$

Using Corollary 3.3, we have that there exists a unique  $z \in W(0, T; \mathbb{V})$  solution to (5.17). Thanks to (3.13c), we have that there is a constant  $C > 0$  such that

$$\|z\|_{W(0, T; \mathbb{V})} \leq \left[ (2 + C \|u_1\|_\infty) e^{\|u_1\|_\infty T} + 1 \right] \|(u_1 - u_2)q(u_2)\|_{L^2(\omega_T)}. \quad (5.18)$$

Since  $\rho(\cdot, T) - \rho^d \in L^\infty(\Omega)$  where  $\rho$  is the weak solution of (1.2), we know that  $q = q(u_2)$  solution of (3.26) satisfies (3.27), that is,

$$\|q(u_2)\|_{L^\infty(\mathbb{R}^N \times (0, T))} \leq e^{\|u_2\|_\infty T} (\|\rho^0\|_{L^\infty(\Omega)} + \|\rho^d\|_{L^\infty(\Omega)}). \quad (5.19)$$

Using the Cauchy-Schwarz inequality and (5.19) in (5.18), we can deduce (5.16).  $\square$

**Lemma 5.12.** *For every  $u \in L^\infty(\omega_T)$ , the linear mapping  $v \mapsto J'(u)v$  can be extended to a linear continuous mapping  $J'(u) : L^2(\omega_T) \rightarrow \mathbb{R}$  given by (5.9).*

*Proof.* Let  $u \in L^\infty(\omega_T)$  and  $v \in L^2(\omega_T)$ . From (5.9), we have that

$$J'(u)v = \int_{\omega_T} (\alpha u + \rho q)v dx dt,$$



where  $\rho$  and  $q$  are solutions of (1.2) and (3.26) with  $v = u$ , respectively. Using (3.21) and (3.27), we have that there is a constant  $C > 0$  independent of  $v$  such that

$$|J'(u)v| \leq C\|v\|_{L^2(\omega_T)}.$$

Thus, the mapping  $v \mapsto J'(u)v$  is linear and continuous on  $L^2(\omega_T)$ .  $\square$

**5.2. Second order necessary and sufficient optimality conditions.** Note that the cost functional  $J$  associated to the optimization problem (4.2) is non-convex and the first order optimality conditions given in Theorem 5.8 are necessary but not sufficient for optimality. The sufficiency requires the use of second order optimality conditions, which is the aim of this section. To proceed, we need the following result.

**Lemma 5.13 (Twice Fréchet differentiability of  $G$ ).** *Let  $\alpha > 0$ ,  $v \in \mathcal{U}$  and  $\rho^0, \rho^d \in L^\infty(\Omega)$ . Let  $u$  be an  $L^\infty$ -local minimum for the problem (4.2). Then, the control-to-state mapping  $G : u \mapsto \rho$  is twice continuously Fréchet differentiable from  $L^\infty(\omega_T)$  into  $W(0, T; \mathbb{V})$ . Moreover,  $G''(u)[w, h] = z$ , where  $w, h \in L^\infty(\omega_T)$  and  $z \in W(0, T; \mathbb{V})$  is the unique weak solution of*

$$\begin{cases} z_t + (-\Delta)^s z &= (uz + hG'(u)w + wG'(u)h)\chi_\omega & \text{in } Q, \\ z &= 0 & \text{in } \Sigma, \\ z(\cdot, 0) &= 0 & \text{in } \Omega. \end{cases} \quad (5.20)$$

*Proof.* According to Lemma 5.5, we have that  $G'(u)w, G'(u)h \in W(0, T; \mathbb{V})$  for all  $w, h \in L^\infty(\omega_T)$ . Therefore,  $(hG'(u)w + wG'(u)h)\chi_\omega \in L^2(\omega_T)$ . In addition,  $u$  being in  $L^\infty(\omega_T)$ , we can deduce from Corollary 3.3 that there exists a unique  $z \in W(0, T; \mathbb{V})$  solution of (5.20).

Now, let  $\lambda > 0$  and set  $z_\lambda := \frac{G'(u + \lambda h)w - G'(u)w}{\lambda}$ . Then  $z_\lambda$  is a solution of

$$\begin{cases} (z_\lambda)_t + (-\Delta)^s z_\lambda &= (uz_\lambda + hG'(u + \lambda h)w + wy_\lambda)\chi_\omega & \text{in } Q, \\ z_\lambda &= 0 & \text{in } \Sigma, \\ z_\lambda(\cdot, 0) &= 0 & \text{in } \Omega, \end{cases}$$

where  $y_\lambda := \frac{\rho(u + \lambda h) - \rho(u)}{\lambda}$ . Define  $p_\lambda := e^{-rt}(z_\lambda - z)$  with  $r > 0$  and  $z$  being the solution of (5.20). Then,  $p_\lambda$  is a solution of

$$\begin{cases} (p_\lambda)_t + (-\Delta)^s p_\lambda + rp_\lambda &= up_\lambda\chi_\omega + g_\lambda & \text{in } Q, \\ p_\lambda &= 0 & \text{in } \Sigma, \\ p_\lambda(\cdot, 0) &= 0 & \text{in } \Omega \end{cases} \quad (5.21)$$

where

$$g_\lambda := [e^{-rt}(h(G'(u + \lambda h)w - G'(u)w) + w(y_\lambda - G'(u)h))] \chi_\omega.$$

Since  $g_\lambda \in L^2(\omega_T)$ , it follows from Corollary 3.3 that there exists a unique  $p_\lambda \in W(0, T; \mathbb{V})$  solution of (5.21). Thanks to the estimate (3.13c), we have that there is a constant  $C = C(N, s, \Omega) > 0$  such that

$$\begin{aligned} \|p_\lambda\|_{W(0, T; \mathbb{V})} &\leq C_1 \|h\|_\infty \|G'(u + \lambda h)w - G'(u)w\|_{L^2(\omega_T)} \\ &\quad + C_1 \|w\|_\infty \|y_\lambda - G'(u)h\|_{L^2(\omega_T)}, \end{aligned} \quad (5.22)$$

where  $C_1 = [(2 + C\|u\|_\infty)e^{\|u\|_\infty T} + 1]$ .

Taking the limit, as  $\lambda \downarrow 0$ , of (5.22) and using Lemmas 5.5 and 5.2, we obtain that  $p_\lambda \rightarrow 0$  strongly in  $W(0, T; \mathbb{V})$ . Hence,  $z_\lambda \rightarrow z$  strongly in  $W(0, T; \mathbb{V})$ , as  $\lambda \downarrow 0$ .  $\square$

**Lemma 5.14 (Twice Fréchet differentiability of  $J$ ).** *Let  $\alpha > 0$  and  $\rho^0, \rho^d \in L^\infty(\Omega)$ . Let  $u \in \mathcal{U}$  be an  $L^\infty$ -local minimum for the minimization problem (4.2) associated to the state  $\rho := G(u)$  solution of (1.2). Let  $q$  be the solution of (3.26) with  $v = u$ . Then, the functional  $J : L^\infty(\omega_T) \rightarrow \mathbb{R}$  is twice continuously Fréchet differentiable and for every  $w, h \in L^\infty(\omega_T)$ , we have that*

$$\begin{aligned} J''(u)[w, h] &= \int_{\omega_T} [hG'(u)w + wG'(u)h]q \, dx \, dt + \int_{\Omega} (G'(u)w)(T)(G'(u)h)(T) \, dx \\ &\quad + \alpha \int_{\omega_T} hw \, dx \, dt. \end{aligned} \quad (5.23)$$

Moreover, the bilinear mapping  $(w, h) \mapsto J''(u)[w, h]$  can be extended to a bilinear continuous mapping  $J''(u) : L^2(\omega_T) \times L^2(\omega_T) \rightarrow \mathbb{R}$  given by (5.23).

*Proof.* The identity (5.23) follows from Lemmas 5.2, 5.13, a straightforward computation and the Lebesgue dominated convergence theorem. Now, let  $w, h \in L^2(\omega_T)$ . Recall that  $G'(u)w$  is a solution of (5.6). Using Cauchy-Schwarz's inequality, (5.7) and (3.27), we obtain that there is a constant  $C = C(N, s, \Omega) > 0$  such that

$$\begin{aligned} |J''(u)[w, h]| &\leq \|q\|_{L^\infty(\mathbb{R}^N \times (0, T))} \|h\|_{L^2(\omega_T)} \|G'(u)w\|_{L^2(Q)} \\ &\quad + \|q\|_{L^\infty(\mathbb{R}^N \times (0, T))} \|w\|_{L^2(\omega_T)} \|G'(u)h\|_{L^2(Q)} \\ &\quad + \|(G'(u)w)(T)\|_{L^2(\Omega)} \|(G'(u)h)(T)\|_{L^2(\Omega)} \\ &\quad + \alpha \|w\|_{L^2(\omega_T)} \|h\|_{L^2(\omega_T)} \\ &\leq \|h\|_{L^2(\omega_T)} \|w\|_{L^2(\omega_T)} (2C_2 \|q\|_{L^\infty(\mathbb{R}^N \times (0, T))} + C_2^2 + \alpha) \\ &\leq C_3 \|w\|_{L^2(\omega_T)} \|h\|_{L^2(\omega_T)} \end{aligned}$$

where

$$C_3 = \left( C_2 e^{\|u\|_\infty T} (\|\rho^0\|_{L^\infty(\Omega)} + \|\rho^d\|_{L^\infty(\Omega)}) + C_2^2 + \alpha \right)$$

with  $C_2 = [(2 + C\|u\|_\infty) e^{\|u\|_\infty T} + 1] e^{\|u\|_\infty T} \|\rho^0\|_{L^\infty(\Omega)}$ . Hence,  $(w, h) \mapsto J''(u)[w, h]$  is a bilinear continuous mapping on  $L^2(\omega_T) \times L^2(\omega_T)$ .  $\square$

Next, we introduce some concepts retrieved from [31]. For a given  $\tau \geq 0$ , we define the set of strongly active constraints  $A_\tau(u)$  by

$$A_\tau(u) := \{(t, x) \in \omega_T : |\alpha u(x, t) + \rho(u)q(x, t)| > \tau\}.$$

The  $\tau$ -critical set associated to a control  $u$  (see e.g. [31]) is defined by

$$\mathcal{C}_\tau(u) = \{v \in L^\infty(\omega_T) : v \text{ fulfills (5.25)}\}, \quad (5.24)$$

that is, for a.e.  $(t, x) \in Q$ , we have that

$$\begin{cases} v(x, t) \geq 0 & \text{if } u(t, x) = m \text{ and } (t, x) \notin A_\tau(u), \\ v(x, t) \leq 0 & \text{if } u(t, x) = M \text{ and } (t, x) \notin A_\tau(u), \\ v(x, t) = 0 & \text{if } (t, x) \in A_\tau(u). \end{cases} \quad (5.25)$$

We have the following observation as in [10].

**Remark 5.15.** We notice that from the differentiability in  $L^\infty(\omega_T)$ , the cone defined in (5.24) is a subset of  $L^\infty(\omega_T)$ . Since  $L^\infty(\omega_T)$  is dense in  $L^2(\omega_T)$  and from Lemma 5.14, the quadratic form  $J''(u)$  is continuous on  $L^2(\omega_T)$ , we have that the second order conditions based on the critical cone in  $L^\infty(\omega_T)$  can be transferred to the extended cone in  $L^2(\omega_T)$ . This cone is convex and closed in  $L^2(\omega_T)$ .

In the rest of the paper, we will adopt the notation  $J''(u)v^2 := J''(u)[v, v]$ .

**Theorem 5.16 (Second order necessary optimality conditions).** *Let  $u \in \mathcal{U}$  be an  $L^\infty$ -local solution of (4.2). Then,  $J''(u)v^2 \geq 0$  for all  $v \in \mathcal{C}_0(u)$ .*

*Proof.* The proof follows as in [31, pp. 246]. Indeed, let  $v \in \mathcal{C}_0(u)$  and  $\kappa \in (0, 1)$ . In general  $u + \kappa v \notin \mathcal{U}$ . We introduce for  $n \in \mathbb{N}$  the set

$$I_n := \left\{ (x, t) \in \omega_T : m + \frac{1}{n} \leq u(x, t) \leq M - \frac{1}{n} \right\}.$$

Consider the functions  $v_n := \chi_n v$ , where

$$\chi_n(x, t) := \begin{cases} 1 & \text{if } (x, t) \in I_n \text{ or } u(x, t) \in \{m, M\} \text{ and } M - m \geq \frac{1}{n}, \\ 0 & \text{if } u(x, t) \in \left(m, m + \frac{1}{n}\right) \cup \left(M - \frac{1}{n}, M\right). \end{cases}$$

Then,  $\chi_n(x, t) = 0$  if  $u(x, t) \in \{m, M\}$ ,  $M - m < 1/n$  and  $u + \kappa v_n \in \mathcal{U}$  for  $\kappa \in (0, 1)$ . Moreover, using that  $u$  is a local optimal control, and that  $J$  is twice Fréchet differentiable (Lemma 5.14), we can deduce from Taylor's theorem that for  $\theta \in (0, 1)$ ,

$$0 \leq \frac{J(u + \kappa v_n) - J(u)}{\kappa} = J'(u)v_n + \frac{1}{2}\kappa J''(u + \theta\kappa v_n)v_n^2. \quad (5.26)$$

Since  $v \in \mathcal{C}_0(u)$ , then  $v_n \in \mathcal{C}_0(u)$ . Hence,  $J'(u)v_n = 0$ . Dividing (5.26) by  $\kappa$  and taking the limit, as  $\kappa \downarrow 0$  (as  $J$  is of class  $\mathcal{C}^2$ ), yields

$$J''(u)v_n^2 \geq 0. \quad (5.27)$$

It remains to prove that, as  $n \rightarrow \infty$ ,  $v_n \rightarrow v$  in  $L^2(\omega_T)$ . First, we note that for a.e.  $(x, t) \in \omega_T$ ,  $v_n(x, t) \rightarrow v(x, t)$  a.e. as  $n \rightarrow \infty$ . In addition  $|v_n| \leq |v|$  a.e. in  $\omega_T$  for all  $n \in \mathbb{N}$ . Using the Lebesgue dominated convergence theorem, we can deduce that as  $n \rightarrow \infty$ ,  $v_n \rightarrow v$  in  $L^2(\omega_T)$ . Taking the limit, as  $n \rightarrow \infty$ , in (5.27) and using the continuity of  $v \rightarrow J''(u)v^2$  in  $L^2(\omega_T)$ , we obtain  $J''(u)v^2 \geq 0$ .  $\square$

**Lemma 5.17.** *Let  $u \in \mathcal{U}$  be an  $L^\infty$  local-control satisfying (5.10). Then, the following assertions hold:*

(a) *The functional  $J : L^\infty(\omega_T) \rightarrow \mathbb{R}$  is of class  $\mathcal{C}^2$ . Furthermore, there exist continuous extensions*

$$J'(u) \in \mathcal{L}(L^2(\omega_T), \mathbb{R}) \quad \text{and} \quad J''(u) \in \mathcal{B}(L^2(\omega_T), \mathbb{R}). \quad (5.28)$$

(b) *For any sequence  $\{(u_k, v_k)\}_{k=1}^\infty \subset \mathcal{U} \times L^2(\omega_T)$  with  $\|u_k - u\|_{L^2(\omega_T)} \rightarrow 0$  and  $v_k \rightharpoonup v$  weakly in  $L^2(\omega_T)$ , as  $k \rightarrow \infty$ , we have that*

$$J'(u)v = \lim_{k \rightarrow \infty} J'(u_k)v_k, \quad (5.29)$$

and

$$J''(u)v^2 \leq \liminf_{k \rightarrow \infty} J''(u_k)v_k^2. \quad (5.30)$$

If  $v = 0$ , then

$$0 = \alpha \liminf_{k \rightarrow \infty} \|v_k\|_{L^2(\omega_T)}^2 \leq \liminf_{k \rightarrow \infty} J''(u_k)v_k^2. \quad (5.31)$$

*Proof.* (a): This part follows from Proposition 5.6, Lemmas 5.12 and 5.14.

(b): We proceed in three steps.

**Step 1.** We show (5.29). Using the Lipschitz continuity of the state mapping  $G$  given in Proposition 5.4, we get that  $G(u_k) \rightarrow G(u)$  in  $W(0, T; \mathbb{V})$ , as  $k \rightarrow \infty$ . From Proposition 5.11, we also obtain that the adjoint state  $q$  solution of (3.26) satisfies  $q(u_k) \rightarrow q(u)$  in  $W(0, T; \mathbb{V})$ , as  $k \rightarrow \infty$ .

We claim that

$$G(u_k)q(u_k) \rightarrow G(u)q(u) \text{ in } L^2(Q), \text{ as } k \rightarrow \infty. \quad (5.32)$$

Indeed, notice that  $G(u_k)$  and  $G(u)$  are solutions to (1.2) with  $v = u_k$  and  $v = u$ , respectively. Therefore, from (3.21), we can deduce that  $G(u_k)$  is bounded in  $L^\infty(Q)$  and  $G(u) \in L^\infty(Q)$ . Thus,  $G(u_k)q(u_k), G(u)q(u) \in L^2(Q)$  and we have that

$$\begin{aligned} \|G(u_k)q(u_k) - G(u)q(u)\|_{L^2(Q)} &\leq \|q(u_k) - q(u)\|_{L^2(Q)} \|G(u_k)\|_{L^\infty(Q)} \\ &\quad + \|G(u_k) - G(u)\|_{L^2(Q)} \|q(u)\|_{L^\infty(Q)}. \end{aligned} \quad (5.33)$$

Taking the limit, as  $k \rightarrow \infty$ , of (5.33) we obtain the claim (5.32). From (5.32) we can deduce that  $\alpha u_k + G(u_k)q(u_k) \rightarrow \alpha u + G(u)q(u)$  in  $L^2(Q)$ , as  $k \rightarrow \infty$ . Using the expression of  $J'$  given in (5.9), we have that

$$\begin{aligned} \lim_{k \rightarrow \infty} J'(u_k)v_k &= \lim_{k \rightarrow \infty} \int_{\omega_T} (\alpha u_k + G(u_k)q(u_k))v_k \, dx \, dt \\ &= \int_{\omega_T} (\alpha u + G(u)q(u))v \, dx \, dt = J'(u)v. \end{aligned}$$

We have shown (5.29).

**Step 2.** We show (5.30). We write

$$\begin{aligned} J''(u_k)v_k^2 &= 2 \int_{\omega_T} v_k(G'(u_k)v_k)q(u_k) \, dx \, dt + \int_{\Omega} |(G'(u_k)v_k)(T)|^2 \, dx \\ &\quad + \alpha \int_{\omega_T} |v_k|^2 \, dx \, dt, \end{aligned} \quad (5.34)$$

where  $G'(u_k)v_k$  is the unique weak solution of (5.6) with  $u = u_k$  and  $w = v_k$ . Since  $G(u_k)$  and  $v_k$  are bounded in  $L^\infty(Q)$  and in  $L^2(\omega_T)$ , respectively, it follows from (5.7) that  $G'(u_k)v_k$  and  $(G'(u_k)v_k)(\cdot, T)$  are bounded in  $W(0, T; \mathbb{V})$  and in  $L^2(\Omega)$ , respectively. Thus, up to a subsequence if necessary, as  $k \rightarrow \infty$ ,  $G'(u_k)v_k$  converges weakly to  $G'(u)v$  in  $L^2((0, T); \mathbb{V})$  and  $(G'(u_k)v_k)(\cdot, T)$  converges weakly to  $(G'(u)v)(\cdot, T)$  in  $L^2(\Omega)$ . Now, thanks to Theorem 2.2, the embedding  $W(0, T; \mathbb{V}) \hookrightarrow L^2(Q)$  is compact. Hence, we obtain that  $G'(u_k)v_k$  converges strongly to  $G'(u)v$  in  $L^2(Q)$ , as  $k \rightarrow \infty$ . Also, since  $\|u_k - u\|_{L^2(\omega_T)} \rightarrow 0$  as  $k \rightarrow \infty$ , we have from Proposition 5.11 that  $q(u_k) \rightarrow q(u)$  in  $L^2((0, T); \mathbb{V})$ , as  $k \rightarrow \infty$ , and  $q(u_k)$  is bounded in  $L^\infty(Q)$  (see Lemma 3.7). Therefore,  $(G'(u_k)v_k)q(u_k)$  converges strongly to  $(G'(u)v)q(u)$  in  $L^2(Q)$ , as  $k \rightarrow \infty$ . Taking the limit, as  $k \rightarrow \infty$ , in (5.34) and using the lower-semicontinuity of the  $L^2$ -norm, we can deduce that

$$\begin{aligned} \lim_{k \rightarrow \infty} J''(u_k)v_k^2 &\geq 2 \lim_{k \rightarrow \infty} \int_{\omega_T} v_k(G'(u_k)v_k)q(u_k) \, dx \, dt \\ &\quad + \liminf_{k \rightarrow \infty} \left[ \int_{\Omega} |(G'(u_k)v_k)(T)|^2 \, dx + \alpha \int_{\omega_T} |v_k|^2 \, dx \, dt \right] \\ &\geq 2 \int_{\omega_T} v(G'(u)v)q(u) \, dx \, dt + \int_{\Omega} |(G'(u)v)(T)|^2 \, dx + \alpha \int_{\omega_T} |v|^2 \, dx \, dt \\ &= J''(u)v^2. \end{aligned}$$

We have shown (5.30).

**Step 3.** If  $v = 0$ , then in (5.34) the first and second terms tend to 0. Hence,  $0 = \alpha \liminf_{k \rightarrow \infty} \|v_k\|_{L^2(\omega_T)}^2 \leq \lim_{k \rightarrow \infty} J''(u_k)v_k^2$ . The proof is finished.  $\square$

**Theorem 5.18 (Second order sufficient optimality conditions).** *Let  $\rho^0, \rho^d \in L^\infty(\Omega)$ . Let  $u \in \mathcal{U}$  be a control satisfying (5.11). Assume that*

$$(6 + C\theta)e^{2\theta T} (\|\rho^0\|_{L^\infty(\Omega)} + \|\rho^d\|_{L^\infty(\Omega)}) \|\rho^0\|_{L^\infty(\Omega)} \leq \frac{\alpha}{2}, \quad (5.35)$$

where  $\theta := \max\{|m|, |M|\}$  and  $C = C(N, s, \Omega) > 0$  is the constant appearing in (5.7). Then, there are two constants  $\gamma > 0$  and  $\beta > 0$  such that

$$J(v) \geq J(u) + \beta \|v - u\|_{L^2(\omega_T)}^2 \quad \forall v \in \mathcal{U} \cap B_\gamma^2(u), \quad (5.36)$$

where  $B_\gamma^2(u)$  is the open ball in  $L^2(Q)$  with center  $u$  and radius  $\gamma$ .

*Proof.* We claim that for every  $\tau > 0$  there is a constant  $\delta > 0$  such that

$$J''(u)v^2 \geq \delta \|v\|_{L^2(\omega_T)}^2 \quad \forall v \in \mathcal{C}_\tau(u). \quad (5.37)$$

Indeed, let  $\tau > 0$  and  $v \in \mathcal{C}_\tau(u)$ . Then,  $v = 0$  in  $A_\tau(u)$ , and it follows from (5.23) that

$$\begin{aligned} J''(u)v^2 &= 2 \int_{\omega_T} v(G'(u)v)q(u)dx dt + \int_{\Omega} |(G'(u)v)(T)|^2 dx + \alpha \int_{\omega_T} |v|^2 dx dt \\ &\geq 2 \int_{\{(x,t) \notin A_\tau(u)\}} v(G'(u)v)q(u)dx dt + \alpha \int_{\omega_T} |v|^2 dx dt. \end{aligned} \quad (5.38)$$

Using the Cauchy Schwartz inequality, (3.28) and (5.7) we obtain

$$\begin{aligned} \left| \int_{\{(x,t) \notin A_\tau(u)\}} v(G'(u)v)q(u)dx dt \right| &\leq \|q\|_{L^\infty(\mathbb{R}^N \times (0,T))} \|G'(u)v\|_{L^2(Q)} \|v\|_{L^2(\omega_T)} \\ &\leq \frac{\Lambda}{2} \|v\|_{L^2(\omega_T)}^2, \end{aligned} \quad (5.39)$$

where  $\Lambda = (6 + C\theta)e^{2\theta T} (\|\rho^0\|_{L^\infty(\Omega)} + \|\rho^d\|_{L^\infty(\Omega)}) \|\rho^0\|_{L^\infty(\Omega)}$ . Combining (5.38) and (5.39), we obtain that

$$J''(u)v^2 \geq (\alpha - \Lambda) \|v\|_{L^2(\omega_T)}^2 \geq \frac{\alpha}{2} \|v\|_{L^2(\omega_T)}^2,$$

which in view of (5.35) gives (5.37) with  $\delta = \alpha/2$ . It follows from Lemma 5.17 and (5.37) that the assumptions in [9, Theorem 2.3] are fulfilled. Hence, (5.36) holds.  $\square$

**Remark 5.19.** In this work, we proved that the cost functional  $J$  is of class  $\mathcal{C}^\infty$  only in  $L^\infty(\omega_T)$ . Hence, the so-called two-norms discrepancy (see e.g. [9]) occurs. The two-norms discrepancy occurs when the functional  $J$  is twice differentiable with respect to one norm (the  $L^\infty$ -norm in our case), but the inequality  $J''(u)v^2 \geq \|v\|_{L^2(\omega_T)}^2$  holds in a weaker norm (the  $L^2$ -norm in our case) in which  $J$  is not twice differentiable. For more details on the topic, we refer to [2, 15, 26] and the references therein.

**5.3. Uniqueness of optimal solutions.** In this section, we show that under additional assumptions on the initial datum and the given target the optimal solutions obtained in Theorem 5.8 are locally unique.

**Theorem 5.20 (Local uniqueness of optimal solutions).** *Let  $\rho^0, \rho^d \in L^\infty(\Omega)$ ,  $\theta := \max\{|m|, |M|\}$  and assume that*

$$3e^{3\theta T} \left( \|\rho^0\|_{L^\infty(\Omega)}^2 + \|\rho^d\|_{L^\infty(\Omega)}^2 \right) < \alpha. \quad (5.40)$$

*Then, the optimal solutions of (1.2)-(1.1) are unique.*

*Proof.* Assume that there are two  $L^\infty$  local-controls  $u$  and  $\bar{u}$  associated to the states  $\rho$  and  $\bar{\rho}$  solutions to (1.2), respectively. We denote by  $q$  and  $\bar{q}$  the corresponding adjoint states solutions of (3.26), respectively. Then,  $\rho - \bar{\rho}$  is a solution of

$$\begin{cases} (\rho - \bar{\rho})_t + (-\Delta)^s(\rho - \bar{\rho}) &= (u - \bar{u})\rho\chi_\omega + \bar{u}(\rho - \bar{\rho})\chi_\omega & \text{in } Q, \\ (\rho - \bar{\rho}) &= 0 & \text{in } \Sigma, \\ (\rho - \bar{\rho})(\cdot, 0) &= 0 & \text{in } \Omega, \end{cases} \quad (5.41)$$

and  $q - \bar{q}$  satisfies

$$\begin{cases} -(q - \bar{q})_t + (-\Delta)^s(q - \bar{q}) &= (u - \bar{u})q\chi_\omega + \bar{u}(q - \bar{q})\chi_\omega & \text{in } Q, \\ (q - \bar{q}) &= 0 & \text{in } \Sigma, \\ (q - \bar{q})(\cdot, T) &= \rho(\cdot, T) - \bar{\rho}(\cdot, T) & \text{in } \Omega. \end{cases} \quad (5.42)$$

Set  $y := e^{-\theta t}(\rho - \bar{\rho})$  and  $z := e^{-\theta(T-t)}(q - \bar{q})$ . Then,  $y$  satisfies

$$\begin{cases} y_t + (-\Delta)^s y + \theta y &= e^{-\theta t}(u - \bar{u})\rho\chi_\omega + \bar{u}y\chi_\omega & \text{in } Q, \\ y &= 0 & \text{in } \Sigma, \\ y(\cdot, 0) &= 0 & \text{in } \Omega, \end{cases} \quad (5.43)$$

and  $z$  is a solution of

$$\begin{cases} -z_t + (-\Delta)^s z + \theta z &= e^{-\theta(T-t)}(u - \bar{u})q\chi_\omega + \bar{u}z\chi_\omega & \text{in } Q, \\ z &= 0 & \text{in } \Sigma, \\ z(\cdot, T) &= \rho(\cdot, T) - \bar{\rho}(\cdot, T) & \text{in } \Omega. \end{cases} \quad (5.44)$$

If we take the duality map between (5.43) and  $y$ , and (5.44) with  $z$ , use Proposition 2.1, and integrate over  $(0, T)$ , we obtain

$$\begin{aligned} \|y\|_{L^2((0,T);V)}^2 &= \int_{\omega_T} \bar{u}y^2 dx dt - \theta \|y\|_{L^2(Q)}^2 - \frac{1}{2} \|\rho(\cdot, T) - \bar{\rho}(\cdot, T)\|_{L^2(\Omega)}^2 \\ &+ \int_{\omega_T} e^{-\theta t}(u - \bar{u})\rho(\rho - \bar{\rho}) dx dt \end{aligned}$$

and

$$\begin{aligned} \|z\|_{L^2((0,T);V)}^2 &= \int_{\omega_T} \bar{u}z^2 dx dt - \theta \|z\|_{L^2(Q)}^2 - \frac{1}{2} \|z(\cdot, 0)\|_{L^2(\Omega)}^2 \\ &+ \int_{\omega_T} e^{-\theta(T-t)}(u - \bar{u})qz dx dt + \frac{1}{2} \|\rho(\cdot, T) - \bar{\rho}(\cdot, T)\|_{L^2(\Omega)}^2. \end{aligned}$$

Adding these two latter identities and using the definition of  $\theta$ , we deduce that

$$\|y\|_{L^2((0,T);V)}^2 + \|z\|_{L^2((0,T);V)}^2 \leq \int_{\omega_T} (u - \bar{u})\rho y dx dt + \int_{\omega_T} (u - \bar{u})qz dx dt.$$

Using the Young inequality, we have that

$$\begin{aligned} \frac{1}{2} \|y\|_{L^2((0,T);V)}^2 + \frac{1}{2} \|z\|_{L^2((0,T);V)}^2 &\leq \frac{1}{2} \|\rho\|_{L^\infty(\mathbb{R}^N \times [0,T])}^2 \|u - \bar{u}\|_{L^2(\omega_T)}^2 \\ &+ \frac{1}{2} \|q\|_{L^\infty(\mathbb{R}^N \times [0,T])}^2 \|u - \bar{u}\|_{L^2(\omega_T)}^2. \end{aligned} \quad (5.45)$$

Using (3.21) and (3.28), we get from (5.45) that

$$\|\rho - \bar{\rho}\|_{L^2((0,T);V)}^2 + \|q - \bar{q}\|_{L^2((0,T);V)}^2 \leq 3e^{4\theta T} \left( \|\rho^0\|_{L^\infty(\Omega)}^2 + \|\rho^d\|_{L^\infty(\Omega)}^2 \right) \|u - \bar{u}\|_{L^2(\omega_T)}^2 \quad (5.46)$$

because  $y := e^{-\theta t}(\rho - \bar{\rho})$  and  $z := e^{-\theta(T-t)}(q - \bar{q})$ .

Using (5.13), the fact that

$$\max\{f, g\} = \frac{f + g + |f - g|}{2}, \quad \min\{f, g\} = \frac{f + g - |f - g|}{2}$$

and a simple calculation, we can deduce that

$$|u - \bar{u}| \leq \frac{1}{\alpha} |(q - \bar{q})\rho + \bar{q}(\rho - \bar{\rho})| \quad \text{a.e. in } \omega_T.$$

Hence,

$$|u - \bar{u}|^2 \leq \frac{1}{\alpha^2} |(q - \bar{q})\rho + \bar{q}(\rho - \bar{\rho})|^2 \leq \frac{1}{\alpha^2} (|\rho|^2 + |\bar{q}|^2)(|q - \bar{q}|^2 + |\rho - \bar{\rho}|^2).$$

Using (3.21) and (3.28) we obtain that

$$|u - \bar{u}|^2 \leq \frac{3e^{2\theta T}}{\alpha^2} \left( \|\rho^0\|_{L^\infty(\Omega)}^2 + \|\rho^d\|_{L^\infty(\Omega)}^2 \right) (|q - \bar{q}|^2 + |\rho - \bar{\rho}|^2)$$

This implies that

$$\begin{aligned} \|u - \bar{u}\|_{L^2(\omega_T)}^2 &\leq \frac{3e^{2\theta T}}{\alpha^2} \left( \|\rho^0\|_{L^\infty(\Omega)}^2 + \|\rho^d\|_{L^\infty(\Omega)}^2 \right) \|q - \bar{q}\|_{L^2((0,T);V)}^2 \\ &\quad + \frac{3e^{2\theta T}}{\alpha^2} \left( \|\rho^0\|_{L^\infty(\Omega)}^2 + \|\rho^d\|_{L^\infty(\Omega)}^2 \right) \|\rho - \bar{\rho}\|_{L^2((0,T);V)}^2. \end{aligned}$$

Using (5.46), we can deduce that

$$\|u - \bar{u}\|_{L^2(\omega_T)}^2 \leq \frac{9e^{6\theta T}}{\alpha^2} \left( \|\rho^0\|_{L^\infty(\Omega)}^2 + \|\rho^d\|_{L^\infty(\Omega)}^2 \right)^2 \|u - \bar{u}\|_{L^2(\omega_T)}^2. \quad (5.47)$$

Choosing  $\rho^0, \rho^d$  in (5.47) such that  $\alpha^2 > 9e^{6\theta T} \left( \|\rho^0\|_{L^\infty(\Omega)}^2 + \|\rho^d\|_{L^\infty(\Omega)}^2 \right)^2$ , we get

$$\|u - \bar{u}\|_{L^2(\omega_T)}^2 < \|u - \bar{u}\|_{L^2(\omega_T)}^2.$$

This implies that  $u = \bar{u}$ . Finally, using this latter result in (5.41) and the uniqueness of solutions, we can deduce that  $\rho = \bar{\rho}$ . Similarly, using (5.42) we get that  $q = \bar{q}$ . Thus, the optimal solutions of (1.2)-(1.1) are unique. The proof is finished.  $\square$

**Remark 5.21.** (a) We notice that given  $\alpha > 0$ , we can choose  $\rho^0, \rho^d \in L^\infty(\Omega)$  such that both conditions (5.35) and (5.40) are satisfied.  
 (b) In (5.35) and (5.40), one may also let  $\rho^0, \rho^d \in L^\infty(\Omega)$  be arbitrary and choose the penalization parameter  $\alpha > 0$  such that both conditions are satisfied.

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