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# Bilinear optimal control for a fractional diffusive equation

Cyrille Kenne\*      Gisèle Mophou †      Pascal Zongo‡

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## Abstract

We consider a bilinear optimal control for an evolution equation involving space fractional Laplacian operator of order  $0 < s < 1$ . We first give some existence and uniqueness results for different equations considered in our work. Then, we consider an optimal control problem which consist to bring the state of our model at final time to a desired state. We show that this optimal control problem has a solution that we characterize using the Euler-Lagrange first order optimality conditions. Finally, we establish some weak maximum principle results that allow us to prove the uniqueness of the optimal control.

Mathematics Subject Classification. 49J20, 49K20, 35S15, 35B50, 46E35

**Key-words** : Fractional Laplacian, bilinear system, optimal control, optimality system, maximum principle.

## 1 Introduction

Let  $\Omega \subset \mathbb{R}^N$  be an open bounded domain with a Lipschitz continuous boundary  $\partial\Omega$  and  $\omega$  be an open subset of  $\Omega$ . We consider the following space fractional diffusion model:

$$\begin{cases} \rho_t + (-\Delta)^s \rho = v\rho\chi_\omega & \text{in } Q := \Omega \times (0, T), \\ \rho = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ \rho(0, \cdot) = \rho^0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where  $T > 0$  and  $0 < s < 1$ . The function  $\rho^0 \in \mathcal{C}(\bar{\Omega})$  is such that  $\rho^0(x) \geq 0$  a.e. in  $\Omega$  and the control  $v \in L^2(\omega_T)$ . The function  $\chi_\omega$  denote the characteristic function of the set  $\omega$ .

In this paper, we want to bring the state  $\rho = \rho(v)$  at final time  $T$  to a given state  $\rho^d$  by acting on the control  $v$ . In other words, we are interested in the following optimal control problem:

$$\inf_{v \in \mathcal{U}} J(v), \quad (1.2)$$

where

$$\mathcal{U} = \left\{ w \in L^2(\omega \times (0, T)) : -M \leq w \leq 0, M \in \mathbb{R}, M > 0 \right\}, \quad (1.3)$$

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$$J(v) = \|\rho(\cdot, T; v) - \rho^d\|_{L^2(\Omega)}^2 + \alpha \|v\|_{L^2(\omega \times (0, T))}^2$$

with  $\alpha > 0$  and  $\rho^d \in \mathcal{C}(\bar{\Omega})$ .

Bilinear systems are used to describe many processes in biology, ecology and engineering. These systems which are nonlinear due the product between the input and the state variable are gained in interest to many researchers. We refer to Bruni *et al.* [5] for instance. Optimal control of such systems has been widely investigated for controls depending only on time, or on time and space. Thus, in the case of control depending only on time, Bradley *et al.* [3] proved the existence and uniqueness of a bilinear optimal control. Actually, the control which acts as a multiplier of a velocity term is a positive uniformly bounded function of time. The uniqueness of the optimal control were achieved for a time  $T$  sufficiently small. Addou *et al.* [1] studied a bilinear optimal control of a system governed by a fourth-order parabolic operator. The bilinearity appeared in the form of the scalar product of the vector of controls and the gradient of the state. The authors proved under suitable hypotheses the existence of an optimal control that they characterized with an optimality system. Then, assuming that the initial state is small enough, they obtained the uniqueness of the optimal control. In [10], an optimal bilinear control of an abstract Schrödinger equation. The existence of an optimal control depending only on time is proved and the first order optimality system is derived. Zerrik *et al.* [22] considered a regional quadratic control problem for distributed bilinear systems. They proved that an optimal control exists and gave an optimality system that characterized such controls. Considering a control depending on time and space, Lenhart *et al.* [12] studied an optimal control of linear heat equation with convective boundary condition in which the heat transfer is took as the control. They proved the existence and uniqueness of the optimal control and the solution of the optimality system. The results were achieved by means of compactness and maximum principle results. Zerrik *et al.* [8] investigated a constrained regional control problem of a bilinear plate equation. They proved using some compactness results the existence of an optimal control that they characterized with an optimality system. The cases of time or space dependent control were also discussed. We refer to [4, 13, 17, 18, 20, 23, 24, 26, 25], for more literature on bilinear optimal control involving Partial differential equations of integer order.

Actually, because bilinear controlled Partial differential equations are nonlinear, the most challenging issue in controlling such models is not only to find appropriate compactness results to obtain the existence of an optimal control, but also to prove the uniqueness the optimal control. This latter result is obtained by improving the regularity of solutions to the optimality systems.

In this paper, we are concerned with the bilinear optimal control of a fractional diffusive equation. More precisely, we aim to drive the state of (1.1) at final time to a desired state by acting on the system with a localized control depending on time and space. We first prove that for a given initial condition and control, system (1.1) has a unique weak solution. This allows to define our quadratic cost function. Then we prove using some compactness results that our control problem admits an optimal control that we characterize with a first order optimality system. Finally, we establish some weak maximum principle results that we use to prove the uniqueness of the optimal control. As far as we know the results we present are new.

The rest of the paper is organized as follows. In Section 2, we give some definitions on fractional Laplacian and some known results. In Section 3, we first prove the existence and uniqueness of weak solution to bilinear fractional diffusive system. Then, under some assumptions on the data, we establish appropriate maximum principle results for a class of bilinear diffusive equation. We prove in Section 4 that the bilinear optimal control (1.2) has an optimal control that we characterize with optimality system. We then prove that if the parameter of the cost function  $\alpha$  is large enough, the optimal control is unique.

## 2 Preliminaries

Let us recall the definition of the fractional Laplacian. Given  $0 < s < 1$ , we let the space

$$\mathcal{L}_s^1(\mathbb{R}^N) := \left\{ w : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable and } \int_{\mathbb{R}^N} \frac{|w(x)|}{(1+|x|)^{N+2s}} dx < \infty \right\}.$$

For  $w \in \mathcal{L}_s^1(\mathbb{R}^N)$  and  $\varepsilon > 0$ , we set

$$(-\Delta)_\varepsilon^s w(x) := C_{N,s} \int_{\{y \in \mathbb{R}^N : |x-y| > \varepsilon\}} \frac{w(x) - w(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where  $C_{N,s}$  is a normalization constant given by

$$C_{N,s} := \frac{s 2^{2s} \Gamma\left(\frac{2s+N}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}. \quad (2.1)$$

The fractional Laplacian  $(-\Delta)^s$  is defined by the following singular integral:

$$(-\Delta)^s w(x) := C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{w(x) - w(y)}{|x-y|^{N+2s}} dy = \lim_{\varepsilon \downarrow 0} (-\Delta)_\varepsilon^s w(x), \quad x \in \mathbb{R}^N, \quad (2.2)$$

provided that the limit exists for a.e.  $x \in \mathbb{R}^N$ . Let  $\Omega \subset \mathbb{R}^N$  be an arbitrary open and bounded set.

Given  $0 < s < 1$ , we let

$$\mathbb{H}^s(\Omega) := \left\{ w : \mathbb{R}^N \rightarrow \mathbb{R} \text{ Lebesgue measurable s.t. } w|_\Omega \in L^2(\Omega) \text{ and } \int \int_{\mathbb{R}^{2N} \setminus \Sigma^2} \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s}} dx dy < \infty \right\}.$$

Then endowed with the norm defined by

$$\|w\|_{\mathbb{H}^s(\Omega)} := \left( \int_\Omega |w(x)|^2 dx + \int \int_{\mathbb{R}^{2N} \setminus \Sigma^2} \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s}} dx dy \right)^{\frac{1}{2}},$$

$\mathbb{H}^s(\Omega)$  is an Hilbert space.

We set

$$\mathbb{H}_0^s(\Omega) := \left\{ w \in \mathbb{H}^s(\mathbb{R}^N) : w = 0 \text{ in } \Sigma \right\}.$$

Then,  $\mathbb{H}_0^s(\Omega)$  endowed with norm

$$\|w\|_{\mathbb{H}_0^s(\Omega)} = \left( \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(w(x) - w(y))^2}{|x-y|^{N+2s}} dx dy \right)^{1/2}, \quad (2.3)$$

is an Hilbert space (see e.g. [19, Lemma 7]). Let  $\mathbb{H}^{-s}(\Omega) := (\mathbb{H}_0^s(\Omega))^*$  be the dual space of  $\mathbb{H}_0^s(\Omega)$  with respect to the pivot space, so that we have the following continuous embeddings (see e.g. [2]):

$$\mathbb{H}_0^s(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow \mathbb{H}^{-s}(\Omega). \quad (2.4)$$

Next, for  $\rho \in H^s(\mathbb{R}^N)$  we introduce the *nonlocal normal derivative*  $\mathcal{N}_s$  given by

$$\mathcal{N}_s \rho(x) := C_{N,s} \int_{\Omega} \frac{\rho(x) - \rho(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N \setminus \bar{\Omega}, \quad (2.5)$$

where  $C_{N,s}$  is the constant given in (2.1).

The following integration by parts formula is contained in [7, 21] for smooth functions. Let  $\rho \in H^s(\mathbb{R}^N)$  be such that  $(-\Delta)^s \rho \in L^2(\Omega)$  and  $\mathcal{N}_s \rho \in L^2(\mathbb{R}^N \setminus \Omega)$ . Then for every  $\psi \in H^s(\mathbb{R}^N)$ , the following identity

$$\begin{aligned} \frac{C_{N,s}}{2} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(\rho(x) - \rho(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} dx dy \\ = \int_{\Omega} \psi (-\Delta)^s \rho dx + \int_{\mathbb{R}^N \setminus \Omega} \psi \mathcal{N}_s \rho dx, \end{aligned} \quad (2.6)$$

holds.

Hence, if  $\rho, \psi \in H^s(\mathbb{R}^N)$  with  $(-\Delta)^s \rho, (-\Delta)^s \psi \in L^2(\Omega)$  and  $\mathcal{N}_s \rho, \mathcal{N}_s \psi \in L^2(\mathbb{R}^N \setminus \Omega)$ . Then the following identity holds,

$$\begin{aligned} \int_{\Omega} \psi (-\Delta)^s \rho dx &= \int_{\Omega} \rho (-\Delta)^s \psi dx + \int_{\mathbb{R}^N \setminus \Omega} \rho \mathcal{N}_s \psi dx \\ &- \int_{\mathbb{R}^N \setminus \Omega} \psi \mathcal{N}_s \rho dx. \end{aligned} \quad (2.7)$$

Observing that

$$\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2 = (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega),$$

we have that if  $\rho = 0$  in  $\mathbb{R}^N \setminus \Omega$  or  $\psi = 0$  in  $\mathbb{R}^N \setminus \Omega$ , then

$$\begin{aligned} \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(\rho(x) - \rho(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} dx dy \\ = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\rho(x) - \rho(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

Hence (2.3) can be rewritten as:

$$\|w\|_{\mathbb{H}_0^s(\Omega)} = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(w(x) - w(y))^2}{|x - y|^{N+2s}} dx dy \right)^{1/2} \quad \forall w \in \mathbb{H}_0^s(\Omega). \quad (2.8)$$

For suitable function  $\rho$  and  $\psi$ , we set

$$\mathcal{F}(\rho, \psi) := \int \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(\rho(x) - \rho(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} dx dy. \quad (2.9)$$

**Remark 2.1**

(a) Note that if  $N > 2s$ , then there exists  $C_0 = C(N, s) > 0$  depending only on  $N$  and  $s$  such that for any  $\rho \in \mathbb{H}_0^s(\Omega)$ ,

$$\|\rho\|_{L^r(\Omega)} \leq C_0 \|\rho\|_{\mathbb{H}_0^s(\Omega)}, \quad (2.10)$$

where  $r = \frac{2N}{N-2s}$  (see e.g. [19, Lemma 6 a])). Therefore,

$$\|\rho\|_{L^2(\Omega)} \leq C_0 \|\rho\|_{\mathbb{H}_0^s(\Omega)}, \quad (2.11)$$

because  $\Omega$  is an open bounded subset of  $\mathbb{R}^N$  and  $r > 2$ . Moreover, the following holds true,

$$\|\rho\|_{L^2((0,T);L^2(\Omega))} \leq C_0 \|\rho\|_{L^2((0,T);\mathbb{H}_0^s(\Omega))}. \quad (2.12)$$

(b) It is well known that the continuous injection

$$\mathbb{H}_0^s(\Omega) \hookrightarrow L^2(\Omega), \quad (2.13)$$

is also compact, see [6, 9].

We first start with these compactness results.

**Theorem 2.1** [14, Theorem 5.1, Page 58] Let  $B_0, B, B_1$  be three Banach spaces such that  $B_0 \subset B \subset B_1$ , with  $B_i$  being reflexive,  $i = 1, 2$ . Assume that the embedding  $B_0 \hookrightarrow B$  is compact and set

$$\mathcal{W} = \left\{ \rho \in L^2((0, T); B_0); \frac{\partial \rho}{\partial t} \in L^2((0, T); B_1) \right\},$$

with  $T < \infty$ . Then,  $\mathcal{W}$  is compactly embedded in  $L^2((0, T); B)$ .

Let us recall the following result given in [15, Page 37].

**Theorem 2.2** Let  $(F, \|\cdot\|_F)$  be a Hilbert space. Let  $\Phi$  be a subspace of  $F$  endowed with a pre-Hilbert scalar product  $((\cdot, \cdot))$ , with associated norm  $\|\cdot\|$ . Moreover, let  $\mathcal{E} : F \times \Phi \rightarrow \mathbb{C}$  be a sesquilinear form. Assume that the following hypotheses hold:

1. The embedding  $\Phi \hookrightarrow F$  is continuous, that is, there is a constant  $C_1 > 0$  such that

$$\|\varphi\|_F \leq C_1 \|\varphi\| \quad \forall \varphi \in \Phi. \quad (2.14)$$

2. For all  $\varphi \in \Phi$ , the mapping  $u \mapsto E(u, \varphi)$  is continuous on  $F$ .

3. There is a constant  $C_2 > 0$  such that

$$\mathcal{E}(\varphi, \varphi) \geq C_2 \|\varphi\|^2 \quad \text{for all } \varphi \in \Phi. \quad (2.15)$$

If  $\varphi \mapsto L(\varphi)$  is a semi linear continuous form on  $\Phi$ , then there exists a function  $u \in F$  verifying

$$\mathcal{E}(u, \varphi) = L(\varphi) \quad \text{for all } \varphi \in \Phi.$$

Let denote by  $\mathbb{X}'$  the dual of  $\mathbb{X}$ . Set

$$W((0, T); \mathbb{X}) := \left\{ \zeta \in L^2((0, T); \mathbb{X}) : \zeta_t \in L^2((0, T); \mathbb{X}') \right\}. \quad (2.16)$$

Then  $W(0, T; \mathbb{X})$  endowed with the norm given by

$$\|\psi\|_{W((0, T); \mathbb{X})}^2 = \|\psi\|_{L^2(0, T; \mathbb{X})}^2 + \|\psi_t\|_{L^2(0, T; \mathbb{X}')}^2, \quad \forall \psi \in W((0, T); \mathbb{X}), \quad (2.17)$$

is a Hilbert space. Moreover, if

$$\mathbb{X} \subset \mathbb{Y} \subset \mathbb{X}',$$

then by [16] we have that the continuous embedding

$$W((0, T); \mathbb{X}) \hookrightarrow \mathcal{C}([0, T]; \mathbb{Y}). \quad (2.18)$$

### 3 Existence and maximum principle results

In this sections, we give some useful results for the study of bilinear optimal control.

#### 3.1 Existence results

Set

$$\mathbb{V} := \left\{ \varphi \in \mathbb{H}_0^s(\Omega) : (-\Delta)^s \varphi \in L^2(\Omega) \right\}. \quad (3.1)$$

Then  $\mathbb{V}$  endowed with the norm (2.8) is a Hilbert space.

Consider the following system:

$$\begin{cases} \rho_t + (-\Delta)^s \rho = f + v\rho\chi_\omega & \text{in } Q = \Omega \times (0, T), \\ \rho = 0 & \text{in } \Sigma = (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ \rho(0, \cdot) = \rho^0 & \text{in } \Omega. \end{cases} \quad (3.2)$$

**Definition 3.1** Let  $f \in L^2(Q)$ ,  $v \in \mathcal{U}$  and  $\rho^0 \in L^2(\Omega)$ . Let  $\mathcal{F}(\cdot, \cdot)$  be defined as in (2.9). We say that a function  $\rho \in L^2((0, T); \mathbb{V})$  is a weak solution to (3.2), if the following equality

$$\begin{aligned} & - \int_Q \phi_t \rho dx dt + \frac{C_{N,s}}{2} \int_0^T \mathcal{F}(\rho, \phi) dt - \int_{\omega_T} v \rho \phi dx dt \\ & = \int_Q f \phi dx dt + \int_\Omega \rho^0 \phi(0) dx, \end{aligned} \quad (3.3)$$

holds, for every

$$\phi \in H(Q) = \left\{ z \in L^2((0, T); \mathbb{V}) : z_t \in L^2(Q) \text{ and } z(\cdot, T) = 0 \text{ in } \Omega \right\}.$$

**Remark 3.1** Note also that if  $\varphi \in H(Q)$ , then  $\varphi_t \in L^2(Q) \hookrightarrow L^2((0, T); \mathbb{V}')$ . Consequently,  $\varphi \in W(0, T; \mathbb{V})$ . Therefore,  $\varphi(\cdot, 0)$  and  $\varphi(\cdot, T)$  exist and belong to  $L^2(\Omega)$ .

**Theorem 3.1** *Let  $f \in L^2(Q)$ ,  $v \in \mathcal{U}$  and  $\rho^0 \in L^2(\Omega)$ . Let also  $\mathcal{F}(\cdot, \cdot)$  be defined as in (2.9). Then, there exists a unique weak solution  $\rho \in \mathcal{H} := L^2((0, T); \mathbb{V}) \cap \mathcal{C}([0, T]; L^2(\Omega))$  to (3.2) in the sense of Definition 3.1. In addition, the following estimates hold*

$$\|\rho\|_{\mathcal{C}([0, T]; L^2(\Omega))}^2 \leq e^{T \frac{(1+2M)}{2}} \left[ \|\rho^0\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q)}^2 \right] \quad (3.4)$$

and

$$\|\rho\|_{L^2((0, T); \mathbb{V})}^2 \leq \frac{2}{C_{N,s}} e^{T \frac{(1+2M)}{2}} \left[ \|\rho^0\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q)}^2 \right]. \quad (3.5)$$

**Proof.** We proceed in three steps.

**Step 1.** We show the estimates (3.4)-(3.5).

Make the change of variable  $z(x, t) = e^{-rt} \rho(x, t)$ ,  $(x, t) \in Q$ , for some  $r > 0$  where  $\rho$  is solution to (3.2), we obtain that  $z$  is solution to

$$\begin{cases} z_t + (-\Delta)^s z + rz &= e^{-rt} f + vz\chi_\omega & \text{in } Q, \\ z &= 0 & \text{in } \Sigma, \\ z(\cdot, 0) &= \rho^0 & \text{in } \Omega. \end{cases} \quad (3.6)$$

Multiplying the first equation in (3.6) by  $z \in L^2((0, T); \mathbb{V})$ , using the integration by parts (2.6) and Cauchy's inequality, we obtain

$$\begin{aligned} & \frac{1}{2} \|z(t)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \mathcal{F}(z(t), z(t)) + r \|z(t)\|_{L^2(\Omega)}^2 \\ &= \int_{\omega} e^{-rt} f(t) z(t) \, dx \, dt + \int_{\omega} vz^2(t) \, dx \\ &\leq \frac{1}{2} \|f(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|z(t)\|_{L^2(\Omega)}^2 + M \|z\|_{L^2(\Omega)}^2. \end{aligned}$$

Hence choosing  $r = \frac{1}{2} + M$  in this latter identity, we deduce that

$$\frac{1}{2} \|z(t)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \|z(t)\|_{\mathbb{V}}^2 \leq \frac{1}{2} \|f(t)\|_{L^2(\Omega)}^2. \quad (3.7)$$

Now, integrating this latter inequality on  $(0, \tau)$ , with  $\tau \in [0, T]$ , we obtain

$$\frac{1}{2} \|z(\tau)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{4} \int_0^\tau \|z(t)\|_{\mathbb{V}}^2 \, dt \leq \frac{1}{2} \|f\|_{L^2(Q)}^2 + \frac{1}{2} \|\rho^0\|_{L^2(\Omega)}^2,$$

from which we deduce that

$$\begin{aligned} \sup_{\tau \in [0, T]} \|\rho(\cdot, \tau)\|_{L^2(\Omega)}^2 &\leq e^{T \frac{(1+2M)}{2}} \left[ \|\rho^0\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q)}^2 \right], \\ \int_0^T \|\rho(\cdot, t)\|_{\mathbb{V}}^2 \, dt &\leq \frac{2}{C_{N,s}} e^{T \frac{(1+2M)}{2}} \left[ \|\rho^0\|_{L^2(\Omega)}^2 + \|f\|_{L^2(Q)}^2 \right]. \end{aligned}$$

**Step 2.** We prove existence of  $\rho$  solution to (3.2) by means of Theorem 2.2.

To this end, we recall that the norm on  $L^2((0, T); \mathbb{V})$  is given by

$$\|\rho\|_{L^2((0, T); \mathbb{V})}^2 = \int_0^T \|\rho(\cdot, t)\|_{\mathbb{V}}^2 \, dt,$$



and we consider the norm defined on  $H(Q)$  by

$$\|\rho\|_{H(Q)}^2 := \|\rho\|_{L^2((0,T);\mathbb{V})}^2 + \|\rho(\cdot, 0)\|_{L^2(\Omega)}^2, \quad \forall \rho \in H(Q).$$

Therefore, it is clear that for any  $\rho \in H(Q)$ , we have

$$\|\rho\|_{L^2((0,T);\mathbb{V})} \leq \|\rho\|_{H(Q)}.$$

This shows that we have the continuous embedding  $H(Q) \hookrightarrow L^2((0,T);\mathbb{V})$ .

Now, let  $\varphi \in H(Q)$  and consider the bilinear form  $\mathcal{E}(\cdot, \cdot)$  defined on  $L^2((0,T);\mathbb{V}) \times H(Q)$  by:

$$\mathcal{E}(\rho, \varphi) = - \int_Q \varphi_t \rho \, dx \, dt + \frac{C_{N,s}}{2} \int_0^T \mathcal{F}(\rho, \varphi) \, dt - \int_{\omega_T} v \rho \varphi \, dx \, dt, \quad (3.8)$$

where we recall that  $\mathcal{F}(\cdot, \cdot)$  has been defined in (2.9). Using Cauchy Schwarz inequality and (2.12), we get that

$$\begin{aligned} |\mathcal{E}(\rho, \varphi)| &\leq \|\rho\|_{L^2(Q)} \|\varphi_t\|_{L^2(Q)} + \frac{C_{N,s}}{2} \left( \int_0^T \mathcal{F}(\rho, \rho) \, dt \right)^{1/2} \left( \int_0^T \mathcal{F}(\varphi, \varphi) \, dt \right)^{1/2} \\ &\quad + M \|\rho\|_{L^2(Q)} \|\varphi\|_{L^2(Q)} \\ &\leq \|\rho\|_{L^2(Q)} \|\varphi_t\|_{L^2(Q)} + \frac{C_{N,s}}{2} \|\rho\|_{L^2((0,T);\mathbb{V})} \|\varphi\|_{L^2((0,T);\mathbb{V})} \\ &\quad + MC_0^2 \|\rho\|_{L^2((0,T);\mathbb{V})} \|\varphi\|_{L^2((0,T);\mathbb{V})} \\ &\leq \left( C_0 \|\varphi_t\|_{L^2(Q)} + \left( MC_0^2 + \frac{C_{N,s}}{2} \right) \|\varphi\|_{L^2((0,T);\mathbb{V})} \right) \|\rho\|_{L^2((0,T);\mathbb{V})} \\ &\leq C \|\rho\|_{L^2((0,T);\mathbb{V})}. \end{aligned}$$

This means that there is a constant  $C = C(C_0, \varphi, C_{N,s}, M) > 0$  such that

$$|\mathcal{E}(\rho, \varphi)| \leq C \|\rho\|_{L^2((0,T);\mathbb{V})}.$$

Consequently, for every fixed  $\varphi \in H(Q)$ , the functional  $\rho \mapsto \mathcal{E}(\rho, \varphi)$  is continuous on  $L^2((0,T);\mathbb{V})$ .

Next, we have that for every  $\varphi \in H(Q)$ ,

$$\begin{aligned} \mathcal{E}(\varphi, \varphi) &= - \int_Q \varphi_t \varphi \, dx \, dt + \frac{C_{N,s}}{2} \int_0^T \mathcal{F}(\varphi, \varphi) \, dt - \int_{\omega_T} v \varphi^2 \, dx \, dt \\ &\geq \frac{1}{2} \|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \int_0^T \mathcal{F}(\varphi, \varphi) \, dt \\ &\geq \frac{1}{2} \|\varphi(\cdot, 0)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \|\varphi\|_{L^2((0,T);\mathbb{V})}^2 \\ &\geq \frac{1}{2} \min(1, C_{N,s}) \|\varphi\|_{H(Q)}^2. \end{aligned}$$

Finally, let us consider the functional  $L(\cdot) : H(Q) \rightarrow \mathbb{R}$  defined by

$$L(\varphi) =: \int_Q f \varphi \, dx \, dt + \int_{\Omega} \rho^0 \varphi(\cdot, 0) \, dx.$$

Then using Cauchy Schwarz inequality, (2.12), (3.5) and the fact that  $v \in \mathcal{U}$ , we obtain after straightforward calculations that

$$|L(\varphi)| \leq C(\rho^0, C_0, f) \|\varphi\|_{H(Q)},$$

where

$$C(\rho^0, C_0, f) = \|\rho^0\|_{L^2(\Omega)} + C_0\|f\|_{L^2(Q)}.$$

Therefore,  $L(\cdot)$  is continuous on  $H(Q)$ . Thus, it follows from Theorem 2.2 that there exists  $\rho \in L^2((0, T); \mathbb{V})$  such that

$$\mathcal{E}(\rho, \varphi) = L(\varphi), \quad \forall \varphi \in H(Q). \quad (3.9)$$

We have shown that the system (3.2) has a solution  $\rho \in L^2((0, T); \mathbb{V})$  in the sense of Definition 3.1.

**Step 3.** We prove the uniqueness of  $\rho$  solution to (3.2).

Assume that there exist  $z_1$  and  $z_2$  solutions to (3.2) with the same right hand side  $f, v$  and initial datum  $\rho^0$ . Set  $z := e^{-rt}(z_1 - z_2)$ . Then  $z$  satisfies

$$\begin{cases} z_t + (-\Delta)^s z + rz = vz\chi_\omega & \text{in } Q, \\ z = 0 & \text{in } \Sigma, \\ z(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \quad (3.10)$$

So, if we multiply the first equation of (3.10) by  $z$ , then use the integration by parts (2.6) and the fact that  $v \in \mathcal{U}$ , we obtain that,

$$\frac{1}{2}\|z(\cdot, T)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \int_0^T \mathcal{F}(z, z) dt + r\|z\|_{L^2(Q)}^2 \leq M\|z\|_{L^2(Q)}^2.$$

Choosing  $r = M$  in this latter inequality, we can deduce that

$$\frac{1}{2}\|z(\cdot, T)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2}\|z\|_{L^2((0, T); \mathbb{H}_0^s(\Omega))}^2 \leq 0.$$

Hence,  $z = 0$  in  $\mathbb{R}^N$ . Thus,  $y_1 = y_2$  in  $\mathbb{R}^N$  and we have shown uniqueness.

This completes the proof. ■

**Corollary 3.1** *Let  $f \in L^2(Q)$  and  $\rho^0 \in \mathbb{H}_0^s(\Omega)$ . Then, there exists a unique weak solution  $\rho \in \mathbb{K} := \mathcal{H} \cap H^1((0, T); L^2(\Omega)) \cap \mathcal{C}([0, T]; \mathbb{H}_0^s(\Omega))$  to (3.2) in the sense of Definition 3.1. In addition, the following estimates hold*

$$\begin{aligned} \|\rho_t\|_{L^2(Q)}^2 &\leq \|f\|_{L^2(Q)}^2 \left( 2 + \frac{4MC_0}{C_{N,s}} e^{T \frac{(1+2M)}{2}} \right) \\ &+ \|\rho^0\|_{\mathbb{H}_0^s(\Omega)}^2 \left( \frac{C_{N,s}}{2} + \frac{4MC_0^2}{C_{N,s}} e^{T \frac{(1+2M)}{2}} \right) \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} \sup_{\tau \in [0, T]} \|\rho(\cdot, \tau)\|_{\mathbb{H}_0^s(\Omega)}^2 &\leq \|f\|_{L^2(Q)}^2 \left( \frac{4}{C_{N,s}} + \frac{8MC_0}{C_{N,s}^2} e^{T \frac{(1+2M)}{2}} \right) \\ &+ \|\rho^0\|_{\mathbb{H}_0^s(\Omega)}^2 \left( 1 + \frac{8MC_0^2}{C_{N,s}^2} e^{T \frac{(1+2M)}{2}} \right). \end{aligned} \quad (3.12)$$

**Proof.** If  $\rho^0 \in \mathbb{H}_0^s(\Omega)$ , then  $\rho^0 \in L^2(\Omega)$ . It follows from Theorem 3.1 that there exists a unique weak solution  $\rho \in \mathcal{H}$  to (3.2) in the sense of Definition 3.1. Moreover (3.5) holds true. To complete the proof of Corollary 3.1, we need to prove (3.11)-(3.12). So, if we multiply the first equation in (3.2) by  $\rho_t$  and use the integration by parts (2.6), we have that for all  $t > 0$ ,

$$\|\rho_t(t)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \mathcal{F}(\rho_t(t), \rho(t)) = \int_{\Omega} f(t)\rho_t(t) dx + \int_{\omega} v(t)\rho(t)\rho_t(t) dx.$$

Integrating this latter identity over  $(0, \tau)$  with  $\tau \in [0, T]$ , then using Young's inequality and the fact that  $v \in \mathcal{U}$ , we obtain that

$$\begin{aligned}
& \int_0^\tau \|\rho_t(t)\|_{L^2(\Omega)}^2 dt + \frac{C_{N,s}}{2} \int_0^\tau \mathcal{F}(\rho_t(t), \rho(t)) dt \\
&= \int_0^\tau \int_\Omega f(t) \rho_t(t) dx dt + \int_0^\tau \int_\omega v(t) \rho(t) \rho_t(t) dx dt \\
&\leq \frac{\delta}{2} \|f\|_{L^2(Q)}^2 + \frac{1}{\delta} \int_0^\tau \|\rho_t(t)\|_{L^2(\Omega)}^2 dt + \frac{\delta M}{2} \|\rho\|_{L^2(Q)}^2,
\end{aligned} \tag{3.13}$$

for some  $\delta > 0$ . Observing that

$$\begin{aligned}
\int_0^\tau \mathcal{F}(\rho_t(t), \rho(t)) dt &= \int_0^\tau \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\rho_t(x) - \rho_t(y))(\rho(x) - \rho(y))}{|x - y|^{N+2s}} dx dy dt \\
&= \int_0^\tau \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\rho(x, t) \rho_t(x, t) + \rho(y, t) \rho_t(y, t))}{|x - y|^{N+2s}} dx dy dt \\
&\quad - \int_0^\tau \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\rho(x, t) \rho_t(y, t) + \rho(y, t) \rho_t(x, t))}{|x - y|^{N+2s}} dx dy dt \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\rho^2(x, \tau) - \rho^2(x, 0) + \rho^2(y, \tau) - \rho^2(y, 0))}{2|x - y|^{N+2s}} dx dy \\
&\quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\rho(x, \tau) \rho(y, \tau) - \rho(y, 0) \rho(x, 0))}{|x - y|^{N+2s}} dx dy \\
&= \frac{1}{2} \|\rho(\cdot, \tau)\|_{\mathbb{H}_0^s(\Omega)}^2 - \frac{1}{2} \|\rho^0\|_{\mathbb{H}_0^s(\Omega)}^2,
\end{aligned}$$

relation (3.13) becomes

$$\begin{aligned}
\int_0^\tau \|\rho_t(t)\|_{L^2(\Omega)}^2 dt + \frac{C_{N,s}}{4} \|\rho(\cdot, \tau)\|_{\mathbb{H}_0^s(\Omega)}^2 &\leq \frac{\delta}{2} \|f\|_{L^2(Q)}^2 + \frac{1}{\delta} \int_0^\tau \|\rho_t(t)\|_{L^2(\Omega)}^2 dt \\
&\quad + \frac{\delta M}{2} \|\rho\|_{L^2(Q)}^2,
\end{aligned}$$

for any  $\tau \in [0, T]$ . Choosing  $\delta = 2$  in this latter equality and using (2.12), we have that,

$$\begin{aligned}
& \frac{1}{2} \int_0^\tau \|\rho_t(t)\|_{L^2(\Omega)}^2 dt + \frac{C_{N,s}}{4} \|\rho(\cdot, \tau)\|_{\mathbb{H}_0^s(\Omega)}^2 \\
&\leq \|f\|_{L^2(Q)}^2 + M \|\rho\|_{L^2(Q)}^2 + \frac{C_{N,s}}{4} \|\rho^0\|_{\mathbb{H}_0^s(\Omega)}^2 \\
&\leq \|f\|_{L^2(Q)}^2 + \frac{C_{N,s}}{4} \|\rho^0\|_{\mathbb{H}_0^s(\Omega)}^2 + MC_0 \|\rho\|_{L^2((0,T);V)}^2,
\end{aligned} \tag{3.14}$$

for any  $\tau \in [0, T]$ . Hence,

$$\frac{1}{2} \int_0^\tau \|\rho_t(t)\|_{L^2(\Omega)}^2 dt \leq \|f\|_{L^2(Q)}^2 + \frac{C_{N,s}}{4} \|\rho^0\|_{\mathbb{H}_0^s(\Omega)}^2 + MC_0 \|\rho\|_{L^2((0,T);V)}^2$$

and

$$\frac{C_{N,s}}{4} \|\rho(\cdot, \tau)\|_{\mathbb{H}_0^s(\Omega)}^2 \leq \|f\|_{L^2(Q)}^2 + \frac{C_{N,s}}{4} \|\rho^0\|_{\mathbb{H}_0^s(\Omega)}^2 + MC_0 \|\rho\|_{L^2((0,T);V)}^2,$$

which in view of (3.5) and (2.11) give for any  $\tau \in [0, T]$ ,

$$\begin{aligned}
\frac{1}{2} \int_0^\tau \|\rho_t(t)\|_{L^2(\Omega)}^2 dt &\leq \|f\|_{L^2(Q)}^2 + \frac{C_{N,s}}{4} \|\rho^0\|_{\mathbb{H}_0^s(\Omega)}^2 \\
&\quad + \frac{2MC_0}{C_{N,s}} e^{T \frac{(1+2M)}{2}} \left[ C_0 \|\rho^0\|_{\mathbb{H}_0^s(\Omega)}^2 + \|f\|_{L^2(Q)}^2 \right]
\end{aligned}$$

and

$$\begin{aligned} \frac{C_{N,s}}{4} \|\rho(\cdot, \tau)\|_{\mathbb{H}_0^s(\Omega)}^2 &\leq \|f\|_{L^2(Q)}^2 + \frac{C_{N,s}}{4} \|\rho^0\|_{\mathbb{H}_0^s(\Omega)}^2 \\ &+ \frac{2MC_0}{C_{N,s}} e^{T\frac{(1+2M)}{2}} \left[ C_0 \|\rho^0\|_{\mathbb{H}_0^s(\Omega)}^2 + \|f\|_{L^2(Q)}^2 \right]. \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^T \|\rho_t(t)\|_{L^2(\Omega)}^2 dt &\leq \|f\|_{L^2(Q)}^2 \left( 2 + \frac{4MC_0}{C_{N,s}} e^{T\frac{(1+2M)}{2}} \right) \\ &+ \|\rho^0\|_{\mathbb{H}_0^s(\Omega)}^2 \left( \frac{C_{N,s}}{2} + \frac{4MC_0^2}{C_{N,s}} e^{T\frac{(1+2M)}{2}} \right) \end{aligned}$$

and

$$\begin{aligned} \sup_{\tau \in [0, T]} \|\rho(\cdot, \tau)\|_{\mathbb{H}_0^s(\Omega)}^2 &\leq \|f\|_{L^2(Q)}^2 \left( \frac{4}{C_{N,s}} + \frac{8MC_0}{C_{N,s}^2} e^{T\frac{(1+2M)}{2}} \right) \\ &+ \|\rho^0\|_{\mathbb{H}_0^s(\Omega)}^2 \left( 1 + \frac{8MC_0^2}{C_{N,s}^2} e^{T\frac{(1+2M)}{2}} \right). \end{aligned}$$

This completes the proof.  $\blacksquare$

**Corollary 3.2** *Let  $v \in \mathcal{U}$  and  $\rho^0 \in \mathcal{C}(\bar{\Omega})$ . Let also  $\mathcal{F}(\cdot, \cdot)$  be defined as in (2.9). Then, there exists a unique weak solution  $\rho \in \mathcal{H} := L^2((0, T); \mathbb{V}) \cap \mathcal{C}([0, T]; L^2(\Omega))$  to (1.1) in the sense of Definition 3.1. In addition, the following estimates hold*

$$\|\rho\|_{\mathcal{C}([0, T]; L^2(\Omega))}^2 \leq e^{T\frac{(1+2M)}{2}} \|\rho^0\|_{L^2(\Omega)}^2 \quad (3.15)$$

and

$$\|\rho\|_{L^2((0, T); \mathbb{V})}^2 \leq \frac{2}{C_{N,s}} e^{T\frac{(1+2M)}{2}} \|\rho^0\|_{L^2(\Omega)}^2. \quad (3.16)$$

**Proof.** Since  $\rho^0 \in \mathcal{C}(\bar{\Omega})$ , we have that  $\rho^0 \in L^2(\Omega)$ . It then suffices to apply Theorem 3.1 with  $f = 0$  to obtain the results.  $\blacksquare$

### 3.2 Maximum principle results

**Theorem 3.2** *Let  $\rho_0 \in \mathcal{C}(\bar{\Omega})$  be such that  $\rho^0(x) > 0$  almost everywhere in  $\Omega$  and  $v \in \mathcal{U}$ . Then the weak solution of (1.1) is positive almost everywhere in  $\mathbb{R}^N \times [0, T]$ .*

**Proof.** Let  $\rho^-(x, t) = \min(0, \rho(x, t))$  where  $\rho \in L^2((0, T); \mathbb{V}) \cap \mathcal{C}([0, T]; L^2(\Omega))$  is the weak solution of (1.1). Denote the support of  $\rho^-$  by  $\mathcal{O} = \text{supp}\rho^-$ . Then we have that

$$\begin{aligned} \rho^- &= \rho \chi_{\mathcal{O}}, \\ \rho^-(x, 0) &= \min(0, \rho^0(x)) = 0 \text{ in } \Omega, \\ \rho_t^- &= \rho_t \chi_{\mathcal{O}} \end{aligned}$$

and  $\rho^- \in L^2((0, T); \mathbb{V}) \cap \mathcal{C}([0, T]; L^2(\Omega))$ .

Let  $\psi \in \mathbb{V}$ . Multiplying the first equation in (1.1) by  $\psi$  and using the integration by parts (2.6), we have for any  $t > 0$ ,

$$\int_{\Omega} \rho_t(t) \psi(t) dx + \frac{C_{N,s}}{2} \mathcal{F}(\rho(t), \psi(t)) = \int_{\Omega} v(t) \rho(t) \psi(t) dx.$$

Taking  $\psi = \rho^-(t)$  in this latter identity yields

$$\int_{\Omega} \rho_t^-(t) \rho^-(t) dx + \frac{C_{N,s}}{2} \mathcal{F}(\rho^-(t), \rho^-(t)) = \int_{\Omega} v(t) (\rho^-(t))^2 dx, \quad (3.17)$$

where  $\mathcal{F}$  is defined as in (2.9) and which according to the support of  $\rho^-$  can be rewritten as

$$\mathcal{F}(\rho^-(t), \rho^-(t)) = \int \int_{(\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2) \cap \mathcal{O}^2} \frac{(\rho(x, t) - \rho(y, t))^2}{|x - y|^{N+2s}} dx dy.$$

Observing that (3.17) can be rewritten as

$$\frac{1}{2} \frac{d}{dt} \|\rho^-(t)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \mathcal{F}(\rho^-(t), \rho^-(t)) = \int_{\Omega} v(t) (\rho^-(t))^2 dx$$

and integrating this latter identity over  $(0, \tau)$  with  $\tau \in [0, T]$ , we deduce that

$$\frac{1}{2} \|\rho^-(\tau)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \int_0^{\tau} \mathcal{F}(\rho^-(t), \rho^-(t)) dt \leq 0,$$

because  $\rho^-(x, 0) = 0$  a.e. in  $\Omega$ . Hence, we deduce

$$\sup_{\tau \in [0, T]} \|\rho^-(\tau)\|_{L^2(\Omega)}^2 = 0$$

and

$$\int_0^T \mathcal{F}(\rho^-(t), \rho^-(t)) dt = \|\rho^-\|_{L^2((0, T); \mathbb{H}_0^s(\Omega))}^2 = 0.$$

It then follows from the two identities above that  $\rho^- = 0$  almost everywhere in  $\mathcal{O}$  and thus  $\rho^- = 0$  almost everywhere in  $\Omega \times [0, T]$ . Consequently,  $\rho \geq 0$  almost everywhere in  $\mathbb{R}^N \times [0, T]$ . ■

**Remark 3.2** Note that  $\rho^0 \in L^2(\Omega)$  is enough to obtain the positivity of  $\rho$ . We have the same observation for a control in  $L^\infty(Q)$ .

The following results can be viewed as maximum principle for class of bilinear fractional diffusion equations. We need these results for the continuity of the solution to (1.1) as well as for the uniqueness of the optimal control.

**Theorem 3.3** Let  $\psi^T \in L^\infty(\Omega)$  be such that  $\psi^T \geq 0$  almost everywhere in  $\Omega$ . Let  $\psi^d \in \mathcal{C}(\bar{\Omega})$  and  $v \in \mathcal{U}$ . Then the weak solution of

$$\begin{cases} \psi_t + (-\Delta)^s \psi &= v \psi \chi_\omega & \text{in } Q, \\ \psi &= 0 & \text{in } \Sigma, \\ \psi(\cdot, 0) &= \psi^T - \psi^d & \text{in } \Omega, \end{cases} \quad (3.18)$$

satisfies

$$\begin{aligned} & \min \left( 0, -\|\psi^T\|_{L^\infty(\Omega)} - \sup_{x \in \Omega} \psi^d(x) \right) \\ & \leq \psi(x, t) \leq \max \left( 0, \|\psi^T\|_{L^\infty(\Omega)} - \inf_{x \in \Omega} \psi^d(x) \right) \quad \text{a.e. in } \mathbb{R}^N \times [0, T]. \end{aligned} \quad (3.19)$$

**Proof.** In view of Theorem 3.1, we that  $\psi \in L^2((0, T); \mathbb{V}) \cap \mathcal{C}([0, T]; L^2(\Omega))$  because  $\psi^T - \psi^d \in L^2(\Omega)$  and  $v \in \mathcal{U}$ . To continue with the proof of Theorem 3.3, we will proceed in two steps.

**Step 1.** We prove that  $\psi(x, t) \leq \max\left(0, \|\psi^T\|_{L^\infty(\Omega)} - \inf_{x \in \Omega} \psi^d(x)\right)$  a.e. in  $\mathbb{R}^N \times [0, T]$ .

Set  $K = \max\left(0, \|\psi^T\|_{L^\infty(\Omega)} - \inf_{x \in \Omega} \psi^d(x)\right)$  and  $w(x, t) = K - \psi(x, t)$  for any  $(x, t) \in \mathbb{R}^N \times (0, T)$ . Then  $K \geq 0$  and

$$\begin{aligned} w(x, 0) &= K - (\psi^T(x) - \psi^d(x)) \\ &\geq \|\psi^T\|_{L^\infty(\Omega)} - \inf_{x \in \Omega} \psi^d(x) - (\psi^T(x) - \psi^d(x)) \\ &\geq 0, \end{aligned}$$

for any  $x \in \Omega$ . Moreover,  $w$  satisfies

$$\begin{cases} w_t + (-\Delta)^s w &= v w \chi_\omega - v K \chi_\omega & \text{in } Q = \Omega \times (0, T), \\ w &= K & \text{in } \Sigma = (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ w(0, \cdot) &= K - (\psi^T - \psi^d) & \text{in } \Omega. \end{cases} \quad (3.20)$$

Then we consider the function  $w^-(x, t) = \min(0, w(x, t))$  for any  $(x, t) \in \mathbb{R}^N \times (0, T)$  and we denote the support of  $w^-$  by  $\mathcal{O} = \text{supp } w^-$ . Since  $K \geq 0$ , we have that  $w^- = 0$  on  $\Sigma$  and  $w^- \in L^2((0, T); \mathbb{V}) \cap \mathcal{C}([0, T]; L^2(\Omega))$ . If we multiply the first equation in (3.20) by  $w^-$  and use the integration by parts (2.6), we have for any  $t > 0$ ,

$$\begin{aligned} \int_{\Omega} w_t(t) w^-(t) dx + \frac{C_{N,s}}{2} \mathcal{F}(w(t), w^-(t)) &= \int_{\Omega} v(t) w(t) w^-(t) dx \\ &\quad - K \int_{\Omega} v(t) w^-(t) dx, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w^-(t)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \mathcal{F}(w^-(t), w^-(t)) &= \int_{\Omega} v(t) (w^-(t))^2 dx \\ &\quad - K \int_{\Omega} v(t) w^-(t) dx, \end{aligned}$$

with

$$\mathcal{F}(w^-(t), w^-(t)) = \int \int_{(\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2) \cap \mathcal{O}^2} \frac{(w(x, t) - w(y, t))^2}{|x - y|^{N+2s}} dx dy.$$

Hence,  $v$  being in  $\mathcal{U}$ ,  $w^-$  negative and  $K \geq 0$ , we deduce that

$$\frac{1}{2} \frac{d}{dt} \|w^-(t)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \mathcal{F}(w^-(t), w^-(t)) \leq 0.$$

Integrating this latter identity over  $(0, \tau)$  with  $s \in [0, T]$  and using the fact that  $w^-(x, 0) = 0$  a.e. in  $\Omega$ , we deduce that

$$\frac{1}{2} \|w^-(\tau)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \int_0^\tau \mathcal{F}(w^-(t), w^-(t)) dt \leq 0.$$

This implies that

$$\sup_{\tau \in [0, T]} \|w^-(\tau)\|_{L^2(\Omega)}^2 = 0$$

and

$$\int_0^T \mathcal{F}(w^-(t), w^-(t)) dt = \|w^-\|_{L^2((0,T); \mathbb{H}_0^s(\Omega))}^2 = 0.$$

So,  $w^- = 0$  almost everywhere in  $\mathcal{O}$  and hence  $w^- = 0$  almost everywhere in  $\mathbb{R}^N \times [0, T]$ . Consequently,  $K - \psi(x, t) \geq 0$  almost everywhere in  $\mathbb{R}^N \times [0, T]$ . This means that

$$\psi(x, t) \leq \max\left(0, \|\psi^T\|_{L^\infty(\Omega)} - \inf_{x \in \Omega} \psi^d(x)\right) \text{ a.e. in } \mathbb{R}^N \times [0, T].$$

**Step 2.** We prove that  $\min\left(0, -\|\psi^T\|_{L^\infty(\Omega)} - \sup_{x \in \Omega} \psi^d(x)\right) \leq \psi(x, t)$  a.e. in  $\mathbb{R}^N \times [0, T]$ .

Set  $L = \min\left(0, -\|\psi^T\|_{L^\infty(\Omega)} - \sup_{x \in \Omega} \psi^d(x)\right)$  and  $\zeta(x, t) = L - \psi(x, t)$  for any  $(x, t) \in \mathbb{R}^N \times [0, T]$ .

Then  $L \leq 0$  and  $\zeta$  satisfies

$$\begin{cases} \zeta_t + (-\Delta)^s \zeta &= v\zeta\chi_\omega - L v\zeta\chi_\omega & \text{in } Q = \Omega \times (0, T), \\ \zeta &= L & \text{in } \Sigma = (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ \zeta(0, \cdot) &= L - (\psi^T - \psi^d) & \text{in } \Omega. \end{cases} \quad (3.21)$$

We consider the function  $\zeta^+(x, t) := \max(0, \zeta(x, t))$  and denote the support of  $\zeta^+$  by  $\mathcal{O}_1 = \text{supp}\zeta^+$ . Since  $L \leq 0$ , we have on the one hand,  $\zeta^+ = 0$  on  $\Sigma$  and  $\zeta^+ \in L^2((0, T); \mathbb{V}) \cap \mathcal{C}([0, T]; L^2(\Omega))$ , and on the other hand,

$$\begin{aligned} \zeta(x, 0) &= L - \psi^T(x) - \psi^d(x) \\ &\leq -\|\psi^T\|_{L^\infty(\Omega)} - \sup_{x \in \Omega} \psi^d(x) - \psi^T(x) + \psi^d(x) \\ &\leq 0 \end{aligned}$$

for any  $x \in \Omega$ , because  $\psi^T \geq 0$ . This means that  $\zeta(x, 0) \leq 0$  and  $\zeta^+(\cdot, 0) = \max(0, \zeta(\cdot, 0)) = 0$  in  $\Omega$ .

If we multiply the first equation in (3.21) by  $\zeta^+$  and use the integration by parts (2.6), we have for any  $t > 0$

$$\begin{aligned} \int_\Omega \zeta_t(t) \zeta^+(t) dx + \frac{C_{N,s}}{2} \mathcal{F}(\zeta(t), \zeta^+(t)) &= \int_\Omega v(t) \zeta(t) \zeta^+(t) dx \\ &- L \int_\Omega v(t) \zeta^+(t) dx, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\zeta^+(t)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \mathcal{F}(\zeta^+(t), \zeta^+(t)) &= \int_\Omega v(t) (\zeta^+(t))^2 dx \\ &- L \int_\Omega v(t) \zeta^+(t) dx. \end{aligned}$$

Hence, using the fact that  $v \in \mathcal{U}$ ,  $\zeta^+$  positive and  $L \leq 0$  we obtain that

$$\frac{1}{2} \frac{d}{dt} \|\zeta^+(t)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \mathcal{F}(\zeta^+(t), \zeta^+(t)) \leq 0.$$

Integrating this latter identity over  $(0, \tau)$  with  $\tau \in [0, T]$ , we have for any  $\tau \in [0, T]$ ,

$$\frac{1}{2} \|\zeta^+(\tau)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \int_0^\tau \mathcal{F}(\zeta^+(t), \zeta^+(t)) dt \leq 0$$

because  $\zeta^+(x, 0) = 0$ . This implies that

$$\sup_{\tau \in [0, T]} \|\zeta^+(\tau)\|_{L^2(\Omega)}^2 = 0$$

and

$$\int_0^T \mathcal{F}(\zeta^+(t), \zeta^+(t)) dt = \|\zeta^+\|_{L^2((0, T); \mathbb{H}_0^s(\Omega))}^2 = 0.$$

Consequently,  $\zeta^+ = 0$  almost everywhere in  $\mathcal{O}_1$  and hence  $\zeta^+ = 0$  almost everywhere in  $\mathbb{R}^N \times [0, T]$ . This means that  $L - \psi \leq 0$  almost everywhere in  $\mathbb{R}^N \times [0, T]$ . ■

**Remark 3.3** Note that theorem 3.3 holds because  $\psi^T \in L^\infty(\Omega)$  with  $\psi^T \geq 0$  a.e. in  $\Omega$ .

**Remark 3.4** Note that  $\psi^d$  being in  $\mathcal{C}(\bar{\Omega})$ , it follows from (3.19) that

$$\|\psi\|_{L^\infty(\mathbb{R}^N \times (0, T))} \leq \|\psi^T\|_{L^\infty(\Omega)} + \|\psi^d\|_{\mathcal{C}(\Omega)}. \quad (3.22)$$

**Corollary 3.3** Let  $\rho^0 \in \mathcal{C}(\bar{\Omega})$  and  $v \in \mathcal{U}$ . Then the weak solution of (1.1) satisfies

$$\min \left( 0, \inf_{x \in \Omega} \rho^0(x) \right) \leq \rho(x, t) \leq \max \left( 0, \sup_{x \in \Omega} \rho^0(x) \right) \quad \text{a.e. in } \mathbb{R}^N \times [0, T]. \quad (3.23)$$

Moreover,

$$\|\rho\|_{L^\infty(\mathbb{R}^N \times (0, T))} \leq \|\rho^0\|_{\mathcal{C}(\Omega)}. \quad (3.24)$$

**Proof.** We apply Theorem 3.3 and (3.22), with  $\psi^T = 0$  in  $\Omega$  and  $\rho^0 = -\psi^d$ . ■

The following results are useful for the study of the optimal control problem (1.1)-(1.3). The proofs follow similar arguments as in [11].

**Proposition 3.1** Let  $v \in \mathcal{U}$  and  $\rho^0 \in \mathcal{C}(\bar{\Omega})$ . Let also  $\mathcal{F}(\cdot, \cdot)$  be defined as in (2.9). Let  $\rho = \rho(v)$  be the solution of (1.1). Then the application  $v \mapsto \rho(v)$  is a continuous function from  $\mathcal{U}$  onto  $L^2((0, T); \mathbb{V})$ .

**Proof.** Let  $v_1, v_2 \in L^2(\omega_T)$ . Set  $z := e^{-rt}(\rho(v_1) - \rho(v_2))$ , with  $r > 0$  and  $\rho(v_1)$  and  $\rho(v_2)$  being solutions of (1.1) with  $v = v_1$  and  $v = v_2$  respectively, we have that  $z$  satisfies

$$\begin{cases} z_t + (-\Delta)^s z + rz &= (v_1 z + e^{-rt}(v_1 - v_2)\rho(v_2))\chi_\omega & \text{in } Q, \\ z &= 0 & \text{in } \Sigma, \\ z(0) &= 0 & \text{in } \Omega. \end{cases} \quad (3.25)$$

So, if we multiply the first equation of (3.25) by  $z$  and integrate over  $Q$ , then use the integration by parts (2.6), we obtain that

$$\begin{aligned} & \frac{1}{2} \|z(\cdot, T)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \|z\|_{L^2((0, T); \mathbb{V})}^2 + r \|z\|_{L^2(Q)}^2 \\ &= \int_{\omega_T} (v_1 z + e^{-rt}(v_1 - v_2)\rho(v_2)) z \, dx \, dt. \end{aligned} \quad (3.26)$$

Observing that  $\rho^0 \in \mathcal{C}(\bar{\Omega})$ , using Corollary 3.3, we have from (3.24) that

$$\|\rho\|_{L^\infty(Q)} \leq \|\rho^0\|_{\mathcal{C}(\Omega)}. \quad (3.27)$$



Therefore, using Cauchy's inequality, the fact that  $v_1 \in \mathcal{U}$  and (3.27), we deduce that

$$\begin{aligned} \frac{C_{N,s}}{2} \|z\|_{L^2((0,T);\mathbb{V})}^2 + r \|z\|_{L^2(Q)}^2 &\leq \frac{1}{2} \|v_1 - v_2\|_{L^2(\omega_T)}^2 \|\rho^0\|_{\mathcal{C}(\Omega)}^2 + \frac{1}{2} \|z\|_{L^2(Q)}^2 \\ &+ M \|z\|_{L^2(Q)}^2. \end{aligned}$$

By choosing  $r = \frac{1}{2} + M$ , this latter inequality lead us to

$$\|z\|_{L^2((0,T);\mathbb{V})}^2 \leq \frac{1}{C_{N,s}} \|v_1 - v_2\|_{L^2(\omega_T)}^2 \|\rho^0\|_{\mathcal{C}(\Omega)}^2.$$

As  $v_1 \rightarrow v_2$ , we deduce that  $z \rightarrow 0$  strongly in  $L^2((0,T);\mathbb{V})$ . Hence  $\rho(v_1) \rightarrow \rho(v_2)$  as  $v_1 \rightarrow v_2$  and, the application  $v \mapsto \rho(v)$  is a continuous map from  $L^2(\omega_T)$  onto  $L^2((0,T);\mathbb{V})$ . This completes the proof.  $\blacksquare$

**Proposition 3.2** *Let  $v, w \in \mathcal{U}$  and  $\rho^0 \in \mathcal{C}(\bar{\Omega})$ . Let also  $\mathcal{F}(\cdot, \cdot)$  be defined as in (2.9). Let also  $\rho = \rho(v)$  be the solution of (1.1) and  $\lambda > 0$ . Set  $z_\lambda := \frac{\rho(v + \lambda w) - \rho(v)}{\lambda}$ . Then  $(z_\lambda)$  converges strongly in  $L^2((0,T);\mathbb{V})$  as  $\lambda \rightarrow 0$  to  $z$ , solution to*

$$\begin{cases} z_t + (-\Delta)^s z &= (vz + w\rho(v))\chi_\omega & \text{in } Q, \\ z &= 0 & \text{in } \Sigma, \\ z(0) &= 0 & \text{in } \Omega. \end{cases} \quad (3.28)$$

**Proof.**  $z_\lambda$  is solution to the problem

$$\begin{cases} (z_\lambda)_t + (-\Delta)^s z_\lambda &= (vz_\lambda + w\rho(v + \lambda w))\chi_\omega & \text{in } Q, \\ z_\lambda &= 0 & \text{in } \Sigma, \\ z_\lambda(0) &= 0 & \text{in } \Omega. \end{cases}$$

Define  $p_\lambda := e^{-rt}(z_\lambda - z)$ , with  $r > 0$ . Then  $p_\lambda$  is a solution to

$$\begin{cases} (p_\lambda)_t + (-\Delta)^s p_\lambda + r p_\lambda &= [v p_\lambda + e^{-rt} w(\rho(v + \lambda w) - \rho(v))]\chi_\omega & \text{in } Q, \\ p_\lambda &= 0 & \text{in } \Sigma, \\ p_\lambda(0) &= 0 & \text{in } \Omega. \end{cases} \quad (3.29)$$

Now, if we multiply the first equation of (3.29) by  $p_\lambda$  and integrate by parts over  $Q$ , we obtain using the integration by parts (2.6) that,

$$\begin{aligned} \frac{1}{2} \|p_\lambda(\cdot, T)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \|p_\lambda\|_{L^2((0,T);\mathbb{V})}^2 + r \|p_\lambda\|_{L^2(Q)}^2 \\ = \int_{\omega_T} (v p_\lambda + e^{-rt} w(\rho(v + \lambda w) - \rho(v))) p_\lambda \, dx \, dt. \end{aligned}$$

Using Cauchy's inequality and (2.11), we deduce that

$$\begin{aligned} \frac{C_{N,s}}{2} \|p_\lambda\|_{L^2((0,T);\mathbb{V})}^2 + r \|p_\lambda\|_{L^2(Q)}^2 &\leq \frac{1}{2} M^2 \|\rho(v + \lambda w) - \rho(v)\|_{L^2(Q)}^2 + \frac{1}{2} \|p_\lambda\|_{L^2(Q)}^2 \\ &+ M \|p_\lambda\|_{L^2(Q)}^2 \\ &\leq \frac{C_0}{2} M^2 \|\rho(v + \lambda w) - \rho(v)\|_{L^2((0,T);\mathbb{V})}^2 \\ &+ \frac{1}{2} \|p_\lambda\|_{L^2(Q)}^2 + M \|p_\lambda\|_{L^2(Q)}^2. \end{aligned}$$

By choosing  $r = \frac{1}{2} + M$ , this latter inequality lead us to

$$\|p_\lambda\|_{L^2((0,T);\mathbb{V})}^2 \leq \frac{C}{C_{N,s}} M^2 \|\rho(v + \lambda w) - \rho(v)\|_{L^2((0,T);\mathbb{V})}^2.$$

By taking the limit as  $\lambda \rightarrow 0$  in this latter inequality and using the Proposition 3.1, we obtain that  $p_\lambda \rightarrow 0$  strongly in  $L^2((0,T);\mathbb{V})$ . Hence  $z_\lambda \rightarrow z$  strongly in  $L^2((0,T);\mathbb{V})$  as  $\lambda \rightarrow 0$ . This completes the proof. ■

**Lemma 3.1** *Let  $(\rho^n)_n$  be a sequence of  $L^2((0,T);\mathbb{V})$  such that  $\rho_t^n \in L^2(Q)$  and*

$$\rho^n \rightharpoonup \rho \text{ weakly in } L^2((0,T);\mathbb{V}) \text{ as } n \rightarrow \infty.$$

*Then*

$$\rho^n \rightarrow \rho \text{ strongly in } L^2(Q) \text{ as } n \rightarrow \infty.$$

**Proof.** If  $\rho^n \in L^2((0,T);\mathbb{V})$  then  $\rho^n \in L^2((0,T);\mathbb{H}_0^s(\Omega))$  and

$$\rho^n \rightharpoonup \rho \text{ weakly in } L^2((0,T);\mathbb{H}_0^s(\Omega)) \text{ as } n \rightarrow \infty.$$

Therefore, we have from (2.13) and Theorem 2.1 that

$$\rho^n \rightarrow \rho \text{ Strongly in } L^2(Q) \text{ as } n \rightarrow \infty.$$

■

## 4 Resolution of the optimization problem

### 4.1 Existence and characterization of an optimal control

In this section, we are interested in the following optimal control problem:

$$\inf_{v \in \mathcal{U}} J(v), \tag{4.1}$$

where

$$J(v) = \|\rho(T) - \rho^d\|_{L^2(\Omega)}^2 + \alpha \|v\|_{L^2(\omega_T)}^2, \tag{4.2}$$

with  $\alpha > 0$ ,  $\rho^d \in \mathcal{C}(\bar{\Omega})$  and  $\rho = \rho(v) \in L^2((0,T);\mathbb{V}) \cap \mathcal{C}([0,T];L^2(\Omega))$  is solution to (1.1).

**Theorem 4.1** *Let  $\alpha > 0$ ,  $v \in \mathcal{U}$  and  $\rho^0, \rho^d \in \mathcal{C}(\bar{\Omega})$ . Then there exists a solution  $u \in \mathcal{U}$  of the optimal control problem (4.1)-(4.2).*

**Proof.** Notice that  $J(v) \geq 0$  for all  $v \in \mathcal{U}$ . Let  $(v^n)_n \subset \mathcal{U}$  be a minimizing sequence such that

$$\lim_{n \rightarrow \infty} J(v^n) \rightarrow \min_{v \in \mathcal{U}} J(v).$$

Since the functional  $J \geq 0$ , such a minimizing sequence always exists. Let  $\rho^n = \rho(v^n)$  be the state associated to the control  $v^n$ , then there is a positive constant  $C$  independent of  $n$  such that

$$\|\rho^n(T)\|_{L^2(\Omega)} \leq C. \tag{4.3}$$

and since  $v^n \in \mathcal{U}$ ,

$$\|v^n\|_{L^2(\omega_T)} \leq M \quad (4.4)$$

Moreover,  $\rho^n$  being solution of (1.1), it follows from (3.16) that

$$\|\rho^n\|_{L^2((0,T);\mathbb{V})} \leq \left(e^{T\frac{(1+2M)}{2}}\right)^{1/2} \|\rho^0\|_{L^2(\Omega)}. \quad (4.5)$$

Consequently, using (2.12), we deduce that

$$\begin{aligned} \|v^n \rho^n\|_{L^2(\omega_T)} &\leq M \|\rho^n\|_{L^2(Q)} \\ &\leq C_0 M \|\rho^n\|_{L^2((0,T);\mathbb{V})} \\ &\leq C_0 M \left(e^{T\frac{(1+2M)}{2}}\right)^{1/2} \|\rho^0\|_{L^2(\Omega)}. \end{aligned} \quad (4.6)$$

From (4.3)-(4.6), we have that there exist  $\eta \in L^2(\Omega)$ ,  $u \in L^2(\omega_T)$ ,  $\beta \in L^2(\omega_T)$  and  $\rho \in L^2((0,T);\mathbb{V})$  such that (up to a subsequence), as  $n \rightarrow \infty$ , we have

$$v^n \rightharpoonup u \text{ weakly in } L^2(\omega_T), \quad (4.7)$$

$$\rho^n(T) \rightharpoonup \eta \text{ weakly in } L^2(\Omega), \quad (4.8)$$

$$\rho^n \rightharpoonup \rho \text{ weakly in } L^2(0,T;\mathbb{V}), \quad (4.9)$$

and

$$v^n \rho^n \rightharpoonup \beta \text{ weakly in } L^2(\omega_T). \quad (4.10)$$

Since  $\mathcal{U}$  is a closed convex subset of  $L^2(\omega_T)$ , we have that  $\mathcal{U}$  is weakly closed and so

$$u \in \mathcal{U}. \quad (4.11)$$

Observing that  $\rho^n \in L^2((0,T);\mathbb{V})$  satisfies (1.1) and that (4.6) holds true, we deduce that  $\rho_t^n = v^n \rho^n - (-\Delta)^s \rho^n \in L^2(Q)$ . It then follows from (4.9) and Lemma 3.1 that

$$\rho^n \rightarrow \rho \text{ strongly in } L^2(Q). \quad (4.12)$$

Therefore, taking (4.7) and (4.12) into account and using the weak-strong convergence, we obtain

$$v^n \rho^n \rightharpoonup u \rho \text{ weakly in } L^1(\omega_T), \quad (4.13)$$

which in view of the continuous embedding of  $L^2(\omega_T)$  into  $L^1(\omega_T)$ , (4.10) and the uniqueness of the weak limit imply that  $\beta = u \rho$ . Hence,

$$v^n \rho^n \rightharpoonup u \rho \text{ weakly in } L^2(\omega_T). \quad (4.14)$$

Now let us show that  $\rho = \rho(u)$ . We recall that  $\rho^n$  is solution to

$$\begin{cases} (\rho^n)_t + (-\Delta)^s \rho^n &= v^n \rho^n \chi_\omega & \text{in } Q, \\ \rho^n &= 0 & \text{in } \Sigma, \\ \rho^n(\cdot, 0) &= \rho^0 & \text{in } \Omega. \end{cases} \quad (4.15)$$

If we multiply the first equation in (4.15) by  $\phi \in H(Q)$  and integrate by parts over  $Q$ , we obtain using the integration by parts (2.6),

$$\begin{aligned} & - \int_Q \phi_t \rho^n dx dt + \frac{C_{N,s}}{2} \int_0^T \mathcal{F}(\rho^n, \phi) dt = \\ & \int_{\omega_T} v^n \rho^n \phi dx dt + \int_{\Omega} \rho^0 \phi(0) dx, \end{aligned} \quad (4.16)$$

where

$$H(Q) = \left\{ z \in L^2((0, T); \mathbb{V}) : z_t \in L^2(Q) \text{ and } z(\cdot, T) = 0 \text{ in } \Omega \right\}.$$

Passing to the limit as  $n \rightarrow \infty$  in (4.16), while using (4.9) and (4.14), we obtain that

$$\begin{aligned} & - \int_Q \phi_t \rho dx dt + \frac{C_{N,s}}{2} \int_0^T \mathcal{F}(\rho, \phi) dt = \\ & \int_{\omega_T} u \rho \phi dx dt + \int_{\Omega} \rho^0 \phi(0) dx, \end{aligned} \quad (4.17)$$

for any  $\phi \in H(Q)$ . Hence, we can deduce that  $\rho = \rho(u)$  is a weak solution to (3.2) with  $v = u$  and  $f = 0$  in the sense of Definition 3.3. In other words,  $\rho = \rho(u)$  is a weak solution to

$$\begin{cases} \rho_t + (-\Delta)^s \rho & = u \rho \chi_{\omega} & \text{in } Q, \\ \rho & = 0 & \text{on } \Sigma, \\ \rho(0) & = \rho^0 & \text{in } \Omega. \end{cases} \quad (4.18)$$

Now let  $\phi \in L^2((0, T); \mathbb{V}) \cap \mathcal{C}([0, T]; L^2(\Omega))$ . If we multiply the first equation in (4.15) by  $\phi$  and we integrate over  $Q$ , we obtain using the integration by parts (2.6),

$$\begin{aligned} \int_{\Omega} \rho^n(T) \phi(T) dx - \int_Q \phi_t \rho^n dx dt + \frac{C_{N,s}}{2} \int_0^T \mathcal{F}(\rho^n, \phi) dt & = \int_{\omega_T} v^n \rho^n \phi dx dt \\ & + \int_{\Omega} \rho^0 \phi(0) dx. \end{aligned} \quad (4.19)$$

Now, passing to the limit in (4.19) as  $n \rightarrow \infty$ , while using (4.8), (4.9) and (4.14), we obtain

$$\begin{aligned} \int_{\Omega} \eta \phi(T) dx - \int_Q \phi_t \rho dx dt + \frac{C_{N,s}}{2} \int_0^T \mathcal{F}(\rho, \phi) dt & = \\ \int_{\omega_T} u \rho \phi dx dt + \int_{\Omega} \rho^0 \phi(0) dx, \end{aligned}$$

which by using again the integration by parts (2.6) gives,

$$\begin{aligned} \int_{\Omega} \rho^0 \phi(0) dx + \int_{\Omega} \phi(T) (\eta - \rho(T)) dx + \int_Q \phi (\rho_t + (-\Delta)^s \rho) dx dt & = \\ \int_{\omega_T} u \rho \phi dx dt + \int_{\Omega} \rho^0 \phi(0) dx, \end{aligned}$$

$$\forall \phi \in L^2((0, T); \mathbb{V}) \cap \mathcal{C}([0, T]; L^2(\Omega)).$$

Since  $\rho = \rho(u)$  is solution of (4.18), we get from this latter identity that

$$\int_{\Omega} \phi(T) (\eta - \rho(T)) dx = 0, \quad \forall \phi \in L^2((0, T); \mathbb{V}) \cap \mathcal{C}([0, T]; L^2(\Omega)). \quad (4.20)$$

Hence, we deduce that

$$\eta = \rho(T) \quad \text{on } \Omega. \quad (4.21)$$

Combining (4.8) and (4.21), we obtain that

$$\rho^n(T) \rightharpoonup \rho(T) \text{ weakly in } L^2(\Omega). \quad (4.22)$$

Moreover, using (4.22), (4.7), (4.11) and the lower semi-continuity of the mapping  $v \mapsto J(v)$  on  $\mathcal{U}$ , it follows that

$$J(u) \leq \liminf_{n \rightarrow \infty} J(v^n) = \inf_{v \in \mathcal{U}} J(v).$$

This implies that  $u$  is a minimizer of  $J(\cdot)$ . This completes the proof.  $\blacksquare$

Next, we characterize the optimality control

**Theorem 4.2** *Let  $\alpha > 0$ ,  $v \in \mathcal{U}$  and  $\rho^0, \rho^d \in \mathcal{C}(\bar{\Omega})$ . Let  $u$  be an optimal control for the minimization problem (4.1). Then there exists  $q \in L^2((0, T); \mathbb{V}) \cap \mathcal{C}([0, T]; L^2(\Omega))$  such that the triple  $(\rho, u, q)$  satisfies*

$$\begin{cases} \rho_t + (-\Delta)^s \rho &= u \rho \chi_\omega & \text{in } Q, \\ \rho &= 0 & \text{on } \Sigma, \\ \rho(0) &= \rho^0 & \text{in } \Omega, \end{cases} \quad (4.23)$$

$$\begin{cases} -q_t + (-\Delta)^s q &= u q \chi_\omega & \text{in } Q, \\ q &= 0 & \text{on } \Sigma, \\ q(T) &= \rho(T) - \rho^d & \text{in } \Omega, \end{cases} \quad (4.24)$$

and

$$u = - \min \left( \max \left( 0, \frac{q}{\alpha} \rho(u) \right), M \right) \quad \text{in } \omega_T. \quad (4.25)$$

**Proof.** We have already shown (4.23) (equivalently (4.18)). To complete the proof of the Theorem 4.2, we write the Euler-Lagrange first order optimality condition that characterizes the optimal control  $u$ . That is,

$$\lim_{\lambda \rightarrow 0} \frac{J(u + \lambda(v - u)) - J(u)}{\lambda} \geq 0, \quad \forall v \in \mathcal{U}. \quad (4.26)$$

After some straightforward calculations and using Proposition 3.2, (4.26) gives

$$\int_{\Omega} z(T)(\rho(T, u) - \rho^d) dx + \alpha \int_{\omega_T} u(v - u) dx dt \geq 0, \quad \forall v \in \mathcal{U}, \quad (4.27)$$

where  $z \in \mathbb{K}$  is the unique solution of

$$\begin{cases} z_t + (-\Delta)^s z &= (uz + (v - u)\rho(u))\chi_\omega & \text{in } Q, \\ z &= 0 & \text{on } \Sigma, \\ z(0) &= 0 & \text{in } \Omega. \end{cases} \quad (4.28)$$

Note that the existence of  $z \in L^2((0, T); \mathbb{V}) \cap \mathcal{C}([0, T]; L^2(\Omega)) \cap H^1((0, T); L^2(\Omega)) \cap \mathcal{C}([0, T]; \mathbb{H}_0^s(\Omega))$  is given by Corollary 3.1, by taking  $f = (v - u)\rho(u)\chi_\omega \in L^2(Q)$ . To interpret (4.27), we use the adjoint

state given by (4.24). Make the change of variable  $t \mapsto T - t$  in (4.24), we have that  $\varphi(x, t) = q(x, T - t)$  satisfies

$$\begin{cases} \varphi_t + (-\Delta)^s \varphi &= \tilde{u} \varphi \chi_\omega & \text{in } Q, \\ \varphi &= 0 & \text{in } \Sigma, \\ \varphi(0) &= \rho(T) - \rho^d & \text{in } \Omega. \end{cases} \quad (4.29)$$

and since  $\rho(T) - \rho^d \in L^2(\Omega)$ , applying Theorem 3.1 with  $f = 0$ , we deduce that, there exists a unique adjoint state  $q \in L^2((0, T); \mathbb{V}) \cap \mathcal{C}([0, T]; L^2(\Omega))$  solution to (4.24) in the sense of Definition 3.1.

So, if we multiply the first equation in (4.28) by  $q$  solution of (4.24), and integrate by parts over  $Q$ , using the integration by parts (2.6), we get

$$\int_{\Omega} z(T)(\rho(T, u) - \rho^d) dx + \int_{\omega_T} (v - u)\rho(u)q ds d\sigma dt = 0, \quad (4.30)$$

which combining with (4.27) gives

$$\int_{\omega_T} (\alpha u - \rho(u)q)(v - u) dx dt \geq 0, \quad \forall v \in \mathcal{U}. \quad (4.31)$$

From which, we deduce (4.25). This completes the proof. ■

## 4.2 Uniqueness of the optimal control

In this section, we are about to prove that the optimal control given in Theorem 4.1 is unique. But before going further we need the following maximum principle result for the adjoint state solution to (4.24).

**Lemma 4.1** *Let  $\rho^0 \in \mathcal{C}(\bar{\Omega})$  be such that  $\rho^0 \geq 0$  a.e. in  $\Omega$ ,  $\rho(\cdot, T)$  be the solution of (1.1) at time  $T$ ,  $\rho^d \in \mathcal{C}(\bar{\Omega})$  and  $v \in \mathcal{U}$ . Then the weak solution of (4.24) satisfies*

$$\begin{aligned} & \min \left( 0, -\|\rho(\cdot, T)\|_{L^\infty(\Omega)} - \sup_{x \in \Omega} \rho^d(x) \right) \\ & \leq q(x, t) \leq \max \left( 0, \|\rho(\cdot, T)\|_{L^\infty(\Omega)} - \inf_{x \in \Omega} \rho^d(x) \right) \text{ a.e. in } \mathbb{R}^N \times [0, T]. \end{aligned} \quad (4.32)$$

Moreover,

$$\|q\|_{L^\infty(\mathbb{R}^N \times (0, T))} \leq \|\rho(\cdot, T)\|_{L^\infty(\Omega)} + \|\rho^d\|_{\mathcal{C}(\Omega)}. \quad (4.33)$$

**Proof.** Make the change of variable  $t \rightarrow T - t$  in (4.24), we have that  $\varphi(x, t) = q(x, T - t)$  satisfies

$$\begin{cases} \varphi + (-\Delta)^s \varphi &= \tilde{u} \varphi \chi_\omega & \text{in } Q, \\ \varphi &= 0 & \text{on } \Sigma, \\ \varphi(T) &= \rho(T) - \rho^d & \text{in } \Omega. \end{cases} \quad (4.34)$$

Since  $\rho^0 \in \mathcal{C}(\bar{\Omega})$  be such that  $\rho^0 \geq 0$  a.e. in  $\Omega$ , we have from Theorem 3.2 that  $\rho(\cdot, T) \geq 0$  a.e. in  $\Omega$ . Therefore, applying Theorem 3.3 and Remark 3.4, with  $\psi^T = \rho(\cdot, T)$  in  $\Omega$  and  $\rho^d = \psi^d$ . we deduce that  $\varphi$  satisfies

$$\begin{aligned} & \min \left( 0, -\|\rho(\cdot, T)\|_{L^\infty(\Omega)} - \sup_{x \in \Omega} \rho^d(x) \right) \\ & \leq \varphi(x, t) \leq \max \left( 0, \|\rho(\cdot, T)\|_{L^\infty(\Omega)} - \inf_{x \in \Omega} \rho^d(x) \right) \text{ a.e. in } \mathbb{R}^N \times [0, T]. \end{aligned}$$

and

$$\|\varphi\|_{L^\infty(\mathbb{R}^N \times (0,T))} \leq \|\rho(\cdot, T)\|_{L^\infty(\Omega)} + \|\rho^d\|_{\mathcal{C}(\Omega)}.$$

Since  $\varphi(x, t) = q(x, T - t)$ , we have that  $q$  satisfies (4.32) and (4.33). ■

**Theorem 4.3** *Let  $\rho^0 \in \mathcal{C}(\bar{\Omega})$  be such that  $\rho^0 \geq 0$  a.e. in  $\Omega$ ,  $\rho(\cdot, T)$  be the solution of (1.1) at time  $T$ ,  $\rho^d \in \mathcal{C}(\bar{\Omega})$  and  $\alpha > \frac{\sqrt{72}}{C_{N,s}} \left( \|\rho^0\|_{\mathcal{C}(\Omega)}^2 + \|\rho^d\|_{\mathcal{C}(\Omega)}^2 \right)$ . Then the control  $u \in \mathcal{U}$  solution of the optimal control problem (4.1) is unique.*

**Proof.** Assume that there exists two controls  $u$  and  $\bar{u}$  associated respectively to the states  $\rho$  and  $\bar{\rho}$  solutions to (4.23). We denote respectively by  $q$  and  $\bar{q}$  the corresponding adjoint states solution to (4.24). Then,  $\rho - \bar{\rho}$  and  $q - \bar{q}$  are respectively solutions to

$$\begin{cases} (\rho - \bar{\rho})_t + (-\Delta)^s(\rho - \bar{\rho}) &= (u - \bar{u})\rho\chi_\omega + \bar{u}(\rho - \bar{\rho})\chi_\omega & \text{in } Q, \\ (\rho - \bar{\rho}) &= 0 & \text{on } \Sigma, \\ (\rho - \bar{\rho})(0) &= 0 & \text{in } \Omega, \end{cases} \quad (4.35)$$

and

$$\begin{cases} -(q - \bar{q})_t + (-\Delta)^s(q - \bar{q}) &= (u - \bar{u})q\chi_\omega + \bar{u}(q - \bar{q})\chi_\omega & \text{in } Q, \\ (q - \bar{q}) &= 0 & \text{on } \Sigma, \\ (q - \bar{q})(T) &= \rho(T) - \bar{\rho}(T) & \text{in } \Omega. \end{cases} \quad (4.36)$$

If we multiply the first equation of (4.35) and (4.36), respectively by  $\rho - \bar{\rho}$  and  $q - \bar{q}$ , integrate by parts over  $Q$ , using the integration parts (2.6) and the fact that  $\bar{u} \in \mathcal{U}$ , we obtain

$$\begin{aligned} & \frac{1}{2} \|\rho(T) - \bar{\rho}(T)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \int_0^T \mathcal{F}(\rho - \bar{\rho}, \rho - \bar{\rho}) dt \\ &= \int_{\omega_T} (u - \bar{u})\rho(\rho - \bar{\rho}) dx dt + \int_{\omega_T} \bar{u}(\rho - \bar{\rho})^2 dx dt \\ &\leq \int_{\omega_T} (u - \bar{u})\rho(\rho - \bar{\rho}) dx dt \end{aligned}$$

and

$$\begin{aligned} & -\frac{1}{2} \|\rho(T) - \bar{\rho}(T)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|q(0) - \bar{q}(0)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \int_0^T \mathcal{F}(q - \bar{q}, q - \bar{q}) dt \\ &= \int_{\omega_T} (u - \bar{u})q(q - \bar{q}) dx dt + \int_{\omega_T} \bar{u}(q - \bar{q})^2 dx dt \\ &\leq \int_{\omega_T} (u - \bar{u})q(q - \bar{q}) dx dt. \end{aligned}$$

Therefore, using the Young's inequality for some  $\delta > 0$ ,

$$\begin{aligned} & \frac{1}{2} \|\rho(T) - \bar{\rho}(T)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2} \|\rho - \bar{\rho}\|_{L^2((0,T);V)}^2 \\ &\leq \frac{\delta}{2} \|\rho - \bar{\rho}\|_{L^2(Q)}^2 + \frac{1}{2\delta} \|\rho\|_{L^\infty(Q)}^2 \|u - \bar{u}\|_{L^2(\omega_T)}^2 \\ &\leq \frac{\delta}{2} \|\rho - \bar{\rho}\|_{L^2((0,T);V)}^2 + \frac{1}{2\delta} \|\rho\|_{L^\infty(\mathbb{R}^N \times [0,T])}^2 \|u - \bar{u}\|_{L^2(\omega_T)}^2, \end{aligned}$$

and

$$\begin{aligned}
& -\frac{1}{2}\|\rho(T) - \bar{\rho}(T)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|q(0) - \bar{q}(0)\|_{L^2(\Omega)}^2 + \frac{C_{N,s}}{2}\|q - \bar{q}\|_{L^2((0,T);\mathbb{V})}^2 \\
& \leq \frac{\delta}{2}\|q - \bar{q}\|_{L^2(Q)}^2 + \frac{1}{2\delta}\|q\|_{L^\infty(Q)}^2\|u - \bar{u}\|_{L^2(\omega_T)}^2 \\
& \leq \frac{\delta}{2}\|q - \bar{q}\|_{L^2((0,T);\mathbb{V})}^2 + \frac{1}{2\delta}\|q\|_{L^\infty(\mathbb{R}^N \times [0,T])}^2\|u - \bar{u}\|_{L^2(\omega_T)}^2.
\end{aligned}$$

Choosing  $\delta = \frac{C_{N,s}}{2}$ , and combining these two latter inequalities, we deduce that

$$\|\rho - \bar{\rho}\|_{L^2((0,T);\mathbb{V})}^2 + \|q - \bar{q}\|_{L^2((0,T);\mathbb{V})}^2 \leq \frac{4}{(C_{N,s})^2}\|u - \bar{u}\|_{L^2(\omega_T)}^2 \times \quad (4.37)$$

$$\left[ \|\rho\|_{L^\infty(\mathbb{R}^N \times [0,T])}^2 + \|q\|_{L^\infty(\mathbb{R}^N \times [0,T])}^2 \right]. \quad (4.38)$$

Using (3.24) and (4.33), it follows that

$$\|\rho - \bar{\rho}\|_{L^2((0,T);\mathbb{V})}^2 + \|q - \bar{q}\|_{L^2((0,T);\mathbb{V})}^2 \leq \frac{12}{(C_{N,s})^2} \left( \|\rho^0\|_{\mathbb{C}(\Omega)}^2 + \|\rho^d\|_{\mathbb{C}(\Omega)}^2 \right) \|u - \bar{u}\|_{L^2(\omega_T)}^2. \quad (4.39)$$

Moreover we write

$$\begin{aligned}
|u - \bar{u}|^2 & \leq \frac{1}{\alpha^2} |(q - \bar{q})\rho + \bar{q}(\rho - \bar{\rho})|^2 \\
& \leq \frac{1}{\alpha^2} \left[ |q - \bar{q}|^2 |\rho|^2 + |\bar{q}|^2 |\rho - \bar{\rho}|^2 \right].
\end{aligned}$$

Hence after straightforward calculations and thanks to (3.24) and (4.33), it follows that

$$\begin{aligned}
\|u - \bar{u}\|_{L^2(\omega_T)}^2 & \leq \frac{2}{\alpha^2} \left[ \|\rho\|_{L^\infty(\mathbb{R}^N \times [0,T])}^2 \|q - \bar{q}\|_{L^2((0,T);\mathbb{V})}^2 + \|q\|_{L^\infty(\mathbb{R}^N \times [0,T])}^2 \|\rho - \bar{\rho}\|_{L^2((0,T);\mathbb{V})}^2 \right] \\
& \leq \frac{6}{\alpha^2} \left( \|\rho^0\|_{\mathbb{C}(\Omega)}^2 + \|\rho^d\|_{\mathbb{C}(\Omega)}^2 \right) \left[ \|q - \bar{q}\|_{L^2((0,T);\mathbb{V})}^2 + \|\rho - \bar{\rho}\|_{L^2((0,T);\mathbb{V})}^2 \right]. \quad (4.40)
\end{aligned}$$

Hence combining (4.39) and (4.40), we deduce that

$$\|u - \bar{u}\|_{L^2(\omega_T)}^2 \leq \frac{72}{\alpha^2 (C_{N,s})^2} \left( \|\rho^0\|_{\mathbb{C}(\Omega)}^2 + \|\rho^d\|_{\mathbb{C}(\Omega)}^2 \right)^2 \|u - \bar{u}\|_{L^2(\omega_T)}^2. \quad (4.41)$$

Choosing  $\alpha$  in (4.41) such that  $\alpha^2 > \frac{72}{(C_{N,s})^2} \left( \|\rho^0\|_{\mathbb{C}(\Omega)}^2 + \|\rho^d\|_{\mathbb{C}(\Omega)}^2 \right)^2$ , we obtain that

$$\|u - \bar{u}\|_{L^2(\omega_T)}^2 \leq 0.$$

This implies that  $u = \bar{u}$ . Therefore the optimal control is unique. ■

## 5 Conclusion

We used a distributed optimal control on a fractional diffusive equation. In contrast to some works on the topic using a control in the whole domain, our control acts on a part of the domain. We established some weak maximum principle results for a class of fractional diffusive equations. We prove the existence and uniqueness of solutions to a quadratic boundary optimal control problem and provide a characterization of the optimal control.



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