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# Gradient Vector Fields of Discrete Morse Functions are Minimum Spanning Forests

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**Abstract.** In this paper, we prove that discrete Morse functions are equivalent to simplicial stacks under reasonable constraints. We also show that, as in Discrete Morse Theory, we can see the GVF of a simplicial stack (seen as a discrete Morse function) as the only relevant information we should consider. Last, but not least, we prove that the Minimum Spanning Forest on the dual graph of a simplicial stack (or a discrete Morse function) is equal to the GVF of the initial function. In other words, the GVF of a discrete Morse function is related to a classic combinatorial minimization problem.

This paper is the sequel of a sequence of papers showing that strong relations exist between different domains: Topology, Discrete Morse Theory, Topological Data Analysis and Mathematical Morphology.

**Keywords:** Topological Data Analysis, Mathematical Morphology, Discrete Morse Theory, Simplicial Stacks, Minimum Spanning Forest

## 1 Introduction

We present here several important results relating Mathematical Morphology [13] (MM) to Discrete Morse Theory [9] (DMT). This strengthens previous works highlighting links between MM and topology. In [4,5], it is demonstrated that watersheds are included in skeletons on pseudomanifolds of arbitrary dimension. Recently (see [1,2]), MM and Topological Data Analysis [16,12] (TDA) have been shown to be strongly related: the *dynamics* [10], used in MM to compute markers for watershed image-segmentation, is equivalent to the *persistence*, a fundamental tool from Persistent Homology [8].

In this paper, the first main result links the spaces used in MM and in TDA: the main mathematical spaces used in DMT, *discrete Morse functions* [15] (DMF), are equivalent (under some light constraints) to spaces well-known in MM and called *simplicial stacks* [6,4,5]. Simplicial stacks are a class of valued simplicial complex whose upper threshold sets are also complexes. Indeed, in a DMF, the values locally increase when we increase the dimension of the face we are observing; in a simplicial stack, it is the opposite. Without surprise, we can then observe that, under some constraints, any DMF is the inverse of a simplicial stack, and conversely.

In TDA, it is usual to consider that the main information conveyed by a DMF is its gradient vector field (GVF), naturally obtained by pairing neighbor faces with same values. Two DMFs with the same GVF are then considered to be equivalent. Using the very same principle on simplicial stacks, we can go further, and consider that a GVF encodes not only a class of DMFs but also the corresponding class of simplicial stacks.

The relation between TDA and MM in the context of DMFs and stacks is not limited to the previous observations. In [6], the authors proved that a watershed-cut is a Maximum Spanning Forests (MSF) cut in the dual graph of a simplicial stack. We prove here that such MSF is the GVF of the simplicial stack (seen as a DMF), the converse being also true. Relations between watersheds and Morse theory have long been informally known [7], but this is the first time that a rigorous theoretical link is exhibited in the discrete setting. Furthermore, as far as we know, this is the first time that a concept from Discrete Morse Theory is proved to be a (classic) combinatorial optimization problem.

The plan of this paper is the following. Section 2 recalls the mathematical background necessary to our proofs. Section 3 shows the equivalence between DMF's and simplicial stacks. Section 4 shows that MSFs and GVFs are equivalent. Section 5 concludes the paper.

## 2 Mathematical background

### 2.1 Simplicial complexes, graphs and pseudomanifolds

We call (*abstract*) *simplex* any finite nonempty set of arbitrary elements. The *dimension* of a simplex  $x$ , denoted by  $\dim(x)$ , is the number of its elements minus one. In the following, a simplex of dimension  $d$  will also be called a *d-simplex*. If  $x$  is a simplex, we set  $\text{Clo}(x) = \{y | y \subseteq x, y \neq \emptyset\}$ . A finite set  $X$  of simplices is a *cell* if there exists  $x \in X$  such that  $X = \text{Clo}(x)$ .

If  $X$  is a finite set of simplices, we write  $\text{Clo}(X) = \{\text{Clo}(x) | x \in X\}$ , the set  $\text{Clo}(X)$  is called the (*simplicial*) *closure* of  $X$ . A finite set  $X$  of simplices is a (*simplicial*) *complex* if  $X = \text{Clo}(X)$ . Let  $X$  be a complex. Any element in  $X$  is a face of  $X$  and we call *d-face* of  $X$  any face of  $X$  whose dimension is  $d$ . Any *d-face* of  $X$  that is not included in any  $(d+1)$ -face of  $X$  is called a (*d-*)*facet* of  $X$  or a *maximal face* of  $X$ . The dimension of  $X$ , written  $\dim(X)$ , is the largest dimension of its faces:  $\dim(X) = \max\{\dim(x) | x \in X\}$ . If  $d$  is the dimension of  $X$ , we say that  $X$  is *pure* whenever the dimension of all its facets equals  $d$ .

A *graph*  $G$  is a pure 1-dimensional simplicial complex. In this paper, we denote the vertices (the 0-dimensional elements) of a graph  $G$  by  $V(G)$ , and the edges (the 1-dimensional elements) by  $E(G)$ .

Let  $X$  be a set of simplices, and let  $d \in \mathbb{N}$ . Let  $\pi = \langle x_0, \dots, x_l \rangle$  be an ordered sequence of  $d$ -simplices in  $X$ . The sequence  $\pi$  is a *d-path* from  $x_0$  to  $x_l$  in  $X$  if  $x_{i-1} \cap x_i$  is a  $(d-1)$ -simplex in  $X$ , for any  $i \in \{1, \dots, l\}$ . Two  $d$ -simplices  $x$  and  $y$  in  $X$  are said to be *d-linked* for  $X$  if there exists a  $d$ -path from  $x$  to  $y$  in  $X$ . We say that the set  $X$  is *d-connected* if any two  $d$ -simplices in  $X$  are  $d$ -linked for  $X$ .

Let  $X$  be a set of simplices, and let  $\pi = \langle x_0, \dots, x_l \rangle$  be  $d$ -path in  $X$ . The  $d$ -path  $\pi$  is said *simple* if for any two distinct  $i$  and  $j$  in  $\{0, \dots, l\}$ ,  $x_i \neq x_j$ . It can be easily seen that  $X$  is  $d$ -connected if and only if, for any two  $d$ -simplices  $x$  and  $y$  of  $X$ , there exists a simple  $d$ -path from  $x$  to  $y$  in  $X$ .

A complex  $X$  of dimension  $d$  is said to be a  $d$ -pseudomanifold if (1)  $X$  is pure, (2) any  $(d-1)$ -face of  $X$  is included in exactly two  $d$ -faces of  $X$ , and (3)  $X$  is  $d$ -connected. Let  $\mathbb{M}$  be a  $d$ -pseudomanifold. Let  $x \in \mathbb{M}$ , the star of  $X$  (in  $\mathbb{M}$ ), denoted by  $\text{St}(x)$ , is the set of all simplices of  $\mathbb{M}$  that include  $x$ , i.e.,  $\text{St}(x) = \{y \in \mathbb{M} \mid x \subseteq y\}$ . If  $A$  is a subset of  $\mathbb{M}$ , the set  $\text{St}(A) = \cup_{x \in A} \text{St}(x)$  is called the *star* of  $A$  (in  $\mathbb{M}$ ). A set  $A$  of simplices of  $\mathbb{M}$  is a star (in  $\mathbb{M}$ ) if  $A = \text{star}(A)$ .

Let  $d \geq 1$  be an integer. Let  $\mathbb{M}$  be a  $d$ -pseudo-manifold,  $a$  a  $d$ -face of  $\mathbb{M}$ , and  $b$  a  $(d-1)$ -face of  $\mathbb{M}$  with  $b \prec a$ . We denote by  $\text{Opp}_a(b)$  the unique face  $h \in \mathbb{M}$  of dimension  $d$  which satisfies  $h \cap a = b$  (and covers  $b$ ). We call it the *opposite* of  $a$  relatively to  $b$ .

## 2.2 Simplicial stacks

Let  $\mathbb{M}$  be a  $n$ -pseudomanifold. Let  $F$  be a mapping  $\mathbb{M} \rightarrow \mathbb{Z}$ . For any face  $h$  of  $\mathbb{M}$ , the value  $F(h)$  is called the *altitude* of  $F$  at  $h$ . For  $k \in \mathbb{Z}$ , the  $k$ -section of  $F$ , denoted by  $[F \geq k]$  is equal to  $\{h \in \mathbb{M} ; F(h) \geq k\}$ . We say that a subset  $A$  of  $\mathbb{M}$  is a *minimum* of  $F$  at altitude  $k \in \mathbb{Z}$  when  $A$  is a connected component of  $[F \leq k] := \{h \in \mathbb{M} ; F(h) \leq k\}$  and  $A \cap [F \leq k-1] = \emptyset$ . In the following, we denote by  $M_-(F)$  the union of all minima of  $F$ . A *simplicial stack*  $F$  on  $\mathbb{M}$  is a map from  $\mathbb{M}$  to  $\mathbb{Z}$  which satisfies that any of its  $k$ -section is a (possibly empty) simplicial complex. In other words, a map  $F$  is a simplicial stack if, for any two faces  $\sigma$  and  $\tau$  of  $\mathbb{M}$  such that  $\sigma \subseteq \tau$ ,  $F(\sigma) \geq F(\tau)$ .

Let  $\sigma$  be any face of  $\mathbb{M}$ . When  $\sigma$  is a free face for  $[F \geq F(\sigma)]$ , we say that  $\sigma$  is a *free face* for  $F$ . If  $\sigma$  is a free face for  $F$ , there exists a unique face  $\tau$  in  $[F \geq F(\sigma)]$  such that  $(\sigma, \tau)$  is a free pair for  $[F \geq F(\sigma)]$ , and we say that  $(\sigma, \tau)$  is a *free pair* for  $F$ . Let  $(\sigma, \tau)$  be a free pair for  $F$ , then it is also a free pair for  $[F \geq F(\sigma)]$ . Thus,  $\tau$  is a face of  $[F \geq F(\sigma)]$ , and we have  $\sigma \subseteq \tau$ . Therefore, we have  $F(\tau) \geq F(\sigma)$  and  $F(\tau) \leq F(\sigma)$  (since  $F$  is a stack), which imply that  $F(\tau) = F(\sigma)$ . Let  $\mathbb{M}_{\text{sub}} \subseteq \mathbb{M}$ , the *indicator function* of  $\mathbb{M}_{\text{sub}}$ , denoted by  $1_{\mathbb{M}_{\text{sub}}} : \mathbb{M} \rightarrow \{0, 1\}$ , is the mapping such that  $1_{\mathbb{M}_{\text{sub}}}(\sigma)$  is equal to 1 when  $\sigma$  belongs to  $\mathbb{M}_{\text{sub}}$  and is equal to 0 when  $\sigma$  belongs to  $\mathbb{M} \setminus \mathbb{M}_{\text{sub}}$ . The *lowering* of  $F$  at  $\mathbb{M}_{\text{sub}}$  is the map  $F - 1_{\mathbb{M}_{\text{sub}}}$  from  $\mathbb{M}$  into  $\mathbb{Z}$ . Let  $(\sigma^{(d-1)}, \tau^{(d)})$  be a free pair for  $F$ . The map  $F - 1_{\{\sigma, \tau\}}$  is called an *elementary  $d$ -collapse* of  $F$ . Thus, this elementary  $d$ -collapse is obtained by removing 1 to the values of  $F$  at  $\sigma$  and  $\tau$ . Note that the obtained mapping is still a simplicial stack. If a simplicial stack  $F_2$  is the result of a series of elementary  $d$ -collapses on  $F$ , then we say that  $F_2$  is a  *$d$ -collapse* of  $F$ . If, furthermore, there is no free pair  $(\sigma^{(d-1)}, \tau^{(d)})$  for  $F_2$ , then  $F_2$  is an *ultimate  $d$ -collapse* of  $F$ .

**Proposition 1** (Sets and ultimate collapses [4]). *Let  $X$  be a subcomplex of the pseudomanifold  $\mathbb{M}$ . If the dimension of  $X$  is equal to  $n \geq 0$ , then necessarily*

there exists a free  $n$ -pair for  $X$ . In other words, the ultimate  $n$ -collapse of  $X$  is of dimension lower than the one of  $\mathbb{M}$ .

Following Prop. 1, we say that an ultimate  $n$ -collapse is *thin*. The *divide* of a simplicial stack  $F$  is the set of all faces of  $\mathbb{M}$  which do not belong to any minimum of  $F$ . Note that since the union of all minima of  $F$ ,  $M_-(F)$ , is a star (and then open), the divide is a simplicial complex.

### 2.3 Discrete Morse functions

We recall that a function  $F : A \rightarrow B$  is said to be  $2 - 1$  when, for every  $b \in B$ , there exist at most two values  $a_1, a_2 \in A$  such that  $f(a_1) = f(a_2) = b$ . Let  $\mathbb{K}$  be a simplicial complex. A function  $f : \mathbb{K} \rightarrow \mathbb{Z}$  is called *weakly increasing* if  $F(\sigma) \leq F(\tau)$  whenever the two faces  $\sigma, \tau$  of  $\mathbb{K}$  satisfy  $\sigma \subseteq \tau$ . A *basic discrete Morse function*  $F : \mathbb{K} \rightarrow \mathbb{Z}$  is a weakly discrete Morse function which is at most  $2 - 1$  and satisfies the property that if  $f(\sigma) = f(\tau)$ , then  $\sigma \subseteq \tau$  or  $\tau \subseteq \sigma$ . Let  $F : \mathbb{K} \rightarrow \mathbb{Z}$  be a basic discrete Morse function. A simplex  $\sigma$  of  $\mathbb{K}$  is said to be *critical* when  $F$  is injective on  $\sigma$ . Otherwise,  $\sigma$  is called *regular*. When  $\sigma$  is a critical simplex,  $F(\sigma)$  is called a *critical value*. If  $\sigma$  is a regular simplex,  $F(\sigma)$  is called a *regular value*. Two basic discrete Morse functions  $f, g$  defined on a same simplicial complex  $\mathbb{K}$  are said to be *Forman-equivalent* when for any two faces  $\sigma, \tau \in \mathbb{K}$  satisfying  $\sigma \prec \tau$ ,  $f(\sigma) < f(\tau)$  if and only if  $g(\sigma) < g(\tau)$ .

Let  $F$  be a basic discrete Morse function on  $\mathbb{K}$ . The *induced GVF*  $\overrightarrow{\text{grad}}$  of  $F$  is defined by  $\overrightarrow{\text{grad}}(F) := \{(\sigma, \tau) ; \sigma, \tau \in \mathbb{K}, \sigma \prec \tau, f(\sigma) = f(\tau)\}$ . If  $(\sigma, \tau)$  belongs to  $\overrightarrow{\text{grad}}(F)$ , then it is called a *vector* or *arrow* whose  $\sigma$  is the *tail* and  $\tau$  is the *head*. The vector  $(\sigma, \tau)$  will be sometimes denoted by  $\overrightarrow{\sigma\tau}$ .

Let  $\mathbb{K}$  be a simplicial complex. A *discrete vector field* on  $\mathbb{K}$  is a set of arrows in  $\mathbb{K}$  satisfying that every simplex of  $\mathbb{K}$  is in at most in one of its elements. Naturally, every GVF is a discrete vector field. Let  $V$  be a discrete vector field on a simplicial complex  $\mathbb{K}$ . A *gradient path* is a sequence of simplices<sup>3</sup>:  $(\tau_{-1}^{(p+1)}, \sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \tau_1^{(p+1)}, \dots, \sigma_{k-1}^{(p)}, \tau_{k-1}^{(p+1)}, \sigma_k^{(p)})$ , of  $\mathbb{K}$ , beginning at either a critical simplex  $\tau_{-1}^{(p+1)}$  or a regular simplex  $\sigma_0^{(p)}$ , such that  $(\sigma_\ell^{(p)}, \tau_\ell^{(p+1)})$  belongs to  $V$  and  $\tau_{\ell-1}^{(p+1)} \succ \sigma_\ell^{(p)}$  for  $0 \leq \ell \leq k - 1$ . If  $k \neq 0$ , then this path is said to be *non-trivial*. Note that the last simplex does not need to be in a pair in  $V$ . A gradient path is said to be *closed* if  $\sigma_k^{(p)} = \sigma_0^{(p)}$ .

**Theorem 2** (Theorem 2.51 p.61 of [15]). *A discrete vector field  $V$  is the GVF of a discrete Morse function (using the Forman definition) iff the discrete vector field  $V$  contains no non-trivial closed paths.*

In other words, there exists no basic discrete Morse function whose discrete vector field is  $V$  when  $V$  contains a non-trivial closed path.

**Theorem 3** (Theorem 2.53 p.62 of [15]). *Two discrete Morse functions  $f$  and  $g$  defined on a complex  $\mathbb{K}$  are Forman-equivalent iff  $f$  and  $g$  induce the same GVF.*

<sup>3</sup> The superscripts correspond to the dimensions of the faces.

The consequence is that any two Forman-equivalent discrete Morse functions defined on a simplicial complex have the same critical simplices.

Let  $\mathbb{K}$  be a simplicial complex and suppose that there is a pair of simplices  $(\sigma, \tau)$  of  $\mathbb{K}$  with  $\sigma \prec \tau$  such that the only coface of  $\sigma$  is  $\tau$ . Then  $\mathbb{K} \setminus \{\sigma, \tau\}$  is a simplicial complex called an *elementary collapse* of  $\mathbb{K}$ . The action of collapsing is denoted by  $\mathbb{K} \searrow \mathbb{K} \setminus \{\sigma, \tau\}$ . For an elementary collapse, such a pair  $\{\sigma, \tau\}$  is called a *free pair*. Note that elementary collapses preserve simple homotopy type.

## 2.4 Watersheds of simplicial stacks

Let  $A$  and  $B$  be two empty open sets in  $\mathbb{M}$ . We say that  $B$  is an extension of  $A$  if  $A \subseteq B$ , and if each connected component of  $B$  includes exactly one connected component of  $A$ . We also say that  $B$  is an extension of  $A$  if  $A = B = \emptyset$ . Let  $X$  be a subcomplex of the pseudomanifold  $\mathbb{M}$  and let  $Y$  be a collapse of  $X$ , then the complement of  $Y$  in  $\mathbb{M}$  is an extension of the complement of  $X$  in  $\mathbb{M}$ . Let  $A$  be a nonempty open set in a pseudomanifold  $\mathbb{M}$  and let  $X$  be a subcomplex of  $\mathbb{M}$ . We say that  $X$  is a *cut* for  $A$  if the complement of  $X$  is an extension of  $A$  and if  $X$  is minimal for this property. Observe that there can be several distinct cuts for a same open set  $A$  and, in this case, these distinct cuts do not necessarily contain the same number of faces. Let  $\pi = \langle x_0, \dots, x_\ell \rangle$  be a path in  $\mathbb{M}$ . We say that the path  $\pi$  is *descending* for  $F$  if for any  $i \in \{1, \dots, \ell\}, F(x_i) \leq F(x_{i-1})$ .

Let  $X$  be a subcomplex of the pseudomanifold  $\mathbb{M}$ . We assume that  $X$  is a cut for  $M_-(F)$ . We say that  $X$  is a *watershed-cut* of  $F$  if for any  $x \in X$ , there exists two descending paths  $\pi_1 = \langle x, x_0, \dots, x_\ell \rangle$  and  $\pi_2 = \langle x, y_0, \dots, y_m \rangle$  such that (1)  $x_\ell$  and  $y_m$  are simplices of two distinct minima of  $F$ ; and (2)  $x_i \notin X, y_j \notin X$ , for any  $i \in \{0, \dots, \ell\}$  and  $j \in \{0, \dots, m\}$ .

Several equivalent definitions of the watershed are given in [4]. In particular, watershed-cuts are equivalent to ultimate-n collapses, and thus, by Prop. 1, they are thin divides. In this paper, we focus on a definition relying on combinatorial optimization, more precisely on the minimum spanning tree. For that, we need a notion of “dual graph”.

The dual graph  $\mathcal{G} = (V, E, F_{\mathcal{G}})$ : starting from a subset  $A$  of  $\mathbb{M}$ , we define the *dual edge-weighted graph*  $\mathcal{G}$  such that its vertex set is composed of all  $n$ -simplices of  $A$  and its edge set is composed of all the pairs  $(x, y)$  such that  $x, y$  are  $n$ -faces of  $A$  and  $x \cap y$  is a  $(n - 1)$ -face of  $A$ . The valuation of the edges of  $\mathcal{G}$  is made as follows: for two distinct  $n$ -faces  $x, y$  in  $A$  sharing a  $(n-1)$ -face in  $A$ ,  $F_{\mathcal{G}}(\{x, y\}) = F(x \cap y)$ .

Let  $A$  and  $B$  be two non-empty subgraphs of the dual graph  $\mathcal{G}_{\mathbb{M}}$  of  $\mathbb{M}$ . We say that  $B$  is a forest relative to  $A$  when (1)  $B$  is an extension of  $A$ ; and (2) for any extension  $C \subseteq B$  of  $A$ , we have  $C = B$  whenever  $B$  and  $C$  share the same vertices. We say that  $B$  is a *spanning forest relative to*  $A$  for  $\mathcal{G}_{\mathbb{M}}$  if  $B$  is a forest relative to  $A$  and if  $B$  and  $\mathcal{G}_{\mathbb{M}}$  share the same vertices. Informally speaking, the second condition imposes that we cannot remove any edge from  $B$  while keeping an extension of  $A$  that has the same vertex set as  $B$ . The *weight* of  $A$  is defined as:  $F_{\mathcal{G}}(A) := \sum_{u \in E(\mathcal{G}(A))} F_{\mathcal{G}}(u)$ . Let  $A$  and  $B$  be two subgraphs of  $\mathcal{G}_{\mathbb{M}}$ . We say

that  $B$  is a *minimum spanning forest (MSF)* relative to  $A$  for  $F_G$  in  $\mathcal{G}_M$  if  $B$  is a spanning forest relative to  $A$  and if the weight of  $B$  is less than or equal to the weight of any other spanning forest relative to  $A$ .

Let  $A$  be a subgraph of  $\mathcal{G}_M$ , and let  $X$  be a set of edges of  $\mathcal{G}_M$ . We say that  $X$  is an *MSF cut* for  $A$  if there exists an MSF  $B$  relative to  $A$  such that  $X$  is the set of all edges of  $\mathcal{G}_M$  adjacent to two distinct connected components of  $B$ . If  $X$  is a set of  $(n-1)$ -faces of  $\mathbb{M}$ , we set  $\text{Edges}(X) = \{\{x, y\} \in E(\mathcal{G}_M) \mid x \cap y \in X\}$ .

**Theorem 4** (Theorem 16 p. 10 [4]). *Let  $X$  be a set of  $(n-1)$ -faces of  $\mathbb{M}$ . The complex resulting from the closure of  $X$  is a watershed-cut of  $F$  iff  $\text{Edges}(X)$  is an MSF cut for the dual graph of the minima of  $F$ .*

In other words, to compute the watershed of a stack  $F$ , it is sufficient to compute in  $\mathcal{G}_M$  an MSF cut relative to the graph associated with the minima of  $F$ . Different algorithms used to compute MSF cuts are detailed in [4].

### 3 Morse functions and simplicial stacks are equivalent

In this section, we show that  $2-1$  simplicial stacks are basic discrete Morse functions, and conversely.

#### 3.1 Fundamental propositions for Morse functions and stacks

**Proposition 5.** *Let  $F$  be a  $2-1$  simplicial stack defined on a simplicial complex  $\mathbb{K}$  of rank  $n \geq 0$ . Then  $F_2 = -F$  is a basic discrete Morse function.*

**Proof:** For any two faces  $\sigma, \tau$  of  $\mathbb{K}$  with  $\sigma \prec \tau$ , we know that  $F(\sigma) \geq F(\tau)$ , thus  $F_2 := -F$  satisfies:

$$F_2(\sigma) \leq F_2(\tau).$$

Furthermore,  $F$  is  $2-1$  and thus  $F_2$  is  $2-1$  too by bijection of the mapping  $v \rightarrow -v$ . So,  $F_2$  is a basic discrete Morse function.  $\square$

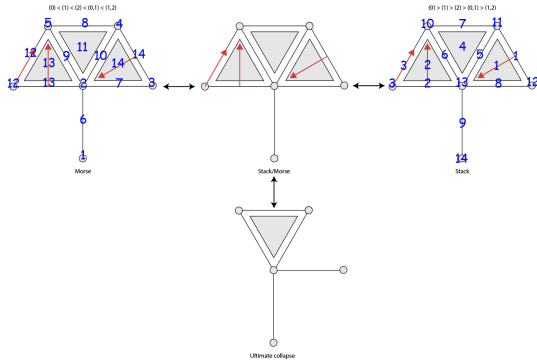
**Proposition 6.** *Let  $F$  be a basic discrete Morse function defined on a simplicial complex  $\mathbb{K}$  of rank  $n \geq 0$ . Then  $F_2 = -F$  is a  $2-1$  simplicial stack.*

**Proof:** We follow exactly the same reasoning as described in the proof of Proposition 5.  $\square$

#### 3.2 Using the GVF

For  $A$  and  $B$  two sets of real values, we say that  $A \triangleleft B$  when for any  $a \in A$  and any  $b \in B$ , we have the relation  $a < b$ . A simple valuation method to obtain a basic discrete Morse function starting from a GVF (obtained for example from any DMF) defined on a simplicial complex is the following:

$$\mathbb{V}(0) \triangleleft \mathbb{V}(1) \triangleleft \dots \triangleleft \mathbb{V}(n) \triangleleft \mathbb{V}(0, 1) \triangleleft \dots \triangleleft \mathbb{V}(n-1, n) \quad (1)$$



**Fig. 1.** The GVF encodes the possible collapses in both discrete Morse functions and simplicial stacks.

where  $\mathbb{V}(k)$  is the set of values described by the faces of dimensions  $k$  and  $\mathbb{V}(k, k + 1)$  is the set of values described by the matched faces of dimensions  $k$  and  $k + 1$  (the matching corresponding to the GVF).

The same approach for simplicial stacks simply leads to the opposite:

$$\mathbb{V}(n - 1, n) \triangleleft \dots \triangleleft \mathbb{V}(0, 1) \triangleleft \mathbb{V}(n) \triangleleft \dots \triangleleft \mathbb{V}(0). \tag{2}$$

Hence, relying on Prop. 5, we define the *gradient vector field of a basic 2 – 1 simplicial stack F* as the GVF of the DMF  $-F$  it corresponds to.

As stated in Th. 3, two basic DMF’s are Forman-equivalent when they induce the same GVF. In other words, at each GVF corresponds a class of DMF’s. Using Proposition 6 and Proposition 5, we have a bijection between the space of basic DMF’s and the space of 2 – 1 simplicial stacks. This leads to the following proposition:

**Proposition 7.** *For any given GVF, there exists a class  $\mathbb{BD}$  of basic DMF’s and a class  $\mathbb{SS}$  of 2 – 1 simplicial stacks, bijective to  $\mathbb{BD}$ , whose induced vector field is equal to this GVF.*

In other words, to each class of basic DMF corresponds a class of 2 – 1 simplicial stacks, and both have the same GVF. This proposition is depicted in Fig. 1 where we show at the bottom the collapse of a valued complex induced by a given GVF.

**3.3 Some conventions for practical usage**

**Conventions for discrete Morse functions** In practice, we will always consider that discrete Morse functions are defined on a subcomplex  $\mathbb{K}$  of a pseudo-manifold  $\mathbb{M}$ . This way, it will be easy to convert it into a simplicial stack. Indeed, starting from a discrete Morse function  $F : \mathbb{K} \subseteq \mathbb{M} \rightarrow \mathbb{N}^*$ , we define the new function  $F_{\mathbb{M}}$  which is set at  $F(\sigma)$  when  $\sigma \in \mathbb{K}$  and at 0 when  $\sigma \in \mathbb{M} \setminus \mathbb{K}$ . Then,



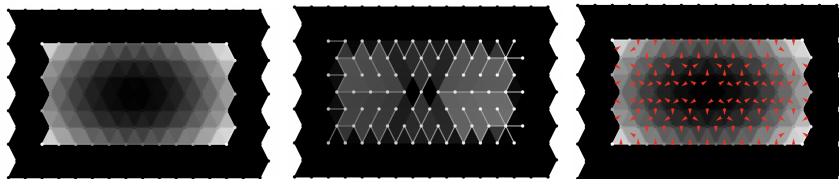


Fig. 2. An initial map, its reordered version, and its GVF.

we define the new function  $F_2 := \max(F_{\mathbb{M}}) + 1 - F_{\mathbb{M}}$ , which is a stack and is defined all over the pseudomanifold  $\mathbb{M}$ .

**Conventions for simplicial stacks** Starting from a simplicial stack  $F : \mathbb{M} \rightarrow \mathbb{N}$  defined on a pseudomanifold  $\mathbb{M}$  where minima are set at 0, we can easily remove the minima from the pseudomanifold to obtain  $\mathbb{K} := \mathbb{M} \setminus M_-(F)$  which is a complex since minima of a stack are stars. Then we define  $F_{\mathbb{K}}$  as the restriction of  $F$  to  $\mathbb{K}$ , and we deduce easily the following discrete Morse function:  $F_2 := \max(F_{\mathbb{K}}) + 1 - F_{\mathbb{K}}$ .

**Stacks are 2 – 1 except on minima** The reader will have noticed that when we assume that minima are set at 0, we can lose the property that stacks are 2 – 1. However, in this paper, we will consider that stacks are 2 – 1 except on the minima, which are plateaus of any size. We tolerate this property since it does not change the property that we can switch between stacks and discrete Morse functions, and furthermore, the gradient is not defined on minima since minima are critical faces.

## 4 The minimum spanning forest of a stack is its GVF

Need of a reevaluation algorithm: since any valued simplicial complex  $F : \mathbb{K} \rightarrow \mathbb{R}$  we start from is not always a stack, we may need to reevaluate it while preserving its GVF. For this aim, we design an algorithm inspired from Prim’s algorithm for minimum spanning tree [14] with a priority queue. We start from the minima of  $F$ , then we propagate in the direction of the steepest gradients of the initial function  $F$  without creating cycles (due to space constraint, a detailed version of this reevaluation will be published in an extended version of the present paper). The main goal is to obtain a total order between the  $n$ -faces of  $\mathbb{K}$  and to output a 2 – 1 simplicial stack. We call the new map the *reordered* version of  $F$ . Note that this reevaluation algorithm has been designed for discretizations of smooth functions, and is not aimed at making a basic DMF from a regular DMF.

### 4.1 Preamble: the (natural) extension of a GVF

For a given GVF  $\overrightarrow{\text{grad}}$  on a complex  $\mathbb{K}$  and for any vector  $\overrightarrow{ab}$  of  $\overrightarrow{\text{grad}}$ , we denote by  $\text{Ext}(\overrightarrow{ab})$  the edge equal to  $cb$  where  $c = \text{Opp}_a(b)$ . By extension, we denote by

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**Algorithm 1:** Computing the MSF from the gradient.
 

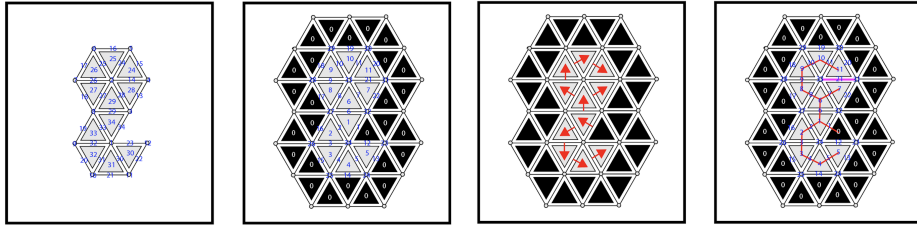
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```

begin
  /* The vertices correspond to the  $n$ -faces of the pseudomanifold. */
   $V = \text{Dual}((\mathbb{M})_n)$ 
   $E = \{\}$ 
  for  $m_1 \cap m_2 \in (\mathbb{M})_{n-1}$  s.t.  $m_1, m_2 \in (M_-(F))_n$  do
     $\lfloor \text{PUSH}(m_1 \cap m_2, E)$ 
  for  $\vec{ab} \in \overrightarrow{\text{grad}}$  do
     $c \leftarrow \text{Opp}_a(b)$ 
     $\lfloor \text{PUSH}(cb, E)$ 
return  $E$ 

```

---



**Fig. 3.** From left to right: we start from a Morse function, we compute its equivalent simplicial stack (up to the minus sign and up to the added zero used to extend the support of the function to avoid border effects). Then, we deduce its MSF and the GVF of the initial Morse function. More than a new numerical scheme used to compute gradients, it shows that **the MSF is a GVF** of both a discrete Morse function and the corresponding simplicial stack.

$\text{Ext}(\overrightarrow{\text{grad}})$  the set:

$$\text{Ext}(\overrightarrow{\text{grad}}) = \bigcup_{\vec{ab} \in \overrightarrow{\text{grad}}} \text{Ext}(\vec{ab}). \quad (3)$$

We call  $\text{Ext}(\overrightarrow{\text{grad}})$  the (*natural*) *extension* of the GVF  $\overrightarrow{\text{grad}}$ .

Figure 2 shows an initial map on a simplicial complex, which is revalued thanks to the reevaluation algorithm. We obtain a  $2-1$  simplicial stack on which we can compute the GVF. Algorithm 1 returns  $E = \text{Ext}(\overrightarrow{\text{grad}})$ , by converting each vector of the GVF into an edge. The set  $E$  of all edges is a forest thanks to Th. 2. We see that  $E$  is the *forest induced* by the GVF  $\overrightarrow{\text{grad}}$ .

#### 4.2 The extension of a GVF is equal to its MSF

**Proposition 8.** *Let  $\mathbb{K}$  be a  $n$ -dimensional simplicial complex, let  $F : \mathbb{K} \rightarrow \mathbb{R}$  be a  $2-1$  simplicial stack, and let  $\overrightarrow{\text{grad}}$  be the GVF of  $F$ . Then, any path*

$\pi = (h^k)_{k \in [0, N]}$  following  $\overrightarrow{\text{grad}}$  is increasing, that is, for any  $k \in [0, N - 1]$ ,  $F(h^k) \leq F(h^{k+1})$ .

**Proof:** Let  $\pi$  some path following  $\overrightarrow{\text{grad}}$ , and let us assume without generality that  $\pi(0)$  is a  $n$ -face of  $\mathbb{K}$ . We know that the  $(n - 1)$ -face  $\pi(2k + 1)$  is paired with the  $n$ -face  $\pi(2k + 2)$  in  $\overrightarrow{\text{grad}}$  for any  $k \in [0, (N - 1)/2 - 1]$ , which means that  $F(\pi(2k + 1)) = F(\pi(2k + 2))$ . We also know that  $F$  is a stack, and then  $F$  decreases when we increase the dimension of the face, so for any  $k \in [0, (N - 1)/2 - 1]$ ,  $F(\pi(2k)) \leq F(\pi(2k + 1))$ . The proof is done.  $\square$

**Proposition 9** (MST Lemma [11,3]). *Let  $G = (V, E, F)$  be some valued graph. Let  $v \in V$  be any vertex in  $G$ . The minimum spanning tree for  $G$  must contain the edge  $vw$  that is the minimum weighted edge incident on  $v$ .*

**Theorem 10.** *Let  $\mathbb{K}$  be a  $n$ -dimensional simplicial complex, let  $F : \mathbb{K} \rightarrow \mathbb{R}_+$  be a  $2 - 1$  simplicial stack, and let  $M_-(F)$  be the minima of  $F$ . Now, let  $\overrightarrow{\text{grad}}$  be the GVF of  $F$  starting from the minima of  $F$  and covering all the  $n$ -faces of  $\mathbb{K}$ . We assume furthermore that  $F$  is strictly positive on any face of  $\mathbb{K} \setminus M_-(F)$ . Then, the MSF relative to  $M_-(F)$  of the graph  $G$  dual to  $\mathbb{K}$  is equal to the spanning forest induced by its GVF  $\overrightarrow{\text{grad}}$ .*

**Proof:** We assume without loss of generalization that  $M_-(F)$  is equal to  $\{\{m\}\}$ , that is,  $F$  admits only one minimum  $\{m\}$  and it is isolated.

For any  $v$  in  $V(G) \setminus \{m\}$ , there exists at least one vector in  $\overrightarrow{\text{grad}}$  which can be written  $\overrightarrow{av}$  since, by hypothesis,  $\overrightarrow{\text{grad}}$  covers  $V$ . Since  $\overrightarrow{\text{grad}}$  is a GVF of a  $2 - 1$  mapping, we have at most one vector in  $\overrightarrow{\text{grad}}$  which is incident to  $v$  (otherwise, we could have at least three faces in  $\mathbb{K}$  whose image is the same, and  $F$  would not be a  $2 - 1$  mapping anymore). So, we have exactly one vector  $\overrightarrow{av}$  incident to  $v$  in  $\overrightarrow{\text{grad}}$ .

The  $(n - 1)$ -face corresponding to the edge  $e = \text{Ext}(av)$  is paired with the  $n$ -face corresponding to the vertex  $v$ . This can be written  $F(\text{Dual}(e)) = F(v)$  in an equivalent manner. Since  $F$  is a simplicial stack, the other edges which can be written  $vb$  are either critical (with  $F(vb) > \max(F(v), F(b))$ ), or not critical but with  $vb$  paired with  $b$  (thus  $F(b) = F(vb) > F(v)$ ). Therefore, the edge  $av$  is the lowest cost edge incident to  $v$ :

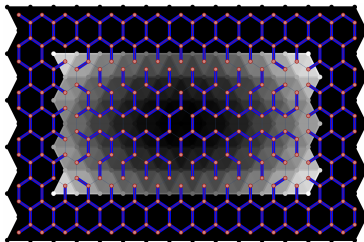
$$F(av) < \{F(bv) ; bv \in E(G), b \neq a\} \quad (4)$$

and thus belongs to the MST of  $F$  by Proposition 9.

Generalizing this property to every  $v \in V \setminus \{m\}$ , we obtain that all the lowest cost edges of these vertices, that is,  $\text{Ext}(\overrightarrow{\text{grad}})$ , belong to the MSF of  $F$ :

$$\text{Ext}(\overrightarrow{\text{grad}}) \subseteq \text{MSF}(F). \quad (5)$$

Since  $\text{Ext}(\overrightarrow{\text{grad}})$  covers  $V$  by hypothesis, it is then equal to the MST of  $F$ : any added edge would increase the total weight of the tree. The proof is done for the MST case.



**Fig. 4.** Comparison between the MSF induced from the GVF (in blue) and the MSF computed directly on the dual graph (in red) using the Prim algorithm: they are equal.

To generalize the proof to an MSF, it is sufficient to add an artificial node  $n_\infty$  to  $V$  and to connect it to all the vertices of  $M_-(F)$ , to apply Proposition 9 as before, and to remove the artificial node  $n_\infty$ . We obtain one more time that the MSF relative to  $M_-(F)$  is equal to the spanning forest induced by its GVF  $\overrightarrow{\text{grad}}$ . A summary of this result is depicted in Figure 3.  $\square$

**Corollary 11.** *The spanning forest  $\mathcal{LN}\mathcal{D}$  induced by the GVF of a simplicial 2-1 stack  $F : \mathbb{K} \rightarrow \mathbb{R}$  computed by Algorithm 1 is equal to the MSF of its reevaluation  $F'$  (see Algorithm 1). In other words, the minimum spanning forest of (the reevaluation of)  $F$  is the GVF of  $F$ .*

**Proof:** Using Algorithm 1 and starting from a valued complex  $(\mathbb{K}, F)$ , we obtain a (reordered) 2 – 1 simplicial stack  $F'$ . Then, we can easily compute its GVF  $\overrightarrow{\text{grad}}$  by pairing the faces which are neighbors and whose image by  $F'$  is the same. Thanks to Theorem 8, we know that the spanning forest induced by  $\overrightarrow{\text{grad}}$  is equal to the MSF of  $F'$ . The proof is done.  $\square$

In Figure 4, an illustration of Corollary 11, we see that the extension of the GVF  $\overrightarrow{\text{grad}}$  of  $F'$  is equal to the MSF of the same function.

Using Th. 4 (asserting that an MSF cut is a watershed-cut) and Th. 10, we can conclude that the cut of the extension of the GVF is also a watershed-cut. This leads to the following corollary.

**Corollary 12.** *Let  $\mathbb{K}$  be a  $n$ -dimensional simplicial complex and let  $F : \mathbb{K} \rightarrow \mathbb{R}_+$  be a 2 – 1 simplicial stack. Then, the watershed-cut of  $F$  is provided equivalently by the MSF of  $F$  or by the GVF of  $F$ .*

## 5 Conclusion

In this paper, we stressed the equivalence of fundamental concepts that exist both in Discrete Topology and in Mathematical Morphology: discrete Morse functions are equivalent to simplicial stacks, GVFs in the Morse sense are applicable to simplicial stacks, and the GVF of a simplicial stack induces the Minimum Spanning Forest of its dual graph. In the extended version of this paper, we extend

the results to regular DMF, and we show how to use the watershed to define, for the first time, a discrete Morse-Smale complex. In the future, we will continue looking for strong relations linking DMT (and specifically TDA) and MM.

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