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Social unacceptability for simple voting procedures

Ahmad Awde, Mostapha Diss, Eric Kamwa, Julien Yves Rolland and Abdelmonaim Tlidi

Abstract A candidate is said to be socially acceptable if the number of voters who rank her among the most preferred half of the candidates is at least as large as the number of voters who rank her among the least preferred half (Mahajne and Volij, 2018). For every voting profile, there always exists at least one socially acceptable candidate. This candidate may not be elected by some well-known voting rules, which may even lead in some cases to the election of a socially unacceptable candidate, the latter being a candidate such that the number of voters who rank her among the most preferred half of the candidates is strictly less than the number of voters who rank her among the least preferred half. In this paper, our contribution is twofold. First, since the existence of a socially unacceptable candidate is not always guaranteed, we determine the probabilities of the existence of such a candidate. Then, we evaluate how often the Plurality rule, the Negative Plurality rule, the Borda rule and their two-round versions can elect a socially unacceptable candidate. We perform our calculations under both the Impartial Culture and the Impartial Anonymous Culture,

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two assumptions which are widely used when studying the likelihood of voting events. Our results show that as the number of candidates increases, it becomes almost assured to have at least one socially unacceptable candidate; in some cases, the probability that half of the candidates in the running are socially unacceptable approaches or even exceeds 50%. It also comes out that the extent to which a socially unacceptable candidate is selected depends strongly on the voting rule, the underlying distribution of voters' preferences, the number of voters, and the number of competing candidates.

Keywords: Voting, Social Unacceptability, Scoring Rules, Probability, Paradoxes.

JEL classification: D71, D72.

1 Introduction

Assuming that voters have strict rankings (without indifference) on all running candidates, Mahajne and Volij (2018) introduce the concept of a socially acceptable candidate, which is a candidate whom at least half of the voters rank higher than at least half of the candidates in their rankings. They show that there always exists at least one socially acceptable candidate for every preference profile; however, such a candidate may not be elected under some scoring rules, with the exception of a new scoring rule, the half accepted-half rejected (HAHR) rule. Furthermore, Mahajne and Volij (2019) show that a socially acceptable candidate may not be a q -Condorcet winner¹ and they identify some restricted preference domains that guarantee that any q -Condorcet winner is socially acceptable as a function of the threshold q . Diss and Mahajne (2020) extend the concept of social acceptability to multi-winner elections, i.e., when the goal is to select a given group of candidates, and perform the same analysis as Mahajne and Volij (2019) in that context.

Among the wide range of existing voting rules, scoring rules and scoring runoff rules are the most common in the literature and in practice. Under scoring rules, each voter ranks all of the alternatives from her most preferred to her least preferred candidate; then, points are awarded to candidates according to their position in voters' rankings. The score of a given candidate is defined by the total number of points received by that candidate taking into account all of the voters; the overall winner of the election is the candidate having the highest score. The most popular scoring rules are the Plurality rule, the Negative Plurality rule and the Borda rule. Although these rules are quite popular, they suffer from a number of limitations. For an overview of these limitations, the reader can refer, among others, to the work of Felsenthal (2012), Gehrlein and Lepelley (2017, 2011), Nurmi (1999) or to the recent book by Diss and Merlin (2021).

It is well known from the debate between Borda (1781) and Condorcet (1785) that scoring rules may lead to the Condorcet winner paradox, i.e., they may fail to

¹ A q -Condorcet winner is a candidate who is preferred to each of the other candidates in pairwise majority comparisons by a fraction q of the total number of voters with $1/2 \leq q < 1$.

elect the Condorcet winner. When she exists, the Condorcet winner is a candidate who is preferred to each of the other competitors by a majority of voters. Scoring rules may also lead to the Condorcet loser paradox, as they may elect the Condorcet loser, a candidate defeated in pairwise comparisons by each of the other candidates. Even worse, a candidate ranked first by an absolute majority of the voters may not be elected (the absolute majority winner paradox) whereas a candidate ranked last by an absolute majority of the voters may be elected (the absolute majority loser paradox).² Avoiding the aforementioned paradoxes can be considered as an attempt to guarantee the election of a candidate supported by an “acceptable” majority of the electorate or to avoid the victory of a candidate who would be supported by only a given “minority”. Note that the existence of a Condorcet winner is not always guaranteed, whereas this is always the case for at least one socially acceptable candidate.

As mentioned above, a socially acceptable candidate may not win under some scoring rules; these rules may in some cases instead select a socially unacceptable candidate. Note that unlike the assured existence of at least one socially acceptable candidate for any preference profile, the existence of a socially unacceptable candidate is not always guaranteed. If there is at least one socially unacceptable candidate, the fact is that a majority consensus leans in favor of the socially acceptable candidate(s) over the socially unacceptable candidate(s). It therefore appears that in such a case the election of a socially unacceptable candidate corresponds to an undesirable scenario. Since such scenarios can occur under some scoring rules, our goal in this paper is to find out if this is frequent or not. More exactly, we evaluate the probability that the Plurality rule, the Negative Plurality rule, the Borda rule, and their two-round versions select a socially unacceptable candidate. Prior to that, we determine the probabilities of existence of socially unacceptable candidates. We focus on elections with a number of candidates in the set $\{3, 4, 5, 6, 10, 15\}$ for some values of the number of voters between 10 and 100000. We perform our analysis by running simulations under both the Impartial Culture and the Impartial Anonymous Culture, which are two widely used assumptions when studying the likelihood of voting events. We will say more about these assumptions in the sequel.

Our results show that as the number of candidates increases, it becomes almost assured to have at least one socially unacceptable candidate; in some case, the probability that half of the candidates in the running are socially unacceptable approaches or even exceeds 50%. It also turns out that the extent to which a socially unacceptable candidate is selected depends strongly on the voting rule, the underlying distribution of voters’ preferences, the number of voters, and the number of competing candidates.

The rest of the paper is organized as follows: Section 2 is devoted to definitions and to the basic notations that sets the framework for our analysis; in this section, we say a few words on our simulation methodology. We present our probability results in Section 3 for each of the two preference models under consideration. Section 4 concludes.

² See for instance, Diss et al. (2018).

2 Definitions

2.1 Preferences and social unacceptability

Let $A = \{a_1, \dots, a_K\}$ be a set of K ($K \geq 3$) candidates and $N = \{1, \dots, n\}$ a set of n ($n \geq 2$) voters. We denote by R the set of binary relations on A , and P the subset of complete, transitive, and antisymmetric binary relations on A . A preference profile is a mapping $\pi = (\succ_1, \dots, \succ_n)$ of preference relations on A to the voters in N . For each voter, $i \in N$, \succ_i represents i 's preference relation over the candidates in A . We denote by P^n the set of preference profiles. A *voting situation* is a $K!$ -tuple $\tilde{n} = (n_1, n_2, \dots, n_{K!})$ that indicates the total number n_t of voters casting each of the $K!$ complete linear orders such that $\sum_{t=1}^{K!} n_t = n$.

Let $\pi \in P^n$ be a preference profile, for any subset $C \subseteq P$ of preference relations, $\mu_\pi(C) = |\{i \in N : \succ_i \in C\}|$ is the number of voters whose preferences are in C . For any preference profile $\pi \in P^n$, the rank of a candidate a in the preference relation \succ is defined as follows: $rank_\succ(a) = K - |\{a' \in A : a > a'\}|$. Candidates whose ranks in the preference relation \succ are less than $(K + 1)/2$ are said to be placed *above the line*, those whose ranks in the preference relation \succ are greater than $(K + 1)/2$ are said to be placed *below the line*, and those whose ranks in the preference relation \succ are equal to $(K + 1)/2$ are said to be placed *on the line*. In the preference ranking $a_1 > a_2 > a_3 > a_4 > a_5$, for instance, candidates a_1 and a_2 are above the line while candidates a_4 and a_5 are below the line, and candidate a_3 is ranked on the line. It is obvious that there are no candidates on the line when the number of candidates is even.

We can now define the concept of social (un)acceptability as introduced by Mahajne and Volij (2018) for single-winner elections.

Definition 1 Let $\pi \in P^n$ be a profile of preference relations, and let $a \in A$ be a given candidate. We say that a is socially unacceptable with respect to π if the number of voters that place her below the line is strictly greater than the number of voters that place her above the line, otherwise a is called socially acceptable. Formally, a is socially unacceptable with respect to π if and only if

$$\mu_\pi\left(\left\{\succ : rank_\succ(a) < (K + 1)/2\right\}\right) < \mu_\pi\left(\left\{\succ : rank_\succ(a) > (K + 1)/2\right\}\right).$$

For illustration, let us consider the following simple example.

Example 1 Consider an election such that $A = \{a_1, a_2, a_3, a_4\}$ and $N = \{1, \dots, 7\}$ with the following rankings:

$$\begin{array}{lll} a_1 \succ_1 a_2 \succ_1 a_3 \succ_1 a_4, & a_1 \succ_2 a_2 \succ_2 a_4 \succ_2 a_3, & a_1 \succ_3 a_3 \succ_3 a_4 \succ_3 a_2, \\ a_2 \succ_4 a_3 \succ_4 a_1 \succ_4 a_4, & a_2 \succ_5 a_4 \succ_5 a_1 \succ_5 a_3, & a_3 \succ_6 a_4 \succ_6 a_1 \succ_6 a_2, \\ a_4 \succ_7 a_3 \succ_7 a_1 \succ_7 a_2. \end{array}$$

In this election, a_1 and a_4 are socially unacceptable candidates since they are ranked below the line by four voters while only three voters rank them above the line; a_2

and a_3 are socially acceptable candidates since they are ranked above the line by four voters while only three voters rank them below the line. Notice that if we erase $>_4$ in the above preference profile, we end up with a profile where there is no socially unacceptable candidate.

2.2 Voting rules

Given a preference profile $\pi \in P^n$, a scoring voting rule is characterized by a list $S = \{S_1, S_2, \dots, S_K\}$ meaning that each voter $i \in N$ assigns S_t points to the candidate that is ranked t -th in her preference relation with $t = 1, \dots, K$. The scoring rule associated with the list S chooses the candidate(s) having the maximum total score. The well-known scoring rules under consideration in this paper are the following:

Plurality rule (PR): The Plurality score of a given candidate is the total number of voters who rank this candidate at the top of their rankings. In other words, the Plurality rule corresponds to the list $S = (1, 0, \dots, 0, 0)$.

Negative Plurality rule (NPR): This rule picks the candidate with the lowest number of last places in voters' rankings; then, the vector of scores is given by $S = (1, 1, \dots, 1, 0)$.

Borda rule (BR): This rule gives $K - t$ points to a candidate each time she is ranked t -th; given the vector of scores $S = (K - 1, K - 2, \dots, 1, 0)$, the Borda score of a candidate is the sum of the points received.

We also consider a runoff version of each of the above scoring rules. Runoff rules are variously defined. Some will eliminate candidates one by one and others by blocks. In the case of a one-by-one elimination, the candidate with the lowest score is eliminated in each round; this is for example the case with the Baldwin rule (Baldwin, 1926) which at each round eliminates the candidate with the lowest Borda score. For block eliminations, the candidate(s) whose score does not meet a given threshold are eliminated in each round; this is the case for the Nanson rule (Nanson, 1883) or Kim-Roush voting rule (Kim and Roush, 1996).³ For simplicity, we define runoff rules here as involving only two rounds: only the two candidates with the highest scores in the first round qualify for the second round. The winner will therefore be the candidate who wins the majority duel that governs the second round. The runoff rules that we consider in the paper at hand are therefore defined as follows:

Plurality with runoff (PRR): A majority duel pits the two candidates with the highest plurality scores against each other and the one who wins this duel is declared the winner.

³ At each round of the Nanson rule, all the candidates with less than the average Borda score are eliminated. The Kim-Roush voting rule eliminates those candidates having a score higher than the average Negative Plurality score.

Negative Plurality with runoff (NPRR): The two candidates who have been ranked last by the voters the fewest times in their rankings find themselves in a majority duel in the second round. The winner of this duel is declared the winner of NPRR.

Borda with runoff (BRR): The winner under this rule is the candidate who wins the majority duel between the two candidates with the highest Borda scores in the first round.

Note that Mahajne and Volij (2018) introduced a scoring rule, called the half accepted-half rejected (HAHR), which always selects a socially acceptable candidate (and never selects a socially unacceptable candidate). HAHR assigns 1 point to the candidates placed above the line, -1 point to candidates below the line, and a score of 0 to the candidate (if there is one) on the line. Given a ranking t in a voter's preference relation, HAHR is formally defined by the scores S_t defined as follows:

$$S_t = \begin{cases} 1 & \text{if } t < \frac{K+1}{2} \\ 0 & \text{if } t = \frac{K+1}{2} \\ -1 & \text{if } t > \frac{K+1}{2} \end{cases}$$

The HAHR winner is the candidate with the highest total score. Notice that given a scalar α and β , a $1 \times K$ matrix of ones, the scoring rules associated with the scores S_t and with the scores $\alpha S_t + \beta$ define one and the same rule. It then follows that for K even, HAHR is equivalent to the well-known $\frac{K}{2}$ -approval rule, i.e., each voter casts a vote for half of the competing candidates. For the particular case of $K = 3$, HAHR is equivalent to the BR.

To motivate our subject, let us reconsider the preferences of Example 1. One can check in this example that candidate a_1 who is socially unacceptable is elected under each of our scoring rules and also under the two-round rules. Thus, our voting rules in this example elect a socially unacceptable candidate even though there are two socially acceptable candidates. Example 1 can be used to construct profiles with any number of candidates such that our voting rules elect a socially unacceptable candidate.

2.3 Probability models and simulation methodology

2.3.1 Probability models

When computing theoretical probabilities of voting events, assumptions are needed on the distribution of voting situations or profiles. This is made through statistical models. In this paper, we address our subject by assuming the following two models which are among the most common in the literature: the Impartial Culture (IC) and the Impartial Anonymous Culture (IAC).

IC was first introduced in the social choice literature by Guilbaud (1952); this model assumes that every preference profile is equally likely to occur; each voter chooses her preference according to a uniform probability distribution and gives a probability of $\frac{1}{K!}$ for each ranking to be chosen independently. The likelihood of a given voting situation $\tilde{n} = (n_1, n_2, \dots, n_{K!})$ is given by $\frac{n!}{\prod_{i=1}^{K!} n_i!} \times (K!)^{-n}$.

IAC was introduced by Kuga and Nagatani (1974) and Gehrlein and Fishburn (1976); this model assumes that all voting situations with n voters are equally likely to be observed. Under this model, the likelihood of a given voting event is equal to the ratio between the number of voting situations in which the event may occur over the total number of possible voting situations. The total number of possible voting situations with K candidates is given by the polynomial $\binom{n+K!-1}{K!-1}$. To determine the number of voting situations associated with a given voting event, several techniques, mathematical tools and algorithms have been proposed. We can quote in this sense, the work of Barvinok (1994), Barvinok and Pommersheim (1999), Bruns and Ichim (2018), Bruns et al. (2019), Bruns and Söger (2015), Cervone et al. (2005), Clauss and Loechner (1998), Huang and Chua (2000), Schürmann (2013), Verdoolaege et al. (2004), Wilson and Pritchard (2007).

The two models have been used over time in an impressive number of works. For a non-exhaustive overview of the use and technical developments around these two models, the reader can refer to the recent books by Diss and Merlin (2021) and Gehrlein and Lepelley (2017, 2011). Note that the techniques and algorithms mentioned above have a limit, namely the number of candidates. They are adapted for calculations involving at most four candidates; depending on the case, the calculation time can be quite variable (from one second to almost weeks). Since our analysis involves voting situations with more than four candidates, we have opted for simulations to get around these limitations. We present in the following the methodology that supports these simulations.

2.3.2 Simulation methodology

Simulations under IC and IAC models have been produced using a handcrafted Python framework.⁴ The simulation method has been based on the PrefLib code by Mattei and Walsh (2013): a reference library on preference data and algorithms for computational social choice. IC and IAC models have been extracted with modifications restricted to the random number generator. We then added the evaluations required for the socially unacceptable observations. A thorough selection has been performed to certify that no pre-existing preference data is used by any included algorithms. As such, the Python simulations are only based on random generation of preferences following IC or IAC models. The 128-bit implementation of the O’Neill’s permutation congruential generator has been used as a pseudo-random

⁴ Code base is available at <https://plmlab.math.cnrs.fr/jrolland/social-unacceptability-frezed-code>

generator (O’Neill, 2014) producing double-precision floating-point numbers with a period of 2^{128} .

The simulations proceed as follows: A number of candidates K , a number of voters n and a model (IC or IAC) is selected. A number \mathcal{I} of preferences is produced randomly (the specific number can be different, as discussed below). Storing selected candidates and preferences, the set of socially unacceptable candidates selected for different values of the input parameters can be built. For each simulation, the results of the six voting rules described in Section 2.2 are investigated and the probabilities of observing the different cardinal values is computed. Results are discussed in Section 3.

Due to the formulation of the algorithms, and the intrinsic behaviour of IC and IAC models, computation time increases sharply with values of K and n . As for the number \mathcal{I} of generations, this is the main contributor in the computation times and its value restricts the maximum precision that can be expected for the probabilities of interest. To assess that dependency, we used the case of the IC model, $K = 10$ candidates, $n = 100$ voters and computed a test probability for increasing values of \mathcal{I} preference generations. The variable tested is the probability to observe a set of socially unacceptable candidates selected with a cardinal value of 5. The results of this numerical calibration are shown in table 1.⁵

The effect of \mathcal{I} on computation time is linear, as expected for a single threaded implementation. Following those observations and due to time constraints, it has been decided to restrict some cases of the parametric study to a lower value of \mathcal{I} .⁶ Concerning precision, if $\mathcal{I} = 1000000$ is taken as a reference, a reduction of generations by a factor of 10 leads to a relative difference of only 0.06% and a factor of 100 scales to a relative difference of 0.13%. The same study applied to other values of n and K shows similar results.

Table 1 Computation times t (in seconds) and precision of the simulations over the number \mathcal{I} of preferences generated.

\mathcal{I}	10	100	1000	10000	100000	1000000
$\log(t)$	-0.62	0.34	1.32	2.32	3.32	4.56
E_{rel}	21.26%	28.61%	3.41%	0.13%	0.06%	—

In regards of computational resources, memory consumption is not a factor in these CPU-bound simulations, neither are the memory bus speed or width. It is to be noted, however, that as of now no optimization aiming for a reduction of computation times has been implemented in the Python framework. Computations for this study have been performed on two infrastructures: the LmB local computing facilities

⁵ Precision is expressed as relative error E_{rel} to the most precise case, in percentages.

⁶ All simulations for $K = 15$ as well as the specific case of IAC model, $n = 100000$ and $K \in \{6, 10\}$ have been performed with a limited $\mathcal{I} = 100000$ generations.

(Intel Xeon CPU - 3.20GHz) and supercomputer facilities of the Mésocentre de calcul de Franche-Comté (Intel Xeon CPU - 2.67GHz).

3 Results

3.1 The existence probability of socially unacceptable candidates

It is useful to note that with K candidates in the running, the number of socially unacceptable candidates can vary from 0 to $K - 1$. Since there is always at least one socially acceptable candidate, it is therefore impossible to have a voting situation in which all candidates are socially unacceptable. In this section, we investigate the proportion of voting situations in which one or more socially unacceptable candidates exist.

Considering voting situations with a number of candidates $K \in \{3, 4, 5, 6, 10, 15\}$, we were able to compute the probabilities of existence of socially unacceptable candidates as a function of the number of unacceptable socially acceptable candidates for some values of the number of voters n between 10 and 100000. Given K , let us denote by $\theta(K)$ the number of socially unacceptable candidates that exists. For space constraints, detailed results of numerical probabilities are provided in online supplementary material.⁷ Figures 1 and 2 present in a synthetic way the results that we obtain under both IAC and IC. For better readability, the colors indicate in each case the number of socially unacceptable candidates.

First of all, we point out some general observations that emerge from our calculations. Whether under IC or IAC, the probabilities show similarities in their trend. Under both models, we notice that as n increases, the probabilities tend to decrease for values of $\theta(K)$ such that $\theta(K) < \frac{K}{2}$ and tend to increase when $\theta(K) > \frac{K}{2}$. For given K and $\theta(K)$, the existence probabilities tend to decrease as the size n of the electorate increases. We also observe that for any given value of $K \in \{3, 4, 5, 15\}$ and n , the probabilities tend to increase for a $\theta(K)$ lower than $\theta(K)^* = \lfloor \frac{K}{2} \rfloor$ the value for which the probability of existence is maximum, and then decrease beyond that.⁸ We have a similar pattern for $K \in \{6, 15\}$ except that $\theta(K)^*$ is sometimes $\theta(K)^* = \frac{K}{2} - 1$ or $\theta(K)^* = \frac{K}{2}$. More precisely, when $K = 6$, we get $\theta(K)^* = 2$ when $n \in \{10, 20\}$ while $\theta(K)^* = 3$ for the other values of n ; for $K = 10$, we get $\theta(K)^* = 4$ when $n \in \{10, 20, 50\}$ and $\theta(K)^* = 5$ for the other values of n . We also notice that for an even number of candidates, the probability of existence evolves both with the parity of the number of voters and with the parity of the number of socially unacceptable candidates.

⁷ This supplementary material is available at https://plmlab.math.cnrs.fr/jrolland/social-unacceptability-frozen-code/-/blob/master/Supplementary_materials/tables.pdf.

⁸ $\lfloor \frac{K}{2} \rfloor$ is the least integer greater than or equal to $\frac{K}{2}$.

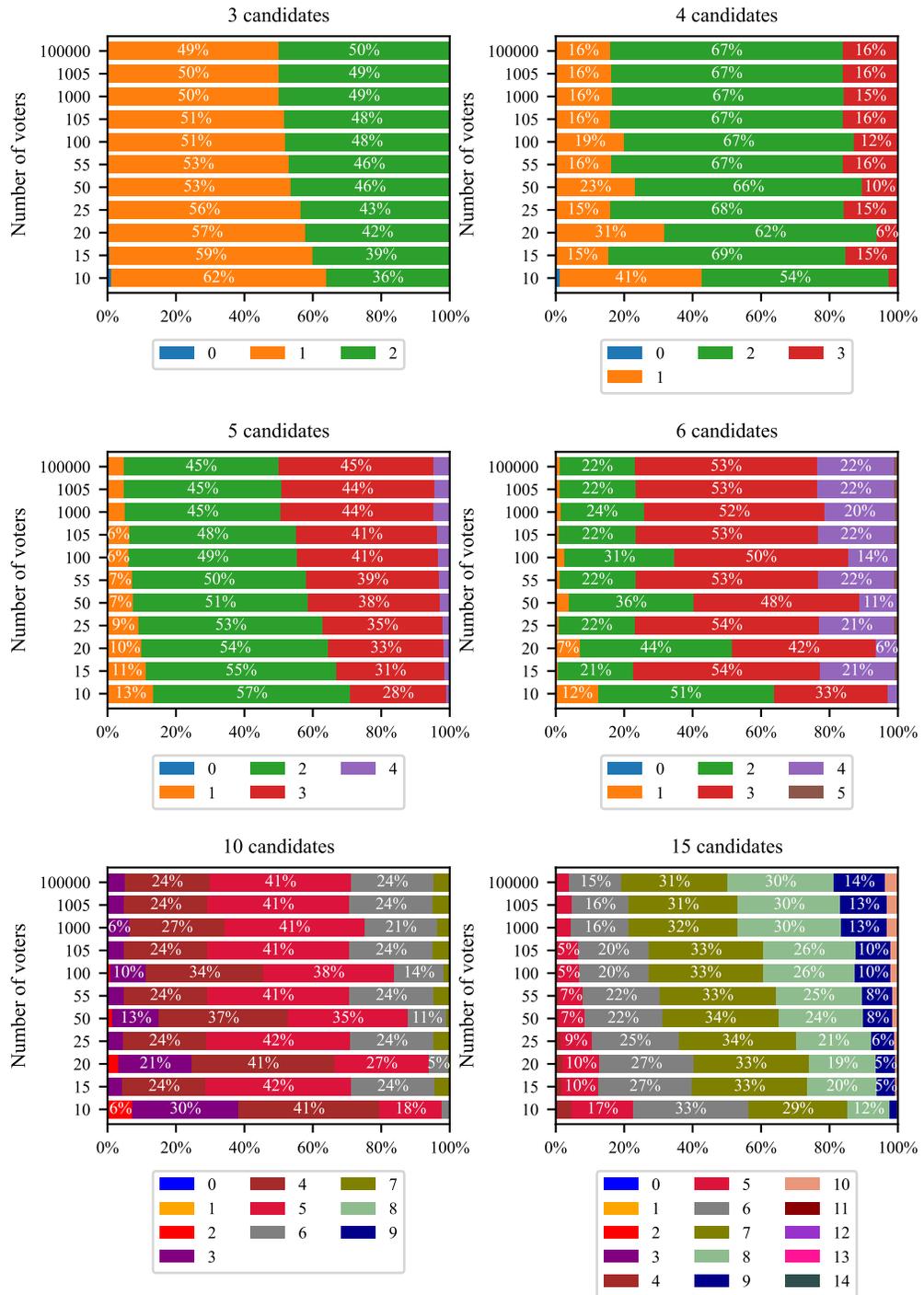


Fig. 1 Existence probability of socially unacceptable candidates under IAC model.

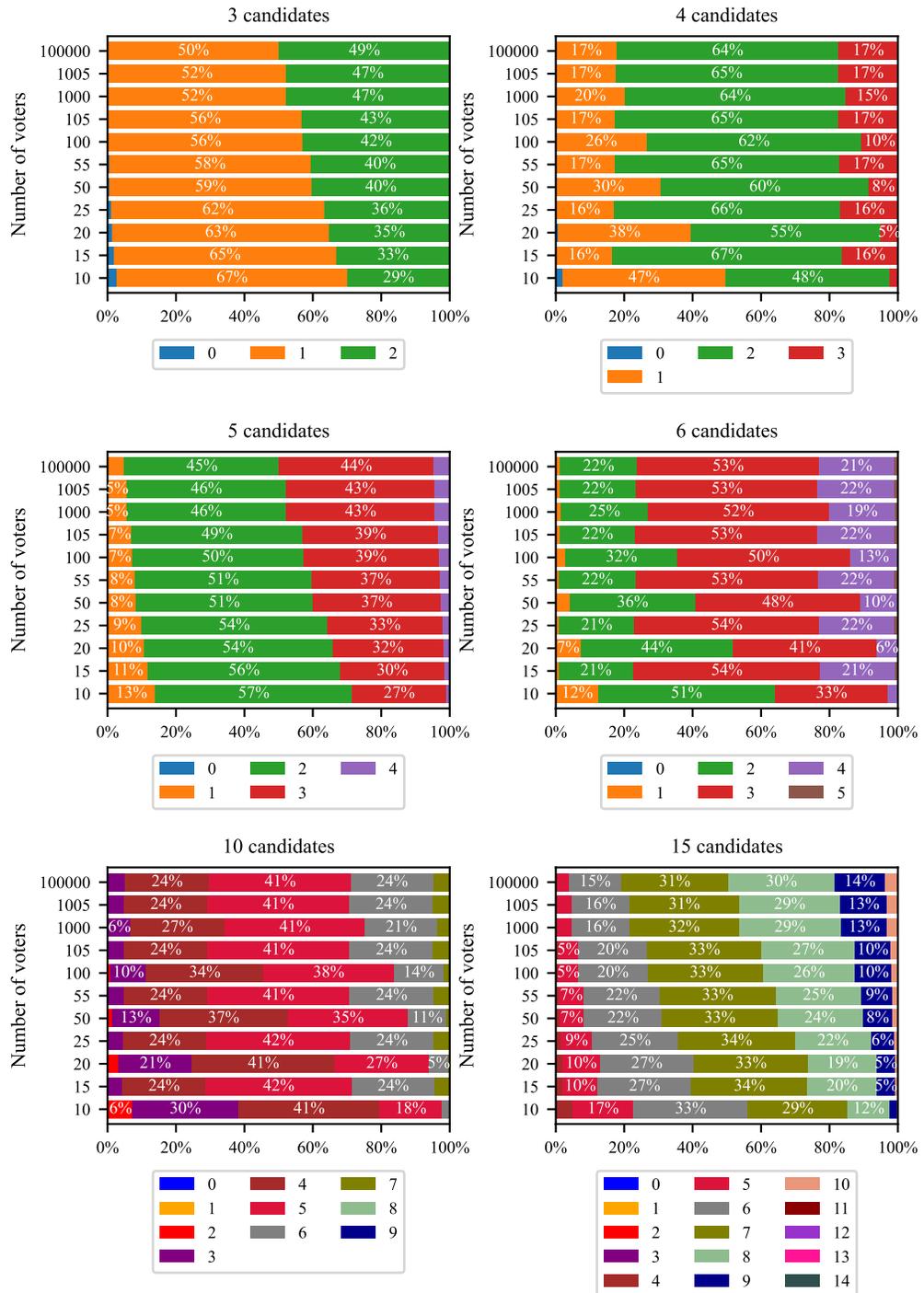


Fig. 2 Existence probability of socially unacceptable candidates under IC model.

We now turn to some probabilities that the reader will find online through the link provided above. If we look at the case of three-candidate elections ($K = 3$), we find that with ten voters there is about a 1.1% chance under IAC and about a 2.65% chance under IC that no socially unacceptable candidate exists. Thus, with such an electorate, there is a nearly 97% chance that there is at least one socially unacceptable candidate. This last conclusion is valid for all other values of n that we consider. We also note that for $K = 3$, with at least 50 voters, we reach and even exceed the 40% chance that 2 of the 3 candidates are socially unacceptable. For $K \in \{4, 6, 10\}$, there is at least a 98% chance that there is at least one socially unacceptable candidate; furthermore, for $n \in \{15, 25, 55, 105, 1005, 100000\}$ the existence probabilities are very low or even zero. For $K = 15$, our results indicate that there are always at least 3 (2 for some values of n) socially unacceptable candidates; this number is at most 12 (11 for some values of n) under IAC and at most 13 (or even 11 or 12 for some values of n) under IC.

In sum, it seems to emerge from our calculations that for voting situations with less than ten candidates, half of the candidates in contention may (in nearly 50% of the cases or even more) be socially unacceptable; also, as the number of candidates increases, it becomes almost assured to have at least one socially unacceptable candidate.

3.2 The probability that some voting rules select a socially unacceptable candidate

The existence probabilities obtained in the previous section led us to the conclusion that the existence of at least one socially unacceptable candidate is almost assured for the studied voting situations. Earlier, we argued that the election of a socially unacceptable candidate is problematic because of the consensus on her status compared to that of a socially acceptable candidate. It was therefore important to verify whether the election of a socially unacceptable candidate is a rare oddity or not. As such, we computed the probabilities that a socially unacceptable candidate would be elected under PR, NPR, BR, PRR, NPRR and BRR introduced and defined above. Figures 3 and 4 show the variation of these probabilities under IAC and IC, respectively.

Tables 2 and 3 provide the amplitudes of variation of the probabilities for each value of K under each model; for a given K , the amplitude of variation measures the difference between the highest and lowest probability. These tables show that for each value of K , the amplitudes are more pronounced under IC than under IAC. It appears that NPR is the rule, under IAC, which for $K \in \{5, 6, 10, 15\}$ presents the highest amplitudes of variation; it is also the case under IC for $K \in \{4, 5, 6, 10, 15\}$. Except for $K = 4$, the amplitudes are the lowest with BR. For $K = 4$, the amplitudes are lowest under IAC with NPRR and under IC with PRR.

The first observation that emerges under both IC and IAC is that for all values of K and n , NPR appears among the voting rules to be the most likely to lead to the election of a socially unacceptable candidate; it is followed by PR, which dominates

Rules	Number of candidates					
	3	4	5	6	10	15
PR	0.0124	0.0786	0.0128	0.1109	0.0887	0.0600
BR	0	0.0342	0.0056	0.0240	0.0114	0.0025
NPR	0.0048	0.0742	0.0761	0.1456	0.3572	0.6448
PRR	0.0250	0.0303	0.0392	0.0338	0.0434	0.0426
BRR	0.0144	0.0324	0.0142	0.0296	0.0164	0.0036
NPRR	0.0121	0.0278	0.0341	0.0374	0.0303	0.0510

Table 2 Amplitudes of variation of the probabilities under IAC model.

Rules	Number of candidates					
	3	4	5	6	10	15
PR	0.0345	0.0922	0.0208	0.1114	0.0907	0.0567
BR	0	0.0405	0.0057	0.0241	0.0114	0.0021
NPR	0.0256	0.0980	0.0812	0.1497	0.3567	0.6425
PRR	0.0379	0.0284	0.0406	0.0328	0.0444	0.0413
BRR	0.0249	0.0377	0.0148	0.0296	0.0164	0.0034
NPRR	0.0328	0.0449	0.0407	0.0385	0.0307	0.0500

Table 3 Amplitudes of variation of the probabilities under IC model.

NPRR, and then PRR. BR stands out as the rule least likely to lead to the election of a socially unacceptable candidate. Let us take a closer look at the behavior of each of our rules by stressing the salient points.

With PR, for each value of K , the probabilities tend to evolve according to the parity of the number of voters. More precisely, except for $K = 3$ or $K = 5$, they tend to increase with even n and to decrease with odd n . For $K = 3$, the probabilities increase from 4.4% under IAC (5.9% under IC) with $n = 10$ to nearly 5.5% under IAC (9.4% under IC) with $n = 100000$. For $K = 5$, we notice that the probabilities increase for $n \in \{10, 20, 15, 25\}$ under IAC and start decreasing for the other values of n whereas under IC we note a growth according to the parity of n . If we consider large values of n , the probabilities under PR tend to increase with K : we go from 5.5% under IAC (9.4% under IC) for $K = 3$ to nearly 30% (under IAC and IC) for $K = 15$. Thus, with PR, an increase in the number of candidates in the running can translate into a greater possibility of electing a socially unacceptable candidate.

Under BR, the evolution of probabilities follows an almost similar pattern under IAC and IC. The zero probabilities for $K = 3$ are well in line with the fact that BR never elects a socially unacceptable candidate since it is equivalent to HAHR. For $K = 4$, the probabilities tend to increase when n is even and to decrease when n is odd. For the other values of K , the probabilities all tend to increase according to the parity of n . We also note that it is for $K = 4$ that the probabilities are the highest under BR. We cannot clearly say, as we did with PR, how the probabilities

evolve when we increase K . The fact is that when we go from $K = 4$ to $K = 5$, the probabilities tend on average to decrease significantly; they tend on average to increase significantly when we go from $K = 5$ to $K = 6$ and then we note a decrease with $K = 10$ and $K = 15$.

The pattern of probabilities obtained with NPR differs for some values of K depending on whether one is under IAC or IC. For instance, when $K = 3$, while the probabilities decrease with the parity of n under IC, they tend under IAC to decrease with n even and to increase for n odd; when $K = 4$, the probabilities decrease with the parity of n under IAC while they tend under IC to decrease with n odd and to increase for n even. When $K = 10$, the probabilities decrease with the parity of n under IAC while under IC they tend to increase with the parity of n . Even if the probabilities tend on average to increase when going from $K = 3$ to $K = 4$, and to decrease slightly when going from $K = 4$ to $K = 5$, the tendency that seems to emerge is the following: the more the number of candidates increases, the more the possibility of electing a socially unacceptable candidate increases. For all values of n considered, the probabilities seem the highest for $K = 15$: we approach 98% for $n = 10$ and 31% for $n = 100000$.

In contrast to one-shot scoring rules, each of the runoff rules has, for a given K , a very similar behavior under IAC and IC: the probabilities obtained under each of these models are quite close. Under BRR, Tables 2 and 3 indicate that the variation of amplitudes of probabilities is highest for $K = 4$ and lowest when $K = 15$. The probabilities tend to increase with the parity of n for each value of K : for all values of n , the probabilities are higher when $K = 4$ where they average around 6% and they are lower when $K = 15$ where they average around 0.5%. Also, given n , the probabilities tend to evolve with the parity of K . With $K = 4$, the probabilities grow from about 3.3% with 10 voters to nearly 7% with 100000 voters; when $K = 15$, they grow from about 0.19% with 10 voters to nearly 0.5% with 100000 voters. As for the variation of the amplitudes of the probabilities, it is the highest for $K = 4$ and lowest when $K = 15$.

With PRR, when $K = 3$ or 6, the probabilities seem to increase for odd n and to decrease for even n . For other values of K , they simply tend to increase with the parity of n . They also tend on average to increase with K . We note that the probabilities are on average highest for $K = 15$ where they grow from almost 14.7% with 10 voters to about 16.9% with 100000 voters. The probabilities are lowest for $K = 3$ where we reach a little over 4% with 100000 voters. As can be seen from Tables 2 and 3, with PRR, variation in amplitude tends to increase with the value of K , with a notable anomaly on $K = 3$ for the IC model.

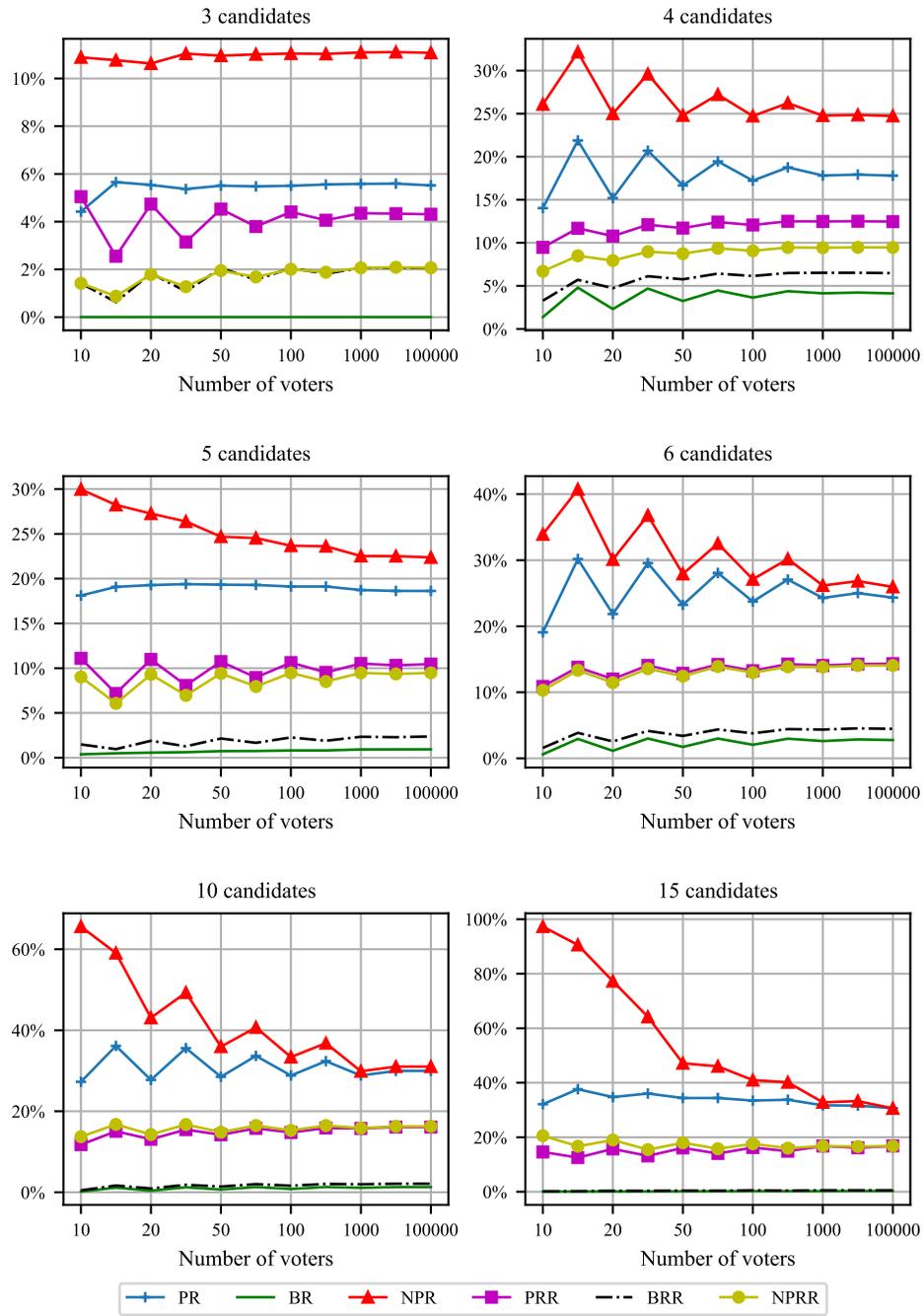


Fig. 3 Probability that some voting rules elect a socially unacceptable candidate under IAC model.

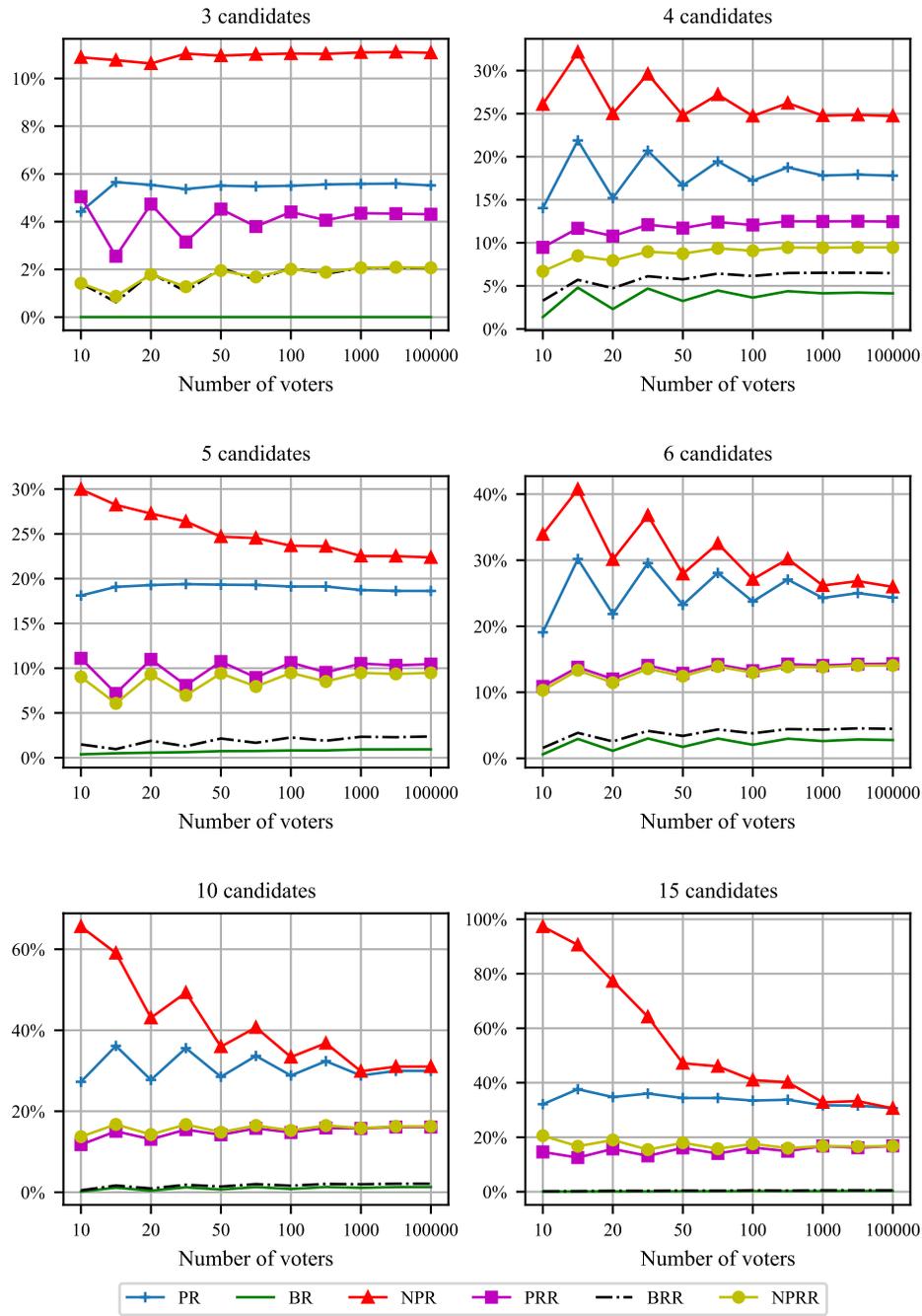


Fig. 4 Probability that some voting rules elect a socially unacceptable candidate under IC model.

Under IC, the probabilities with NPRR show a pattern of evolution quite close to what we have with PRR. The observation is the same under IAC except for $K = 3$ where the pattern is quite close to that of BRR. However, the probabilities are still higher than with BRR. From the similarity in the pattern of evolution of probabilities between NPRR and PRR, it is only for $K = 10$ and $K = 15$ that NPRR has higher probabilities than PRR. We notice with NPRR that except for $K = 15$ where the probabilities decrease with the parity of n , the probabilities tend to increase with the parity of n for all other values of K . Also, given n , they tend to increase with K .

To summarize, it appears that – for the rules under consideration – the probabilities of electing a socially unacceptable candidate tend to evolve both with the size of the electorate (notably its parity) and the number of candidates; BR stands out as the least likely to elect a socially unacceptable candidate.

4 Concluding remarks

The objective of this paper is, on the one hand, to account for the probabilities of the existence of socially unacceptable candidates. On the other hand, to determine the propensities of some popular scoring rules to elect such candidates when they exist. Using simulations under IAC and IC, we investigated voting situations with a number of candidates between 3 and 15 candidates and an electorate between 10 and 100000 voters. Our analysis reveals under both IC and IAC that the probability of existence tends to decrease as the size of the electorate increases. Furthermore, there is a high probability that in some cases nearly half of the candidates will be socially unacceptable. As for the propensity of the voting rules to elect a socially unacceptable candidate, the Negative Plurality rule emerges as the most likely while the Borda rule is the least likely. In addition, the probabilities tend to evolve according to the parity of the number of voters.

This paper has then allowed us to highlight another property of the Borda rule and of its two-round version. They are the least likely to elect a socially unacceptable candidate when she exists compared to the other scoring rules and their two-round versions considered in the paper at hand. Note that a possible extension of this paper could therefore be to question the variation of our probabilities when the socially unacceptable candidate that is elected is a Condorcet winner (resp. a Condorcet loser) given that such a candidate exists.

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