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On Two Voting systems that combine approval and preferences: Fallback Voting and Preference Approval Voting

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Abstract

Preference Approval Voting (PAV) and Fallback Voting (FV) are two voting rules that combine approval and preferences. They were first introduced by [Brams and Sanver \(2009\)](#). Under PAV, voters rank the candidates and indicate which ones they approve of; with FV, they rank only those candidates they approve of. In this paper, we supplement the work of [Brams and Sanver \(2009\)](#) by exploring some other normative properties of FV and PAV. We show among other that FV and PAV satisfy and fail the same criteria; they possess two properties that AV does not: Pareto optimality and the fact of always electing the absolute Condorcet winner when he exists. For three-candidate elections and a very large electorate, we compare FV and PAV to other voting rules by evaluating the probabilities of satisfying the Condorcet majority criteria. We find that PAV performs better than the Borda rule. We also find that in terms of agreement, FV and PAV are closer to scoring rules than to Approval voting. Our analysis is performed under the Impartial Anonymous Culture assumption.

Keywords: Approval Voting, Rankings, Condorcet, Properties, Impartial and Anonymous Culture

JEL Classification: D71 , D72

1 Introduction

In the context of the election of a single winner, two major blocks or groups of voting rules dominate both literature and practice. On the one hand, there are scoring rules based on rankings and, on the other hand, rules based on evaluation or approval. Scoring rules (SCR) most often require voters to rank (all or some of) the candidates in contention; on the basis of these rankings, points are awarded to candidates according to their positions. The total number of points received defines their score for the rule under consideration and the winner is the candidate with the highest score. Among the best-known SCR are the Plurality Rule (PR), the Negative Plurality Rule (NPR) and the Borda Rule (BR). Popularized by [Brams and Fishburn \(1978\)](#), Approval Voting (AV) is today among the most popular voting rules: voters simply indicate which of the candidates they consider as acceptable. Under AV, each voter is then free to approve as many candidates as he wishes; the winner(s) is the candidate(s) with the most approval. AV has been touted by some as a serious alternative to SCR, and several organizations have since adopted it for their decision-making processes (see [Regenwetter and Tsetlin, 2004](#)).

Many works have analyzed the virtues and weaknesses of both SCR and AV; see for instance [Felsenthal \(2012\)](#) or the *Handbook of Approval Voting* edited by [Laslier and Sanver \(2010\)](#). No consensus seems to emerge in the literature on a possible superiority of one rule over the others since the arguments put forward in favour of one or the other rule being varied and numerous. [Brams and Sanver \(2009\)](#) introduced two voting rules that seem to reconcile ranking-based decisions with approval-based decisions:¹ *Preference Approval Voting* (PAV) and *Fallback Voting* (FV). Under PAV, voters rank all the running candidates and distinguish the ones they approve of from those they disapprove of. When no candidate gets a majority of approvals (greater than half of the number of voters), PAV picks the AV winner; when more than one candidates are approved by more than half of the electorate, PAV picks the ones who is majority preferred among them –or, in case of a majority cycle among these candidates, it picks the one with the highest number of approvals among them. Under FV, voters first indicate all the candidates they approve of (this can range from no candidates to all) and then they rank only these candidates; each level of rankings (of the approved candidates) is considered and if at a given level, a majority of voters agree on one highest-ranked candidate, this candidate is the FV winner. Notice that FV is called “Majoritarian Approval Compromise” in [Sanver \(2010\)](#) and it is an adaptation of the Majoritarian Compromise of [Sertel \(1986\)](#); [Sertel and Yilmaz \(1999\)](#). The procedures implemented under PAV or FV to determine the winner are defined in such a way as to satisfy both the principle of the “most approved” and that of the “most preferred”.

[Brams and Sanver \(2009\)](#) have highlighted several desirable properties and drawbacks of FV and PAV. They showed among others that – FV, PAV,

¹Notice that the first formal introduction of this framework in terms of ordinal versus cardinal preferences is made in [Sanver \(2010\)](#).

and AV may all give different winners for the same profile; – a unanimously approved candidate may not be a FV or a PAV winner; – a least-approved candidate may be a FV or PAV winner; – a PAV winner may be different from winners under BR; – FV and PAV may fail to pick the Condorcet winner when he exists. It is striking to note that since PAV and FV were introduced, almost no work has been addressed on these rules as it was the case for AV or SCR. [Kamwa \(2019\)](#) is the only paper to our knowledge that have paid a particular attention at least to PAV; he investigated the propensity of PAV to elect the Condorcet winner or the Condorcet loser² In the first part of this paper, we supplement the work of [Brams and Sanver \(2009\)](#) by exploring other properties of FV and PAV. So, we evaluate FV and PAV on the basis of some other appealing properties of voting rules. We show among others that FV and PAV are Pareto optimal, they always elect the Absolute Condorcet winner when he exists;³; we determine the conditions under which these rules satisfy the reinforcement criterion, and for which they are not vulnerable to the No-show paradox. From our analysis, it appears that FV and PAV satisfy and fail the same criteria and they meet some properties that AV fails.

A great deal of work has been done in recent years on the probabilities for AV to elect the Condorcet Winner (or the Condorcet Loser) when he exists. These works have not failed to make on this basis, comparisons between AV and the three most popular SCR (PR, NPR and BR). We can quote in this sense, the works of [Diss et al. \(2010\)](#); [El Ouafdi et al. \(2020\)](#); [Gehrlein et al. \(2016\)](#); [Gehrlein and Lepelley \(1998, 2015\)](#). The second contribution of this paper will go in a similar direction. This will give us the opportunity to confront AV, FV, PAV with each other and with the SCR. First, we evaluate for voting situations with three candidates and an electorate tending to infinity, the probabilities of agreement between AV, FV and PAV; this analysis is extended to the three scoring rules PR, NPR and BR. We are also interested in the probabilities of satisfaction or violation of the Condorcet criteria. The fact that FV and PAV are based on the principle of the “most preferred” gives them a certain advantage over AV in terms of electing the winner of Condorcet when it exists. It will then be necessary here to be aware of the amplitude of this advantage. To do so, our calculations assume the impartial and anonymous culture assumption. This assumption will be defined later. Our computation analysis teaches us that, for three-candidate elections, as FV and PAV combine approvals and rankings, it tends in terms of agreement to bring them closer to SCR than to AV; more, they perform better in terms of compliance with the Condorcet criteria than some SCR.

The rest of the paper is organized as follows: Section 2 is devoted to basic definitions. Section 3 presents our results on the properties of FV and PAV. In Section 4, we provide our probabilistic results. Section 5 concludes.

²A Condorcet winner (resp. a Condorcet loser) is a candidate who defeats (resp. is defeated by) each of the other candidates in pairwise comparisons.

³An absolute Condorcet winner is a candidate ranked first by more than half of the voters.

2 Notation and definitions

Consider a set of n ($n \geq 2$) non-abstaining individuals $N = \{1, 2, \dots, i, \dots, n\}$ who vote sincerely on $\mathcal{C} = \{a, b, c, \dots, m\}$ a set of m ($m \geq 3$) candidates. We assume that they rank all the candidates strictly (i.e. without indifference) while indicating which ones they approve of.⁴ So, if a voter approves a and rank b ahead of a , this implies that he also approves b . For example, the ranking $\underline{a} \succ \underline{b} \succ c$ (or simply \underline{abc}) means that a is ranked ahead of b which is ahead of c and a and b are both approved while c is disapproved.

As voters inputs are both rankings and approvals, a voting situation is therefore an κ -tuple $\pi = (n_1, n_2, \dots, n_t, \dots, n_\kappa)$ that indicates the total number n_t of voters for each of the κ admissible rankings on \mathcal{C} such that $\sum_{t=1}^{\kappa} n_t = n$; given the assumptions made above, $\kappa = (m + 1)!$. Given π , we denote by $n_{ab}(\pi)$ (or simply n_{ab}), the number of voters who rank a before b . Candidate a is majority preferred to b if $n_{ab} > n_{ba}$. We say that candidate a is the Condorcet winner if $n_{ab} > n_{ba} \quad \forall b \in \mathcal{C} \setminus \{a\}$; candidate a is the Condorcet loser if $n_{ab} < n_{ba} \quad \forall b \in \mathcal{C} \setminus \{a\}$. A candidate a an absolute Condorcet winner (resp. an absolute Condorcet loser) if he is ranked first (resp. last) by more than half of the voters.

Given the rankings and approvals of the voters, we denote by $S^l(a, \pi)$ or simply $S^l(a)$, the total number of approval of candidate a when rankings of level l are considered ($l = 1, 2, \dots, m$); we say that candidate a is majority approved at level l if $S^l(a) > \frac{n}{2}$.

Let us now define each of the voting rules under consideration here.

Approval Voting (AV): Under this rule, voters can vote for (approve of) as many candidates as they wish. We denote by $\mathbf{AV}(a, \pi)$ the total number of approvals for candidate a given π . Candidate a is the AV winner if $\mathbf{AV}(a, \pi) > \mathbf{AV}(b, \pi) \quad \forall b \in \mathcal{C} \setminus \{a\}$. Notice that $\mathbf{AV}(a, \pi) = S^m(a)$.

Preference Approval Voting (PAV): According to [Brams and Sanver \(2009\)](#), PAV is determined by two rules and proceeds as follows:

Rule 1: The PAV winner is the AV winner if

- i. no candidate receives a majority of approval votes (*i.e.* approved by more than half of the electorate)
- ii. exactly one candidate receives a majority of approval votes.

Rule 2: In the case that two or more candidates receive a majority of approval votes,

- i. The PAV winner is the one among these candidates who is preferred by a majority to every other majority-approved candidate.
- ii. In the case of a cycle among the majority-approved candidates, then the AV winner among them is the PAV winner.

⁴A voter may endorse as many candidates as he wishes; thus, he may endorse no candidate as well as all the candidates in the race.

Fallback Voting (FV): Following [Brams and Sanver \(2009\)](#), FV proceeds as follows:

1. Voters indicate all candidates of whom they approve, who may range from no candidate (which a voter does by abstaining from voting) to all candidates. Voters rank only those candidates of whom they approve.
2. The highest-ranked candidate of all voters is considered. If a majority of voters agree on one highest-ranked candidate, this candidate is the FV winner. The procedure stops, and we call this candidate a level 1 winner.
3. If there is no level 1 winner, the next-highest ranked candidate of all voters is considered.⁵ If a majority of voters agree on one candidate as either their highest or their next-highest ranked candidate, this candidate is the FV winner. If more than one candidate receives majority approval, then the candidate with the largest majority is the FV winner. The procedure stops, and we call this candidate a level 2 winner.
4. If there is no level 2 winner, the voters descend - one level at a time - to lower and lower ranks of approved candidates, stopping when, for the first time, one or more candidates are approved of by a majority of voters, or no more candidates are ranked. If exactly one candidate receives majority approval, this candidate is the FV winner. If more than one candidate receives majority approval, then the candidate with the largest majority is the FV winner. If the descent reaches the lowest rank of all voters and no candidate is approved of by a majority of voters, the candidate with the most approval is the FV winner.

Since the second part of the paper will have to include the three most popular scoring rules, we agree to define them at the outset here.

Plurality rule (PR): This rule picks the candidate who is the most ranked at the top. If we denote by $\mathbf{PR}(a, \pi)$ the Plurality score of candidate a . Notice that $\mathbf{PR}(a, \pi) = S^1(a)$. Candidate a is the PR winner if $\mathbf{PR}(a, \pi) > \mathbf{PR}(b, \pi) \quad \forall b \in \mathcal{C} \setminus \{a\}$.

Negative Plurality rule (NPR): under this rule, the winner is the candidate with the lowest number of last places in the voters' rankings; If we denote by $\mathbf{NPR}(a, \pi)$ the number of last places (Negative Plurality score) of candidate a , this candidate is the NPR winner if $\mathbf{NPR}(a, \pi) < \mathbf{NPR}(b, \pi) \quad \forall b \in \mathcal{C} \setminus \{a\}$.

Borda rule (BR): BR gives $K - t$ points to a candidate each time she is ranked t -th; $\mathbf{BR}(a, \pi)$ the Borda score of a candidate is the sum of the points received. Candidate a is the BR winner if $\mathbf{BR}(a, \pi) > \mathbf{BR}(b, \pi) \quad \forall b \in \mathcal{C} \setminus \{a\}$.

In the context of a single winner, it can happen under AV, PR, NPR or BR that several candidates obtain the same score; in such a case, the use of a tie-break rule is necessary. In this paper, the setting is such that we will not need to use a tie-break rule.

⁵Notice that we stop going down when reaching the approval line of a voter which may be differently placed among different voters.

We can now review the properties of FV and PAV.

3 Normative properties of FV and PAV

As mentioned above, [Sanver \(2010\)](#) and [Brams and Sanver \(2009\)](#) have identified a number of properties of FV and PAV. They showed that these rules are monotonic, more precisely, they are approval-monotonic and rank-monotonic. A voting rule is approval-monotonic (resp. rank-monotonic) if a class of voters, by approving of a new candidate (resp. by raising a candidate in their ranking) - without changing their approval of other candidates - never hurts and may help this candidate get elected. In this section, we review other properties that FV and PAV may fulfill or fail.

3.1 Condorcet principle

We know that AV may fail to pick the (absolute) Condorcet winner when he exists ([Felsenthal, 2012](#)). According to [Brams and Sanver \(2009\)](#), FV and PAV may fail to elect the Condorcet winner when he exists; through Propositions [1](#) and [2](#), we refine this result.

Proposition 1 *When AV selects the Condorcet winner, this candidate is also the PAV winner but the reverse is not always true.*

Proof By definition, PAV always elects the AV winner under Rule 1; this may not be the case under Rule 2. So, for the proof, we only need to focus on Rule 2. Assume that candidate a is both the Condorcet winner and the AV winner. Let us also assume that candidate b ($b \neq a$) is the PAV winner under Rule 2; this means that (i) $\mathbf{AV}(b, \pi) > \frac{n}{2} > \mathbf{AV}(a, \pi)$ or (ii) $\mathbf{AV}(a, \pi) > \frac{n}{2}$ and $\mathbf{AV}(b, \pi) > \frac{n}{2}$. It is obvious that (i) clearly contradicts that a is the AV winner. By definition, b cannot win under (ii) since $n_{ab} > n_{ba}$, so a wins. Thus, if AV selects the Condorcet winner, this is also the case for PAV. Let us exhibit a profile to show that the reverse is not always true. Consider the following profile with 3 candidates and 11 voters:

$$5 : \underline{a} \succ \underline{b} \succ c \quad 5 : \underline{b} \succ a \succ c \quad 1 : \underline{c} \succ \underline{a} \succ b$$

With this profile, it is easy to see that b is the AV winner while a is both the Condorcet winner and the PAV winner. \square

Proposition 2 *FV and PAV always elect the absolute Condorcet winner when he exists.*

Proof Assume that candidate a is the Absolute Condorcet winner. As he is ranked first by more than half of the voters, $\mathbf{AV}(a, \pi) > \frac{n}{2}$ and $n_{ab} > n_{ba}$ for all $b \in \mathcal{C} \setminus \{a\}$. Under PAV, if a is the only one to be majority approved, he is obviously elected; if there are more candidates majority approved, a is elected since he is the Condorcet winner. So, PAV always elects the Absolute Condorcet winner. Since a is the absolute

Condorcet winner we get $S^1(a) > \frac{n}{2}$: by definition, he is the FV winner. So, FV always elects the Absolute Condorcet winner. \square

The fact is that when there is an absolute Condorcet winner, this candidate is the winner under both FV, PAV and PR. As will be noted in the rest of the paper, FV and PAV will be insensitive to certain paradoxes on the domain where there is an absolute Condorcet winner. The story is quite different on the domain where there is an (absolute) Condorcet loser. We know that AV can elect the (Absolute) Condorcet loser (see [Felsenthal, 2012](#)). We also know from [Kamwa \(2019\)](#) that PAV may pick the Condorcet loser when he exists. To our knowledge, nothing is known concerning FV. Propositions 3 tell us more on this.

Proposition 3 *FV and PAV may elect the (absolute) Condorcet loser when he exists. When PAV elects the (absolute) Condorcet loser, this candidate is also the AV winner but the reverse is not true. When FV elects the absolute Condorcet loser, this candidate is also the AV winner but the reverse is not always true.*

Proof Consider the following profile with 3 candidates and 11 voters.

$$5 : \underline{a} \succ b \succ c \quad 3 : \underline{b} \succ c \succ a \quad 3 : \underline{c} \succ b \succ a$$

With this profile it is easy to see that a is the Absolute Condorcet loser and that he is both the AV winner, the FV winner and the PAV winner. So, FV and PAV may elect the (absolute) Condorcet loser when he exists.

By definition, PAV can elect a (absolute) Condorcet loser only under Rule 1; as Rule 1 of PAV is equivalent to AV, it follows that for a given profile, if PAV elects the (absolute) Condorcet loser, he is also the AV winner. In the following profile, a is both the Absolute Condorcet loser and the AV winner but b is the PAV winner.

$$5 : \underline{a} \succ b \succ c \quad 3 : \underline{a} \succ c \succ b \quad 5 : \underline{b} \succ \underline{c} \succ \underline{a} \quad 4 : \underline{c} \succ b \succ a$$

Assume that candidate a is the Absolute Condorcet loser; it follows that $S^l(a) < \frac{n}{2}$ for $1 \leq l \leq m - 1$. Candidate a cannot be the FV winner on this range. He can only be elected at $l = m$; if so, this means that he gets the highest score of AV, then he is also the AV winner. The above profile is sufficient to show that the reverse is not true since a is both the absolute Condorcet loser and the AV winner while c is the FV winner. \square

3.2 Efficient compromise

The efficient compromise axiom was introduced by [Özkal-Sanver and Sanver \(2004\)](#) as a trade-off between the quantity and quality of support that an candidate may receive; the quantity refers to the number of voters supporting a candidate, and the quality of support is defined in terms of a candidate's rank in the order of voters' preferences. According to [Merlin at al. \(2019\)](#), at any profile, the efficient compromises are candidates receiving the highest quantity of support at some efficient level of quality. A voting rule is said to

satisfy the efficient compromise if and only if it always picks efficient compromises. Following [Özkal-Sanver and Sanver \(2004\)](#), the Plurality rule meets the efficient compromise; this is also the case for the q -Approval Fallback Bargaining⁶ for any $q \in \{1, 2, \dots, n\}$ while the Borda rule and all the Condorcet consistent rules does not. More, [Merlin et al. \(2019\)](#) showed that if the set of efficient compromise contains only one candidate, all the scoring rules will pick this candidate. Proposition 4 tells us that AV and PAV may fail the efficient compromise axiom except on the domain where there is a absolute majority winner.

Proposition 4 *FV, PAV and AV does not satisfy the efficient compromise axiom. FV and PAV always meet the efficient compromise axiom over the domain where there is an absolute Condorcet winner.*

Proof To show that FV, PAV and AV does not meet the efficient compromise, let us consider the following profile⁷ with four candidates $\{a, b, c, d\}$ and seven voters;

$$\begin{array}{lll} 1 : \underline{a} \succ \underline{b} \succ d \succ c & 2 : \underline{a} \succ c \succ d \succ b & 2 : \underline{b} \succ c \succ d \succ a \\ 1 : \underline{c} \succ \underline{b} \succ d \succ a & 1 : \underline{d} \succ c \succ b \succ a & \end{array}$$

With this profile, the reader can check that $\{a, c, d\}$ is the set of efficient compromises while b is the winner under both AV, FV and PAV.

Notice that an absolute Condorcet winner exists he is also an efficient compromise. As already pointed above, when there is an absolute Condorcet winner, both FV and PAV are equivalent to PR; since PR meet the efficient compromise, it will pick the absolute Condorcet winner, as does FV and PAV. \square

3.3 Socially (un)acceptability

In the search for a certain consensus around a candidate, [Mahajne and Volij \(2018\)](#) have introduced the concept of social acceptability. They say that a candidate is *socially acceptable* if the number of voters who rank him among their most preferred half of the candidates is at least as large as the number of voters who rank him among the least preferred half. [Mahajne and Volij \(2018\)](#) showed that there always exist at least one socially acceptable candidate in any profile; and they show that there exists a unique scoring rule that always elects such a candidate, the *Half Accepted-Half Rejected* rule (HAHR).⁸ In contrast to a socially acceptable candidate, a candidate is said to be *socially unacceptable* if the number of individuals who rank him among their least preferred half of the candidates is at least as large as the number of voters who rank him among the most preferred half.

⁶ q -Approval Fallback Bargaining winners are the candidates receiving the support of q voters at the highest possible quality.

⁷This profile is adapted from [Özkal-Sanver and Sanver \(2004\)](#).

⁸For m even, HAHR is equivalent to $\frac{m}{2}$ -approval rule.

Proposition 5 *AV, FV and PAV may not select a socially acceptable candidate and they may select socially unacceptable candidate. Following Proposition 2, over the domain where there is an absolute Condorcet winner, FV and PAV always select a socially acceptable candidate and never select a socially unacceptable candidate.*

Proof Consider the following profile with 3 candidates and 6 voters.

$$2 : \underline{a} \succ c \succ b \qquad 1 : \underline{a} \succ \underline{b} \succ c \qquad 1 : \underline{b} \succ a \succ c \qquad 2 : \underline{c} \succ \underline{b} \succ a$$

In this profile, a is a socially acceptable candidate while b is socially unacceptable and he is the winner under both AV, FV and PAV.

It is obvious that, when he exists, an absolute Condorcet winner is also a socially acceptable candidate. By Proposition 2, FV and PAV always select this candidate. Such a candidate cannot be socially unacceptable; he is still elected in presence of a socially unacceptable candidate. But, this may not be the case for AV: to see this, just add a voter with $\underline{a} \succ \underline{b} \succ c$; it follows that a is the absolute majority winner and therefore socially acceptable but the AV winner is b . \square

3.4 Cancellation property

Before going further, let us raise a point about PAV. By definition, Rule 2 of PAV relies on pairwise comparisons to decide the winner; what if all the majority duels between the majority-approved candidates end up in tie? In such a case, should all candidates be declared elected, or only the one(s) with the highest AV score? This situation does not seem to have been taken into account by [Brams and Sanver \(2009\)](#). In such a scenario, not choosing all the candidates involved implies a violation of the cancellation criterion. The cancellation condition requires that when all the majority comparisons end up in a tie, all the candidates should be selected ([Young, 1974](#)). It should be admitted that it is a bit difficult to apply the cancellation property to AV, because this rule does not fundamentally depend on rankings. Proposition 6 tells us that when FV and PAV fail the cancellation criterion; this also the case of AV when it is based on rankings.

Proposition 6 *AV, FV and PAV do not meet the cancellation property.*

Proof Consider the following profile with 3 candidates and 4 voters.

$$2 : \underline{a} \succ \underline{b} \succ c \qquad 2 : \underline{c} \succ \underline{b} \succ a$$

We can see in this profile that all the pairwise comparison end up in ties while candidate b is the winner of both AV, FV and PAV. So, AV, FV and PAV fail the cancellation property. \square

3.5 Pareto optimality

In a given voting situation, candidate a Pareto dominates candidate b if if all the voters strictly prefer a to b . A candidate is said to be *Pareto-optimal* if

there is no another candidate that dominates him. According to [Felsenthal \(2012\)](#), it is not tolerable the election of a candidate a when there is another candidate b that all voters rank before him. [Felsenthal \(2012\)](#) drives the point home by arguing that a voting rule that can elect a Pareto dominated candidate should be disqualified no matter how low the frequency. A voting rule meets the Pareto criterion if for all voting profile it never elects a Pareto dominated candidate. According to [Felsenthal \(2012\)](#), AV may elect a Pareto dominated candidate. This is not the case for FV and PAV following Proposition 7.

Proposition 7 *FV and PAV meet the Pareto criterion.*

Proof Given π , assume that a is the PAV winner and that he is Pareto-dominated by b . As b Pareto-dominates a , if a voter approve a this is also the case for b ; it follows that $\mathbf{AV}(b, \pi) \geq \mathbf{AV}(a, \pi)$ and b is majority preferred to a since $n_{ba} = n$. If a wins under Rule 1i of PAV, this means that $\mathbf{AV}(b, \pi) < \mathbf{AV}(a, \pi) < \frac{n}{2}$ which contradicts $\mathbf{AV}(b, \pi) \geq \mathbf{AV}(a, \pi)$. If a wins under Rule 1ii of PAV, this leads to $\mathbf{AV}(a, \pi) > \frac{n}{2}$ and $\mathbf{AV}(b, \pi) < \frac{n}{2}$ which contradicts $\mathbf{AV}(b, \pi) \geq \mathbf{AV}(a, \pi)$. If a wins under Rule 2 of PAV, the following three cases can be considered : (i) $\mathbf{AV}(a, \pi) > \frac{n}{2}$, $\mathbf{AV}(b, \pi) > \frac{n}{2}$ and $n_{ab} > n_{ba}$ or (ii) $\mathbf{AV}(a, \pi) > \frac{n}{2} > \mathbf{AV}(b, \pi)$, $\mathbf{AV}(c, \pi) > \frac{n}{2}$ and $n_{ac} > n_{ca}$ for $c \in \mathcal{C} \setminus \{a, b\}$ or (iii) $\mathbf{AV}(a, \pi) > \mathbf{AV}(b, \pi) > \frac{n}{2}$. It turns out that (i) contradicts $n_{ba} = n$ while (ii) and (iii) contradicts $\mathbf{AV}(b, \pi) \geq \mathbf{AV}(a, \pi)$. Thus, b cannot win: PAV meets the Pareto criterion.

Given π , assume that a is the FV winner and that he is Pareto-dominated by b . By definition, as b Pareto-dominates a we get $S^1(b) > S^1(a)$, and $S^l(b) \geq S^l(a)$ for all $l > 1$. Candidate a wins at a level l implies that $\frac{n}{2} > S^l(a) > S^l(b)$ or $S^l(a) > \frac{n}{2} > S^l(b)$ or $S^l(a) > S^l(b) > \frac{n}{2}$; these conditions all contradict that $S^l(b) \geq S^l(a)$. So, b cannot be the winner: FV never elects a Pareto dominated candidate. \square

3.6 The reinforcement condition

According to the *reinforcement condition*⁹ ([Myerson, 1995](#)) when an electorate is divided in two disjoints groups of voters N_1 ($|N_1| = n_1$) and N_2 ($|N_2| = n_2$) such that $N_1 \cap N_2 = \emptyset$ and $N_1 \cup N_2 = N$ ($|N| = n_1 + n_2 = n$), and the winner is the same for each group, this outcome will remain unchanged when both groups of voters are merged. It is known that AV, PR, NPR and BR meet the reinforcement condition (see [Felsenthal, 2012](#)). Proposition 8 tells us that FV and PAV does not meet the reinforcement condition and it characterizes when this is (not) the case.

Proposition 8 *Assume that an electorate is divided in two disjoints groups of voters N_1 and N_2 such that the winner is the same for each groups.*

⁹This condition is also known as the *Separability axiom* in [Smith \(1973\)](#) or the *Consistency axiom* in [Young \(1975\)](#).

- Considering that PAV is defined by four rules (Rule 1i, Rule 1ii, Rule 2i and Rule 2ii), it always meets the criterion if the winner in each of the two groups of voters is determined by Rule 1i or Rule 1ii; this is also the case when the winner is determined in one group by Rule 1i and in the other group by Rule 1ii. In the other cases, PAV may fail the reinforcement condition.
- FV meets reinforcement condition if the winner in each group is determined at the same level of rankings. In the other cases, it may fail the reinforcement condition.

Proof See Appendix □

3.7 Homogeneity

Given the voting outcome on a voting profile, if duplicating this profile λ times ($\lambda > 1$, $\lambda \in \mathbb{N}$) changes the result, we say that the homogeneity property is not satisfied. The violation of the homogeneity property is a major challenge for collective decision rules (Nurmi, 2004). It is obvious that AV is homogeneous since duplicating a population also duplicates the approvals in the same magnitude. Proposition 9 tells us what about FV and PAV.

Proposition 9 *FV and PAV are homogeneous.*

Proof Suppose we duplicate a profile π , λ times. On the resulting profile, given a candidate x , we have $S^l(x, \lambda\pi) = \lambda S^l(x, \pi)$, $\mathbf{AV}(x, \lambda\pi) = \lambda \mathbf{AV}(x, \pi)$ and $n_{xy}(\lambda\pi) = \lambda n_{xy}(\pi)$. It then follows that if a candidate wins under FV at level l in π , he also wins at the same level in $\lambda\pi$; we reach the same conclusion with PAV. Thus, duplicating a profile does not change the outcome under FV and PAV. □

3.8 The No-show paradox and the truncation paradox

The No-Show paradox describes a situation under which some voters may do better to abstain than to vote since abstaining may result in the victory of a more preferable or desirable candidate (Doron and Kronick, 1977; Fishburn and Brams, 1983). The Plurality rule, the Borda rule and Approval voting are among the few voting rules not vulnerable to the No-Show paradox (Felsenthal, 2012). It is known that the vulnerability of a voting rule to the No-Show paradox leads to its vulnerability to the truncation paradox but the reverse is not always true Nurmi (1987). The Truncation paradox occurs when some voters may reach a more preferred outcome by submitting a sincere but incomplete ranking (Fishburn and Brams, 1983, 1984). According to Brams (1982), AV is sensitive to the Truncation paradox; this is also the case for NPR and BR but not for PR.¹⁰ Proposition 10 characterizes the vulnerability of FV and PAV to the No-Show paradox.

¹⁰For more details on the truncation paradox and its occurrence under the scoring rules, we refer to Kamwa (2019); Kamwa and Moyouwou (2021).

Proposition 10 *PAV is vulnerable to the No-show paradox only when the winner is determined by Rule 2i. FV is not vulnerable to the No-show paradox only when the winner is determined at level $l = 1$ or $l = m$. Thus, FV and PAV are vulnerable to the Truncation paradox.*

Proof See Appendix. □

3.9 Independence of clones

Following Tideman (1987), a proper subset of two or more candidates, S , is a set of clones if no voter ranks any candidate outside of S as either tied with any element of S or between any two elements of S . A voting rule is said to be independent of clones if and only if the following two conditions are met when clones are on the ballot:

1. A candidate that is a member of a set of clones wins if and only if some member of that set of clones wins after a member of the set is eliminated from the ballot.
2. A candidate that is not a member of a set of clones wins if and only if that candidate wins after any clone is eliminated from the ballot.

Tideman (1987) shows that AV is not generally independent of clones. Nonetheless, he points out that applying the concept of clones to AV is somewhat problematic because clones are defined in terms voters' rankings. This problem does not arise with FV and PAV, which are defined in terms of rankings. As Tideman (1987) points out, when talking about clones in an approval setting, it is obvious to assume that if a voter approves a candidate a and not his clone b , that voter will approve b if a comes to withdraw; based on this, he showed that AV is not independent of clones. Since there are situations in which FV or PAV coincide with AV, it follows that in these situations FV and PAV may be vulnerable to cloning.

Proposition 11 *FV and PAV are not independent of clones. Nonetheless, they are independent of clones on the domain where there is an absolute Condorcet winner.*

Proof To show that FV and PAV are not independent of clones, let us consider the following profile:

$$4 : \underline{a} \succ b \succ c \qquad 3 : \underline{b} \succ c \succ a \qquad 2 : \underline{c} \succ b \succ a$$

In this profile, candidate a is both the AV, FV and PAV winner. Candidates b and c are clones; if one of them withdraws, the other becomes the winner for each rule given that if a voter approves candidate b and not his clone c , that voter will approve c if b comes to withdraw. So, FV and PAV are not independent of clones.

Now, let assume that there is absolute Condorcet winner, say a ; following Proposition 2, this candidate is elected by both FV and PAV. The withdrawal of one or

more candidates (clones) does not change a 's status as the absolute Condorcet winner; Proposition 2 applies and a remains the winner. Thus, over the domain where there is an absolute Condorcet winner, FV and PAV are independent of clones. \square

3.10 Independence criterion and Spoiler effect

According to Sanver (2010), a social choice satisfies independence if and only if it does not admit any spoiler; a spoiler is a candidate $x \notin \mathcal{C}$ such that its presence as an alternative can change the social choice without x being chosen. Sanver (2010) showed that – under FV, for any number of voters, there may be a spoiler who is approved by only one voter; – under PAV, a candidate is a spoiler only if he is socially qualified as good.

Proposition 12 (Sanver, 2010) *AV satisfies independence while PAV and FV fail it.*

The independence criterion as introduced by Sanver (2010) obviously refers us to the spoiler effect even though these two concepts differ in their definitions. In single-winner elections, the spoiler effect occurs if the removal of a non-winning candidate (called a spoiler) changes the election result (Kaminski, 2018; Miller, 2017): a spoiler turns a winner into a non-winner and a non-winner into a winner. The independence criterion assumes that the spoiler is a new candidate (introduced without any chance of winning) while the classical conception of the spoiler assumes that it is an element of the original set of candidates. It is therefore obvious that a conjunction of Propositions 11 and 12 leads to Proposition 13.

Proposition 13 *AV, PAV and FV are sensitive to the spoiler effect.*

We are now going to summarize all the above results in Table 1. In this table, we recap our findings about PAV, AV and FV as well as those of Brams and Sanver (2009) and Sanver (2010); besides FV and PAV, we include PR, NPR and BR. The fact that these rules satisfy or does not meet one of the criteria retained here comes from results of the literature (see Nurmi, 1987, 1999; Felsenthal, 2012). In this table, a “Yes” means that the voting rule meets the supposed criterion and a “No” if it does not.

On the basis of the normative criteria used in our analysis, it appears FV and PAV satisfy and fail the same criteria; they possess two properties that AV does not: Pareto optimality and the fact of always electing the absolute Condorcet winner when he exists. AV for its part, meets two criteria that FV and PAV do not: reinforcement and non-vulnerability to the No-show paradox. Approval-based rules, compared to score-based rules, satisfy fewer criteria. Another way to compare these sets of rules would be to check the frequencies for the criteria they violate. This is what we try to do in the next section with particular attention to the Condorcet criteria.

Table 1 Normative properties of the rules

Criteria	Rules					
	AV	FV	PAV	PR	NPR	BR
Condorcet winner	No	No	No	No	No	No
Absolute Condorcet winner	No	Yes	Yes	Yes	No	No
Condorcet loser	No	No	No	No	No	Yes
Absolute Condorcet loser	No	No	No	No	Yes	Yes
Pareto optimality	No	Yes	Yes	Yes	Yes	Yes
Efficient compromise	No	No	No	Yes	No	No
Social acceptability	No	No	No	No	No	No
Social unacceptability	No	No	No	No	No	No
Cancellation	No	No	No	No	No	Yes
Reinforcement	Yes	No	No	Yes	Yes	Yes
Homogeneity	Yes	Yes	Yes	Yes	Yes	Yes
No-Show	Yes	No	No	Yes	Yes	Yes
Truncation	No	No	No	Yes	No	No
Monotonicity	Yes	Yes	Yes	Yes	Yes	Yes
Independent of clones	No	No	No	No	No	No
Independence criterion	Yes	No	No	No	No	No
Spoiler effect	No	No	No	No	No	No

4 computational analysis

Our aim in this section is to evaluate, for voting situations with three candidates and an electorate tending to infinity, the probabilities of some voting events. We investigate the agreement between AV, FV and PAV. We are also interested in the probabilities of satisfaction or violation of the Condorcet criteria. We extend our analysis to the three popular scoring rules (PR, NPR and BR) and consider comparisons with FV and PAV. This extension is justified by the fact that we have pointed out above that in some configurations, FV is very close to PR. For our computations, we assume the impartial and anonymous culture assumption that we will present later.

Before going further, we need to present the rankings and approvals in the particular case of three candidates. For the sake of simplicity, we rule out the possibilities of approving nothing; so, given his ranking, a voter may approve at least one candidate and at most all the running candidates. So, given $\mathcal{C} = \{a, b, c\}$, the 18 possible types of preferences on \mathcal{C} are reported in Table 2. Then, a voting situation is the 18-tuple $\pi = (n_1, n_2, \dots, n_t, \dots, n_{18})$ such that $\sum_{t=1}^{18} n_t = n$.

Table 2 The 18 types of rankings and approvals on $\mathcal{C} = \{a, b, c\}$

$\underline{a} \succ b \succ c$	(n_1)	$\underline{a} \succ \underline{b} \succ c$	(n_7)	$\underline{a} \succ \underline{b} \succ \underline{c}$	(n_{13})
$\underline{a} \succ c \succ b$	(n_2)	$\underline{a} \succ \underline{c} \succ b$	(n_8)	$\underline{a} \succ \underline{c} \succ \underline{b}$	(n_{14})
$\underline{b} \succ a \succ c$	(n_3)	$\underline{b} \succ \underline{a} \succ c$	(n_9)	$\underline{b} \succ \underline{a} \succ \underline{c}$	(n_{15})
$\underline{b} \succ c \succ a$	(n_4)	$\underline{b} \succ \underline{c} \succ a$	(n_{10})	$\underline{b} \succ \underline{c} \succ \underline{a}$	(n_{16})
$\underline{c} \succ a \succ b$	(n_5)	$\underline{c} \succ \underline{a} \succ b$	(n_{11})	$\underline{c} \succ \underline{a} \succ \underline{b}$	(n_{17})
$\underline{c} \succ b \succ a$	(n_6)	$\underline{c} \succ \underline{b} \succ a$	(n_{12})	$\underline{c} \succ \underline{b} \succ \underline{a}$	(n_{18})

Given the labels of Table 2, the approval scores $S^l(\cdot)$ at level l are provided in Table 3; with three candidates, l varies from 1 to 3.

Table 3 The approval scores $S^l(\cdot)$

	Candidates		
	a	b	c
$S^1(\cdot)$	$n_1 + n_2 + n_7 + n_8 + n_{13} + n_{14}$	$n_3 + n_4 + n_9 + n_{10} + n_{15} + n_{16}$	$n_5 + n_6 + n_{11} + n_{12} + n_{17} + n_{18}$
$S^2(\cdot)$	$n_1 + n_2 + n_7 + n_8 + n_9$ $+n_{11} + n_{13} + n_{14} + n_{15} + n_{17}$	$n_3 + n_4 + n_7 + n_9 + n_{10}$ $+n_{12} + n_{13} + n_{15} + n_{16} + n_{18}$	$n_5 + n_6 + n_8 + n_{10} + n_{11}$ $+n_{12} + n_{14} + n_{16} + n_{17} + n_{18}$
$S^3(\cdot)$	$n_1 + n_2 + n_7 + n_8 + n_9 + n_{11}$ $+n_{13} + n_{14} + n_{15} + n_{16} + n_{17} + n_{18}$	$n_3 + n_4 + n_7 + n_9 + n_{10} + n_{12}$ $+n_{13} + n_{14} + n_{15} + n_{16} + n_{17} + n_{18}$	$n_5 + n_6 + n_8 + n_{10} + n_{11} + n_{12}$ $+n_{13} + n_{14} + n_{15} + n_{16} + n_{17} + n_{18}$

Notice that $S^3(a) = \mathbf{AV}(a, \pi)$. Candidate a is the AV winner if the conditions described by Eq. 1 are met.

$$\begin{cases} S^3(a) > S^3(b) \\ S^3(a) > S^3(c) \end{cases} \quad (1)$$

Recall that $S^1(\cdot) = \mathbf{PR}(\cdot, \pi)$. We provide in Table 4, the scores of the candidates under NPR and BR.¹¹

Table 4 Scores of the candidates under NPR and BR

	Candidates		
	a	b	c
$\mathbf{NPR}(\cdot, \pi)$	$n_4 + n_6 + n_{10} + n_{12} + n_{16} + n_{18}$	$n_2 + n_5 + n_8 + n_{11} + n_{14} + n_{17}$	$n_1 + n_3 + n_7 + n_9 + n_{13} + n_{15}$
$\mathbf{BR}(\cdot, \pi)$	$2(n_1 + n_2 + n_7 + n_8 + n_{13} + n_{14})$ $+n_3 + n_5 + n_9 + n_{11} + n_{15} + n_{17}$	$2(n_3 + n_4 + n_9 + n_{10} + n_{15} + n_{16})$ $+n_1 + n_6 + n_7 + n_{12} + n_{13} + n_{18}$	$2(n_5 + n_6 + n_{11} + n_{12} + n_{17} + n_{18})$ $+n_2 + n_4 + n_8 + n_{10} + n_{14} + n_{16}$

Given π , if candidate a is the FV winner, the following scenarios are possible:

- Candidate a is the only majority approved candidate at $l = 1$; this is fully described by Eq 2.
- No one wins at $l = 1$ and a is the only candidate majority approved at $l = 2$. In this case, we get Eq 3.
- No one wins at $l = 1$ and a with b (or c) are majority approved at $l = 2$; a gets more approvals than b (or c) at this stage. This situation is characterized by Eq 4 or Eq 5.
- No one wins at $l = 1$ and a , b and c are majority approved at $l = 2$. At this stage, a gets more approvals than b and c . In this case, we get Eq 6.
- There is no winner at both $l = 1$ and $l = 2$ and a is the only candidate who is majority approved at $l = 3$. This situation is characterized by Eq 7.
- There is no winner at $l = 1$ and $l = 2$: a with b (or c) are majority approved at $l = 3$. In this case, we get Eq. 8 or Eq 9.

¹¹With three candidates, Borda rule gives 2 to a candidate each time he is ranked first, 1 point when he is second and 0 when he is ranked last.

- No candidate is majority approved at $l = 1, 2$ but they are all majority approved at $l = 3$; a gets more approvals than b and c . This situation is characterized by Eq. 10.
- No candidate is majority approved at $l = 1, 2, 3$ and a gets more approvals than b and c . This situation is characterized by Eq. 11.

$$\left\{ \begin{array}{l} S^1(a) > \frac{2}{3} \\ S^1(b) < \frac{2}{3} \\ S^1(c) < \frac{2}{3} \end{array} \right. \quad (2) \quad \left\{ \begin{array}{l} S^1(a) < \frac{2}{3} \\ S^1(b) < \frac{2}{3} \\ S^1(c) < \frac{2}{3} \\ S^2(a) > \frac{2}{3} \\ S^2(b) < \frac{2}{3} \\ S^2(c) < \frac{2}{3} \end{array} \right. \quad (3) \quad \left\{ \begin{array}{l} S^1(a) < \frac{2}{3} \\ S^1(b) < \frac{2}{3} \\ S^1(c) < \frac{2}{3} \\ S^2(b) > \frac{2}{3} \\ S^2(c) < \frac{2}{3} \\ S^2(a) > S^2(b) \end{array} \right. \quad (4)$$

$$\left\{ \begin{array}{l} S^1(a) < \frac{2}{3} \\ S^1(b) < \frac{2}{3} \\ S^1(c) < \frac{2}{3} \\ S^2(b) < \frac{2}{3} \\ S^2(c) > \frac{2}{3} \\ S^2(a) > S^2(c) \end{array} \right. \quad (5) \quad \left\{ \begin{array}{l} S^1(a) < \frac{2}{3} \\ S^1(b) < \frac{2}{3} \\ S^1(c) < \frac{2}{3} \\ S^2(b) > \frac{2}{3} \\ S^2(c) > \frac{2}{3} \\ S^2(a) > S^2(b) \\ S^2(a) > S^2(c) \end{array} \right. \quad (6) \quad \left\{ \begin{array}{l} S^1(a) < \frac{2}{3} \\ S^1(b) < \frac{2}{3} \\ S^1(c) < \frac{2}{3} \\ S^2(a) < \frac{2}{3} \\ S^2(b) < \frac{2}{3} \\ S^2(c) < \frac{2}{3} \\ S^3(a) > \frac{2}{3} \\ S^3(b) < \frac{2}{3} \\ S^3(c) < \frac{2}{3} \end{array} \right. \quad (7)$$

$$\left\{ \begin{array}{l} S^1(a) < \frac{2}{3} \\ S^1(b) < \frac{2}{3} \\ S^1(c) < \frac{2}{3} \\ S^2(a) < \frac{2}{3} \\ S^2(b) < \frac{2}{3} \\ S^2(c) < \frac{2}{3} \\ S^3(b) > \frac{2}{3} \\ S^3(c) < \frac{2}{3} \\ S^3(a) > S^3(b) \end{array} \right. \quad (8) \quad \left\{ \begin{array}{l} S^1(a) < \frac{2}{3} \\ S^1(b) < \frac{2}{3} \\ S^1(c) < \frac{2}{3} \\ S^2(a) < \frac{2}{3} \\ S^2(b) < \frac{2}{3} \\ S^2(c) < \frac{2}{3} \\ S^3(b) < \frac{2}{3} \\ S^3(c) > \frac{2}{3} \\ S^3(a) > S^3(c) \end{array} \right. \quad (9) \quad \left\{ \begin{array}{l} S^1(a) < \frac{2}{3} \\ S^1(b) < \frac{2}{3} \\ S^1(c) < \frac{2}{3} \\ S^2(a) < \frac{2}{3} \\ S^2(b) < \frac{2}{3} \\ S^2(c) < \frac{2}{3} \\ S^3(a) > \frac{2}{3} \\ S^3(b) > S^3(a) \\ S^3(a) > S^3(c) \end{array} \right. \quad (10)$$

$$\left\{ \begin{array}{l} S^1(a) < \frac{2}{3} \\ S^1(b) < \frac{2}{3} \\ S^1(c) < \frac{2}{3} \\ S^2(a) < \frac{2}{3} \\ S^2(b) < \frac{2}{3} \\ S^2(c) < \frac{2}{3} \\ S^3(a) < \frac{2}{3} \\ S^3(b) > S^3(a) \\ S^3(a) > S^3(c) \end{array} \right. \quad (11)$$

If we assume that candidate a is the PAV winner, the following five scenarios are possible:

- no candidate gets a majority of approvals and a gets the highest number of approvals; this is fully described by Eq. 12.
- only a gets a majority of approvals; in this case, we get Eq. 13.
- a and b (or c) get a majority of approvals and a is majority preferred to b (or to c); this leads to Eq. 14 (or Eq. 15).
- all the three candidates get a majority of approvals and a majority dominates b and c ; this is described by Eq. 16.
- all the three candidates get a majority of approvals, there is a majority cycle and a gets the highest number of approvals; we thus get Eq. 17 or 18.

$$\left\{ \begin{array}{l} S^3(a) < \frac{n}{2} \\ S^3(a) > S^3(b) \\ S^3(a) > S^3(c) \end{array} \right. \quad (12) \quad \left\{ \begin{array}{l} S^3(a) > \frac{n}{2} \\ S^3(b) < \frac{n}{2} \\ S^3(c) < \frac{n}{2} \end{array} \right. \quad (13) \quad \left\{ \begin{array}{l} S^3(a) > \frac{n}{2} \\ S^3(b) > \frac{n}{2} \\ S^3(c) < \frac{n}{2} \\ n_{ab} > n_{ba} \end{array} \right. \quad (14)$$

$$\left\{ \begin{array}{l} S^3(a) > \frac{n}{2} \\ S^3(b) < \frac{n}{2} \\ S^3(c) > \frac{n}{2} \\ n_{ac} > n_{ca} \end{array} \right. \quad (15) \quad \left\{ \begin{array}{l} S^3(a) > \frac{n}{2} \\ S^3(b) > \frac{n}{2} \\ S^3(c) > \frac{n}{2} \\ n_{ab} > n_{ba} \\ n_{ac} > n_{ca} \end{array} \right. \quad (16) \quad \left\{ \begin{array}{l} S^3(a) > S^3(b) \\ S^3(a) > S^3(c) \\ S^3(b) > \frac{n}{2} \\ S^3(c) > \frac{n}{2} \\ n_{ab} > n_{ba} \\ n_{bc} > n_{cb} \\ n_{ca} > n_{ac} \end{array} \right. \quad (17)$$

$$\left\{ \begin{array}{l} S^3(a) > S^3(b) \\ S^3(a) > S^3(c) \\ S^3(b) > \frac{n}{2} \\ S^3(c) > \frac{n}{2} \\ n_{ba} > n_{ab} \\ n_{ac} > n_{ca} \\ n_{cb} > n_{bc} \end{array} \right. \quad (18)$$

When computing the likelihood of voting events, the impartial and anonymous culture (IAC) assumption introduced by [Kuga and Nagatani \(1974\)](#) and [Gehrlein and Fishburn \(1976\)](#) is one of the most widely used assumptions in social choice theory literature. Under this assumption, all voting situations are equally likely to be observed; it follows that the probability of a given event is calculated according to the ratio between the number of voting situations in which the event occurs and the total number of possible voting situations. For a given voting event, the number of voting situations can be reduced to the solutions of a finite system of linear constraints with rational coefficients. The appropriate mathematical tools to find these solutions are the Ehrhart polynomials. For a non-exhaustive overview of these techniques and algorithms, we refer to the recent books by [Diss and Merlin \(2021\)](#) and [Gehrlein and Lepelley \(2011, 2017\)](#). As in this paper we deal with situations where the number of voters tends to infinity, finding the limiting probabilities under IAC is reduced to the computation of volumes of convex polytopes ([Bruns and Söger, 2015](#); [Schürmann, 2013](#)). For our computations, we use the software *Normaliz* ([Bruns and Ichim, 2018](#); [Bruns et al., 2019](#)).¹² It should be noted that the calculations are relatively simple to implement under *Normaliz* because it is enough to enter the conditions describing an event and the algorithm returns the volume of the corresponding polytope which is the probability of this event.

¹²To more on *Normaliz*, we refer the reader to the paper of [Bruns and Söger \(2015\)](#) or the website dedicated to this algorithm, <https://www.normaliz.uni-osnabrueck.de>.

4.1 Agreement between the rules

As we recalled above, [Brams and Sanver \(2009\)](#) showed that both PAV and FV can (for the same voting profile) elect a different candidate than the AV winner or even more so the least-approved candidate. [Brams and Sanver \(2009\)](#) also shows that for the same preference profile AV, FV and PAV can elect completely different candidates. Our goal here is to determine the frequencies with which each of these events can occur.¹³

First of all, let us look at situations where two rules coincide. Let us take the case where FV and AV agree on candidate a as the winner. We denote by $P(AV = FV = a)$ the limiting probability of this event. This probability is in fact equal to a sum of volumes of polytopes to take into account the different scenarios that can occur under FV as described above. For example, the case where the winner of AV is the same as the winner of FV at level $l = 1$ is described by the inequalities of Eq. 1 and 2; we denote this volume obtained by $\mathcal{V}_{1 \cap 2}(\pi)$. In a similar way, we determine the volumes $\mathcal{V}_{1 \cap j}(\pi)$ for $j = 3, 4, \dots, 11$. Thus, we obtain

$$\begin{aligned} P(AV = FV = a) &= \sum_{j=2}^{j=11} \mathcal{V}_{1 \cap j}(\pi) \\ &= \frac{3864518350115}{15850845241344} \end{aligned}$$

We can therefore deduce $\mathcal{P}(FV = AV)$ the probability of agreement between AV and FV as follows

$$\mathcal{P}(FV = AV) = 3P(FV = AV = a) = \frac{3864518350115}{5283615080448}$$

Note that the calculation of $(FV = PAV = a)$ requires to review $7 \times 10 = 70$ possible configurations; among these configurations only 29 are possible because of the incompatibilities between the conditions. Proceeding in a similar way and including the scoring rules in our analysis, we obtain:

¹³For reasons of space, we cannot present the detailed calculations here. These calculation details are available upon request.

$$\begin{aligned}
\mathcal{P}(FV = PAV) &= \frac{20645280898898557}{28682781685383168}, & \mathcal{P}(PAV = AV) &= \frac{405549109}{603979776}, \\
\mathcal{P}(FV = PR) &= \frac{858426742033860211}{1299967445355724800}, & \mathcal{P}(PAV = PR) &= \frac{15393646886073191531}{19258776968232960000}, \\
\mathcal{P}(FV = NPR) &= \frac{29078653154282273}{41085390865563648}, & \mathcal{P}(PAV = NPR) &= \frac{3367171932047414983}{5135673858195456000}, \\
\mathcal{P}(FV = BR) &= \frac{356641532074024159}{462316319539200000}, & \mathcal{P}(PAV = BR) &= \frac{2966266305301241}{3424565329920000}, \\
\mathcal{P}(AV = PR) &= \frac{590913882103}{979552051200}, & \mathcal{P}(AV = BR) &= \frac{23515466951}{36578304000}, \\
\mathcal{P}(AV = NPR) &= \frac{5661560137}{9795520512}, & \mathcal{P}(NPR = BR) &= \frac{20645280898898557}{28682781685383168}, \\
\mathcal{P}(PR = BR) &= \frac{54057569}{68024448}, & \mathcal{P}(PR = NPR) &= \frac{4615849949}{8707129344}.
\end{aligned}$$

Table 5 Limiting probabilities of agreement

	FV	PAV	AV	PR	NPR	BR
FV	1	0.7474874	0.7314156	0.6603448	0.7077614	0.7714232
PAV	0.7474874	1	0.6714614	0.7993055	0.6556436	0.8661731
AV	0.7314156	0.6714614	1	0.6032491	0.5779744	0.6428802
PR	0.6603448	0.7993055	0.6032491	1	0.5301230	0.7946785
NPR	0.7077614	0.6556436	0.5779744	0.5301230	1	0.7197796
BR	0.7714232	0.8661731	0.6428802	0.7946785	0.7197796	1

We summarize our results in Table 5. It turns out that among the rules under consideration, BR is the one with the highest probability of agreement with each of the other rules. In at least 66% of the cases, FV agrees with each of the other rules and it tends to agree more with PAV (74.75%) than with AV (73.14%) , more with NPR (70.77%) than with PR (66.03%). PAV tends to more agree with PR (79.93%) than with AV (67.15%) or with NPR (65.56%). Not surprisingly, AV tends to agree more with FV and PAV than with scoring rules. Regarding scoring rules, PR and BR tend to agree the most with PAV. The general observation is therefore that as FV and PAV combine approvals and rankings it tends, in terms of agreement, to bring them closer to scoring rules than to AV.

Using the same approach as above, we were able to determine $\mathcal{P}(FV = AV = PAV)$ the limiting probability that AV, FV, and PAV agree on the same profile.

$$\mathcal{P}(FV = AV = PAV) = \frac{38878305102793}{66045188505600} \approx 0.5886621870$$

Thus, for the same voting profile, PAV, AV and FV elect the same winner in about 58.86% of cases; they therefore diverge in about 41.13% of cases.

4.2 FV and PAV may pick the least-approved candidate

Let us now look at the cases where FV and PAV can elect the least-approved candidate. Let us assume on π that candidate a is the least approved; this leads to Eq. 19.

$$\begin{cases} S^3(a) < S^3(b) \\ S^3(a) < S^3(c) \end{cases} \quad (19)$$

As pointed out by [Brams and Sanver \(2009\)](#), a least-approved candidate may be a PAV winner under rule 2i; in our framework, this event is fully characterized by the inequalities of Eq. 15 and 19. We then need to compute the volume $\mathcal{V}_{15 \cap 19}(\pi)$ that we multiply by 3 to find $\mathcal{P}(PAV = LAV)$ the limiting probability that PAV elects a least approved candidate as follows:

$$\mathcal{P}(FV = LAV) = 3\mathcal{V}_{15 \cap 19}(\pi) = \frac{6095207}{75497472} \approx 0.0807339$$

So, it is thus in nearly 8.07% of cases that PAV can lead to the election of a least approved candidate. What about FV? Since at level $l = m$, $S^l(\cdot)$ is equal to the AV score, it is obvious that FV cannot elect a least approved candidate at this level. It follows then that with three candidates, FV can elect a least approved candidate only at $l = 1$ or $l = 2$; this corresponds to Eq. 2 to 6. Thus, $\mathcal{P}(FV = LAV)$ the limiting probability that FV elects a least approved candidate is computed as follows:

$$\begin{aligned} \mathcal{P}(FV = LAV) &= 3 \left(\sum_{j=2}^6 \mathcal{V}_{19 \cap j}(\pi) \right) \\ &= \frac{262005663203}{5283615080448} \approx 0.04958833 \end{aligned}$$

FV would therefore be almost half as likely to elect a least approved candidate than PAV. This result would tend to confirm the fact that in terms of agreement, AV coincides more with FV than PAV.

4.3 A unanimously approved candidate may not win under FV or PAV

As [Brams and Sanver \(2009\)](#) notes, there may be times when FV and PAV do not elect a unanimously approved candidate. This marks another point of dissonance between these rules and AV. By definition, this can only occur with PAV under Rule 2i. Let us assume on $\mathcal{C} = \{a, b, c\}$ that b is unanimously approved. In our framework, this is tantamount to:

$$n_1 + n_2 + n_5 + n_6 + n_8 + n_{11} = 0 \quad (20)$$

If b and c are both unanimously approved, we get

$$n_{10} + n_{12} + n_{13} + n_{14} + n_{15} + n_{16} + n_{17} + n_{18} = n \quad (21)$$

Given Eq. 20, situations where a is the PAV winner while b (resp. c) is unanimously approved occur when Eq. 14 or 16 (resp. Eq. 15 or 16) hold. The case where both b and c are unanimously approved while a is the PAV winner can only occur if Eq. 16 holds. Then, $\mathcal{P}(PAV \neq Uap)$ the limiting probability that PAV fails to elect a unanimously approved candidate is computed as follows:

$$\begin{aligned} \mathcal{P}(PAV \neq Uap) &= 3 \left[2 \left(\mathcal{V}_{20 \cap 14}(\pi) + \mathcal{V}_{20 \cap 16}(\pi) \right) - \mathcal{V}_{21 \cap 16}(\pi) \right] \\ &= 3 \left[2 \left(\frac{5}{512} + \frac{313}{4096} \right) - \frac{7}{64} \right] \\ &= \frac{387}{2048} \approx 0.1889648 \end{aligned}$$

FV may fail to pick a unanimous approved candidate, when Eq. 2 or 4 or 5 or 6. Then, $\mathcal{P}(FV \neq Uap)$ the limiting probability that FV fails to elect a unanimously approved candidate is computed as follows:¹⁴

$$\begin{aligned} \mathcal{P}(PAV \neq Uap) &= 3 \left(2 \sum_{j=2}^6 \mathcal{V}_{20 \cap j}(\pi) - \sum_{j=2}^6 \mathcal{V}_{21 \cap j}(\pi) \right) \\ &= 3 \left[2 \left(\frac{67}{2048} + 0 + \frac{2573}{262144} + \frac{3}{2048} + \frac{36212845}{1719926784} \right) \right. \\ &\quad \left. - \left(\frac{1}{16} + 0 + \frac{9}{2048} + \frac{9}{2048} + \frac{5635}{186624} \right) \right] \\ &= \frac{12304397}{143327232} \approx 0.08584828 \end{aligned}$$

From the above, we note that in almost 18.89% of the cases, PAV may not elect a unanimously approved candidate while this is the case in only 8.58% of the cases for FV.

4.4 The election of the Condorcet winner

According to Ju (2010) and Xu (2010), when voters have dichotomous preferences, AV always elects the Condorcet winner when he exists. This is not always the case as voters have rather strict preferences or when indifference are allowed as shown by Diss et al. (2010); Gehrlein and Lepelley (1998, 2015); Kamwa (2019). Considering three-candidate elections with a certain degree of

¹⁴We notice that there is an incompatibility between Eq. 20 (or Eq. 21) and the conditions of Eq. 3. So, with these conditions FV does not fail to pick a unanimous approved candidate.

indifference under the *extended impartial culture condition*,¹⁵ Diss et al. (2010) and Gehrlein and Lepelley (2015) conclude that: AV is more likely to elect the Condorcet winner than both PR and NPR; BR performs better than AV. Gehrlein and Lepelley (2015) and El Ouafdi et al. (2020) reach a quite similar conclusion when considering the extended impartial anonymous culture condition.¹⁶ When it comes to electing the Absolute Condorcet winner when he exists, El Ouafdi et al. (2020) shows in their framework that AV does less well than BR but better than NPR.¹⁷

Almost nothing is known about the propensity of FV and PAV to elect the Condorcet winner when it exists. Kamwa (2019) investigates the limiting Condorcet efficiency of PAV in three-candidate elections while assuming the extended impartial culture condition; he find that PAV tends to performs better than AV. Considering the framework developed in this paper, our objective in this section is to compute the Condorcet efficiency of AV, FV, PAV, PR, NPR and BR. Then, we will look at their propensity to elect the Absolute Condorcet winner when it exists.

Recall that candidate a is the Condorcet winner if he is majority preferred to both b and c ; using our notation, this is equivalent to Eq. 22.

$$\begin{cases} n_{ab} > n_{ba} \\ n_{ac} > n_{ca} \end{cases} \quad (22)$$

Using the conditions of Eq. 22, Normaliz gives us the probability $P(a = CW)$ that a is the winner of Condorcet over π .

$$P(a = CW) = \frac{20129}{65536}$$

In the same way, we determined the probability that a is the Absolute Condorcet winner (i.e. $S^1(a) > \frac{n}{2}$):

$$P(a = ACW) = \frac{4701}{65536}$$

We therefore deduce $\mathcal{P}(CW)$ the existence probability of the Condorcet winner and $\mathcal{P}(ACW)$ that of the Absolute Condorcet winner:

$$\mathcal{P}(CW) = 3P(a = CW) = \frac{60387}{65536} \approx 0.9214325$$

$$\mathcal{P}(ACW) = 3P(a = ACW) = \frac{14103}{65536} \approx 0.2151947$$

¹⁵Under the impartial culture condition (Guilbaud, 1952) it is assumed that each voter chooses his preference (randomly and independently) on the basis of a uniform probability distribution across all strict orders. The extended impartial culture condition allows dichotomous preferences with complete indifference between two or more candidates.

¹⁶The extended anonymous impartial culture condition allows dichotomous preferences with complete indifference between two or more candidates.

¹⁷Recall that PR always elects the Absolute Condorcet winner when he exists.

To determine $\mathcal{CE}(R)$ the Condorcet efficiency of a given rule R , the methodology is the following: we determine the volume of the polytope describing the situation in which a is the Condorcet winner and the winner of R ; this volume will then be divided by $P(a = CW)$ to obtain the desired probability. This procedure allows us to obtain :

$$\mathcal{CE}(AV) = \frac{7491383}{11594304}; \quad \mathcal{CE}(PAV) = \frac{69380155}{226492416};$$

$$\mathcal{CE}(FV) = \frac{77089920161}{330225942528}; \quad \mathcal{CE}(BR) = \frac{39814829}{44022123};$$

$$\mathcal{CE}(PR) = \frac{8906796973}{10697375889}; \quad \mathcal{CE}(NPR) = \frac{14904579328717}{21908225820672};$$

Proceeding as in the case of the Condorcet efficiency, we determine $\mathcal{ACE}(R)$ the probability that the rule R elects the Absolute Condorcet winner when he exists. It is known that PR always elects the the Absolute Condorcet winner when he exists. Following Prop. 2, this is also the case for FV and PAV; so, $\mathcal{ACE}(PR) = \mathcal{ACE}(FV) = \mathcal{ACE}(PAV) = 1$. For the other rules, we get:

$$\mathcal{ACE}(AV) = \frac{13158985}{16246656}; \quad \mathcal{ACE}(NPR) = \frac{700614205919}{959348790144}; \quad \mathcal{ACE}(BR) = \frac{10223639}{10281087};$$

The second and fourth columns of Table 6 allow a better visualization of the results obtained in terms of (Absolute) Condorcet efficiency.

Table 6 Voting rules and the limiting probabilities of the Condorcet principle

	$\mathcal{CE}(R)$	$\mathcal{CL}(R)$	$\mathcal{ACE}(R)$	$\mathcal{ACL}(R)$
AV	0.6461261	0.0898578	0.8099504	0.0293452
FV	0.7600535	0.0300712	1	0.0001114
PAV	0.9973310	0.0001028	1	0.0000249
PR	0.8326151	0.0340336	1	0.0139566
NPR	0.6803188	0.0359394	0.7303018	0
BR	0.9044277	0	0.9944123	0

It turns out that PAV is the best performing rule in terms of Condorcet efficiency; it is followed by BR. FV performs better than AV but worse than PR. It turns out that AV is in our framework, the rule with the lowest Condorcet efficiency; here it therefore does worse than the scoring rules. The fact that AV performs worse than PR and NPR here contrasts with what [Diss et al. \(2010\)](#); [Gehrlein and Lepelley \(2015\)](#) and [El Ouafdi et al. \(2020\)](#) achieve in their different settings. As to elect the Absolute Condorcet winner when he exists, AV performs worse than BR but better than NPR. This conclusion is in agreement with what [El Ouafdi et al. \(2020\)](#) obtains in their setting.

To refine the comparisons, we could better assess how each of the above probabilities react to the proportion $\alpha = \frac{n_1+n_2+n_3+n_4+n_5+n_6}{n}$ of voters who approve only one candidate. When $\alpha = 1$, AV and PR are equivalent. For some values of α , we report in Table 7 the probabilities $\mathcal{CE}(R, \alpha)$ and $\mathcal{ACE}(R, \alpha)$ as functions of α .

As can be seen in Table 7, the probabilities evolve well as a function of α . We note that for $\alpha = 1$, $\mathcal{CE}(AV, 1) = \mathcal{CE}(FV, 1) = \mathcal{CE}(PAV, 1) = \mathcal{CE}(PR, 1)$ and $\mathcal{ACE}(AV, 1) = 1$. So, if the electorate is made only of voters who approves

Table 7 Some computed values of $\mathcal{CE}(R, \alpha)$, $\mathcal{ACE}(R, \alpha)$, $\mathcal{CL}(R, \alpha)$ and $\mathcal{ACL}(R, \alpha)$

		α							
Rules		0	1/4	1/3	1/2	2/3	3/4	1	
$\mathcal{CE}(R, \alpha)$	AV	0.5384051	0.5916563	0.6396992	0.7544142	0.8361545	0.8590698	0.8814815	
	FV	0.7818569	0.7379713	0.7497197	0.8356119	0.8517430	0.8648282	0.8814815	
	PAV	1	0.9999867	0.9998151	0.9911858	0.9297356	0.8965582	0.8814815	
	PR	0.8484781	0.8330283	0.8276168	0.8376708	0.8620404	0.8713166	0.8814815	
	NPR	0.6639015	0.6798728	0.6855510	0.6750562	0.6493909	0.6397497	0.6296296	
	BR	0.9061312	0.9045864	0.9037684	0.9050865	0.9089313	0.9101536	0.9111111	
$\mathcal{ACE}(R, \alpha)$	AV	0.5482718	0.7196296	0.8260673	0.9647654	0.9961465	0.9989699	1	
	FV	1	1	1	1	1	1	1	
	PAV	1	1	1	1	1	1	1	
	PR	1	1	1	1	1	1	1	
	NPR	0.6786731	0.7309078	0.7398935	0.7234537	0.6791482	0.6578623	0.6080247	
	BR	0.9856676	0.9946450	0.9948858	0.9941102	0.9911774	0.9877319	0.9629629	
$\mathcal{CL}(R, \alpha)$	AV	0.1291352	0.1080288	0.0907788	0.0552262	0.0360703	0.0322064	0.0296296	
	FV	0.0266157	0.0314631	0.0306498	0.0533053	0.0315597	0.0315432	0.0296296	
	PAV	0	0.0000000	0.0000000	0.0000345	0.0122605	0.0265217	0.0296296	
	PR	0.0328691	0.0338361	0.0347429	0.0333069	0.0299841	0.0293066	0.0296296	
	NPR	0.0348903	0.0357271	0.0365456	0.0352852	0.0324935	0.0318894	0.0314815	
	BR	0	0	0	0	0	0	0	
$\mathcal{ACL}(R, \alpha)$	AV	0.0506465	0.0380844	0.0294946	0.0135855	0.0091007	0.0104586	0.0246913	
	FV	0	0	0	0	0.0065876	0.0101787	0.0246912	
	PAV	0	0	0	0	0.0025317	0.0089947	0.0246913	
	PR	0.0191904	0.0136628	0.0145583	0.0134981	0.0116721	0.0124432	0.0246913	
	NPR	0	0	0	0	0	0	0	
	BR	0	0	0	0	0	0	0	

exactly on candidates, PR, AV, PAV and FV performs the same on the (Absolute) Condorcet winner criterion. PAV seems to dominate all other rules in terms of Condorcet efficiency for $0 \leq \alpha < 3/4$; for $3/4 \leq \alpha \leq 1$, BR dominates. PR tends to dominate FV for $0 \leq \alpha < 1/2$ while we get the reserve for $1/2 \leq \alpha \leq 1$. PR also dominates AV for all α ; AV is also dominated by NPR for $0 \leq \alpha < 1/2$. As to elect the absolute Condorcet winner, AV tends to dominate NPR for $\alpha \geq 1/3$ and it dominates BR for $\alpha \geq 2/3$.

4.5 The election of the Condorcet loser

Let us assume on $\mathcal{C} = \{a, b, c\}$ that a is the Condorcet loser (resp. the absolute Condorcet loser); using the labels of Table 2, this is equivalent to Eq. 23 (resp. Eq. 24).

$$\begin{cases} n_{ab} < n_{ba} \\ n_{ac} < n_{ca} \end{cases} \quad (23) \quad \mathbf{NPR}(a, \pi) > \frac{n}{2} \quad (24)$$

For our voting situations with three candidates, $\mathcal{P}(CL)$ the existence probability of the Condorcet loser and $\mathcal{P}(ACL)$ that of the Absolute Condorcet loser are as follows: $\mathcal{P}(CL) = \mathcal{P}(CW)$ and $\mathcal{P}(ACL) = \mathcal{P}(ACW)$.

We know from Prop. 3 that FV and PAV may elect the (Absolute) Condorcet loser when he exists. When a voting rule may elect a Condorcet loser (resp. a Absolute Condorcet loser), it said vulnerable to the Borda paradox (Absolute Majority Loser Paradox). By definition, a (Absolute) Condorcet loser when he exists, can never be elected under rule 2 of PAV; this can only be the case under Rule 1. With FV, the Condorcet loser cannot be elected at level $l = 1$ and the the Absolute Condorcet loser can only be elected at level $l = 3$. We follow the same methodology as for the Condorcet efficiency to determine $\mathcal{CL}(\cdot)$ (resp. $\mathcal{ACL}(\cdot)$) the limiting probability of electing the Condorcet loser (resp. the Absolute Condorcet loser) when he exists. From our computations, we get:¹⁸

$$\begin{aligned} \mathcal{CL}(AV) &= \frac{1041839}{11594304}; & \mathcal{ACL}(AV) &= \frac{476761}{16246656}; \\ \mathcal{CL}(FV) &= \frac{1016677891}{110075314176}; & \mathcal{ACL}(FV) &= \frac{10859}{97479936}; \\ \mathcal{CL}(PAV) &= \frac{14311}{452984832}; & \mathcal{ACL}(PAV) &= \frac{809}{32493312}; \\ \mathcal{CL}(PR) &= \frac{364069916}{10697375889}; & \mathcal{ACL}(PR) &= \frac{104603503}{7494912423}; \\ \mathcal{CL}(NPR) &= \frac{787367474789}{21908225820672}; \end{aligned}$$

It emerges that in our analytical framework, AV is the rule most likely to elect the loser Condorcet when it exists; it does less well than PR and NPR. This result contrasts with what [El Ouafdi et al. \(2020\)](#) or [Gehrlein et al. \(2016\)](#) achieve in their respective frameworks. With a limiting probability of nearly 0.01%, PAV performs significantly better than FV whose probability is raised to nearly 3%. Regarding the election of an Absolute Condorcet loser, our results show that AV is among our rules the most likely to elect such a candidate; PAV performs better than FV which performs better than PR.

We were willing, as we did in the previous section, to refine our findings based on α the proportion of voters who approve of exactly one candidate. The probabilities $\mathcal{CL}(R, \alpha)$ and $\mathcal{ACL}(R, \alpha)$ that we obtained in this regard are provided in Table 7.

For any value of α , NPR appears to be the rule most likely to elect the Condorcet loser. It appears that $\mathcal{CL}(AV, \alpha)$ decreases with α while $\mathcal{CL}(PAV, \alpha)$ increases with α . $\mathcal{CL}(FV, \alpha)$ tends to increase for α going from 0 to reach its maximum at $\alpha = 1/2$ then decreases. $\mathcal{CL}(PR, \alpha)$ and $\mathcal{CL}(NPR, \alpha)$ tend to grow for $0 \leq \alpha < 1/2$ then to decrease for $1/2 \leq \alpha < 1$. For $\alpha = 1$, we find that AV, PR, FV and PAV have the same probability to elect the Condorcet loser.

For $0 \leq \alpha \leq 1/2$, AV appears to be the rule most likely to elect an absolute Condorcet loser; over this interval, $\mathcal{ACL}(AV, \alpha)$ tends to decrease with α . We also note that for $0 \leq \alpha \leq 1/2$, FV and PAV never elect an absolute Condorcet

¹⁸The computation details are available upon request.

loser. For $1/2 < \alpha < 1$, PR is the most likely to elect an absolute Condorcet loser; it is followed by AV while PAV performs better than FV. For $\alpha = 1$, AV, FV, PAV and PR have the same probability (about 2.47%) of electing the absolute Condorcet loser.

5 Concluding remarks

The first objective of this paper was to supplement the analysis of [Brams and Sanver \(2009\)](#) regarding the normative properties of FV and PAV. This is how we managed to show that – FV and PAV are Pareto optimal as they never elect a Pareto dominated candidate; –they are homogeneous; –FV and PV always elect the absolute Condorcet winner when it exists; –on the domain where there is an absolute Condorcet winner, these rules always elect a socially acceptable candidate, they never elect a socially unacceptable candidate and they are resistant to manipulation by clones. Nonetheless, these rules do not meet the cancellation property, the reinforcement criterion and they are vulnerable to the No-show paradox and to the truncation paradox. We managed to find some condition under which these rules always meet the reinforcement criterion or are not sensitive to the No-show paradox. It turns out that FV and PAV satisfy and fail the same criteria; they possess two properties that AV does not: Pareto optimality and the fact of always electing the absolute Condorcet winner when he exists. AV for its part, meets two criteria that FV and PAV do not: reinforcement and non-vulnerability to the No-show paradox.

Even if by definition, there is a certain advantage of FV and PAV over AV regarding the respect of the majority criteria, we wanted to measure the extent of this advantage. Thus, for voting situations with three candidates, we calculated the probabilities that these rules would elect the (absolute) Condorcet winner or the (absolute) Condorcet loser. Our analysis shows that in terms of the election of the Condorcet winner, PAV performs better than BR who dominates FV. When it comes to electing the absolute Condorcet winner, PAV and FV dominate BR, AV and PR. To prevent the election of an (absolute) Condorcet loser, FV and PAV perform better than AV and PR.

Our analysis shows that FV and PAV tend to deliver on the promise of being rules that could reconcile the advocates of score rules with those of approval voting. FV and PAV share the simplicity that characterizes AV, and with scoring rules they share the constraint of ranking candidates, which can be a daunting task when there is a large number of candidates.

Appendices

A.Proof of Proposition 8

Assume that an electorate is divided in two disjoint groups of voters N_1 ($|N_1| = n_1$) and N_2 ($|N_2| = n_2$) such that $N_1 \cap N_2 = \emptyset$ and $N_1 \cup N_2 = N$ ($|N| = n_1 + n_2 = n$).

A.1. PAV and the reinforcement criterion

Since AV meets the reinforcement condition and that it is equivalent to Rule 1 of PAV, it follows that PAV meets the reinforcement condition if the winner in each group is elected by one by Rule 1i or 1ii. To complete the proof, let us show that this is no longer the case in the other configurations. So, consider the following profiles

Profile 1	Profile 2		Profile 3	
1 : $a \succ b \succ c$	1 : $a \succ b \succ c$;	2 : $a \succ c \succ b$;	2 : $a \succ b \succ c$;	2 : $c \succ a \succ b$;
3 : $b \succ a \succ c$	1 : $c \succ a \succ b$;	2 : $c \succ b \succ a$;	1 : $b \succ c \succ a$;	1 : $b \succ a \succ c$;
2 : $c \succ b \succ a$	1 : $b \succ c \succ a$;			

Profile 4	Profile 5	Profile 6	Profile 7	
1 : $a \succ c \succ b$	1 : $a \succ c \succ b$	2 : $a \succ b \succ c$;	2 : $a \succ c \succ b$;	1 : $b \succ a \succ c$;
1 : $b \succ c \succ a$	1 : $b \succ a \succ c$	1 : $b \succ c \succ a$;	2 : $b \succ a \succ c$;	1 : $b \succ c \succ a$;
2 : $c \succ a \succ b$	1 : $c \succ a \succ b$	1 : $b \succ c \succ a$;	3 : $c \succ b \succ a$	
		2 : $c \succ a \succ b$;		

It is easy to check that a is elected in each of the seven profiles: through Rule 1ii in Profile 1, through Rule 2i in Profiles 2, 4 and 5, through Rule 2ii in Profiles 3, 6 and 7. When Profiles 1 and 2 are merged, b wins; this is also the case when Profiles 1 and 3 are merged. When Profiles 2 and 3 or Profiles 4 and 5 or Profiles 6 and 7 are merged, c wins. It follows from the profiles above that if a candidate wins with Rule 1 (1i or 1ii) in one group of voters and with Rule 2 (2i or 2ii) in an other group, he may not win when both groups are merged. We reach the same conclusion if a candidate wins with Rule 2i in one group of voters and with Rule 2ii in an other group, or when a candidate wins through Rule 2i (resp. 2ii) in both groups of voters.

We can give a summary that reflects whether or not the criterion is met as follows:

		N_2			
		Rule 1i	Rule 1ii	Rule 2i	Rule 2ii
N_1	Rule 1i	yes	yes	no	no
	Rule 1ii	yes	yes	no	no
	Rule 2i	no	no	no	no
	Rule 2ii	no	no	no	no

Thus, PAV always meets the criterion if the winner in each of the two groups of voters is determined by Rule 1i or Rule 1ii or both. In the other cases, PAV may fail the reinforcement condition.

A.2. FV and the reinforcement criterion

Assume that candidate a is the FV winner at level l in both groups N_1 and N_2 . Let us denote by $S_j^l(a)$ the l -level score of a in group j ($j = 1, 2$). We distinguish two cases: (i) at l , a is the only majority approved in each group. This means for all other candidate b , we get $S_1^l(a) > \frac{n_1}{2} \geq S_1^l(b)$ and $S_2^l(a) > \frac{n_2}{2} \geq S_2^l(b)$. Consider the profile obtained when both populations are merged and assume

that b wins at a given level r . It is obvious that we get a contradiction for $r \geq l$ since $S_1^l(a) + S_2^l(a) > \frac{n}{2} \geq S_1^l(b) + S_2^l(b)$. Since it is assumed that a is the only majority approved in each group at l , it follows that for all $r < l$, we get $S_j^r(a) \leq \frac{n_j}{2} < S_j^l(a)$ and $S_j^r(b) \leq S_j^l(b) \leq \frac{n_j}{2}$. Therefore, when both groups are merged, it is impossible at level r for b to be majority approved or to score more than a . So, if a is the only majority approved in each group at a given level l , he remains elected when both groups are merged.

(ii) at l , a has the greatest score among the majority approved candidates in each group. This means that $S_1^l(a) > S_1^l(b) > \frac{n_1}{2} \geq S_1^l(c)$ and $S_2^l(a) > S_2^l(b) > \frac{n_2}{2} \geq S_2^l(c)$. What we have in (i) implies that c can never win when the two groups merge.

To complete the proof, let us use some profile to show that when a same FV winner is determined in two groups at two different levels, he may not remain elected when both groups merge.

Profile 1	Profile 2
$2 : \underline{a} \succ \underline{b} \succ c$	$1 : \underline{a} \succ \underline{b} \succ c; \quad 1 : \underline{b} \succ c \succ a;$
$1 : \underline{a} \succ \underline{b} \succ c$	$2 : \underline{a} \succ \underline{b} \succ c; \quad 1 : \underline{c} \succ b \succ a;$
$2 : \underline{b} \succ c \succ a$	$2 : \underline{b} \succ \underline{a} \succ c;$

In Profile 1, a wins at the first level since $S^1(a) = 3$, $S^1(b) = 2$ and $S^1(c) = 0$; he also wins with Profile 2 at level 2 since $S^1(a) = S^1(b) = 3$, $S^1(c) = 1$, $S^2(a) = 5$, $S^2(b) = 4$ and $S^2(c) = 1$. When both profiles are merged, b wins since $S^1(a) = 6$, $S^1(b) = 5$, $S^1(c) = 1$, $S^2(a) = 8$, $S^2(b) = 18$ and $S^2(c) = 1$. So, FV may fail the reinforcement criterion when the winner in both groups is elected at two different levels of preferences.

B. Proof of Proposition 10

B.1. PAV is vulnerable to the No-show paradox only when the winner is determined by Rule 2i.

It is well known that AV is not vulnerable to the No-show paradox (see [Felsenthal, 2012](#)); as rule 1 of PAV is equivalent to AV under, it follows that under Rule 1, PAV is not vulnerable to the No-show paradox.

Consider a voting situation where a is PAV winner. Assume a group of β voters ($\beta \geq 1$) who decide to not show up in order to favor a more preferred candidate b . Obviously, if these voters do not approve of a in the original profile, the maneuver is futile. Thus, suppose that these voters approve of a in the original profile. When they abstain, the AV score of each candidate they approved decreases by β . Let us discuss each of the possible configurations.

Consider the configuration where candidate a was the winner under Rule 2i. First, let us assume that b was not among the majority approved candidates ($\mathbf{AV}(b, \pi) < \frac{n}{2}$) and that he wins after abstention. After abstention, $\mathbf{AV}(b, \pi) - \beta < \frac{n-\beta}{2}$; b cannot win if $\mathbf{AV}(a, \pi) - \beta > \frac{n-\beta}{2}$; if $\mathbf{AV}(b, \pi) - \beta < \frac{n-\beta}{2}$, b wins if $\mathbf{AV}(b, \pi) - \beta > \mathbf{AV}(a, \pi) - \beta$ which is equivalent to $\mathbf{AV}(b, \pi) > \mathbf{AV}(a, \pi)$: this contradicts $\mathbf{AV}(a, \pi) > \frac{n}{2}$. So, it is not possible to favor b . Let us now assume that b was among the majority approved candidates ($\mathbf{AV}(b, \pi) > \frac{n}{2}$) and that

he wins after abstention. It is obvious after abstention that b cannot win if $\mathbf{AV}(a, \pi) - \beta > \frac{n-\beta}{2}$. After abstention, if $\mathbf{AV}(a, \pi) - \beta < \frac{n-\beta}{2}$, two cases are possible:

- If $\mathbf{AV}(a, \pi) > \mathbf{AV}(b, \pi)$, it is not possible to favor b ; let us show how. Given $\mathbf{AV}(a, \pi) > \mathbf{AV}(b, \pi)$, if $\mathbf{AV}(b, \pi) - \beta > \frac{n-\beta}{2}$ and b wins, this means that $\mathbf{AV}(b, \pi) > \mathbf{AV}(a, \pi)$; we get a contradiction. For $\mathbf{AV}(b, \pi) - \beta < \frac{n-\beta}{2}$, b wins if $\mathbf{AV}(b, \pi) - \beta > \mathbf{AV}(a, \pi) - \beta$ which is tantamount to $\mathbf{AV}(b, \pi) > \mathbf{AV}(a, \pi)$: a contradiction.
- If $\mathbf{AV}(a, \pi) < \mathbf{AV}(b, \pi)$,
 - it is possible to favor b since it possible to get $\mathbf{AV}(b, \pi) - \beta > \frac{n-\beta}{2}$ such that b wins as illustrated by the following profile with 3 candidates and 19 voters.

$$\begin{array}{lll} 1 : \underline{a} \succ b \succ c & 6 : \underline{b} \succ a \succ c & 3 : \underline{c} \succ a \succ b \\ 4 : \underline{a} \succ \underline{b} \succ c & 3 : \underline{b} \succ \underline{a} \succ c & 2 : \underline{c} \succ \underline{a} \succ b \end{array}$$

In this profile, $\mathbf{AV}(a, \pi) = 10$, $\mathbf{AV}(b, \pi) = 13$ and $\mathbf{AV}(c, \pi) = 5$. Candidates a and b are majority approved and a wins since $n_{ab} = 10 > n_{ba} = 9$. Assume that the 3 voters with $\underline{b} \succ \underline{a} \succ c$ abstain. In the new profile π' with 16 voters, the scores are $\mathbf{AV}(a, \pi') = 7$, $\mathbf{AV}(b, \pi') = 10$ and $\mathbf{AV}(c, \pi') = 5$: b wins since he now the only majority approved candidates.

- it is possible to favor b since it possible to get $\mathbf{AV}(b, \pi) - \beta < \frac{n-\beta}{2}$ such that b wins as illustrated by the following profile with 3 candidates and 18 voters.

$$\begin{array}{lll} 1 : \underline{a} \succ b \succ c & 3 : \underline{b} \succ a \succ c & 5 : \underline{c} \succ a \succ b \\ 4 : \underline{a} \succ \underline{b} \succ c & 4 : \underline{b} \succ \underline{a} \succ c & 1 : \underline{c} \succ \underline{a} \succ b \end{array}$$

In this profile, $\mathbf{AV}(a, \pi) = 10$, $\mathbf{AV}(b, \pi) = 11$ and $\mathbf{AV}(c, \pi) = 6$. Candidates a and b are majority approved and a wins since $n_{ab} = 11 > n_{ba} = 7$. Assume that the 4 voters with $\underline{b} \succ \underline{a} \succ c$ abstain. In the new profile π' with 14 voters, the scores are $\mathbf{AV}(a, \pi') = 6$, $\mathbf{AV}(b, \pi') = 7$ and $\mathbf{AV}(c, \pi') = 6$: no candidate is majority approved then b wins since he get the highest AV score.

Consider now the configuration where candidate a was the winner under Rule 2ii. If b was not among the majority approved candidates, the same reasoning as above applies and b cannot win after abstention. Let us assume that b was among the majority approved candidates; as a wins, this means that he has the greatest AV scores among the majority approved ($\mathbf{AV}(a, \pi) > \mathbf{AV}(b, \pi)$). Candidate b may win after abstention if the new scores are such that $\frac{n-\beta}{2} < \mathbf{AV}(a, \pi) - \beta < \mathbf{AV}(b, \pi) - \beta$ or $\mathbf{AV}(a, \pi) - \beta < \frac{n-\beta}{2} < \mathbf{AV}(b, \pi) - \beta$; in each case, these conditions lead to $\mathbf{AV}(a, \pi) < \mathbf{AV}(b, \pi)$ which is a

contradiction. Thus, PAV is vulnerable to the No-show paradox only when the winner is determined by Rule 2i.

B.2. FV is not vulnerable to the No-show paradox only when the winner is determined at level $l = 1$ or $l = m$.

Given that a FV winner is determined at $l = 1$, any abstention of voters who do not rank this winner first does not affect the approval of the level $l = 1$. So, for $l = 1$, the no-show paradox never occurs.

Let us assume that a is the FV winner at level $l = m$ and that a group of β voters try to favor a more preferred candidate b by abstaining. Assume at $l = m$ that a is the only majority approved, this means that $S^m(a) > \frac{n}{2} > S^m(b)$; after abstention, we may get i) $S^m(a) - \beta > \frac{n}{2}$ and $S^m(b) - \beta < \frac{n}{2}$ or ii) $S^m(a) - \beta < \frac{n}{2}$ and $S^m(b) - \beta < \frac{n}{2}$. Candidate a remains the winner under i); candidate b wins under ii) if $S^m(a) - \beta < S^m(b) - \beta$ which is equivalent to $S^m(a) < S^m(b)$: this contradicts that a was the only majority approved. Let us now assume that a and b are among the majority approved; since a wins, this means that $S^m(a) > S^m(b) > \frac{n}{2}$. After truncation, we can get $S^m(a) - \beta > S^m(b) - \beta > \frac{n}{2}$ or $\frac{n}{2} > S^m(a) - \beta > S^m(b) - \beta$; in each case, b cannot be the winner. It follows that FV is not sensitive to the No-show paradox when the winner is determined at level $l = 1$ or $l = m$.

Now, let us assume that a is the FV winner at level l ($l \neq 1, m$) and consider the following profile with 12 voters and 3 candidates.

$$\begin{array}{ll} 2 : \underline{a} \succ b \succ c; & 1 : \underline{a} \succ \underline{c} \succ \underline{b}; \\ 4 : \underline{b} \succ \underline{a} \succ c; & 2 : \underline{b} \succ c \succ a; \\ 1 : \underline{c} \succ \underline{a} \succ \underline{b}; & 1 : \underline{c} \succ \underline{b} \succ a; \\ 1 : \underline{c} \succ b \succ a; & \end{array}$$

With this profile, no candidate wins at $l = 1$ since $S^1(a) = S^1(c) = 3$ and $S^1(b) = 6$; at $l = 2$, $S^2(a) = 8$, $S^2(b) = 7$ and $S^2(c) = 4$: candidate a wins. Assume that the 4 voters with $\underline{b} \succ \underline{a} \succ c$ abstain. In the new profile, we get: $S^1(a) = S^1(c) = 3$, $S^1(b) = 2$ at $l = 1$ and no one wins; $S^2(a) = S^2(c) = 4$ and $S^2(b) = 3$ and no one wins. At $l = 3$, $S^3(a) = S^3(c) = 4$ and $S^3(b) = 5$ and b wins; by abstaining, the 4 voters have favored b who is preferred to a . Thus, FV is vulnerable to the no-show paradox when the winner is determined at level of approval $l \neq 1$ and $l \neq m$.

Since the vulnerability of a voting rule to the No-show paradox leads to its vulnerability to the truncation paradox, FV and PAV would therefore be vulnerable to the truncation paradox. Preference truncation is efficient under FV and PAV only if it consists, as shown by [Brams and Sanver \(2009\)](#), in a contraction of the set of approved candidates.

References

Brams S.J. (1982) The AMS nominating system is vulnerable to truncation of preferences. *Notices of the American Mathematical Society* 29: 136-138.

- Brams S.J. and Fishburn P.C. (1978) Approval Voting. *American Political Science Review* 72 (3): 831-847.
- Brams S.J. and Sanver R (2009) Voting systems that combine approval and preferences. In: Brams SJ, Gehrlein WV, Roberts FS (eds) *The mathematics of preference, choice and order*. Springer, Berlin, pp 215–237.
- Bruns W. and Ichim B. (2018) Polytope volume by descent in the face lattice and applications in social choice. <https://arxiv.org/abs/1807.02835>.
- Bruns W., Ichim B. and Söger C. (2019) Computations of volumes and Ehrhart series in four candidates elections, *Annals of Operations Research*, 280: 241-265
- Bruns W. and Söger C.(2015) The computation of generalized Ehrhart series in normaliz. *Journal of Symbolic Computation* 68: 75–86.
- Diss M. and Merlin V. (2021) *Evaluating Voting Systems with Probability Models, Essays by and in honor of William V. Gehrlein and Dominique Lepelley*, Studies in Choice and Welfare, Springer, Berlin.
- Diss M., Merlin V. and Valognes F. (2010) On the condorcet efficiency of approval voting and extended scoring rules for three alternatives. In: Laslier J-F, Sanver RM (eds) *Handbook on approval voting*. Springer, Berlin, pp 255–283.
- Doron G. and Kronick R. (1977) Single Transferable Vote: An example of a Oerverse Social Choice Function. *American Journal of Political Science* 21(2): 303-311.
- El Ouafdi A., Lepelley D., and Hatem Smaoui H. (2020a) On the Condorcet efficiency of evaluative voting (and other voting rules) with trichotomous preferences. *Annals of Operations Research* 289:227–241.
- Felsenthal D.S. (2012). Review of paradoxes afflicting procedures for electing a single candidate, in *Electoral Systems: Paradoxes, Assumptions, and Procedures*, Studies in Choice and Welfare, Felsenthal, DS, & Machover, M. (Eds.), Springer Berlin Heidelberg.
- Felsenthal D.S. (1989) On combining approval with disapproval voting. *Behavioral Science* 34: 53–60.
- Fishburn P.C. and Brams S.J. (1983) Paradoxes of preferential voting. *Mathematics Magazine* 56, 207–214.
- Fishburn P.C. and Brams S.J. (1984) Manipulability of voting by sincere truncation of preferences. *Public Choice* 44, 397–410.

- Gehrlein W.V. and Fishburn P.C. (1976) The probability of the paradox of voting: A computable solution. *Journal of Economic Theory*, 13: 14–25.
- Gehrlein W.V. and Lepelley D. (1998) The Condorcet efficiency of approval voting and the probability of electing the Condorcet loser. *Journal of Mathematical Economics* 29: 271–283.
- Gehrlein W.V. and Lepelley D. (2017) *Elections, Voting Rules and Paradoxical Outcomes*. Springer, Berlin/Heidelberg.
- Gehrlein W.V. and Lepelley D. (2015) The Condorcet Efficiency Advantage that Voter Indifference Gives to Approval Voting Over Some Other Voting Rules. *Group Decis Negot* 24: 243–269.
- Gehrlein W.V. and Lepelley D. (2011) *Voting Paradoxes and Group Coherence*. Springer, Berlin/Heidelberg.
- Gehrlein W.V., Lepelley D. and Moyouwou I. (2016) A note on Approval Voting and electing the Condorcet loser. *Mathematical Social Sciences* 80:115–122.
- Guilbaud G-T. (1952) Les théories de l'intérêt général et le problème logique de l'agrégation. *Economie Appliquée* 5(4): 501-584
- Ju B-G (2010) Collective choices for simple preferences. In: Laslier J-F, Sanver MR (eds) *Handbook on Approval Voting*. Studies in Choice and Welfare, Springer-Verlag Berlin Heidelberg, pp 41–90.
- Kaminski M. (2018) Spoiler effects in proportional representation systems: evidence from eight Polish parliamentary elections, 1991-2015. *Public Choice* 176: 441–460.
- Kamwa E. (2022) Scoring Rules, Ballot Truncation, and the Truncation Paradox.
- Kamwa E. and Moyouwou I. (2021) Susceptibility to manipulation by sincere truncation: the case of scoring rules and scoring runoff systems. In Diss M. and Merlin V. (editors), *Evaluating Voting Systems with Probability Models, Essays by and in honor of William Gehrlein and Dominique Lepelley*, Springer, Berlin.
- Kamwa E. (2019) Condorcet Efficiency of the Preference Approval Voting and the Probability of Selecting the Condorcet Loser. *Theory and Decision* 87(3): 299–320.
- Kuga K. and Nagatani H. (1974) Voter Antagonism and the Paradox of Voting. *Econometrica* 42(6): 1045–1067.

- Laslier J-F. and Sanver R.M. (2010) Handbook on Approval Voting. Studies in Choice and Welfare. Springer Berlin Heidelberg.
- Lepelley D. (1993) On the probability of electing the Condorcet loser. *Mathematical Social Sciences* 25: 105–116.
- Mahajne M. and Volij O. (2018) The socially acceptable scoring rule. *Social Choice and Welfare* 51(2): 223-233.
- Merlin V., Özkal-Sanver I. and Sanver M.R. (2019) Compromise rules revisited. *Group Decision and Negotiation* 28(1):63-78.
- Miller N.R. (2017) Closeness matters: monotonicity failure in IRV elections with three candidates. *Public choice* 173(1): 91–108.
- Myerson R.B. (1995) Axiomatic derivation of scoring rules without the ordering assumption. *Social Choice and Welfare* 12: 59-74.
- Nurmi H.(2004) A comparison of some distance-based choice rules in ranking environments. *Theory and Decision* 57: 5-24.
- Nurmi H. (1999) *Voting Paradoxes and How to Deal with Them*. Springer-Verlag.
- Nurmi H.(1987) *Comparing Voting Systems* D. Reidel Publishing Company.
- Özkal-Sanver I. and Sanver M.R (2004) Efficiency in the degree of compromise: a new axiom for social choice theory. *Group Decision Negotiation* 13:375-380
- Regenwetter M. and Tsetlin I. (2004) Approval voting and positional voting methods: Inference, relationship, examples. *Soc Choice Welfare* 22: 539–566.
- Sanver M.R. (2010) Approval as an intrinsic part of preference. In: Laslier J-F, Sanver RM (eds) Handbook on approval voting. Springer, Berlin, pp. 469–481.
- Schürmann A. (2013) Exploiting polyhedral symmetries in social choice. *Social Choice and Welfare* 40(4): 1097–1110.
- Sertel MR (1986) Lectures notes in microeconomics. Bogazici University, unpublished Manuscript.
- Sertel MR and Yilmaz B. (1999) The majoritarian compromise is majoritarian optimal and subgame perfect implementable. *Social Choice Welf* 16:615–627
- Smith J.H. (1973) Aggregation of preferences with variable electorate. *Econometrica* 41: 1027-1041.

- Tideman T. (1987) Independence of clones as a criterion for voting rules. *Social Choice and Welfare* 4: 185–206.
- Xu Y. (2010) Axiomatizations of approval voting. In: Laslier J-F, Sanver MR (eds) *Handbook on Approval Voting. Studies in Choice and Welfare*, Springer-Verlag Berlin Heidelberg, pp 91–102.
- Young H.P. (1975) Social Choice scoring functions. *SIAM Journal of Applied Mathematics* 28: 824-838.
- Young H.P. (1974) An axiomatization of Borda’s rule. *Journal of Economic Theory* 9: 43-52.