ON THE DYNAMICS OF SHALLOW ICE SHEETS. MODELLING AND ANALYSIS

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Abstract. In this paper we formulate a model describing the evolution of thickness of a shallow ice sheet. The thickness of the ice sheet is constrained to be nonnegative. This renders the problem under consideration an obstacle problem. A rigorous analysis shows that the model is thus governed by a set of variational inequalities that involve nonlinearities in the time derivative and in the elliptic term, and that it admits solutions, whose existence is established by means of a semi-discrete scheme and the penalty method.

Contents

1. Introduction

The study of ice sheets melting and its correlation with the global warming problem has been attracting the interest of experts from all over the branches of science. The mathematical literature related to ice sheets and glaciers is vast and abundant; in this direction we mention, for instance, the papers [25, 26, 28, 29, 34, 41, 42], and the celebrated article by W.D. Hibler [23] to which we refer later on. The existence of solutions to Hibler's model has recently been established by Titi and his associates in the pre-print [33] and by Brandt and his associates in the pre-print [5].

In the recent pre-print [18], Figalli, Ros-Oton and Serra studied the phase transition of ice melting to water as a Stefan problem. The results established in this paper provide a refined understanding of the Stefan problem's singularities and answer some long-standing open questions in the field of free-boundary problems.

In this article we propose and study a mathematical model describing the evolution of the thickness of a shallow ice sheet. The ice thickness of a shallow ice sheet evolves as a consequence of many factors like, for instance, the rate at which snow deposits, the rate at which melting occurs, as well as the velocity at which the glacier slides along the lithosphere. Since the ice thickness level is constrained to remain on or above the lithosphere at all times, the problem under consideration can be regarded as a time-dependent obstacle problem.

In 2002 Calvo, Díaz, Durany, Schiavi and Vázquez [9] studied a simplified version of the evolution of the thickness of a shallow ice sheet. Indeed, in their article, the authors only considered one spatial direction, they assumed the basal velocity to be smooth, and assumed the lithosphere to be flat.

Ten years later, in 2012, Jouvet and Bueler [27] studied the steady (i.e., time-independent) version of the problem considered in [9] where, this time, two spatial directions and a more general lithosphere topography were taken into account.

It appears that the ice thickness evolution as a time-dependent obstacle problem over a two-dimensional spatial domain has not been addressed in the literature yet. The purpose of this article is exactly to address this problem.

This paper is divided into four sections (including this one). In section 2 we present the main notations we shall be using throughout the manuscript, and we formally derive the governing equations for our model.

In section 3 we formulate the "penalized" version corresponding to the obstacle problem introduced in section 2, and we establish the existence of solutions to this model by resorting to a series of new preparatory results as well as the Dubinskii's compactness theorem.

Finally, in section 4, we pass to the limit in the penalty parameter and we recover the actual model corresponding to the model we formally derived in section 2. This model, for which we also define the rigorous concept of solution, will take the form of a set of variational inequalities. The presence of the constraint, which adds a further nonlinearity to the two already considered (the first nonlinearity appears in the evolutionary term, while the second nonlinearity is the p-Laplacian), requires new strategies to be adopted in the limit passage in order to overcome the arising mathematical difficulties. In particular, we will see that vector-valued measure will play a critical role in the analysis.

2. Analytical notation and formal derivation of the model

We denote by $(\mathbb{R}^n, \cdot\,)$ the n-dimensional Euclidean space equipped with its standard inner product. Given an open subset Ω of $\mathbb{R}^n$ notations such as $L^2(\Omega)$, $H^m(\Omega)$, or $H^m_0(\Omega)$, $m \geq 1$, designate the usual Lebesgue and Sobolev spaces, and the notation $\mathcal{D}(\Omega)$ designates the space of all functions that are infinitely differentiable over Ω and have compact support in Ω. The notation $\|\cdot\|_X$ designates the norm in a normed vector space X. The dual space of a vector space X is denoted by X^* and the duality pair between X^* and X is denoted by $\langle \cdot\,, \cdot \rangle_{X^*,X}$. Spaces of vector-valued functions are denoted with boldface letters. Lebesgue spaces defined over a bounded open interval I (cf. [30]), are denoted $L^p(I;X)$, where X is a Banach space and $1 \leq p \leq \infty$. The notation $\|\cdot\|_{L^p(I;X)}$ designates the norm of the Lebesgue space $L^p(I;H)$. Sobolev spaces defined over a bounded open interval I (cf. [30]), are denoted $W^{m,p}(I;X)$, where X is a Banach space, $m \geq 1$ and $1 \leq p \leq \infty$. The notation $\|\cdot\|_{W^{m,p}(I;X)}$ designates the norm of the Sobolev space $W^{m,p}(I;X)$.

A *domain in* $\mathbb{R}^n$ is a bounded and connected open subset Ω of $\mathbb{R}^n$, whose boundary $\partial\Omega$ is Lipschitz-continuous, the set Ω being locally on a single side of $\partial\Omega$, viz. [10].

Let Ω be a domain in $\mathbb{R}^2$ and let $x = (x_1, x_2)$ be a generic point in $\overline{\Omega}$. Let $\boldsymbol{\nu}$ denote the outer unit vector field along the boundary of Ω. Let ∇ denote the gradient operator with respect to the coordinates x_1 and x_2, namely,

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right),$$

and let the symbol $(\nabla \cdot\,)$ denote the divergence operator, namely, for any vector field $\boldsymbol{v} = (v_1, v_2) : \Omega \to \mathbb{R}^2$

$$\nabla \cdot \boldsymbol{v} := \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}.$$

Let $T > 0$ be given and let us consider the time interval $(0, T)$, any time instant in which is denoted by the letter t.

The lithosphere elevation is described by the function $b : \overline{\Omega} \to \mathbb{R}$. Positive values of b are associated with altitudes above the sea level, whereas negative values of b are associated with altitudes below the sea level. We assume, without loss of generality, that the lithosphere topography does not change throughout the observation time.

The elevation of the upper ice surface is described by the function $h : [0, T] \times \overline{\Omega} \to \mathbb{R}$. It is immediate to observe that

$$h \geq b \quad \text{in } [0, T] \times \overline{\Omega}. \tag{2.1}$$

The ice thickness $H := h - b$ is thus nonnegative in $[0,T] \times \overline{\Omega}$. This consideration implies that the problem of studying the evolution of the sea ice thickness can be regarded as an obstacle problem, where the *obstacle* is represented by the lithosphere.

This constraint implies the existence of a *free boundary* [6, 8, 17]. For all, or possibly almost every (a.e. in what follows) $t \in (0,T)$, we define the set Ω_t^+ by:

$$\Omega_t^+ := \{x \in \Omega; h(t,x) > b(x)\} = \{x \in \Omega; H(t,x) > 0\}. \tag{2.2}$$

The set Ω_t^+ denotes the region of Ω which is covered with ice at the time instant t. The corresponding free boundary is the set

$$\Gamma_{f,t} := \Omega \cap \partial\Omega_t^+. \tag{2.3}$$

The variation of the ice thickness H is influenced by two source terms: the surface-mass balance a_s, which is associated with ice accumulation and ablation rate, and the basal melting rate a_b. The function a_b is equal to zero when the basal temperature is smaller than the ice melting point; otherwise, it is greater than zero. The functions a_s and a_b, in general, solely depend on the horizontal location and the surface elevation. The terms a_s and a_b can thus be regarded as functions

$$a_s : [0,T] \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}_0^+,$$
$$a_b : [0,T] \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}_0^+.$$

We define the function $a : [0,T] \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by:

$$a := a_s - a_b,$$

and we recall that this function is *assumed* to be continuous in $[0,T] \times \overline{\Omega}$, strictly positive in a subset of Ω_t^+, strictly negative in $\Omega_t^- := \Omega \setminus (\Omega_t^+ \cup \Gamma_{f,t})$, and nonnegative in the complementary region of Ω_t^+ characterised by ablation [22, 27]. Ice sheets are incompressible, non-Newtonian, gravity driven flows [19, 22]. Ice flows from areas of Ω_t^+ characterised by accumulation (i.e., regions where $a_s > 0$) to areas of Ω_t^+ characterised by ablation (i.e., regions where $a_s \le 0$).

Ice thickness is also influenced by the basal sliding velocity $\boldsymbol{U}_b$, which can be regarded as a given vector field in $\mathbb{R}^2$ solely depending on the horizontal position, i.e.,

$$\boldsymbol{U}_b : \overline{\Omega} \to \mathbb{R}^2.$$

and we recall that (cf., e.g., [22]), when the ice base is frozen, we have $\boldsymbol{U}_b = \boldsymbol{0}$.

The vector field $\boldsymbol{U} : [0,T] \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^2$ denotes the horizontal ice flow velocity; the vector field $\boldsymbol{Q}$ denotes the *volume flux*, defined as the integral of the horizontal ice flow velocity $\boldsymbol{U}$ with respect to the vertical direction (cf., e.g., equation (5.47) of [22]), namely:

$$\boldsymbol{Q} : [0,T] \times \overline{\Omega} \to \mathbb{R}^2 \qquad \text{and} \qquad \boldsymbol{Q} := \int_b^h \boldsymbol{U} \, \mathrm{d}z. \tag{2.4}$$

In the same spirit as Jouvet & Bueler [27], we *assume* that for each $t \in (0,T)$ there is no volume ice flow towards Ω_t^-, i.e.,

$$\boldsymbol{Q} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_{f,t}. \tag{2.5}$$

In view of (2.5), we can naturally extend the volume flux $\boldsymbol{Q}$ by zero outside Ω_t^+, i.e.,

$$\boldsymbol{Q} = \boldsymbol{0} \quad \text{in } \Omega_t^-. \tag{2.6}$$

Besides, once again in the spirit of [27], we have that

$$H = 0 \text{ on } \Gamma_{f,t} \qquad \text{and} \qquad H = 0 \text{ on } \partial\Omega. \tag{2.7}$$

The evolution of the ice thickness is governed by the *ice thickness equation* (cf., e.g., equation (5.55) in [22]), that we recall here below:

$$\frac{\partial H}{\partial t} = -\nabla \cdot \boldsymbol{Q} + a. \tag{2.8}$$

The free boundary $\Gamma_{f,t}$ and the region Ω_t^+ are correctly described *only* through a weak formulation of the problem under consideration. Prior to rigorously stating the weak formulation of the problem under consideration, we have to *formally* recover the boundary value problem associated with (2.8). To this aim we *first* fix a *smooth enough* test function v such that $v \geq b$ in $[0,T] \times \overline{\Omega}$ and, *second*, we multiply (2.8) by $(v-h)$ and integrate over Ω. As a result of this manipulation of (2.8), the following identity holds for all $t \in (0,T)$:

$$\int_\Omega \frac{\partial H}{\partial t}(v-h)\,\mathrm{d}x + \int_\Omega \nabla\cdot\boldsymbol{Q}(v-h)\,\mathrm{d}x = \int_\Omega a(v-h)\,\mathrm{d}x. \tag{2.9}$$

By virtue of the fact that $h \geq b$ in $[0,T]\times\overline{\Omega}$, we can specialise $v = h$. The definition of Ω_t^- in turn implies that $\Omega = \Omega_t^+ \sqcup \Omega_t^- \cup \Gamma_{f,t}$, where the symbol $\sqcup$ denotes the union of two disjoint sets. An application of (2.6) and of the Gauss-Green theorem to (2.9) gives

$$\begin{aligned}
&\int_{\Omega_t^+} \frac{\partial H}{\partial t}(v-h)\,\mathrm{d}x + \int_{\Omega_t^-} \frac{\partial H}{\partial t}(v-h)\,\mathrm{d}x\\
&\quad + \int_{\Omega_t^+} \nabla\cdot\boldsymbol{Q}(v-h)\,\mathrm{d}x + \int_{\Omega_t^-} \nabla\cdot\boldsymbol{Q}(v-h)\,\mathrm{d}x\\
&\quad - \int_{\Omega_t^+} a(v-h)\,\mathrm{d}x - \int_{\Omega_t^-} a(v-h)\,\mathrm{d}x\\
&= \int_{\Omega_t^+} \left(\frac{\partial H}{\partial t} + \nabla\cdot\boldsymbol{Q} - a\right)(v-h)\,\mathrm{d}x + \int_{\Omega_t^-} \frac{\partial H}{\partial t}(v-h)\,\mathrm{d}x\\
&\quad + \underbrace{\int_{\Omega_t^-} \boldsymbol{Q}\cdot\boldsymbol{\nu}(v-h)\,\mathrm{d}\Gamma - \int_{\partial\Omega_t^-} \boldsymbol{Q}\cdot\nabla(v-h)\,\mathrm{d}x}_{=0 \text{ by (2.6) and the fact that } \partial\Omega_t^+=\partial\Omega_t^-} - \int_{\Omega_t^-} a(v-h)\,\mathrm{d}x.
\end{aligned} \tag{2.10}$$

In order to work out the following step of the boundary value problem recovery, let us recall that Ω_t^- denotes the region in Ω where the lithosphere is *not* covered with ice, i.e., we have $H = 0$ in Ω_t^-. Moreover, the ice thickness H in Ω_t^- *either* does not change ($\partial H/\partial t = 0$) *or* increases ($\partial H/\partial t > 0$); equivalently, the ice thickness H *cannot* diminish in Ω_t^-.

On the one hand, in the region Ω_t^+ the ice thickness evolution is governed by (2.8), so that we have

$$\int_{\Omega_t^+} \left(\frac{\partial H}{\partial t} + \nabla\cdot\boldsymbol{Q} - a\right)(v-h)\,\mathrm{d}x = 0, \quad \text{for all } t\in(0,T). \tag{2.11}$$

On the other hand, specialising $v = h+\varphi$ in (2.10), where $\varphi \in \mathcal{D}((0,T)\times\Omega)$, $\varphi \geq 0$ in $[0,T]\times\Omega$, recalling that $a \leq 0$ in Ω_t^-, and recalling the remark made above about the nonnegativeness of $\partial H/\partial t$ in Ω_t^-, we obtain

$$\int_{\Omega_t^-} \left(\frac{\partial H}{\partial t} - a\right)\varphi\,\mathrm{d}x \geq 0, \quad \text{for all } t\in(0,T). \tag{2.12}$$

Putting together (2.9)–(2.12) gives

$$\int_\Omega \left(\frac{\partial H}{\partial t} + \nabla\cdot\boldsymbol{Q} - a\right)\varphi\,\mathrm{d}x \geq 0, \quad \text{for all } \varphi\in\mathcal{D}((0,T)\times\Omega), \varphi\geq 0 \text{ in } [0,T]\times\overline{\Omega}. \tag{2.13}$$

The latter can be straightforwardly changed into:

$$\begin{aligned}
&\frac{\partial H}{\partial t} + \nabla\cdot\boldsymbol{Q} - a = 0, \quad \text{in } \Omega_t^+ = \{x\in\Omega; H(t,x)>0\} \text{ for all } t\in(0,T),\\
&\frac{\partial H}{\partial t} + \nabla\cdot\boldsymbol{Q} - a \geq 0, \quad \text{in } \Omega_t^- = \Omega\setminus(\Omega_t^+\cup\Gamma_{f,t}) \text{ for all } t\in(0,T).
\end{aligned} \tag{2.14}$$

Define $H_0 := h(0) - b$, and observe that $H_0 \geq 0$ by virtue of the observation made at the beginning of this section (section 2). Putting together (2.7) and (2.14) gives the sought free boundary value problem: *Find $H \geq 0$ satisfying:*

$$\begin{cases} \dfrac{\partial H}{\partial t} + \nabla \cdot \boldsymbol{Q} - a = 0, & \text{in } \Omega_t^+ = \{x \in \Omega; H(t,x) > 0\} \text{ for all } t \in (0,T), \\[2ex] \dfrac{\partial H}{\partial t} + \nabla \cdot \boldsymbol{Q} - a \geq 0, & \text{in } \Omega_t^- = \Omega \setminus (\Omega_t^+ \cup (\Omega \cap \partial \Omega_t^+)) \text{ for all } t \in (0,T), \\[2ex] H(t,\cdot) = 0, \quad \text{on } \partial\Omega \text{ for all } t \in (0,T), \\[2ex] H(0,\cdot) = H_0 \geq 0. \end{cases} \tag{2.15}$$

Multiplying the equation in the boundary value problem (2.15) by $(h - b)$, integrating over $(0,T) \times \Omega$, and observing that $h(t,\cdot) = b$ in Ω_t^- gives:

$$\int_0^T \int_\Omega \left(\frac{\partial H}{\partial t} + \nabla \cdot \boldsymbol{Q} - a \right) (h - b) \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{2.16}$$

Let $v = v(t,x)$ be any test function such that $v(t,x) \geq b(x)$ for a.e. $(t,x) \in (0,T) \times \Omega$. Multiplying the equations in the boundary value problem (2.15) by $(v - b)$, and integrating over $(0,T) \times \Omega$ gives:

$$\int_0^T \int_\Omega \left(\frac{\partial H}{\partial t} + \nabla \cdot \boldsymbol{Q} - a \right) (v - b) \, \mathrm{d}x \, \mathrm{d}t \geq 0. \tag{2.17}$$

Combining (2.16) and (2.17) gives:

$$\int_0^T \int_\Omega \left(\frac{\partial H}{\partial t} + \nabla \cdot \boldsymbol{Q} - a \right) (v - h) \, \mathrm{d}x \, \mathrm{d}t \geq 0. \tag{2.18}$$

The arbitrariness of the test function v taken as above finally allows us to write down the weak variational formulation associated with the boundary value problem (2.15): *Find $H \geq 0$ satisfying the variational inequalities:*

$$\int_0^T \int_\Omega \frac{\partial H}{\partial t} (v - h) \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_\Omega \boldsymbol{Q} \cdot \nabla (v - h) \, \mathrm{d}x \, \mathrm{d}t \geq \int_0^T \int_\Omega a(v - h) \, \mathrm{d}x \, \mathrm{d}t, \tag{2.19}$$

for all test function v such that $v \geq b$ for a.e. $(t,x) \in (0,T) \times \Omega$, and *satisfying the following boundary conditions along $\partial\Omega$*

$$H = 0 \quad \text{on } \partial\Omega,$$

and satisfying the following initial condition

$$H(0,x) = H_0(x), \qquad \text{for a.e. } x \in \Omega,$$

where $H_0 = h(0) - b$ is given, nonnegative and nonzero (see (2.1)).

Note that, at least for the time being, the rigorous concept of solution has not been defined yet as we did not specify the regularity of H and of the data. This task will be postponed to forthcoming sections.

Ice is viscous too; its viscosity is described in terms of the Glen power law (cf., e.g., equation (4.16) in [22]) with ice softness coefficient $A(x,z)$ and exponent $2.8 \leq p \leq 5$. The attainable values for p are suggested by laboratory experiments [21]. Regarding the ice softness as a function is motivated by the idea of coupling the ice thickness equation with a thermodynamic model [22, 24].

By equation (5.84) in [22], the horizontal ice flow velocity $\boldsymbol{U}$ can be expressed in terms of the elevation of the upper ice surface h via the following formula:

$$\boldsymbol{U}(\cdot,\cdot,z) = -2(\rho g)^{p-1} \left(\int_b^z A(s)(h - s)^{p-1} \, \mathrm{d}s \right) |\nabla h|^{p-2} \nabla h + \boldsymbol{U}_b. \tag{2.20}$$

Plugging formula (2.20) into (2.4) gives:

$$\begin{aligned}\boldsymbol{Q}(\,\cdot\,,\,\cdot\,,z) &= -2(\rho g)^{p-1}\left(\int_b^h \int_b^z A(s)(h-s)^{p-1}\,\mathrm{d}s\,\mathrm{d}z\right)|\nabla h|^{p-2}\nabla h + (h-b)\boldsymbol{U}_b\\ &= -2(\rho g)^{p-1}\left(\int_b^h A(s)(h-s)^{p}\,\mathrm{d}s\right)|\nabla h|^{p-2}\nabla h + (h-b)\boldsymbol{U}_b,\end{aligned} \tag{2.21}$$

where the latter equality is obtained by an integration by parts with respect to the variable z.

For sake of clarity, the magnitudes involved in the model are listed in the following table.

Variable	**Description**
$A = A(s)$	Ice softness in the Glen power law
b	Lithosphere elevation
g	Gravitational acceleration
H	Ice thickness
h	Upper ice surface elevation
a_s	Accumulation rate function
a_b	Ablation rate function
a	$a = a_s - a_b$ Balance function
$\boldsymbol{U}_b$	Basal sliding velocity
$\boldsymbol{U}$	Horizontal ice flow velocity
$\boldsymbol{Q}$	Ice volume flux
ρ	Ice density
p	Index in the Glen power law

Table 1. Quantities entering the model

As already observed by Jouvet & Bueler [27], the expression of $\boldsymbol{Q}$ in (2.21) exhibits a degenerate behaviour at the free boundary, in the sense that the gradient norm power blows up as h gets close to the free boundary of Ω. This degeneracy does not manifest in other large-scale models like the one proposed by Hibler in the seminal paper [23].

By contrast with the model we are considering, the model proposed by Hibler couples the shallow ice equation with a *mechanical* (hyperbolic) equation whose unknown is the horizontal ice flow velocity $\boldsymbol{U}$. The model we are considering follows the formulation originally proposed by J.W. Glen [20], according to which ice was regarded as a viscous fluid. The model proposed by Hibler, itself inspired by the article of M.D. Coon [12], is on the one hand *less precise* than Glen's model as the ice velocity $\boldsymbol{U}$ has a simplified expression. On the other hand, it achieves the goal of describing in the same instance the behaviours of ice as a solid and as a fluid.

The operation of averaging considerably simplifies the expression of the ice volume flux $\boldsymbol{Q}$, while the coupling of the shallow ice equation for the averaged ice thickness with the mechanical equation spares the effort of expressing the ice flow velocity $\boldsymbol{U}$ in terms of the ice thickness.

In order to overcome the difficulty arising as a result of the gradient degeneracy in the vicinity of the free boundary, we introduce the following transformation, originally suggested in [9]:

$$H := u^{(p-1)/2p}. \tag{2.22}$$

Observe that $H \geq 0$ in $[0,T] \times \overline{\Omega}$ if and only if $u \geq 0$ in $[0,T] \times \overline{\Omega}$. As a result of the transformation (2.22), we *first* obtain, formally,

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial t}(|u|^{\frac{3p-1}{2p}-2}\,u), \tag{2.23}$$

secondly, we obtain

$$\int_b^h A(s)(h-s)^p\,\mathrm{d}s = u^{\frac{(p+1)(p-1)}{2p}} \int_0^1 A(b + u^{\frac{p-1}{2p}}\,s')(1-s')^p\,\mathrm{d}s', \tag{2.24}$$

and, *third*, we obtain:

$$|\nabla h|^{p-2}\nabla h = \left(\frac{p-1}{2p}\right)^{p-1} u^{\frac{(-p-1)(p-2)}{2p}} \left|\nabla u + u^{(p+1)/2p}\nabla b\right|^{p-2} u^{\frac{-p-1}{2p}} \left(\nabla u + u^{(p+1)/2p}\nabla b\right). \tag{2.25}$$

For all $x \in \Omega$ and all $u = u(t,x) \in \mathbb{R}$, we define the vector field $\boldsymbol{\Phi} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^2$ by:

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}(x,u) := -\frac{2p}{p-1} u^{(p+1)/2p}\nabla b. \tag{2.26}$$

Plugging (2.26) into (2.25) gives:

$$|\nabla h|^{p-2}\nabla h = \left(\frac{p-1}{2p}\right)^{p-1} u^{\frac{(-p-1)(p-2)}{2p}} |\nabla u - \boldsymbol{\Phi}|^{p-2} u^{\frac{-p-1}{2p}} (\nabla u - \boldsymbol{\Phi}). \tag{2.27}$$

Finally, for all $x \in \Omega$ and all $u = u(t,x) \in \mathbb{R}$, we define the vector field $\boldsymbol{\Psi} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^2$ by,

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}(x,u) := (h-b)\boldsymbol{U}_b, \tag{2.28}$$

and the function $\tilde{a} : [0,T] \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by:

$$\tilde{a}(t,x,u) := a(t,x,b + u^{(p-1)/2p}). \tag{2.29}$$

Inserting (2.24)–(2.29) into (2.21) gives:

$$-\boldsymbol{Q} = 2\left(\rho g \frac{p-1}{2p}\right)^{p-1} \left[\int_0^1 A(b + u^{(p-1)/2p} s')(1-s')^p \,\mathrm{d}s'\right] |\nabla u - \boldsymbol{\Phi}|^{p-2}(\nabla u - \boldsymbol{\Phi}) - \boldsymbol{\Psi}. \tag{2.30}$$

For sake of brevity, we define the function $\mu : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ along $u = u(x,t)$ by:

$$\mu(x,u) := 2\left(\rho g \frac{p-1}{2p}\right)^{p-1} \left[\int_0^1 A(b + u^{(p-1)/2p} s')(1-s')^p \,\mathrm{d}s'\right]. \tag{2.31}$$

Observe that plugging (2.31) into (2.30) gives:

$$-\boldsymbol{Q} = \mu(x,u)|\nabla u - \boldsymbol{\Phi}|^{p-2}(\nabla u - \boldsymbol{\Phi}) - \boldsymbol{\Psi}. \tag{2.32}$$

Plugging (2.23)–(2.29) and (2.32) into the boundary value problem (2.15), gives that the *new* unknown u satisfies the following unilateral boundary value problem: *Find* $u \geq 0$ defined on $[0,T] \times \overline{\Omega}$ *satisfying:*

$$\begin{cases} \dfrac{\partial}{\partial t}(|u|^{\frac{3p-1}{2p}-2} u) - \nabla \cdot \left(\mu(x,u)|\nabla u - \boldsymbol{\Phi}|^{p-2}(\nabla u - \boldsymbol{\Phi}) - \boldsymbol{\Psi}\right) = \tilde{a}, \text{ in } \Omega_t^+ \text{ for all } t \in (0,T), \\[2ex] \dfrac{\partial}{\partial t}(|u|^{\frac{3p-1}{2p}-2} u) - \nabla \cdot \left(\mu(x,u)|\nabla u - \boldsymbol{\Phi}|^{p-2}(\nabla u - \boldsymbol{\Phi}) - \boldsymbol{\Psi}\right) \geq \tilde{a}, \text{ in } \Omega_t^- \text{ for all } t \in (0,T), \\[2ex] u(t,\cdot) = 0, \quad \text{on } \partial\Omega \text{ for all } t \in (0,T), \\[2ex] u(0,\cdot) = u_0 := H_0^{(2p/(p-1))} \geq 0. \end{cases} \tag{2.33}$$

Similarly to (2.19), the weak formulation associated with the boundary value problem (2.33) is expected to take the following form: *Find* $u \geq 0$ *satisfying the variational inequality:*

$$\begin{aligned} &\int_0^T \int_\Omega \left(\frac{\partial}{\partial t}(|u|^{\frac{3p-1}{2p}-2} u)\right)(v-u)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_0^T \int_\Omega \mu(x,u)|\nabla u - \boldsymbol{\Phi}|^{p-2}(\nabla u - \boldsymbol{\Phi}) \cdot \nabla(v-u)\,\mathrm{d}x\,\mathrm{d}t \\ &\geq \int_0^T \int_\Omega \tilde{a}(v-u)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \boldsymbol{\Psi} \cdot \nabla(v-u)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{2.34}$$

for all test functions v such that $v \geq b$ for a.e. $(t, x) \in (0, T) \times \Omega$, and satisfying the following boundary conditions along $\partial\Omega$

$$u = 0 \quad \text{on } \partial\Omega,$$

and satisfying the following initial condition

$$u(0, x) = u_0(x) = [H_0(x)]^{(2p/(p-1))}, \quad \text{for a.e. } x \in \Omega.$$

Once again, note that, at least for the time being, the rigorous concept of solution has not been defined yet. This task will be carried out in Section 3, where the function spaces where the solutions are going to be sought will be defined as well as the requirements a function has to meet in order to be regarded as a solution.

3. Weak formulation of the unilateral boundary value problem

Let $\Omega \subset \mathbb{R}^2$ be a domain. Let $2.8 \leq p \leq 5$ as suggested by the experimental results in [21]. For the sake of brevity, let

$$\alpha := \frac{3p-1}{2p}, \tag{3.1}$$

and observe that $1 < \alpha < 2 < p + 1$. By the Rellich-Kondrachov theorem (cf., e.g., Lions & Magenes [32] and the references therein) the following chain of embeddings hold:

$$W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathcal{C}^0(\overline{\Omega}) \hookrightarrow L^\alpha(\Omega). \tag{3.2}$$

Define the set

$$K := \{v \in W_0^{1,p}(\Omega); v \geq 0 \text{ in } \overline{\Omega}\}. \tag{3.3}$$

The weak formulation of the unilateral boundary value problem and the definition of the concept of solution will be recovered as a result of a constructive proof, based on the penalty method. The idea of using the penalty method to derive the existence of solutions of time-dependent contact problems was first used by Bock and Jarusek in [2, 3] and was improved by Bock, Jarusek and Silhavy in the recent paper [4].

For sake of simplicity, we make the following assumptions on the geometry of the problem:

- $(H1)$ $b = 0$ in $\overline{\Omega}$, i.e., the lithosphere is flat;
- $(H2)$ The ice softness A is assumed to be independent of the ice sheet height and time. As a result, the function μ defined in (2.31) is independent of the ice sheet height as well. Moreover, there exist two positive constants μ_1 and μ_2 such that $\mu_1 \leq \mu(x) \leq \mu_2$ for all $x \in \overline{\Omega}$;
- $(H3)$ The basal velocity $\boldsymbol{U}_b = \mathbf{0}$;
- $(H4)$ The function $\tilde{a}$ defined in (2.29) is independent of the ice sheet height and is of class $W^{1,p}(0, T; \mathcal{C}^0(\overline{\Omega}))$.

Let us now establish some preparatory results. The first result collects some properties of the negative part operator.

Lemma 3.1. *Let $\omega \subset \mathbb{R}^m$, with $m \geq 1$ an integer, be an open set. The operator $-\{\cdot\} : L^2(\omega) \to L^2(\omega)$ defined by*

$$f \in L^2(\omega) \mapsto -\{f\}^- := -\min\{f, 0\} \in L^2(\omega),$$

is monotone, bounded and Lipschitz continuous with Lipschitz constant equal to 1.

Proof. Let f and g be arbitrarily given in $L^(\omega)$. In what follows, sets of the form $\{f \geq 0\}$ read

$$\{x \in \omega; f(x) \geq 0\}.$$

Recall that the negative part of a function is also given by

$$f^- = \frac{|f| - f}{2}.$$

We have that

$$\begin{aligned}
&\int_\omega \left\{(-\{f\}^-) - (-\{g\}^-)\right\} (f-g)\,\mathrm{d}x\\
&\ge \int_\omega |-\{f\}^-|^2\,\mathrm{d}x + \int_\omega |-\{g\}^-|^2\,\mathrm{d}x - 2\int_{\{f\le 0\}\cap\{g\le 0\}} (-\{f\}^-)(-\{g\}^-)\,\mathrm{d}x\\
&\ge \int_{\{f\le 0\}\cap\{g\le 0\}} \left|(-\{f\}^-) - (-\{g\}^-)\right|^2\,\mathrm{d}x \ge 0,
\end{aligned}$$

and the monotonicity is thus established.

Let us show that the operator under consideration is bounded, in the sense that maps bounded sets onto bounded sets. Let $\mathscr{F} \subset L^2(\omega)$ be bounded. By Hölder's inequality, we have that for each $f \in \mathscr{F}$,

$$\sup_{\substack{v\in L^2(\omega)\\ v\neq 0}} \frac{\left|\int_\omega (-\{f\}^-)v\,\mathrm{d}x\right|}{\|v\|_{L^2(\omega)}} \le \|f\|_{L^2(\omega)},$$

and the boundedness of $\mathscr{F}$ implies the boundedness of the supremum. The boundedness property is thus established.

Finally, to establish the Lipschitz continuity property, evaluate

$$\begin{aligned}
&\left(\int_\omega |(-\{f\}^-) - (-\{g\}^-)|^2\,\mathrm{d}x\right)^{1/2} = \left(\int_\omega \left|\frac{f-|f|}{2} - \frac{g-|g|}{2}\right|^2\,\mathrm{d}x\right)^{1/2}\\
&= \frac{1}{2}\left(\int_\omega |(f-g) - (|f|-|g|)|^2\,\mathrm{d}x\right)^{1/2} \le \frac{1}{2}\{\|f-g\|_{L^2(\omega)} + \||f|-|g|\|_{L^2(\omega)}\}\\
&= \frac{1}{2}\|f-g\|_{L^2(\omega)} + \frac{1}{2}\left(\int_\omega ||f|-|g||^2\,\mathrm{d}x\right)^{1/2} \le \frac{1}{2}\|f-g\|_{L^2(\omega)} + \frac{1}{2}\left(\int_\omega |f-g|^2\,\mathrm{d}x\right)^{1/2}\\
&= \|f-g\|_{L^2(\omega)},
\end{aligned}$$

where the inequality holds thanks to the Minkowski inequality. In conclusion, we have shown that

$$\|(-\{f\}^-) - (-\{g\}^-)\|_{L^2(\omega)} \le \|f-g\|_{L^2(\omega)},$$

and the arbitrariness of f and g gives the desired Lipschitz continuity property. This completes the proof. ◻

The following lemma, which was originally proved using C^*-algebras (cf., e.g., [36]), plays a crucial role in the forthcoming analysis. We hereby provide an alternative proof, which solely makes use of convex analysis tools.

Lemma 3.2. *Let $\alpha > 1$. Then*

$$|x-y|^\alpha \le \left||x|^\alpha - |y|^\alpha\right|, \quad \text{for all } x, y \in \mathbb{R}.$$

Proof. Assume, without loss of generality, that $x \ge y \neq 0$, and let

$$t := \frac{x}{y} \ge 1.$$

Therefore, the sought inequality is equivalent to proving that

$$(t-1)^\alpha \le t^\alpha - 1, \quad \text{for all } t \ge 1.$$

Consider the function $f : [1, \infty) \to \mathbb{R}$ defined by

$$f(t) := (t-1)^\alpha - t^\alpha + 1.$$

Observe that

$$f'(t) = \alpha(t-1)^{\alpha-1} - \alpha t^{\alpha-1} = \alpha((t-1)^{\alpha-1} - t^{\alpha-1}).$$

Since $\alpha - 1 > 0$ and since $0 \le t-1 < t$, the monotonicity of the power operator gives that $(t-1)^{\alpha-1} \le t^{\alpha-1}$, so that $f'(t) \le 0$ for all $t > 1$. Since $f(1) = 0$, we infer that $f(t) \le 0$ for all $t \ge 1$ and the proof is complete. □

Thanks to Lemma 3.2, we can establish the following result, thanks to which we will be able to define a sound initial condition for the variational formulation we will be considering.

Lemma 3.3. *Let Ω be a domain in $\mathbb{R}^2$, let $T > 0$ be given, let $1 < \alpha < 2$, and let $u \in L^\infty(0,T;L^\alpha(\Omega))$ be such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^{\frac{\alpha-2}{2}} u\right) \in L^2(0,T;L^2(\Omega)).$$

Then, we have that $|u|^{\frac{\alpha-2}{2}}u$ is of class $\mathcal{C}^0([0,T];L^2(\Omega))$ and u is of class $\mathcal{C}^0([0,T];L^\alpha(\Omega))$.

Proof. Observe that

$$\int_0^T \int_\Omega \left||u|^{\frac{\alpha-2}{2}}u\right|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega |u|^\alpha \,\mathrm{d}x\,\mathrm{d}t.$$

Since it was assumed that $u \in L^\infty(0,T;L^\alpha(\Omega))$, then the latter integral is finite. Hence, combining the latter with the assumption on the distributional derivative in time gives

$$\left(|u|^{\frac{\alpha-2}{2}}u\right) \in H^1(0,T;L^2(\Omega)) \hookrightarrow \mathcal{C}^0([0,T];L^2(\Omega)).$$

To show that $u \in \mathcal{C}^0([0,T];L^\alpha(\Omega))$, we have to show that for each $t_0 \in [0,T]$,

$$\lim_{t\to t_0} \int_\Omega |u(t)-u(t_0)|^\alpha \,\mathrm{d}x = 0.$$

By Lemma 3.2, we have that the Cauchy-Schwarz inequality and the triangle inequality give

$$\begin{aligned}
&\int_\Omega |u(t)-u(t_0)|^\alpha \,\mathrm{d}x \le \int_\Omega ||u(t)|^\alpha - |u(t_0)|^\alpha| \,\mathrm{d}x\\
&= \int_\Omega \left|\left|\left||u(t)|^{\frac{\alpha-2}{2}}u(t)\right|^{2/\alpha}\right|^\alpha - \left|\left||u(t_0)|^{\frac{\alpha-2}{2}}u(t_0)\right|^{2/\alpha}\right|^\alpha\right| \,\mathrm{d}x = \int_\Omega \left|\left||u(t)|^{\frac{\alpha-2}{2}}u(t)\right|^2 - \left||u(t_0)|^{\frac{\alpha-2}{2}}u(t_0)\right|^2\right| \,\mathrm{d}x\\
&= \int_\Omega \left|\left(\left||u(t)|^{\frac{\alpha-2}{2}}u(t)\right| - \left||u(t_0)|^{\frac{\alpha-2}{2}}u(t_0)\right|\right)\cdot\left(\left||u(t)|^{\frac{\alpha-2}{2}}u(t)\right| + \left||u(t_0)|^{\frac{\alpha-2}{2}}u(t_0)\right|\right)\right| \,\mathrm{d}x\\
&\le \left\|\left||u(t)|^{\frac{\alpha-2}{2}}u(t)\right| - \left||u(t_0)|^{\frac{\alpha-2}{2}}u(t_0)\right|\right\|_{L^2(\Omega)} \left\|\left||u(t)|^{\frac{\alpha-2}{2}}u(t)\right| + \left||u(t_0)|^{\frac{\alpha-2}{2}}u(t_0)\right|\right\|_{L^2(\Omega)}\\
&\le \left\||u(t)|^{\frac{\alpha-2}{2}}u(t) - |u(t_0)|^{\frac{\alpha-2}{2}}u(t_0)\right\|_{L^2(\Omega)} \left\|\left||u(t)|^{\frac{\alpha-2}{2}}u(t)\right| + \left||u(t_0)|^{\frac{\alpha-2}{2}}u(t_0)\right|\right\|_{L^2(\Omega)}.
\end{aligned}$$

Observe that the continuity in $[0,T]$ of $\left(|u|^{\frac{\alpha-2}{2}}u\right)$ implies the uniform boundedness of the second factor, as well as that the first factor tends to zero as $t \to t_0$. This completes the proof. □

The next lemma establishes that the supremum can be interchanged with a monotonically increasing continuous extended-real-valued function.

Lemma 3.4. *Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be a monotonically increasing and continuous function. Then, given any $S \subset \mathbb{R}$,*

$$\sup_{x\in S} f(x) = f(\sup S).$$

Proof. By the definition of supremum, we can find a maximizing sequence $\{s_k\}_{k=1}^\infty \subset S$ for which

$$s_k \to \sup S, \qquad \text{as } k \to \infty.$$

By the assumed continuity of f, we have that

$$\lim_{k\to\infty} f(s_k) = f(\sup S).$$

Since $f(s_k) \le \sup_{x\in S} f(x)$ then the inequality $f(\sup S) \le \sup_{x\in S} f(x)$ is obviously true, on the one hand. On the other hand, since f is monotone, we have that

$$f(x) \le f(\sup S), \qquad \text{for all } x \in S.$$

Therefore, passing to the supremum on the left-hand side, we obtain that

$$\sup_{x\in S} f(x) \le \sup_{x\in S} f(\sup S) = f(\sup S),$$

and the proof is complete. □

The next lemma establishes a convergence property for sequences of functions enjoying the regularities announced in Lemma 3.3.

Lemma 3.5. *Let Ω be a domain in $\mathbb{R}^2$, let $T > 0$ be given, let $1 < \alpha < 2$, and let $\{u_k\}_{k=1}^{\infty} \subset L^{\infty}(0,T;W_0^{1,p}(\Omega))$ be such that*

$$\left\{|u_k|^{\frac{\alpha-2}{2}} u_k\right\}_{k=1}^{\infty} \text{ strongly converges in } \mathcal{C}^0([0,T];L^2(\Omega)).$$

Then $\{u_k\}_{k=1}^{\infty}$ strongly converges in $\mathcal{C}^0([0,T];L^{\alpha}(\Omega))$.

Proof. Since the space $\mathcal{C}^0([0,T];L^{\alpha}(\Omega))$ is complete, it suffices to show that the sequence $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{C}^0([0,T];L^{\alpha}(\Omega))$, i.e., we have to show that

$$\lim_{k,\ell\to\infty} \sup_{t\in[0,T]} \left(\int_{\Omega} |u_k(t) - u_\ell(t)|^{\alpha}\,\mathrm{d}x\right)^{1/\alpha} = 0. \tag{3.4}$$

Using Lemma 3.2, Hölder's inequality and Lemma 3.4, we have that

$$\sup_{t\in[0,T]} \left(\int_{\Omega} |u_k(t) - u_\ell(t)|^{\alpha}\,\mathrm{d}x\right)^{1/\alpha} \le \sup_{t\in[0,T]} \left(\int_{\Omega} \left||u_k(t)|^{\alpha} - |u_\ell(t)|^{\alpha}\right|\,\mathrm{d}x\right)^{1/\alpha}$$

$$= \sup_{t\in[0,T]} \left(\int_{\Omega} \left|\left||u_k(t)|^{\frac{\alpha-2}{2}} u_k(t)\right|^2 - \left||u_\ell(t)|^{\frac{\alpha-2}{2}} u_\ell(t)\right|^2\right|\,\mathrm{d}x\right)^{1/\alpha}$$

$$\le \sup_{t\in[0,T]} \left\{\left(\int_{\Omega} \left||u_k(t)|^{\frac{\alpha-2}{2}} u_k(t) - |u_\ell(t)|^{\frac{\alpha-2}{2}} u_\ell(t)\right|^2 \mathrm{d}x\right)^{1/2} \left(\int_{\Omega} \left||u_k(t)|^{\frac{\alpha-2}{2}} u_k(t) + |u_\ell(t)|^{\frac{\alpha-2}{2}} u_\ell(t)\right|^2 \mathrm{d}x\right)^{1/2}\right\}^{1/\alpha}$$

$$= \left\{\sup_{t\in[0,T]} \left[\left\||u_k(t)|^{\frac{\alpha-2}{2}} u_k(t) - |u_\ell(t)|^{\frac{\alpha-2}{2}} u_\ell(t)\right\|_{L^2(\Omega)} \left\||u_k(t)|^{\frac{\alpha-2}{2}} u_k(t) + |u_\ell(t)|^{\frac{\alpha-2}{2}} u_\ell(t)\right\|_{L^2(\Omega)}\right]\right\}^{1/\alpha}.$$

The assumed strong convergence implies that the second factor is uniformly bounded with respect to $t \in [0,T]$. Hence, the latter term is less or equal than

$$\left\{\left[\sup_{t\in[0,T]} \left\||u_k(t)|^{\frac{\alpha-2}{2}} u_k(t) - |u_\ell(t)|^{\frac{\alpha-2}{2}} u_\ell(t)\right\|_{L^2(\Omega)}\right] \cdot \left[\sup_{t\in[0,T]} \left\||u_k(t)|^{\frac{\alpha-2}{2}} u_k(t) + |u_\ell(t)|^{\frac{\alpha-2}{2}} u_\ell(t)\right\|_{L^2(\Omega)}\right]\right\}^{1/\alpha}.$$

Since the second factor is uniformly bounded with respect to $t \in [0,T]$ and since the first factor tends to zero as $k,\ell \to \infty$, by the assumed strong convergence, we finally obtain the sought convergence (3.4). This completes the proof. □

The next lemma establishes an immersion that will be used in the proof of the existence of the solution.

Lemma 3.6. *Let $T > 0$ be given and let X be a normed vector space with the Radon-Nikodym property (cf., e.g., [40]). Let $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$ denote the duality between $\left(\mathcal{C}^0([0,T];X)\right)^*$ and $\mathcal{C}^0([0,T];X)$, and let $\langle\,\cdot\,,\,\cdot\,\rangle$ denote the duality between X^* and X.*

The mapping

$$\Upsilon : L^1(0,T;X^*) \to \left(\mathcal{C}^0([0,T];X)\right)^*$$

defined by

$$\langle\langle \Upsilon u, v\rangle\rangle := \int_0^T \langle u, v\rangle \,\mathrm{d}t, \qquad \text{where } u \in L^1(0,T;X^*) \text{ and } v \in \mathcal{C}^0([0,T];X),$$

is linear, continuous and injective. Besides, if $\{u_k\}_{k=1}^\infty$ *is bounded in* $L^1(0,T;X^*)$ *then* $\{\Upsilon u_k\}_{k=1}^\infty$ *is bounded in* $\left(\mathcal{C}^0([0,T];X)\right)^*$.

Proof. Let us check that Υ is linear. For each $u_1, u_2 \in L^1(0,T;X^*)$ we have that

$$\langle\langle \Upsilon(u_1+u_2), v\rangle\rangle = \int_0^T \langle u_1+u_2, v\rangle \,\mathrm{d}t = \int_0^T \langle u_1, v\rangle \,\mathrm{d}t + \int_0^T \langle u_2, v\rangle \,\mathrm{d}t = \langle\langle \Upsilon u_1, v\rangle\rangle + \langle\langle \Upsilon u_2, v\rangle\rangle = \langle\langle \Upsilon u_1 + \Upsilon u_2, v\rangle\rangle,$$

for all $v \in \mathcal{C}^0([0,T];X)$.

Similarly, for each $\alpha \neq 0$ (for $\alpha = 0$ the conclusion is immediate) and each $u \in L^1(0,T;X^*)$, we have that

$$\langle\langle \Upsilon(\alpha u), v\rangle\rangle = \int_0^T \alpha\langle u, v\rangle \,\mathrm{d}t = \langle\langle \Upsilon u, \alpha v\rangle\rangle = \langle\langle \alpha\Upsilon u, v\rangle\rangle,$$

for all $v \in \mathcal{C}^0([0,T];X)$, and the sought linearity property is thus proved.

To show the continuity of the mapping Υ, let $\{u_k\}_{k=1}^\infty$ be such that $u_k \to u$ in $L^1(0,T;X^*)$. By Hölder's inequality in Lebesgue-Bochner spaces (cf., e.g., [45]), we have that

$$|\langle\langle \Upsilon u_k - \Upsilon u, v\rangle\rangle| \le \int_0^T |\langle u_k - u, v\rangle| \,\mathrm{d}t \le \|u_k - u\|_{L^1(0,T;X^*)} \|v\|_{L^\infty(0,T;X)} \to 0 \text{ as } k \to \infty.$$

To prove that Υ is injective, assume that $\Upsilon u_1 = \Upsilon u_2$ in $\left(\mathcal{C}^0([0,T];X)\right)^*$ then, for each $v \in \mathcal{C}^0([0,T];X)$, we have that

$$\int_0^T \langle u_1 - u_2, v\rangle \,\mathrm{d}t = 0.$$

In particular, if we consider functions of the form $v(t) := \varphi(t)w$, with $\varphi \in \mathcal{D}(0,T)$ and $w \in X$. Fix $\varphi \in \mathcal{D}(0,T)$ and let $w \in X$ vary arbitrarily. Since X has the Radon-Nikodym property, we have that (cf., e.g., Theorem 8.13 of [30]) the latter becomes

$$0 = \int_0^T \langle u_1 - u_2, v\rangle \,\mathrm{d}t = \left\langle \int_0^T (u_1(t) - u_2(t))\varphi(t) \,\mathrm{d}t, w \right\rangle, \qquad \text{for all } w \in X.$$

This in turn implies that

$$\int_0^T (u_1(t) - u_2(t))\varphi(t) \,\mathrm{d}t = 0 \text{ in } X^*.$$

Since this conclusion is independent of the choice of $\varphi \in \mathcal{D}(0,T)$, an application of the Fundamental Lemma of the Calculus of Variations (cf., e.g., [45]) gives that $u_1 = u_2$ in $L^1(0,T;X^*)$ and the injectivity is thus proved.

Let $\{u_k\}_{k=1}^\infty$ be bounded in $L^1(0,T;X^*)$. For each $v \in \mathcal{C}^0([0,T];X)$, the Hölder inequality in Lebesgue-Bochner spaces (cf. Theorem of [45]) and the assumed uniform boundedness of the sequence $\{u_k\}_{k=1}^\infty$ give:

$$|\langle\langle \Upsilon u_k, v\rangle\rangle| = \left|\int_0^T \langle u_k(t), v(t)\rangle \,\mathrm{d}t\right| \le \int_0^T |\langle u_k(t), v(t)\rangle| \,\mathrm{d}t \le \|u_k\|_{L^1(0,T;X^*)} \|v\|_{L^\infty(0,T;X)} \le C\|v\|_{L^\infty(0,T;X)},$$

for some $C > 0$ independent of k. This shows that each operator $\tilde{\Upsilon}_k : \mathcal{C}^0([0,T];X) \to \mathbb{R}$ defined by $\tilde{\Upsilon}_k := \Upsilon u_k$ for all integers $k \ge 1$ is linear, continuous, and such that, for each $v \in \mathcal{C}^0([0,T];X^*)$

$$\sup_{k\ge 1} |\tilde{\Upsilon}_k(v)| \le C_v,$$

for some constant $C_v > 0$ that solely depends on v. An application of the Banach-Steinhaus theorem (cf., e.g., Theorem 2.2 of [7]) gives that there exists a constant $C > 0$ for which

$$\sup_{k\ge 1} \|\tilde{\Upsilon}_k\|_{(\mathcal{C}^0([0,T];X))^*} \le C,$$

thus proving that the sequence $\{\Upsilon u_k\}_{k=1}^{\infty}$ is uniformly bounded in $(\mathcal{C}^0([0,T];X))^*$. The proof is complete. $\square$

The conclusion of Lemma 3.6 is that the space $L^1(0,T;X^*)$ can be identified with a subspace of $(\mathcal{C}^0([0,T];X))^*$.

The proof of existence of solutions hinges on a compactness result proved by Dubinskii [15] (see also [1] for some improvements and corrections), as well as other results proved by Raviart in the paper [38] that we recall here below.

Lemma 3.7 (Lemma 1.1 of [38]). *Let $1<r<\infty$ and let r' denote the Hölder conjugate exponent of r. The following inequalities hold*

$$
\begin{aligned}
(|\xi|^{r-2}\xi-|\eta|^{r-2}\eta)\xi &\geq \frac{1}{r'}(|\xi|^r-|\eta|^r),\\
(|\xi|^{r-2}\xi-|\eta|^{r-2}\eta)(\xi-\eta) &\geq C\left(|\xi|^{(r-2)/2}\xi-|\eta|^{(r-2)/2}\eta\right)^2, \text{ for some } C=C(r)>0,
\end{aligned}
$$

for all $\xi,\eta\in\mathbb{R}$. $\square$

The first inequality in this lemma is a direct consequence of Young's inequality (cf., e.g., [46]), while the second inequality was proved by Simon in the paper [43].

The next preliminary result we recall, is a generalized integration-by-parts formula, whose proof hinges on the Lebesgue theorem (cf., e.g., Theorem 2.11-3 of [11]).

Lemma 3.8 (Lemma 1.2 of [38]). *Let Ω be a domain in $\mathbb{R}^2$, and let $T>0$. Let α and p be two real numbers greater than 1, and let v be a function such that*

$$
\begin{aligned}
v &\in L^p(0,T;W_0^{1,p}(\Omega))\cap\mathcal{C}^0([0,T];L^\alpha(\Omega)),\\
\frac{\mathrm{d}}{\mathrm{d}t}(|v|^{\alpha-2}v) &\in L^{p'}(0,T;W^{-1,p'}(\Omega)).
\end{aligned}
$$

Therefore, the following formula holds

$$
\int_0^T\left\langle\frac{\mathrm{d}}{\mathrm{d}t}(|v|^{\alpha-2}v),v\right\rangle\,\mathrm{d}t=\frac{\|v(T)\|_{L^\alpha(\Omega)}^\alpha}{\alpha'}-\frac{\|v(0)\|_{L^\alpha(\Omega)}^\alpha}{\alpha'},
$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. $\square$

Finally, we recall Dubinskii's compactness theorem.

Theorem 3.9. *Let A_0 and A_1 be normed linear spaces such that $A_0\hookrightarrow A_1$. Let $S\subset A_0$ be such that $\lambda S\subset S$, for all $\lambda\in\mathbb{R}$.*

Assume that the set S is endowed with the semi-norm $M:S\to\mathbb{R}$, having the following properties:

1. *$M(v)\geq 0$, for all $v\in S$,*
2. *$M(\lambda v)=|\lambda|M(v)$, for all $v\in S$ and all $\lambda\in\mathbb{R}$.*

Assume that the set $\mathscr{M}:=\{v\in S;M(v)\leq 1\}$ is relatively compact in A_0. Consider the semi-normed set

$$
Y:=\left\{u\in L^1_{\mathrm{loc}}(0,T;A_1);\int_0^T[M(u(t))]^{q_0}\,\mathrm{d}t<\infty \text{ and } \int_0^T\left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|_{A_1}^{q_1}\mathrm{d}t<\infty\right\},
$$

with $1\leq q_0\leq\infty$ and $1\leq q_1\leq\infty$, and the pairs $(q_0,q_1)=(1,\infty)$ and $(q_0,q_1)=(\infty,1)$ cannot be attained.

Then Y is relatively compact in $L^{q_0}(0,T;A_0)$. $\square$

A discrete version of this compactness result has been established by Raviart in the paper [37].

Theorem 3.10 (Lemma 1.4 of [38]). *Let A_0 and A_1 be normed linear spaces such that $A_0\hookrightarrow A_1$. Let $S\subset A_0$ be such that $\lambda S\subset S$, for all $\lambda\in\mathbb{R}$.*

Assume that the set S is endowed with the semi-norm $M:S\to\mathbb{R}$, having the following properties:

1. *$M(v)\geq 0$, for all $v\in S$,*

(2) *$M(\lambda v) = |\lambda| M(v)$, for all $v \in S$ and all $\lambda \in \mathbb{R}$.*

Assume that the set $\mathscr{M} := \{v \in S; M(v) \le 1\}$ is relatively compact in A_0.

For each $\ell > 0$, consider the vector $\boldsymbol{v}_\ell := (v_\ell^n)_{n=0}^N$ of elements of S such that

$$\ell \sum_{n=0}^{N} [M(v_\ell^n)]^{q_0} \le c_0, \qquad \text{for some constants } c_0 \ge 0,$$

and

$$\ell \sum_{n=0}^{N-1} \left\| \frac{v_\ell^{n+1} - v_\ell^n}{k} \right\|_{A_1}^{q_1} \le c_1, \qquad \text{for some constants } c_1 \ge 0,$$

where $1 \le q_0 \le \infty$ and $1 \le q_1 \le \infty$, and the pairs $(q_0, q_1) = (1, \infty)$ and $(q_0, q_1) = (\infty, 1)$ cannot be attained.

Then, it is possible to extract a subsequence of the sequence $(\Pi_\ell \boldsymbol{v}_\ell)$ that strongly converges in $L^{q_0}(0, T; A_0)$, where

$$\Pi_k \boldsymbol{u}_k : (0, T) \to A_0 \qquad \text{with} \qquad (\Pi_\ell \boldsymbol{v}_\ell)(t) := v_\ell^{n+1}, \text{ for a.a. } n\ell < t \le (n+1)\ell.$$

$\square$

Let us also recall a result on vector-valued measures proved by Zinger in the paper [47] and later improved by Dinculeanu (see also, e.g., page 182 of [13], and page 380 of [14]).

Theorem 3.11 (Dinculeanu-Zinger theorem)**.** *Let ω be a compact Hausdorff space and let X be a Banach space satisfying the Radon-Nikodym property. Let $\mathcal{F}$ be the collection of Borel sets of ω.*

There exists an isomorphism between $(\mathcal{C}^0(\omega; X))^*$ and the space of the regular Borel measures with finite variation taking values in X^*. In particular, for each $F \in (\mathcal{C}^0(\omega; X))^*$, there exists a unique regular Borel measure $\mu : \mathcal{F} \to X^*$ in $\mathcal{M}(\omega; X^*)$ with finite variation such that

$$\langle\langle \alpha, F \rangle\rangle_X = \int_\omega {}_{X^*}\langle \mathrm{d}\mu, \alpha \rangle_X,$$

for all $\alpha \in \mathcal{C}^0(\omega; X)$. $\square$

We now state the penalized problem, which is suggested by the formal model (2.33). Note that the initial condition makes sense thanks to Lemma 3.3.

Problem $\mathcal{P}_\kappa$. Find a function u_κ that satisfies

$$\begin{aligned} u_\kappa &\in L^\infty(0, T; W_0^{1,p}(\Omega)), \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(|u_\kappa|^{\frac{\alpha-2}{2}} u_\kappa\right) &\in L^2(0, T; L^2(\Omega)), \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(|u_\kappa|^{\alpha-2} u_\kappa\right) &\in L^\infty(0, T; W^{-1,p'}(\Omega)), \end{aligned}$$

satisfying the following variational equations:

$$\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}(|u|^{\alpha-2} u) v \,\mathrm{d}x + \int_\Omega \mu(x, u) |\nabla u|^{p-2} \nabla u \cdot \nabla v \,\mathrm{d}x - \frac{1}{\kappa} \int_\Omega \{u_\kappa\}^- v \,\mathrm{d}x = \int_\Omega \tilde{a}(t, x) v \,\mathrm{d}x, \tag{3.5}$$

for all $v \in W_0^{1,p}(\Omega)$ in the sense of distributions on $(0, T)$, as well as the following initial condition

$$u_\kappa(0) = u_0,$$

for some nonzero prescribed $u_0 \in K$. $\blacksquare$

4. Existence of weak solutions and variational formulation for Problem (2.33)

The recovery of the variational formulation for Problem (2.33) and the existence of solutions for this problem are broken into three parts. We first establish the existence of solutions for Problem $\mathcal{P}_\kappa$ by means of a semi-discrete scheme. For each $0 \le n \le N-1$, consider the semi-discrete problem:

Problem $\mathcal{P}_\kappa^{n+1}$. Given $u_{\kappa,\ell}^n \in W_0^{1,p}(\Omega)$, find a function $u_{\kappa,\ell}^{n+1} \in W_0^{1,p}(\Omega)$ that satisfies the following variational equations:

$$\begin{aligned}(4.1)\qquad &\frac{1}{\ell}\{|u_{\kappa,\ell}^{n+1}|^{\alpha-2}u_{\kappa,\ell}^{n+1} - |u_{\kappa,\ell}^{n}|^{\alpha-2}u_{\kappa,\ell}^{n}\} - \nabla\cdot\left(\mu|\nabla u_{\kappa,\ell}^{n+1}|^{p-2}\nabla u_{\kappa,\ell}^{n+1}\right) - \frac{\{u_{\kappa,\ell}^{n+1}\}^-}{\kappa}\\ &= \frac{1}{\ell}\int_{n\ell}^{(n+1)\ell}\tilde{a}(t)\,\mathrm{d}t \text{ in } W^{-1,p'}(\Omega),\end{aligned}$$

where $u_{\kappa,\ell}^0 := u_0 \in K$, and u_0 is the prescribed element appearing in Problem $\mathcal{P}_\kappa$. ■

The following existence-and-uniqueness result can be established.

Theorem 4.1. *Let $T>0$, $\Omega \subset \mathbb{R}^2$ and p be as in section 3 and let α be as in (3.1). Let $\kappa > 0$ be given, let $N \ge 1$ be an integer, and define $\ell := T/N$. Assume that (H1)–(H4) hold.*

For each $0 \le n \le N-1$, Problem $\mathcal{P}_\kappa^{n+1}$ admits a unique solution $u_{\kappa,\ell}^{n+1} \in W_0^{1,p}(\Omega)$.

Proof. For sake of brevity, define

$$(4.2)\qquad \tilde{a}_\ell^n := \frac{1}{\ell}\int_{n\ell}^{(n+1)\ell}\tilde{a}(t)\,\mathrm{d}t.$$

Consider the operator $A_\kappa : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$(4.3)\qquad A_\kappa(v) := |v|^{\alpha-2}v - \nabla\cdot\left(\mu|\nabla v|^{p-2}\nabla v\right) - \frac{\{v\}^-}{\kappa}, \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

Consider the operator $B_\kappa : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$(4.4)\qquad B_\kappa(v) := -\nabla\cdot\left(\mu|\nabla v|^{p-2}\nabla v\right) - \frac{\{v\}^-}{\kappa}, \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

The operators A_κ and B_κ are hemi-continuous, as each of their terms is hemi-continuous. By Lemma 3.2 and Lemma 3.7, the operators A_κ and B_κ are strictly monotone. Finally, an application of the Poincaré-Friedrichs inequality and the monotonicity of the negative part operator (Lemma 3.2) give:

$$\frac{\langle A_\kappa v, v\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}}{\|v\|_{W_0^{1,p}(\Omega)}} \ge \frac{\|v\|_{L^\alpha(\Omega)}^\alpha + \mu_1\|\nabla v\|_{L^p(\Omega)}^p}{\|v\|_{W_0^{1,p}(\Omega)}} \ge \mu_1 c_0 \|v\|_{W_0^{1,p}(\Omega)}^{p-1},$$

and the term on the right diverges as $\|v\|_{W_0^{1,p}(\Omega)} \to \infty$. This means that the operator A_κ is coercive. With the same reasoning, it can be proved that the operator B_κ is coercive too. An application of the Minty-Browder theorem (cf., e.g., Theorem 9.14-1 of [10]) ensures that the numerical scheme in (4.1) admits a unique solution $u_{\kappa,\ell}^{n+1} \in W_0^{1,p}(\Omega)$. The proof is complete. □

Define the mapping $\Pi_\ell \boldsymbol{u}_\kappa : (0,T) \to W_0^{1,p}(\Omega)$ by

$$(4.5)\qquad \Pi_\ell \boldsymbol{u}_\kappa(t) := u_{\kappa,\ell}^{n+1}, \quad \text{if } n\ell < t \le (n+1)\ell.$$

Next, we discuss the convergence of the sequence $\{\Pi_\ell \boldsymbol{u}_\kappa\}_{\ell>0}$ as $\ell \to 0^+$ or, equivalently, as $N \to \infty$. To this aim, we need to establish some *a priori* estimates.

Theorem 4.2. *Let $T>0$, $\Omega\subset\mathbb{R}^2$ and p be as in section 3 and let α be as in (3.1). Let $\kappa>0$ be given, let $N\ge 1$ be an integer, and define $\ell:=T/N$. Assume that (H1)–(H4) hold.*

Assume that the following stability condition holds: There exists a constant $C_1>0$ independent of κ and N for which

$$\frac{1}{\kappa}\sum_{n=0}^{N-1}\int_\Omega \{u^{n+1}_{\kappa,\ell}\}^-(u^{n+1}_{\kappa,\ell}-u^n_{\kappa,\ell})\,\mathrm{d}x\le C_1, \tag{4.6}$$

for all integers $N\ge 1$.

Then, the following a priori *estimates hold:*

$$\max_{0\le n\le N}\|u^n_{\kappa,\ell}\|_{L^\alpha(\Omega)}\le C, \tag{4.7}$$

$$\ell\sum_{n=0}^{N}\|u^n_{\kappa,\ell}\|^p_{W^{1,p}_0(\Omega)}\le C, \tag{4.8}$$

$$\ell\sum_{n=0}^{N-1}\left\|\frac{|u^{n+1}_{\kappa,\ell}|^{\frac{\alpha-2}{2}}u^{n+1}_{\kappa,\ell}-|u^n_{\kappa,\ell}|^{\frac{\alpha-2}{2}}u^n_{\kappa,\ell}}{\ell}\right\|^2_{L^2(\Omega)}\le C, \tag{4.9}$$

$$\max_{0\le n\le N}\left\||u^n_{\kappa,\ell}|^{\frac{\alpha-2}{2}}u^n_{\kappa,\ell}\right\|_{L^2(\Omega)}\le C, \tag{4.10}$$

$$\max_{0\le n\le N}\|u^n_{\kappa,\ell}\|_{W^{1,p}_0(\Omega)}\le C, \tag{4.11}$$

$$\max_{0\le n\le N-1}\left\|\frac{|u^{n+1}_{\kappa,\ell}|^{\alpha-2}u^{n+1}_{\kappa,\ell}-|u^n_{\kappa,\ell}|^{\alpha-2}u^n_{\kappa,\ell}}{\ell}\right\|_{W^{-1,p'}(\Omega)}\le C\left(1+\frac{1}{\kappa}\right), \tag{4.12}$$

$$\max_{0\le n\le N}\left\||u^n_{\kappa,\ell}|^{\alpha-2}u^n_{\kappa,\ell}\right\|_{L^{\alpha'}(\Omega)}\le C, \tag{4.13}$$

for some $C>0$ independent of κ and N.

Proof. Multiply (4.1) by $u^{n+1}_{\kappa,\ell}$ in the sense of the duality between $W^{-1,p'}(\Omega)$ and $W^{1,p}_0(\Omega)$. We obtain

$$\begin{aligned}&\frac{1}{\ell}\langle|u^{n+1}_{\kappa,\ell}|^{\alpha-2}u^{n+1}_{\kappa,\ell}-|u^n_{\kappa,\ell}|^{\alpha-2}u^n_{\kappa,\ell},u^{n+1}_{\kappa,\ell}\rangle_{W^{-1,p'}(\Omega),W^{1,p}_0(\Omega)}+\int_\Omega\mu(x)|\nabla u^{n+1}_{\kappa,\ell}|^{p-2}\nabla u^{n+1}_{\kappa,\ell}\cdot\nabla u^{n+1}_{\kappa,\ell}\,\mathrm{d}x\\&\quad-\frac{1}{\kappa}\int_\Omega\{u^{n+1}_{\kappa,\ell}\}^-u^{n+1}_{\kappa,\ell}\,\mathrm{d}x=\frac{1}{\ell}\int_\Omega\left(\int_{n\ell}^{(n+1)\ell}\tilde{a}(t)\,\mathrm{d}t\right)u^{n+1}_{\kappa,\ell}\,\mathrm{d}x.\end{aligned}$$

Using the first estimate in Lemma 3.7 with $r=\alpha$, $\xi=u^{n+1}_{\kappa,\ell}$ and $\eta=u^n_{\kappa,\ell}$, the Poincaré-Friedrichs inequality, Bochner's theorem (Theorem 8.9 of [30]) and Young's inequality (cf., e.g., [46]) we get:

$$\begin{aligned}&\frac{1}{\alpha'\ell}(\|u^{n+1}_{\kappa,\ell}\|^\alpha_{L^\alpha(\Omega)}-\|u^n_{\kappa,\ell}\|^\alpha_{L^\alpha(\Omega)})+c_0\mu_1\|u^{n+1}_{\kappa,\ell}\|^p_{W^{1,p}_0(\Omega)}+\frac{1}{\kappa}\|\{u^{n+1}_{\kappa,\ell}\}^-\|^2_{L^2(\Omega)}\\&\le\frac{1}{\ell\varepsilon p'}\left(\int_{n\ell}^{(n+1)\ell}\|\tilde{a}(t)\|^{p'}_{W^{-1,p'}(\Omega)}\,\mathrm{d}t\right)+\frac{\varepsilon}{p}\|u^{n+1}_{\kappa,\ell}\|^p_{W^{1,p}_0(\Omega)}.\end{aligned}$$

Multiplying by $(\ell\alpha')$ both sides and summing over $1 \le n \le s-1$, where $1 \le s \le N$, we have that

$$\sum_{n=0}^{s-1}\left\{\|u_{\kappa,\ell}^{n+1}\|_{L^\alpha(\Omega)}^\alpha + \left(c_0\mu_1 - \frac{\varepsilon}{p}\right)\ell\alpha'\|u_{\kappa,\ell}^{n+1}\|_{W_0^{1,p}(\Omega)}^p + \frac{\ell\alpha'}{\kappa}\|\{u_{\kappa,\ell}^{n+1}\}^{-1}\|_{L^2(\Omega)}^2\right\}$$
$$\le \frac{\alpha'}{\varepsilon p'}\sum_{n=0}^{s-1}\left(\int_{n\ell}^{(n+1)\ell}\|\tilde{a}(t)\|_{W^{-1,p'}(\Omega)}^{p'}\,\mathrm{d}t\right) + \sum_{n=0}^{s-1}\|u_{\kappa,\ell}^n\|_{L^\alpha(\Omega)}^\alpha,$$

and we finally obtain

$$\|u_{\kappa,\ell}^s\|_{L^\alpha(\Omega)}^\alpha + \left(c_0\mu_1 - \frac{\varepsilon}{p}\right)\ell\alpha'\sum_{n=0}^{s-1}\|u_{\kappa,\ell}^{n+1}\|_{W_0^{1,p}(\Omega)}^p + \frac{\ell\alpha'}{\kappa}\sum_{n=0}^{s-1}\|\{u_{\kappa,\ell}^{n+1}\}^{-1}\|_{L^2(\Omega)}^2$$
$$\le \frac{\alpha'}{\varepsilon p'}\sum_{n=0}^{s-1}\left(\int_{n\ell}^{(n+1)\ell}\|\tilde{a}(t)\|_{W^{-1,p'}(\Omega)}^{p'}\,\mathrm{d}t\right).$$

Therefore, there exists $C = C(u_0, \alpha, \varepsilon, p, \Omega) > 0$ such that:

$$\begin{aligned}
&\max_{0\le n\le N}\|u_{\kappa,\ell}^n\|_{L^\alpha(\Omega)} \le C,\\
&\ell\sum_{n=0}^{N}\|u_{\kappa,\ell}^n\|_{W_0^{1,p}(\Omega)}^p \le C,\\
&\frac{\ell}{\kappa}\sum_{n=1}^{N}\|\{u_{\kappa,\ell}^n\}^-\|_{L^2(\Omega)}^2 \le C,
\end{aligned}\tag{4.14}$$

so that the estimates (4.7) and (4.8) are proved.

By (4.7), for each $0 \le n \le N$, we have that

$$\|u_{\kappa,\ell}^n\|_{L^\alpha(\Omega)}^\alpha = \int_\Omega \left||u_{\kappa,\ell}^n|^{\frac{\alpha-2}{2}}u_{\kappa,\ell}^n\right|^2\,\mathrm{d}x = \left\||u_{\kappa,\ell}^n|^{\frac{\alpha-2}{2}}u_{\kappa,\ell}^n\right\|_{L^2(\Omega)}^2.$$

Since the left-hand side of the previous equation is uniformly bounded with respect to N and κ, we immediately infer that

$$\max_{0\le n\le N}\left\||u_{\kappa,\ell}^n|^{\frac{\alpha-2}{2}}u_{\kappa,\ell}^n\right\|_{L^2(\Omega)} \le C,$$

for some $C > 0$ independent of N and κ, thus establishing the estimate (4.10).

In order to recover the estimates (4.9) and (4.11), let us multiply (4.1) by $(u_{\kappa,\ell}^{n+1} - u_{\kappa,\ell}^n)$ in the sense of the duality between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. We get

$$\begin{aligned}
&\frac{1}{\ell}\langle|u_{\kappa,\ell}^{n+1}|^{\alpha-2}u_{\kappa,\ell}^{n+1} - |u_{\kappa,\ell}^n|^{\alpha-2}u_{\kappa,\ell}^n, u_{\kappa,\ell}^{n+1} - u_{\kappa,\ell}^n\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\\
&\quad + \int_\Omega \mu(x)|\nabla u_{\kappa,\ell}^{n+1}|^{p-2}\nabla u_{\kappa,\ell}^{n+1}\cdot\nabla(u_{\kappa,\ell}^{n+1} - u_{\kappa,\ell}^n)\,\mathrm{d}x\\
&\quad - \frac{1}{\kappa}\int_\Omega\{u_{\kappa,\ell}^{n+1}\}^-(u_{\kappa,\ell}^{n+1} - u_{\kappa,\ell}^n)\,\mathrm{d}x = \frac{1}{\ell}\int_\Omega\left(\int_{n\ell}^{(n+1)\ell}\tilde{a}(t)\,\mathrm{d}t\right)(u_{\kappa,\ell}^{n+1} - u_{\kappa,\ell}^n)\,\mathrm{d}x.
\end{aligned}\tag{4.15}$$

For each $1 \le s \le N$, we have that the following identity holds:

$$\begin{aligned}
&\sum_{n=0}^{s-1}\int_\Omega \tilde{a}_\ell^n(u_{\kappa,\ell}^{n+1} - u_{\kappa,\ell}^n)\,\mathrm{d}x = -\sum_{n=0}^{s-2}\int_\Omega(\tilde{a}_\ell^{n+1} - \tilde{a}_\ell^n)u_{\kappa,\ell}^{n+1}\,\mathrm{d}x + \int_\Omega \tilde{a}_\ell^{s-1}u_{\kappa,\ell}^s\,\mathrm{d}x - \int_\Omega \tilde{a}_\ell^0 u_0\,\mathrm{d}x\\
&= \sum_{n=0}^{s-2}\int_{n\ell}^{(n+1)\ell}\int_\Omega\left(\frac{\tilde{a}(t+\ell) - \tilde{a}(t)}{\ell}\right)u_{\kappa,\ell}^{n+1}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \tilde{a}_\ell^{s-1}u_{\kappa,\ell}^s\,\mathrm{d}x - \int_\Omega \tilde{a}_\ell^0 u_0\,\mathrm{d}x.
\end{aligned}\tag{4.16}$$

An application of Lebesgue's inequality, the triangle inequality and Young's inequality [46] gives

$$
\begin{aligned}
&\left|\sum_{n=0}^{s-2}\int_\Omega(\tilde{a}_\ell^{n+1}-\tilde{a}_\ell^{n})u_{\kappa,\ell}^{n+1}\,\mathrm{d}x\right| = \left|\sum_{n=0}^{s-2}\int_{n\ell}^{(n+1)\ell}\int_\Omega\left(\frac{\tilde{a}(t+\ell)-\tilde{a}(t)}{\ell}\right)u_{\kappa,\ell}^{n+1}\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\le \sum_{n=0}^{s-2}\int_{n\ell}^{(n+1)\ell}\left|\int_\Omega\frac{\tilde{a}(t+\ell)-\tilde{a}(t)}{\ell}u_{\kappa,\ell}^{n+1}\,\mathrm{d}x\right|\mathrm{d}t\\
&\le \sum_{n=0}^{s-2}\int_{n\ell}^{(n+1)\ell}\left\|\frac{\tilde{a}(t+\ell)-\tilde{a}(t)}{\ell}\right\|_{W^{-1,p'}(\Omega)}\|u_{\kappa,\ell}^{n+1}\|_{W_0^{1,p}(\Omega)}\,\mathrm{d}t\\
&\le \sum_{n=0}^{s-2}\left[\frac{1}{p'\varepsilon}\int_{n\ell}^{(n+1)\ell}\left\|\frac{\tilde{a}(t+\ell)-\tilde{a}(t)}{\ell}\right\|_{W^{-1,p'}(\Omega)}^{p'}\mathrm{d}t+\frac{\varepsilon\ell}{p}\|u_{\kappa,\ell}^{n+1}\|_{W_0^{1,p}(\Omega)}^p\right].
\end{aligned}
\tag{4.17}
$$

Since $\tilde{a}\in W^{1,p}(0,T;\mathcal{C}^0(\overline{\Omega}))$ (see assumption $(H4)$), an application of the finite difference quotients theory (cf., e.g., Chapter 5 in [16]), we have that the latter term can be estimated as follows:

$$
\begin{aligned}
&\sum_{n=0}^{s-2}\left[\frac{1}{p'\varepsilon}\int_{n\ell}^{(n+1)\ell}\left\|\frac{\tilde{a}(t+\ell)-\tilde{a}(t)}{\ell}\right\|_{W^{-1,p'}(\Omega)}^{p'}\mathrm{d}t+\frac{\varepsilon\ell}{p}\|u_{\kappa,\ell}^{n+1}\|_{W_0^{1,p}(\Omega)}^p\right]\\
&\le \frac{C}{p'\varepsilon}\sum_{n=0}^{s-2}\int_{n\ell}^{(n+1)\ell}\left\|\frac{\mathrm{d}\tilde{a}}{\mathrm{d}t}(t)\right\|_{W^{-1,p'}(\Omega)}^{p'}\mathrm{d}t+C\frac{\varepsilon\ell}{p}\sum_{n=0}^{s-2}\|u_{\kappa,\ell}^{n+1}\|_{W_0^{1,p}(\Omega)}^p\\
&= \frac{C}{p'\varepsilon}\int_0^{(s-1)\ell}\left\|\frac{\mathrm{d}\tilde{a}}{\mathrm{d}t}(t)\right\|_{W^{-1,p'}(\Omega)}^{p'}\mathrm{d}t+C\frac{\varepsilon\ell}{p}\sum_{n=0}^{s-2}\|u_{\kappa,\ell}^{n+1}\|_{W_0^{1,p}(\Omega)}^p.
\end{aligned}
\tag{4.18}
$$

Putting together (4.16)–(4.18) thus gives

$$
\begin{aligned}
&\left|\sum_{n=0}^{s-1}\int_\Omega\tilde{a}_\ell^{n}(u_{\kappa,\ell}^{n+1}-u_{\kappa,\ell}^{n})\,\mathrm{d}x\right| \le \left|\sum_{n=0}^{s-2}\int_{n\ell}^{(n+1)\ell}\int_\Omega\left(\frac{\tilde{a}(t+\ell)-\tilde{a}(t)}{\ell}\right)u_{\kappa,\ell}^{n+1}\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\quad+\left|\int_\Omega\tilde{a}_\ell^{s-1}u_{\kappa,\ell}^{s}\,\mathrm{d}x\right|+\left|\int_\Omega\tilde{a}_\ell^{0}u_0\,\mathrm{d}x\right|\\
&\le \frac{C}{p'\varepsilon}\int_0^{(s-1)\ell}\left\|\frac{\mathrm{d}\tilde{a}}{\mathrm{d}t}(t)\right\|_{W^{-1,p'}(\Omega)}^{p'}\mathrm{d}t+C\frac{\varepsilon\ell}{p}\sum_{n=0}^{s-2}\|u_{\kappa,\ell}^{n+1}\|_{W_0^{1,p}(\Omega)}^p\\
&\quad+\frac{1}{p'\varepsilon}\|\tilde{a}\|_{L^\infty(0,T;W^{-1,p'}(\Omega))}^{p'}+\frac{\varepsilon}{p}\|u_{\kappa,\ell}^{s}\|_{W_0^{1,p}(\Omega)}^p\\
&\quad+\frac{1}{p'\varepsilon}\|\tilde{a}\|_{L^\infty(0,T;W^{-1,p'}(\Omega))}^{p'}+\frac{\varepsilon}{p}\|u_0\|_{W_0^{1,p}(\Omega)}^p\\
&\le \left[\frac{C}{p'\varepsilon}\int_0^{s\ell}\left\|\frac{\mathrm{d}\tilde{a}}{\mathrm{d}t}(t)\right\|_{W^{-1,p'}(\Omega)}^{p'}\mathrm{d}t+\frac{2}{p'\varepsilon}\|\tilde{a}\|_{L^\infty(0,T;W^{-1,p'}(\Omega))}^{p'}\right]\\
&\quad\cdot\left[\underbrace{C\,\frac{\varepsilon\ell}{p}\left(\sum_{n=0}^{s-2}\|u_{\kappa,\ell}^{n}\|_{W_0^{1,p}(\Omega)}^p\right)}_{\text{bounded by (4.8)}}+\frac{\varepsilon}{p}\|u_{\kappa,\ell}^{s}\|_{W_0^{1,p}(\Omega)}^p+\frac{\varepsilon}{p}\|u_0\|_{W_0^{1,p}(\Omega)}^p\right]\\
&\le C(1+\varepsilon\|u_{\kappa,\ell}^{s}\|_{W_0^{1,p}(\Omega)}^p).
\end{aligned}
\tag{4.19}
$$

Summing (4.15) over $0 \le n \le s-1$, with $1 \le s \le N$, applying Lemma 3.7, exploiting (4.19), the assumed stability condition (4.6) and the Poincaré-Friedrichs inequality gives

$$\begin{aligned}
&\ell \sum_{n=0}^{s-1} \left\| \frac{|u_{\kappa,\ell}^{n+1}|^{\frac{\alpha-2}{2}} u_{\kappa,\ell}^{n+1} - |u_{\kappa,\ell}^{n}|^{\frac{\alpha-2}{2}} u_{\kappa,\ell}^{n}}{\ell} \right\|_{L^2(\Omega)}^2 + \frac{\mu_1 c_0}{p} \|u_{\kappa,\ell}^s\|_{W_0^{1,p}(\Omega)}^p \\
&\le \ell \sum_{n=0}^{s-1} \left\| \frac{|u_{\kappa,\ell}^{n+1}|^{\frac{\alpha-2}{2}} u_{\kappa,\ell}^{n+1} - |u_{\kappa,\ell}^{n}|^{\frac{\alpha-2}{2}} u_{\kappa,\ell}^{n}}{\ell} \right\|_{L^2(\Omega)}^2 + \frac{\mu_1}{p} \sum_{n=0}^{s-1} \left\{ \|\nabla u_{\kappa,\ell}^{n+1}\|_{L^p(\Omega)}^p - \|\nabla u_{\kappa,\ell}^{n}\|_{L^p(\Omega)}^p \right\} \\
&\le C(1 + \varepsilon \|u_{\kappa,\ell}^s\|_{W_0^{1,p}(\Omega)}^p),
\end{aligned}$$

so that, in the end, we obtain the following estimate

$$\ell \sum_{n=0}^{s-1} \left\| \frac{|u_{\kappa,\ell}^{n+1}|^{\frac{\alpha-2}{2}} u_{\kappa,\ell}^{n+1} - |u_{\kappa,\ell}^{n}|^{\frac{\alpha-2}{2}} u_{\kappa,\ell}^{n}}{\ell} \right\|_{L^2(\Omega)}^2 + \left(\frac{\mu_1 c_0}{p} - C\varepsilon \right) \|u_{\kappa,\ell}^s\|_{W_0^{1,p}(\Omega)}^p \le C,$$

and the estimates (4.9) and (4.11) straightforwardly follow.

In order to establish the estimate (4.12), we exploit the boundedness of the p-Laplace operator (cf., e.g., Chapter 9 in [10]) and (4.11) so as to be in a position to evaluate

$$\begin{aligned}
&\left\| \frac{|u_{\kappa,\ell}^{n+1}|^{\alpha-2} u_{\kappa,\ell}^{n+1} - |u_{\kappa,\ell}^{n}|^{\alpha-2} u_{\kappa,\ell}^{n}}{\ell} \right\|_{W^{-1,p'}(\Omega)} \\
&\le \left\| \nabla \cdot \left(\mu |\nabla u_{\kappa,\ell}^{n+1}|^{p-2} \nabla u_{\kappa,\ell}^{n+1} \right) \right\|_{W^{-1,p'}(\Omega)} + \frac{1}{\kappa} \|\{u_{\kappa,\ell}^{n+1}\}^-\|_{W^{-1,p'}(\Omega)} + \|\tilde{a}\|_{W^{1,p}(0,T;\mathcal{C}^0(\overline{\Omega}))} \\
&\le C\left(1 + \frac{1}{\kappa}\right), \quad \text{for some } C > 0 \text{ independent of } N,
\end{aligned}$$

for all $0 \le n \le N-1$, thus establishing the estimate (4.12).

By (4.7), for each $0 \le n \le N$, we have that

$$\begin{aligned}
\|u_{\kappa,\ell}^n\|_{L^\alpha(\Omega)} &= \left(\int_\Omega |u_{\kappa,\ell}^n|^\alpha \,\mathrm{d}x \right)^{1/\alpha} = \left(\int_\Omega \left| |u_{\kappa,\ell}^n|^{\alpha-2} u_{\kappa,\ell}^n \right|^{\frac{\alpha}{\alpha-1}} \,\mathrm{d}x \right)^{1/\alpha} = \left(\int_\Omega \left| |u_{\kappa,\ell}^n|^{\alpha-2} u_{\kappa,\ell}^n \right|^{\alpha'} \,\mathrm{d}x \right)^{\alpha'(\alpha-1)} \\
&= \left\| |u_{\kappa,\ell}^n|^{\alpha-2} u_{\kappa,\ell}^n \right\|_{L^{\alpha'}(\Omega)}^{\alpha'/\alpha}.
\end{aligned}$$

Since the first term is uniformly bounded with respect to N and κ, we immediately infer that

$$\max_{0 \le n \le N} \left\| |u_{\kappa,\ell}^n|^{\alpha-2} u_{\kappa,\ell}^n \right\|_{L^{\alpha'}(\Omega)} \le C, \tag{4.20}$$

for some $C > 0$ independent of N and κ, thus establishing (4.13), and completing the proof. □

Remark 4.3. Observe that the following stability condition is sufficient to (4.6):

$$\{u_{\kappa,\ell}^{n+1}\}^- \ge \{u_{\kappa,\ell}^n\}^-, \quad \text{for all } 0 \le n \le N-1. \tag{4.21}$$

Indeed, by Lemma 3.2, we have that for all $0 \le n \le N-1$

$$\begin{aligned}
\int_\Omega \{u_{\kappa,\ell}^n\}^- (u_{\kappa,\ell}^{n+1} - u_{\kappa,\ell}^n) \,\mathrm{d}x &= \int_{\{u_{\kappa,\ell}^n \ge 0\}} \left(\{u_{\kappa,\ell}^{n+1}\}^- - \{u_{\kappa,\ell}^n\}^- \right) (u_{\kappa,\ell}^{n+1} - u_{\kappa,\ell}^n) \,\mathrm{d}x \\
&\quad + \int_{\{u_{\kappa,\ell}^n \le 0\}} \{u_{\kappa,\ell}^{n+1}\}^- (u_{\kappa,\ell}^{n+1} + \{u_{\kappa,\ell}^n\}^-) \,\mathrm{d}x \\
&\le \int_{\{u_{\kappa,\ell}^n \le 0\}} \{u_{\kappa,\ell}^{n+1}\}^- (\{u_{\kappa,\ell}^{n+1}\}^+ - \{u_{\kappa,\ell}^{n+1}\}^- + \{u_{\kappa,\ell}^n\}^-) \,\mathrm{d}x \\
&= \int_{\{u_{\kappa,\ell}^n \le 0\}} \{u_{\kappa,\ell}^{n+1}\}^- (-\{u_{\kappa,\ell}^{n+1}\}^- + \{u_{\kappa,\ell}^n\}^-) \,\mathrm{d}x \le 0,
\end{aligned} \tag{4.22}$$

where the last inequality holds thanks to (4.21). Thanks to (4.22) and the positiveness of κ, we have that :

$$\frac{1}{\kappa}\sum_{n=0}^{s-1}\int_\Omega \{u^n_{\kappa,\ell}\}^-(u^{n+1}_{\kappa,\ell}-u^n_{\kappa,\ell})\,\mathrm{d}x \le 0,$$

for all $1 \le s \le N$, which is a special case of the stability condition (4.6) appearing in the statement of Theorem 4.2.

The condition (4.21) is physically more realistic than the "abstract" stability condition (4.6). The stability condition (4.21) describes a regime of pure melting, which is a common regime glaciers undergo during Spring and Summer. ■

Given $\boldsymbol{v}_{\kappa,\ell} = \{v^n_{\kappa,\ell}\}_{n=0}^N$, the function $D_\ell(\Pi_\ell \boldsymbol{v}_{\kappa,\ell}) : (0,T) \to W^{1,p}_0(\Omega)$ is defined by

$$D_\ell(\Pi_\ell \boldsymbol{v}_{\kappa,\ell})(t) := \frac{v^{n+1}_{\kappa,\ell}-v^n_{\kappa,\ell}}{\ell}, \quad \text{for all } n\ell < t \le (n+1)\ell, \quad 0 \le n \le N-1. \tag{4.23}$$

As a result of the estimates (4.7)-(4.13), we have that

$$\begin{aligned}
&\{\Pi_\ell \boldsymbol{u}_{\kappa,\ell}\}_{\ell>0} \text{ is bounded in } L^\infty(0,T;W^{1,p}_0(\Omega)),\\
&\{B_\kappa(\Pi_\ell \boldsymbol{u}_{\kappa,\ell})\}_{\ell>0} \text{ is bounded in } L^\infty(0,T;W^{-1,p'}(\Omega)),\\
&\{|\Pi_\ell \boldsymbol{u}_{\kappa,\ell}|^{\frac{\alpha-2}{2}}\Pi_\ell \boldsymbol{u}_{\kappa,\ell}\}_{\ell>0} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)),\\
&\{D_\ell(|\Pi_\ell \boldsymbol{u}_{\kappa,\ell}|^{\frac{\alpha-2}{2}}\Pi_\ell \boldsymbol{u}_{\kappa,\ell})\}_{\ell>0} \text{ is bounded in } L^\infty(0,T;L^2(\Omega)),\\
&\{|\Pi_\ell \boldsymbol{u}_{\kappa,\ell}|^{\alpha-2}\Pi_\ell \boldsymbol{u}_{\kappa,\ell}\}_{\ell>0} \text{ is bounded in } L^\infty(0,T;L^{\alpha'}(\Omega)),\\
&\{D_\ell(|\Pi_\ell \boldsymbol{u}_{\kappa,\ell}|^{\alpha-2}\Pi_\ell \boldsymbol{u}_{\kappa,\ell})\}_{\ell>0} \text{ is bounded in } L^\infty(0,T;W^{-1,p'}(\Omega)).
\end{aligned} \tag{4.24}$$

Thanks to (4.24), we can establish the existence of solutions for Problem ($\mathcal{P}_\kappa$).

Theorem 4.4. *Let $T>0$, $\Omega \subset \mathbb{R}^2$ and p be as in section 3 and let α be as in (3.1). Let $\kappa > 0$ be given, let $N \ge 1$ be an integer, and define $\ell := T/N$. Assume that $(H1)$–$(H4)$ hold. The* a priori *estimates (4.24) imply that the following convergence process takes place (recall that B_κ has been defined in (4.4)):*

$$\begin{aligned}
&\Pi_\ell \boldsymbol{u}_{\kappa,\ell} \overset{*}{\rightharpoonup} u_\kappa \text{ in } L^\infty(0,T;W^{1,p}_0(\Omega)),\\
&B_\kappa(\Pi_\ell \boldsymbol{u}_{\kappa,\ell}) \overset{*}{\rightharpoonup} g_\kappa \text{ in } L^\infty(0,T;W^{-1,p'}(\Omega)),\\
&|\Pi_\ell \boldsymbol{u}_{\kappa,\ell}|^{\frac{\alpha-2}{2}}\Pi_\ell \boldsymbol{u}_{\kappa,\ell} \overset{*}{\rightharpoonup} v_\kappa \text{ in } L^\infty(0,T;L^2(\Omega)),\\
&D_\ell(|\Pi_\ell \boldsymbol{u}_{\kappa,\ell}|^{\frac{\alpha-2}{2}}\Pi_\ell \boldsymbol{u}_{\kappa,\ell}) \rightharpoonup \frac{\mathrm{d}v_\kappa}{\mathrm{d}t} \text{ in } L^2(0,T;L^2(\Omega)),\\
&|\Pi_\ell \boldsymbol{u}_{\kappa,\ell}|^{\alpha-2}\Pi_\ell \boldsymbol{u}_{\kappa,\ell} \overset{*}{\rightharpoonup} w_\kappa \text{ in } L^\infty(0,T;L^{\alpha'}(\Omega)),\\
&D_\ell(|\Pi_\ell \boldsymbol{u}_{\kappa,\ell}|^{\alpha-2}\Pi_\ell \boldsymbol{u}_{\kappa,\ell}) \overset{*}{\rightharpoonup} \frac{\mathrm{d}w_\kappa}{\mathrm{d}t} \text{ in } L^\infty(0,T;W^{-1,p'}(\Omega)),\\
&|u^N_{\kappa,\ell}|^{\alpha-2}u^N_{\kappa,\ell} \rightharpoonup \chi_\kappa \text{ in } L^{\alpha'}(\Omega).
\end{aligned} \tag{4.25}$$

Besides, the weak-star limit u_κ recovered in the first convergence of (4.25) is a solution for Problem $\mathcal{P}_\kappa$, and the weak-star limits v_κ and w_κ satisfy

$$\begin{aligned}
v_\kappa &= |u_\kappa|^{\frac{\alpha-2}{2}}u_\kappa,\\
w_\kappa &= |u_\kappa|^{\alpha-2}u_\kappa.
\end{aligned}$$

Proof. The convergence process (4.25) holds by virtue of an application of the Banach-Alaoglu-Bourbaki theorem (cf., e.g., Theorem 3.6 of [7]) to the estimates (4.24) and (4.20). The nontivial part of the proof amounts to identifying the weak-star limits v_κ and w_κ and to showing that the weak-star limit u_κ solves Problem $\mathcal{P}_\kappa$.

To begin with, we show that $w_\kappa = |u_\kappa|^{\alpha-2}u_\kappa$. Given any $u \in W_0^{1,p}(\Omega)$, we look for a number $\beta \in \mathbb{R}$ for which

$$\left||u|^{\alpha-2}u\right|^{\beta-2}|u|^{\alpha-2}u = u.$$

Simple algebraic manipulations transform the latter into

$$|u|^{(\alpha-2)(\beta-2)+(\beta-2)+(\alpha-2)}u = u,$$

and we observe that a sufficient condition insuring this is that β satisfies

$$(\alpha-1)(\beta-2)+(\alpha-2)=0,$$

which is equivalent to writing

$$\beta = 2+\frac{2-\alpha}{\alpha-1} = \frac{2\alpha-2+2-\alpha}{\alpha-1} = \frac{\alpha-}{\alpha-1} = \alpha'.$$

Let $v := |u|^{\alpha-2}u$ and observe that if $\beta=\alpha'$ then $|v|^{\beta-2}v \in W_0^{1,p}(\Omega)$ so that the set

$$S := \{v; (|v|^{\alpha'-2}v) \in W_0^{1,p}(\Omega)\}$$

is non-empty. define the semi-norm

$$M(v) := \left\|\nabla(|v|^{\alpha'-2}v)\right\|_{L^p(\Omega)}^{\frac{1}{\alpha'-1}}, \quad \text{for all } v \in S,$$

and define the set

$$\mathscr{M} := \{v \in S; M(v) \le 1\}.$$

An application of the Poincaré-Friedrichs inequality gives that there exists a constant $c_0 = c_0(\Omega) > 0$ such that

$$\begin{aligned}1 &\ge M(v) \ge c_0^{\frac{1}{\alpha'-1}}\left\||v|^{\alpha'-2}v\right\|_{W_0^{1,p}(\Omega)}^{\frac{1}{\alpha'-1}} \ge c_0^{\frac{1}{\alpha'-1}}\left\||v|^{\alpha'-2}v\right\|_{L^p(\Omega)}^{\frac{1}{\alpha'-1}}\\ &= c_0^{\frac{1}{\alpha'-1}}\left(\int_\Omega \left||v|^{\alpha'-2}v\right|^p \,\mathrm{d}x\right)^{1/(p(\alpha'-1))} = c_0^{\frac{1}{\alpha'-1}}\left(\int_\Omega |v|^{(\alpha'-1)p}\,\mathrm{d}x\right)^{1/(p(\alpha'-1))} = c_0^{\frac{1}{\alpha'-1}}\|v\|_{L^{(\alpha'-1)p}(\Omega)}.\end{aligned}$$

Let $\{v_k\}_{k=1}^\infty$ be a sequence in $\mathscr{M}$. Since, by the Rellich-Kondrašov theorem, we have that $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ we obtain that, up to passing to a subsequence, there exists an element $w \in L^p(\Omega)$ such that

$$(|v_k|^{\alpha'-2}v_k) \to w, \quad \text{in } L^p(\Omega), \quad \text{as } k \to \infty. \tag{4.26}$$

Since $1<\alpha<2$ and $2.8 \le p \le 5$, then $\alpha'>2$ and it thus results that $1<p'<p<(\alpha'-1)p<\infty$ and that $\{v_k\}_{k=1}^\infty$ is bounded in $L^{p'}(\Omega)$. The reflexivity of $L^{p'}(\Omega)$ puts us in a position to apply the Banach-Eberlein-Smulian theorem (cf., e.g., Theorem 5.14-4 of [10]) and extract a subsequence, still denoted $\{v_k\}_{k=1}^\infty$, that weakly converges to an element $v \in L^{p'}(\Omega)$. Consider the mapping

$$v \in L^{p'}(\Omega) \mapsto (|v|^{\alpha'-2}v) \in L^p(\Omega),$$

and observe that this mapping is hemi-continuous and monotone, being the mapping $\xi \in \mathbb{R} \to (|\xi|^{\alpha'-2}\xi) \in \mathbb{R}$, with $\alpha'>2$ thanks to (3.1), continuous and monotone. Therefore, an application of Theorem 9.13-2 of [10] gives that $w=|v|^{\alpha'-2}v \in L^p(\Omega)$. Therefore, the convergence (4.26) reads:

$$(|v_k|^{\alpha'-2}v_k) \to w = (|v|^{\alpha'-2}v), \quad \text{in } L^p(\Omega), \quad \text{as } k \to \infty. \tag{4.27}$$

In order to show that $\mathscr{M}$ is relatively compact in $L^{(\alpha'-1)p}(\Omega)$, we have to show that every sequence $\{v_k\}_{k=1}^\infty \subset \mathscr{M}$ admits a convergent subsequence in $L^{(\alpha'-1)p}(\Omega)$. We will see that any of the subsequences satisfying (4.27) will serve for this purpose. In this direction, let $\{v_k\}_{k=1}^\infty$ denote one of the subsequences satisfying (4.27) and let $v \in L^{(\alpha'-1)p}(\Omega)$ be the weak limit. Since $1<\alpha<2$

by (3.1), then $\alpha' > 2$ and, therefore, an application of Lemma 3.2, the triangle inequality and (4.27) gives:

$$\left(\int_\Omega |v_k - v|^{(\alpha'-1)p} \,\mathrm{d}x\right)^{1/((\alpha'-1)p)} \le \left(\int_\Omega \left||v_k|^{\alpha'-1} - |v|^{\alpha'-1}\right|^p \,\mathrm{d}x\right)^{1/((\alpha'-1)p)}$$
$$= \left(\int_\Omega \left|\left||v_k|^{\alpha'-2} v_k\right| - \left||v|^{\alpha'-2} v\right|\right|^p \,\mathrm{d}x\right)^{1/((\alpha'-1)p)} \le \left(\int_\Omega \left||v_k|^{\alpha'-2} v_k - |v|^{\alpha'-2} v\right|^p \,\mathrm{d}x\right)^{1/((\alpha'-1)p)}$$
$$= \left\||v_k|^{\alpha'-2} v_k - |v|^{\alpha'-2} v\right\|_{L^p(\Omega)}^{1/(\alpha'-1)} \to 0 \text{ as } k \to \infty.$$

The latter shows that

$$v_k \to v, \quad \text{in } L^{(\alpha'-1)p}(\Omega) \text{ as } k \to \infty,$$

in turn implying that the set $\mathscr{M}$ is relatively compact in $L^{(\alpha'-1)p}(\Omega)$, as it was to be proved. The established relative compactness of the set $\mathscr{M}$ in $L^{(\alpha'-1)p}(\Omega)$ and the sixth convergence in the process (4.25) (which in turn implies that the time-derivatives in the sense of distributions are uniformly bounded) allow us apply Dubinskii's compactness theorem (Theorem 3.10) with $A_0 = L^{(\alpha'-1)p}(\Omega)$, $A_1 = W^{-1,p'}(\Omega)$, $q_0 = q_1 = 2$, so that

$$|\Pi_\ell \boldsymbol{u}_{\kappa,\ell}|^{\alpha-2} \Pi_\ell \boldsymbol{u}_{\kappa,\ell} \to w_\kappa, \quad \text{in } L^2(0,T; L^{(\alpha'-1)p}(\Omega)) \text{ as } \ell \to 0, \tag{4.28}$$

where, once again, the monotonicity of $\xi \in \mathbb{R} \mapsto |\xi|^{\alpha-2}\xi$, the first convergence in the process (4.25) and Theorem 9.13-2 of [10] imply that

$$w_\kappa = |u_\kappa|^{\alpha-2} u_\kappa.$$

Second, we show that $v_\kappa = |u_\kappa|^{\frac{\alpha-2}{2}} u_\kappa$. Given any $u \in W_0^{1,p}(\Omega)$, we look for a number $\beta \in \mathbb{R}$ for which

$$\left||u|^{\frac{\alpha-2}{2}} u\right|^{\frac{\beta-2}{2}} |u|^{\frac{\alpha-2}{2}} u = u.$$

We observe that a sufficient condition insuring this is that β satisfies

$$\left(\frac{\alpha-2}{2} + 1\right) \frac{\beta-2}{2} + \frac{\alpha-2}{2} = 0,$$

which is equivalent to writing

$$\beta = 2 + 2\left(\frac{2-\alpha}{2} \frac{2}{\alpha}\right) = \frac{4}{\alpha}.$$

Let $v := |u|^{\frac{\alpha-2}{2}} u$ and observe that if $\beta = 4/\alpha$ then $|v|^{\frac{\beta-2}{2}} v \in W_0^{1,p}(\Omega)$ so that the set

$$\tilde{S} := \{v; (|v|^{\frac{(4/\alpha)-2}{2}} v) \in W_0^{1,p}(\Omega)\}$$

is non-empty. define the semi-norm

$$\tilde{M}(v) := \left\|\nabla(|v|^{\frac{(4/\alpha)-2}{2}} v)\right\|_{L^p(\Omega)}^{\frac{\alpha}{2}} = \left\|\nabla(|v|^{\frac{2-\alpha}{\alpha}} v)\right\|_{L^p(\Omega)}^{\frac{\alpha}{2}}, \quad \text{for all } v \in \tilde{S},$$

and define the set

$$\tilde{\mathscr{M}} := \{v \in \tilde{S}; \tilde{M}(v) \le 1\}.$$

An application of the Poincaré-Friedrichs inequality gives that there exists a constant $c_0 = c_0(\Omega) > 0$ such that

$$1 \ge \tilde{M}(v) \ge c_0^{\frac{\alpha}{2}} \left\||v|^{\frac{2-\alpha}{\alpha}} v\right\|_{W_0^{1,p}(\Omega)}^{\frac{\alpha}{2}} \ge c_0^{\frac{\alpha}{2}} \left\||v|^{\frac{2-\alpha}{\alpha}} v\right\|_{L^p(\Omega)}^{\frac{\alpha}{2}} = c_0^{\frac{\alpha}{2}} \left(\int_\Omega \left||v|^{\frac{2-\alpha}{\alpha}} v\right|^p \,\mathrm{d}x\right)^{\alpha/(2p)}$$
$$= c_0^{\frac{\alpha}{2}} \left(\int_\Omega |v|^{\frac{2p}{\alpha}} \,\mathrm{d}x\right)^{\alpha/(2p)} = c_0^{\frac{\alpha}{2}} \|v\|_{L^{\frac{2p}{\alpha}}(\Omega)}.$$

Let $\{v_k\}_{k=1}^{\infty}$ be a sequence in $\tilde{\mathscr{M}}$. Since, by the Rellich-Kondrašov theorem, we have that $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ we obtain that, up to passing to a subsequence, there exists an element $w \in L^p(\Omega)$ such that

$$(|v_k|^{\frac{2-\alpha}{2}} v_k) \to w, \quad \text{in } L^p(\Omega), \quad \text{as } k \to \infty. \tag{4.29}$$

Since $1 < \alpha < 2$ and $2.8 \le p \le 5$, it thus results that $1 \le p' < 2 < p < \frac{2p}{\alpha} < 2p < \infty$ and that $\{v_k\}_{k=1}^{\infty}$ is bounded in $L^{\frac{2p}{\alpha}}(\Omega)$. The reflexivity of $L^{\frac{2p}{\alpha}}(\Omega)$ puts us in a position to apply the Banach-Eberlein-Smulian theorem (cf., e.g., Theorem 5.14-4 of [10]) and extract a subsequence, still denoted $\{v_k\}_{k=1}^{\infty}$, that weakly converges to an element $v \in L^{p'}(\Omega)$. Consider the mapping

$$v \in L^{p'}(\Omega) \mapsto (|v|^{\frac{2-\alpha}{2}} v) \in L^p(\Omega),$$

and observe that this mapping is hemi-continuous and monotone, being the mapping $\xi \in \mathbb{R} \to (|\xi|^{\frac{2-\alpha}{2}} \xi) \in \mathbb{R}$ continuous and monotone. Therefore, an application of Theorem 9.13-2 of [10] gives that $w = |v|^{\frac{2-\alpha}{\alpha}} v \in L^p(\Omega)$. Therefore, the convergence (4.26) reads:

$$(|v_k|^{\frac{2-\alpha}{\alpha}} v_k) \to w = (|v|^{\frac{2-\alpha}{\alpha}} v), \quad \text{in } L^p(\Omega), \quad \text{as } k \to \infty. \tag{4.30}$$

In order to show that $\tilde{\mathscr{M}}$ is relatively compact in $L^{\frac{2p}{\alpha}}(\Omega)$, we have to show that every sequence $\{v_k\}_{k=1}^{\infty} \subset \tilde{\mathscr{M}}$ admits a convergent subsequence in $L^{\frac{2p}{\alpha}}(\Omega)$. We will see that any of the subsequences satisfying (4.27) will serve for this purpose. In this direction, let $\{v_k\}_{k=1}^{\infty}$ denote one of the subsequences satisfying (4.27) and let $v \in L^{\frac{2p}{\alpha}}(\Omega)$ be the weak limit. An application of Lemma 3.2, the triangle inequality and (4.27) gives:

$$\begin{aligned}
&\left(\int_\Omega |v_k - v|^{\frac{2p}{\alpha}} \,\mathrm{d}x\right)^{\alpha/(2p)} \le \left(\int_\Omega \left||v_k|^{\frac{2}{\alpha}} - |v|^{\frac{2}{\alpha}}\right|^p \,\mathrm{d}x\right)^{\alpha/(2p)}\\
&= \left(\int_\Omega \left|\left||v_k|^{\frac{2-\alpha}{\alpha}} v_k\right| - \left||v|^{\frac{2-\alpha}{\alpha}} v\right|\right|^p \,\mathrm{d}x\right)^{\alpha/(2p)} \le \left(\int_\Omega \left||v_k|^{\frac{2-\alpha}{\alpha}} v_k - |v|^{\frac{2-\alpha}{\alpha}} v\right|^p \,\mathrm{d}x\right)^{\alpha/(2p)}\\
&= \left\||v_k|^{\frac{2-\alpha}{\alpha}} v_k - |v|^{\frac{2-\alpha}{\alpha}} v\right\|_{L^p(\Omega)}^{\alpha/2} \to 0 \text{ as } k \to \infty.
\end{aligned}$$

The latter shows that

$$v_k \to v, \quad \text{in } L^{\frac{2p}{\alpha}}(\Omega) \text{ as } k \to \infty,$$

in turn implying that the set $\tilde{\mathscr{M}}$ is relatively compact in $L^{\frac{2p}{\alpha}}(\Omega)$, as it was to be proved. The established relative compactness of the set $\tilde{\mathscr{M}}$ in $L^{\frac{2p}{\alpha}}(\Omega)$ and the fourth convergence in the process (4.25) (which in turn implies that the time-derivatives in the sense of distributions are uniformly bounded) allow us apply Dubinskii's compactness theorem (Theorem 3.10) with $A_0 = L^{\frac{2p}{\alpha}}(\Omega)$, $A_1 = L^2(\Omega)$, $q_0 = q_1 = 2$, so that

$$|\Pi_\ell \boldsymbol{u}_{\kappa,\ell}|^{\frac{\alpha-2}{2}} \Pi_\ell \boldsymbol{u}_{\kappa,\ell} \to v_\kappa, \quad \text{in } L^2(0,T;L^{\frac{2p}{\alpha}}(\Omega)) \text{ as } \ell \to 0, \tag{4.31}$$

where, once again, the monotonicity of $\xi \in \mathbb{R} \mapsto |\xi|^{\frac{\alpha-2}{2}} \xi$, the first convergence in the process (4.25) and Theorem 9.13-2 of [10] imply that

$$v_\kappa = |u_\kappa|^{\frac{\alpha-2}{2}} u_\kappa.$$

We are left to show that the weak-star limit u_κ is a solution for Problem $\mathcal{P}_\kappa$. Let $v \in \mathcal{D}(\Omega)$ and let $\psi \in \mathcal{C}^1([0,T])$. For each $0 \le n \le N-1$, multiply (4.1) by $\{v\psi(n\ell)\}$, getting

$$\begin{aligned}
&\frac{\psi(n\ell)}{\ell}\int_\Omega \{|u^{n+1}_{\kappa,\ell}|^{\alpha-2}u^{n+1}_{\kappa,\ell} - |u^{n}_{\kappa,\ell}|^{\alpha-2}u^{n}_{\kappa,\ell}\} v \,\mathrm{d}x \\
&\quad + \psi(n\ell)\int_\Omega \mu|\nabla u^{n+1}_{\kappa,\ell}|^{p-2}\nabla u^{n+1}_{\kappa,\ell}\cdot\nabla v\,\mathrm{d}x - \psi(n\ell)\int_\Omega \frac{\{u^{n+1}_{\kappa,\ell}\}^-}{\kappa} v\,\mathrm{d}x \\
&= \int_\Omega \left(\frac{1}{\ell}\int_{n\ell}^{(n+1)\ell}\tilde{a}(t)\,\mathrm{d}t\right) v\psi(n\ell)\,\mathrm{d}x.
\end{aligned} \tag{4.32}$$

Multiplying (4.32) by ℓ and summing over $0 \le n \le N-1$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{N-1}\ell\int_\Omega \frac{|u^{n+1}_{\kappa,\ell}|^{\alpha-2}u^{n+1}_{\kappa,\ell} - |u^{n}_{\kappa,\ell}|^{\alpha-2}u^{n}_{\kappa,\ell}}{\ell} v\psi(n\ell)\,\mathrm{d}x \\
&\quad + \sum_{n=0}^{N-1}\ell\int_\Omega \mu|\nabla u^{n+1}_{\kappa,\ell}|^{p-2}\nabla u^{n+1}_{\kappa,\ell}\cdot\nabla(\psi(n\ell)v)\,\mathrm{d}x \\
&\quad - \frac{1}{\kappa}\sum_{n=0}^{N-1}\ell\int_\Omega \{u^{n+1}_{\kappa,\ell}\}^- v\psi(n\ell)\,\mathrm{d}x = \sum_{n=0}^{N-1}\ell\int_\Omega\left(\frac{1}{\ell}\int_{n\ell}^{(n+1)\ell}\tilde{a}(t)\,\mathrm{d}t\right) v\psi(n\ell)\,\mathrm{d}x.
\end{aligned} \tag{4.33}$$

For the sake of brevity, define $\psi_\ell(t) := \psi(n\ell)$, $n\ell \le t \le (n+1)\ell$ and $0 \le n \le N-1$. Equation (4.33) can be thus re-arranged as follows:

$$\begin{aligned}
&\int_0^T\int_\Omega D_\ell(|\Pi_\ell \boldsymbol{u}_{\kappa,\ell}|^{\alpha-2}\Pi_\ell\boldsymbol{u}_{\kappa,\ell}) v\,\mathrm{d}x\psi_\ell(t)\,\mathrm{d}t \\
&\quad - \int_0^T\int_\Omega \nabla\cdot\left(\mu|\nabla(\Pi_\ell\boldsymbol{u}_{\kappa,\ell})|^{p-2}\nabla(\Pi_\ell\boldsymbol{u}_{\kappa,\ell})\right) v\,\mathrm{d}x\psi_\ell(t)\,\mathrm{d}t \\
&\quad - \frac{1}{\kappa}\int_0^T\int_\Omega \{\Pi_\ell\boldsymbol{u}_{\kappa,\ell}\}^- v\,\mathrm{d}x\psi_\ell(t)\,\mathrm{d}t = \int_0^T\left(\int_\Omega \tilde{a}(t)v\,\mathrm{d}x\right)\psi_\ell(t)\,\mathrm{d}t.
\end{aligned} \tag{4.34}$$

Letting $\ell \to 0$ and exploiting the convergence process (4.25) and the Riemann integrability of ψ, we obtain:

$$\begin{aligned}
&\int_0^T\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(|u_\kappa|^{\alpha-2}u_\kappa\right), v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\psi(t)\,\mathrm{d}t + \int_0^T\int_\Omega g_\kappa(t)v\,\mathrm{d}x\psi(t)\,\mathrm{d}t \\
&= \int_0^T\int_\Omega \tilde{a}(t)v\,\mathrm{d}x\psi(t)\,\mathrm{d}t.
\end{aligned} \tag{4.35}$$

Let us rearrange the first term on the left-hand side of equation (4.33) as follows:

$$
\begin{aligned}
&\frac{1}{\ell}\sum_{n=0}^{N-1}\ell\int_\Omega\{|u_{\kappa,\ell}^{n+1}|^{\alpha-2}u_{\kappa,\ell}^{n+1}-|u_{\kappa,\ell}^{n}|^{\alpha-2}u_{\kappa,\ell}^{n}\}v\psi(n\ell)\,\mathrm{d}x\\
&=\int_\Omega\Bigg\{\left[|u_{\kappa,\ell}^{1}|^{\alpha-2}u_{\kappa,\ell}^{1}-|u_0|^{\alpha-2}u_0\right]v\psi(0)\\
&\quad+\left[|u_{\kappa,\ell}^{2}|^{\alpha-2}u_{\kappa,\ell}^{2}-|u_{\kappa,\ell}^{1}|^{\alpha-2}u_{\kappa,\ell}^{1}\right]v\psi(\ell)\\
&\quad+\dots\\
&\quad+\left[|u_{\kappa,\ell}^{N-1}|^{\alpha-2}u_{\kappa,\ell}^{N-1}-|u_{\kappa,\ell}^{N-2}|^{\alpha-2}u_{\kappa,\ell}^{N-2}\right]v\psi((N-1)\ell)\\
&\quad\left[|u_{\kappa,\ell}^{N}|^{\alpha-2}u_{\kappa,\ell}^{N}-|u_{\kappa,\ell}^{N-1}|^{\alpha-2}u_{\kappa,\ell}^{N-1}\right]v\psi(T)\Bigg\}\,\mathrm{d}x\\
&=\int_\Omega-|u_0|^{\alpha-2}u_0v\psi(0)\,\mathrm{d}x\\
&\quad+\int_\Omega\Bigg\{\left[-|u_{\kappa,\ell}^{1}|^{\alpha-2}u_{\kappa,\ell}^{1}v(\psi(\ell)-\psi(0))\right]+\left[-|u_{\kappa,\ell}^{2}|^{\alpha-2}u_{\kappa,\ell}^{2}v(\psi(2\ell)-\psi(\ell))\right]\\
&\quad+\dots\\
&\quad+\left[-|u_{\kappa,\ell}^{N-2}|^{\alpha-2}u_{\kappa,\ell}^{N-2}v(\psi((N-1)\ell)-\psi((N-2)\ell))\right]\\
&\quad+\left[-|u_{\kappa,\ell}^{N-1}|^{\alpha-2}u_{\kappa,\ell}^{N-1}v(\psi(T)-\psi((N-1)\ell))\right]\Bigg\}\,\mathrm{d}x\\
&\quad+\int_\Omega|u_{\kappa,\ell}^{N}|^{\alpha-2}u_{\kappa,\ell}^{N}v\psi(T)\,\mathrm{d}x\\
&=-\sum_{n=0}^{N-1}\ell\int_\Omega|u_{\kappa,\ell}^{n}|^{\alpha-2}u_{\kappa,\ell}^{n}v\left[\frac{\psi(n\ell)-\psi((n-1)\ell)}{\ell}\right]\,\mathrm{d}x\\
&\quad+\int_\Omega|u_{\kappa,\ell}^{N}|^{\alpha-2}u_{\kappa,\ell}^{N}v\psi(T)\,\mathrm{d}x-\int_\Omega|u_0|^{\alpha-2}u_0v\psi(0)\,\mathrm{d}x.
\end{aligned}
$$

Therefore, equation (4.33) can be thoroughly re-arranged as follows:

$$
\begin{aligned}
&\int_\Omega|u_{\kappa,\ell}^{N}|^{\alpha-2}u_{\kappa,\ell}^{N}v\psi(T)\,\mathrm{d}x-\int_\Omega|u_0|^{\alpha-2}u_0v\psi(0)\,\mathrm{d}x\\
&\quad-\sum_{n=0}^{N-1}\ell\int_\Omega|u_{\kappa,\ell}^{n}|^{\alpha-2}u_{\kappa,\ell}^{n}v\left[\frac{\psi(n\ell)-\psi((n-1)\ell)}{\ell}\right]\,\mathrm{d}x\\
&\quad+\sum_{n=0}^{N-1}\ell\int_\Omega\mu|\nabla u_{\kappa,\ell}^{n+1}|^{p-2}\nabla u_{\kappa,\ell}^{n+1}\cdot\nabla(\psi(n\ell)v)\,\mathrm{d}x\\
&\quad-\frac{1}{\kappa}\sum_{n=0}^{N-1}\ell\int_\Omega\{u_{\kappa,\ell}^{n+1}\}^{-}v\psi(n\ell)\,\mathrm{d}x=\sum_{n=0}^{N-1}\ell\int_\Omega\left(\frac{1}{\ell}\int_{n\ell}^{(n+1)\ell}\tilde{a}(t)\,\mathrm{d}t\right)v\psi(n\ell)\,\mathrm{d}x.
\end{aligned}
\tag{4.36}
$$

Letting $\ell\to 0$ in (4.36) thus gives:

$$
\begin{aligned}
&-\int_0^T\int_\Omega|u_\kappa|^{\alpha-2}u_\kappa v\,\mathrm{d}x\frac{\mathrm{d}\psi}{\mathrm{d}t}\,\mathrm{d}t+\int_0^T\langle g_\kappa(t),v\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\psi(t)\,\mathrm{d}t\\
&\quad+\int_0^T\int_\Omega[\chi_\kappa\psi(T)-|u_0|^{\alpha-2}u_0\psi(0)]v\,\mathrm{d}x=\int_0^T\int_\Omega\tilde{a}(t)v\,\mathrm{d}x\psi(t)\,\mathrm{d}t.
\end{aligned}
\tag{4.37}
$$

Observe that an application of the Sobolev embedding theorem (cf., e.g., Theorem 6.6-1 of [10]) and an integration by parts in (4.35) give:

$$\begin{aligned}
&-\int_0^T \int_\Omega |u_\kappa|^{\alpha-2} u_\kappa v \,\mathrm{d}x \frac{\mathrm{d}\psi}{\mathrm{d}t} \,\mathrm{d}t + \langle |u_\kappa(T)|^{\alpha-2} u_\kappa(T), \psi(T) v\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \\
&\quad - \int_\Omega |u_\kappa(0)|^{\alpha-2} u_\kappa(0) \psi(0) v \,\mathrm{d}x + \int_0^T \langle g_k(t), v\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \psi(t) \,\mathrm{d}t \\
&= \int_0^T \int_\Omega \tilde{a}(t) v \,\mathrm{d}x \psi(t) \,\mathrm{d}t.
\end{aligned} \tag{4.38}$$

Comparing equations (4.37) and (4.38) gives

$$\begin{aligned}
&\langle |u_\kappa(T)|^{\alpha-2} u_\kappa(T), \psi(T) v\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} - \int_\Omega |u_\kappa(0)|^{\alpha-2} u_\kappa(0) \psi(0) v \,\mathrm{d}x \\
&= \int_0^T \int_\Omega [\chi_\kappa \psi(T) - |u_0|^{\alpha-2} u_0 \psi(0)] v \,\mathrm{d}x
\end{aligned} \tag{4.39}$$

Since $\psi \in \mathcal{C}^1([0,T])$ is arbitrarily chosen, let us specialize ψ in (4.39) in a way such that $\psi(0) = 0$. We obtain

$$\langle |u_\kappa(T)|^{\alpha-2} u_\kappa(T) - \chi_\kappa, \psi(T) v\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} = 0, \quad \text{for all } v \in \mathcal{D}(\Omega). \tag{4.40}$$

Since the duality in (4.40) is continuous with respect to v, and since $\mathcal{D}(\Omega)$ is, by definition, dense in $W_0^{1,p}(\Omega)$, we immediately infer that:

$$|u_\kappa(T)|^{\alpha-2} u_\kappa(T) = \chi_\kappa \in L^{\alpha'}(\Omega). \tag{4.41}$$

Thanks to Lemma 3.3, it is immediate to observe that

$$|\chi_\kappa|^{\alpha'-2} \chi_\kappa = \left| |u_\kappa(T)|^{\alpha-2} u_\kappa(T) \right|^{\alpha'-2} \chi_\kappa = |u_\kappa(T)|^{2-\alpha} \left[|u_\kappa(T)|^{\alpha-2} u_\kappa(T) \right] = u_\kappa(T) \in L^\alpha(\Omega).$$

Let us now specialize ψ in (4.39) in a way such that $\psi(T) = 0$. We obtain

$$\int_\Omega \left(|u_\kappa(0)|^{\alpha-2} u_\kappa(0) - |u_0|^{\alpha-2} u_0 \right) \psi(0) v \,\mathrm{d}x = 0, \quad \text{for all } v \in \mathcal{D}(\Omega). \tag{4.42}$$

Since the integration in (4.42) is continuous with respect to v, and since $\mathcal{D}(\Omega)$ is, by definition, dense in $W_0^{1,p}(\Omega)$, we immediately infer that:

$$|u_\kappa(0)|^{\alpha-2} u_\kappa(0) = |u_0|^{\alpha-2} u_0,$$

so that the injectivity of the monotone and hemi-continuous operator $\xi \mapsto |\xi|^{\alpha-2}\xi$ in turn implies that:

$$u_\kappa(0) = u_0 \in K. \tag{4.43}$$

The last thing to check is that $g_\kappa = B_\kappa(u_\kappa)$. For each $0 \le n \le N-1$, multiply (4.1) by $u_{\kappa,\ell}^{n+1}$ and apply Lemma 3.7, thus getting

$$\begin{aligned}
&\frac{1}{\alpha'} \sum_{n=0}^{N-1} \left\{ \left\| |u_{\kappa,\ell}^{n+1}|^{\alpha-2} u_{\kappa,\ell}^{n+1} \right\|_{L^{\alpha'}(\Omega)}^{\alpha'} - \left\| |u_{\kappa,\ell}^{n}|^{\alpha-2} u_{\kappa,\ell}^{n} \right\|_{L^{\alpha'}(\Omega)}^{\alpha'} \right\} \\
&\quad + \sum_{n=0}^{N-1} \ell \left\langle B_\kappa(u_{\kappa,\ell}^{n+1}), u_{\kappa,\ell}^{n+1} \right\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \\
&\le \sum_{n=0}^{N-1} \ell \int_\Omega \left(\frac{1}{\ell} \int_{n\ell}^{(n+1)\ell} \tilde{a}(t \,\mathrm{d}t \right) u_{\kappa,\ell}^{n+1} \,\mathrm{d}x,
\end{aligned}$$

which in turn implies:

$$\begin{aligned}
&\frac{1}{\alpha'}\left\||u^N_{\kappa,\ell}|^{\alpha-2}u^N_{\kappa,\ell}\right\|^{\alpha'}_{L^{\alpha'}(\Omega)}\\
&\quad+\int_0^T\langle B_\kappa(\Pi_\ell\boldsymbol{u}_{\kappa,\ell}),\Pi_\ell\boldsymbol{u}_{\kappa,\ell}\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\\
&\leq\int_0^T\int_\Omega\tilde{a}(t)\Pi_\ell\boldsymbol{u}_{\kappa,\ell}\,\mathrm{d}x\,\mathrm{d}t\frac{1}{\alpha'}\left\||u_0|^{\alpha-2}u_0\right\|^{\alpha'}_{L^{\alpha'}(\Omega)}.
\end{aligned}\tag{4.44}$$

We now exploit a trick developed by Minty [35] which is, by now, classical. Passing to the lim inf as $\ell\to 0$ in (4.44) and keeping in mind the convergence process (4.25) as well as the identities (4.41)–(4.43) gives, on the one hand:

$$\begin{aligned}
&\frac{1}{\alpha'}\|u_\kappa(T)\|^\alpha_{L^\alpha(\Omega)}+\liminf_{\ell\to 0}\int_0^T\langle B_\kappa(\Pi_\ell\boldsymbol{u}_{\kappa,\ell}),\Pi_\ell\boldsymbol{u}_{\kappa,\ell}\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,\mathrm{d}t\\
&=\frac{1}{\alpha'}\left\||u_\kappa(T)|^{\alpha-2}u_\kappa(T)\right\|^{\alpha'}_{L^{\alpha'}(\Omega)}+\liminf_{\ell\to 0}\int_0^T\langle B_\kappa(\Pi_\ell\boldsymbol{u}_{\kappa,\ell}),\Pi_\ell\boldsymbol{u}_{\kappa,\ell}\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,\mathrm{d}t\\
&\leq\int_0^T\int_\Omega\tilde{a}(t)u_\kappa\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\alpha'}\left\||u_0|^{\alpha-2}u_0\right\|^{\alpha'}_{L^{\alpha'}(\Omega)}=\int_0^T\int_\Omega\tilde{a}(t)v\,\mathrm{d}x\psi(t)\,\mathrm{d}t+\frac{1}{\alpha'}\|u_0\|^\alpha_{L^\alpha(\Omega)}.
\end{aligned}\tag{4.45}$$

On the other hand, the specializations $v=u_\kappa$ and $\psi\equiv 1$ in (4.35), and an application of Lemma 3.8 give:

$$\begin{aligned}
&\frac{1}{\alpha'}\|u_\kappa(T)\|^\alpha_{L^\alpha(\Omega)}+\int_0^T\langle g_\kappa(t),v\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\psi(t)\,\mathrm{d}t\\
&=\int_0^T\int_\Omega\tilde{a}(t)v\,\mathrm{d}x\psi(t)\,\mathrm{d}t+\frac{1}{\alpha'}\|u_0\|^\alpha_{L^\alpha(\Omega)}.
\end{aligned}\tag{4.46}$$

Combining (4.45) and (4.46) gives:

$$\liminf_{\ell\to 0}\int_0^T\langle B_\kappa(\Pi_\ell\boldsymbol{u}_{\kappa,\ell}),\Pi_\ell\boldsymbol{u}_{\kappa,\ell}\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\leq\int_0^T\langle g_\kappa(t),v\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\psi(t)\,\mathrm{d}t,\tag{4.47}$$

Let $w\in L^p(0,T;W_0^{1,p}(\Omega))$. An application of the strict monotonicity of the operator B_κ defined in (4.4) and (4.47) gives:

$$\begin{aligned}
&\int_0^T\langle g_\kappa-B_\kappa w,u_\kappa-w\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,\mathrm{d}t\\
&\geq\liminf_{\ell\to 0}\int_0^T\langle B_\kappa(\Pi_\ell\boldsymbol{u}_{\kappa,\ell})-B_\kappa w,\Pi_\ell\boldsymbol{u}_{\kappa,\ell}-w\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,\mathrm{d}t\geq 0.
\end{aligned}\tag{4.48}$$

Let $\lambda>0$ and specialize $w=u_\kappa-\lambda v$ in (4.48), where v is arbitrarily chosen in $L^p(0,T;W_0^{1,p}(\Omega))$. We obtain that:

$$\int_0^T\langle g_\kappa-B_\kappa(u_\kappa-\lambda v)),\lambda v\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,\mathrm{d}t\geq 0.\tag{4.49}$$

Dividing (4.49) by $\lambda>0$ and letting $\lambda\to 0^+$ gives:

$$\int_0^T\langle g_\kappa-B_\kappa u_\kappa),v\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,\mathrm{d}t\geq 0.\tag{4.50}$$

By the arbitrariness of $v\in L^p(0,T;W_0^{1,p}(\Omega))$, we obtain that:

$$g_\kappa=B_\kappa u_\kappa\in L^\infty(0,T;W^{-1,p'}(\Omega)),\tag{4.51}$$

which thus implies that u_κ is a solution of Problem $\mathcal{P}_\kappa$. This completes the proof. □

The next step, which constitutes the main novelty of this paper, is the passage to the limit as $\kappa \to 0^+$ as well as the recovery of the actual model governing the variation of shallow ice sheets in time.

The estimates (4.7), (4.10), (4.12) and (4.13) in turn imply that there exists a constant $C > 0$ independent of κ such that

$$\begin{aligned}
\|u_\kappa\|_{L^\infty(0,T;W_0^{1,p}(\Omega))} &\le C,\\
\left\||u_\kappa|^{\frac{\alpha-2}{2}}u_\kappa\right\|_{L^\infty(0,T;L^2(\Omega))} &\le C,\\
\left\|\frac{\mathrm{d}}{\mathrm{d}t}(|u_\kappa|^{\frac{\alpha-2}{2}}u_\kappa)\right\|_{L^2(0,T;L^2(\Omega))} &\le C,\\
\||u_\kappa|^{\alpha-2}u_\kappa\|_{L^\infty(0,T;L^{\alpha'}(\Omega))} &\le C,\\
\left\|\frac{\mathrm{d}}{\mathrm{d}t}(|u_\kappa|^{\alpha-2}u_\kappa)\right\|_{L^\infty(0,T;W^{-1,p'}(\Omega))} &\le C\left(1+\frac{1}{\kappa}\right).
\end{aligned}$$

By the Banach-Alaoglu-Bourbaki theorem (cf., e.g., Theorem 3.6 of [7]) we infer that, up to passing to a subsequence still denoted by $\{u_\kappa\}_{\kappa>0}$, the following convergences hold:

$$\begin{aligned}
u_\kappa &\overset{*}{\rightharpoonup} u, \text{ in } L^\infty(0,T;W_0^{1,p}(\Omega)),\\
|u_\kappa|^{\frac{\alpha-2}{2}}u_\kappa &\overset{*}{\rightharpoonup} v, \text{ in } L^\infty(0,T;L^2(\Omega)),\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(|u_\kappa|^{\frac{\alpha-2}{2}}u_\kappa\right) &\rightharpoonup \frac{\mathrm{d}v}{\mathrm{d}t}, \text{ in } L^\infty(0,T;L^2(\Omega)),\\
|u_\kappa|^{\alpha-2}u_\kappa &\overset{*}{\rightharpoonup} w, \text{ in } L^\infty(0,T;L^{\alpha'}(\Omega)).
\end{aligned} \tag{4.52}$$

Using the same argument as in Theorem 4.4, an application of Dubinskii's theorem (Theorem 3.9) with $A_0 := L^{\frac{2p}{\alpha}}(\Omega)$, $A_1 := L^2(\Omega)$, $q_0 = \infty$ and $q_1 = 2$ gives that:

$$\left\{|u_\kappa|^{\frac{\alpha-2}{2}}u_\kappa\right\}_{\kappa>0} \text{ strongly converges in } L^\infty(0,T;L^{\frac{2p}{\alpha}}(\Omega)).$$

Moreover, by Lemma 2 of [1], it can be established that:

$$\left\{|u_\kappa|^{\frac{\alpha-2}{2}}u_\kappa\right\}_{\kappa>0} \text{ strongly converges in } \mathcal{C}^0([0,T];L^2(\Omega)).$$

Therefore, an application of Lemma 3.5 gives that:

$$u_\kappa \to u, \quad \text{in } \mathcal{C}^0([0,T];L^\alpha(\Omega)). \tag{4.53}$$

The monotonicity argument in Theorem 4.4 and the third convergence in (4.52) in turn imply that:

$$v = |u|^{\frac{\alpha-2}{2}}u \in H^1(0,T;L^2(\Omega)).$$

Define the linear and continuous operator $L_0 : \mathcal{C}^0([0,T];L^\alpha(\Omega)) \to L^\alpha(\Omega)$ by:

$$L_0(v) := v(0), \quad \text{for all } v \in \mathcal{C}^0([0,T];L^\alpha(\Omega)).$$

By (4.53) and the continuity of L_0, we have that $u_\kappa(0) \to u(0)$ in $L^\alpha(\Omega)$. However, since $u_\kappa(0) = u_0$ for all $\kappa > 0$, we immediately deduce that $u(0) = u_0 \in K$ and so that the weak-star limit u satisfies the expected initial condition.

Note that the variational equations in Problem $\mathcal{P}_\kappa$ take the following equivalent form:

$$\frac{\mathrm{d}}{\mathrm{d}t}(|u_\kappa(t)|^{\alpha-2}u_\kappa(t)) - \nabla\cdot\left(\mu|\nabla u_\kappa(t)|^{p-2}\nabla u_\kappa(t)\right) - \frac{1}{\kappa}\{u_\kappa(t)\}^- = \tilde{a}(t), \quad \text{in } W^{-1,p'}(\Omega). \tag{4.54}$$

Integrating (4.54) in $(0,T)$ gives:

$$
\begin{aligned}
&-\frac{1}{\kappa}\int_0^T \{u_\kappa(t)\}^- \,\mathrm{d}t = \int_0^T \tilde{a}(t)\,\mathrm{d}t + \int_0^T \nabla\cdot\left(\mu|\nabla u_\kappa(t)|^{p-2}\nabla u_\kappa(t)\right)\,\mathrm{d}t - \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}(|u_\kappa(t)|^{\alpha-2}u_\kappa(t))\,\mathrm{d}t\\
&= \int_0^T \tilde{a}(t)\,\mathrm{d}t + \int_0^T \nabla\cdot\left(\mu|\nabla u_\kappa(t)|^{p-2}\nabla u_\kappa(t)\right)\,\mathrm{d}t - \left[|u_\kappa(T)|^{\alpha-2}u_\kappa(T) - |u_0|^{\alpha-2}u_0\right]\\
&= \int_0^T \tilde{a}(t)\,\mathrm{d}t + \int_0^T \nabla\cdot\left(\mu|\nabla u_\kappa(t)|^{p-2}\nabla u_\kappa(t)\right)\,\mathrm{d}t - \left[|u_\kappa(T)|^{\alpha-2}u_\kappa(T) - u_0^{\alpha-1}\right], \quad \text{in } W^{-1,p'}(\Omega),
\end{aligned}
$$

where the last equality holds since $u_0 \geq 0$ in $\overline{\Omega}$, being $u_0 \in K$ by assumption.

Let $v \in W_0^{1,p}(\Omega)$ be arbitrarily chosen in a way such that $\|v\|_{W_0^{1,p}(\Omega)} = 1$. By Stampacchia's theorem (cf. the seminal paper [44]) we have that $|v| \in W_0^{1,p}(\Omega)$ as well. We have that:

$$
\begin{aligned}
&\frac{1}{\kappa}\left|\left\langle \int_0^T \{u_\kappa(t)\}^-\,\mathrm{d}t, v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\right| \leq \left|\left\langle \int_0^T \tilde{a}(t)\,\mathrm{d}t, v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\right|\\
&\quad + \left|\left\langle \int_0^T \nabla\cdot\left(\mu|\nabla u_\kappa(t)|^{p-2}\nabla u_\kappa(t)\right)\,\mathrm{d}t, v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\right|\\
&\quad + \left|\int_\Omega \underbrace{|u_\kappa(T)|^{\alpha-2}u_\kappa(T)}_{\in L^{\alpha'}(\Omega) \text{ by } (4.41)} v\,\mathrm{d}x\right| + \int_\Omega u_0^{\alpha-1}|v|\,\mathrm{d}x\\
&\leq \int_0^T \|\tilde{a}(t)\|_{W^{-1,p'}(\Omega)}\|v\|_{W_0^{1,p}(\Omega)}\,\mathrm{d}t + T\left\|\nabla\cdot\left(\mu|\nabla u_\kappa|^{p-2}\nabla u_\kappa\right)\right\|_{L^\infty(0,T;W^{-1,p'}(\Omega))}\|v\|_{W_0^{1,p}(\Omega)}\\
&\quad + \left\||u_\kappa(T)|^{\alpha-2}u_\kappa(T)\right\|_{L^{\alpha'}(\Omega)}\|v\|_{L^\alpha(\Omega)} + \|u_0^{\alpha-1}\|_{L^{\alpha'}(\Omega)}\|v\|_{L^\alpha(\Omega)}\\
&\leq \|\tilde{a}\|_{W^{1,p}(0,T;\mathcal{C}^0(\overline{\Omega}))} + T\left\|\nabla\cdot\left(\mu|\nabla u_\kappa|^{p-2}\nabla u_\kappa\right)\right\|_{L^\infty(0,T;W^{-1,p'}(\Omega))} + \|u_\kappa(T)\|_{L^\alpha(\Omega)}^{\alpha-1} + \|u_0\|_{L^\alpha(\Omega)}^{\alpha-1}.
\end{aligned}
$$

Thanks to Lemma 3.5 and the boundedness of the p-Laplacian, the latter term is uniformly bounded with respect to κ. By the arbitrariness of $v \in W_0^{1,p}(\Omega)$ with $\|v\|_{W_0^{1,p}(\Omega)}$, we deduce that:

$$
\sup_{\substack{v\in W_0^{1,p}(\Omega)\\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \frac{1}{\kappa}\left|\left\langle \int_0^T \{u_\kappa(t)\}^-\,\mathrm{d}t, v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\right| \leq C, \quad \text{for all } \kappa > 0,
$$

for some $C > 0$ independent of κ or, equivalently,

$$
\left\|\frac{1}{\kappa}\int_0^T \{u_\kappa(t)\}^-\,\mathrm{d}t\right\|_{W^{-1,p'}(\Omega)} \leq C, \quad \text{for all } \kappa > 0, \tag{4.55}
$$

for some $C > 0$ independent of κ. By Bochner's theorem (cf., e.g., Theorem 8.9 of [30]), we have that the following inequality is always true:

$$
\left\|\frac{1}{\kappa}\int_0^T \{u_\kappa(t)\}^-\,\mathrm{d}t\right\|_{W^{-1,p'}(\Omega)} \leq \frac{1}{\kappa}\int_0^T \|\{u_\kappa(t)\}^-\|_{W^{-1,p'}(\Omega)}\,\mathrm{d}t.
$$

We now show that the inverse inequality holds true too. Observe that, by Stampacchia's theorem, we have that $\{u_\kappa(t)\}^- \in W_0^{1,p}(\Omega)$, so that $\{u_\kappa(t)\}^- \geq 0$ in $\overline{\Omega}$, for a.a. $t \in (0,T)$. Therefore, we have:

$$
\langle \{u_\kappa(t)\}^-, v\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} = \int_\Omega \{u_\kappa(t)\}^- v\,\mathrm{d}x \leq \int_\Omega \{u_\kappa(t)\}^-|v|\,\mathrm{d}x, \quad \text{for all } v \in W_0^{1,p}(\Omega),
$$

for a.a. $t \in (0,T)$. Therefore, for a.a. $t \in (0,T)$, the supremum

$$\sup_{\substack{v \in W_0^{1,p}(\Omega)\\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \left| \langle \{u_\kappa(t)\}^-, v\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \right|$$

is attained for functions $v \in W_0^{1,p}(\Omega)$ with unitary norm that are either greater or equal than zero in $\overline{\Omega}$, or less or equal than zero in $\overline{\Omega}$. In view of this remark, we have that:

$$\begin{aligned}
&\sup_{\substack{v \in W_0^{1,p}(\Omega)\\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \int_0^T \left| \left\langle \frac{\{u_\kappa(t)\}^-}{\kappa}, v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \right| \,\mathrm{d}t = \sup_{\substack{v \in W_0^{1,p}(\Omega)\\ \|v\|_{W_0^{1,p}(\Omega)}=1\\ v\geq 0 \text{ in } \overline{\Omega}}} \int_0^T \left| \left\langle \frac{\{u_\kappa(t)\}^-}{\kappa}, v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \right| \,\mathrm{d}t\\
&= \sup_{\substack{v \in W_0^{1,p}(\Omega)\\ \|v\|_{W_0^{1,p}(\Omega)}=1\\ v\geq 0 \text{ in } \overline{\Omega}}} \int_0^T \left\langle \frac{\{u_\kappa(t)\}^-}{\kappa}, v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \,\mathrm{d}t \leq \sup_{\substack{v \in W_0^{1,p}(\Omega)\\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \left| \int_0^T \left\langle \frac{\{u_\kappa(t)\}^-}{\kappa}, v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \,\mathrm{d}t \right|\\
&\leq \sup_{\substack{v \in W_0^{1,p}(\Omega)\\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \int_0^T \left| \left\langle \frac{\{u_\kappa(t)\}^-}{\kappa}, v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \right| \,\mathrm{d}t,
\end{aligned}$$

so that the last two inequalities are, actually, equalities.

Let us now show that

$$\int_0^T \left(\sup_{\substack{v \in W_0^{1,p}(\Omega)\\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \left| \left\langle \frac{\{u_\kappa(t)\}^-}{\kappa}, v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \right| \right) \mathrm{d}t \leq \sup_{\substack{v \in W_0^{1,p}(\Omega)\\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \int_0^T \left| \frac{\{u_\kappa(t)\}^-}{\kappa} \right| \mathrm{d}t. \tag{4.56}$$

Let $\{v_k\}_{k=1}^\infty \subset W_0^{1,p}(\Omega)$, $\|v_k\|_{W_0^{1,p}(\Omega)} = 1$ be such that the following convergence in $\mathbb{R}$ holds

$$\lim_{k\to\infty} \left| \left\langle \frac{\{u_\kappa\}^-}{\kappa}, v_k\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \right| = \sup_{\substack{v \in W_0^{1,p}(\Omega)\\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \left| \left\langle \frac{u_\kappa}{\kappa}, v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \right|, \tag{4.57}$$

for a.a. $t \in (0,T)$. Observe that, for each $k \geq 1$, we always have that

$$\int_0^T \left| \left\langle \frac{u_\kappa}{\kappa}, v_k\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \right| \mathrm{d}t \leq \sup_{\substack{v \in W_0^{1,p}(\Omega)\\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \int_0^T \left| \left\langle \frac{\{u_\kappa(t)\}^-}{\kappa}, v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \right| \mathrm{d}t. \tag{4.58}$$

Since the convergence in (4.57) holds for a.a. $t \in (0,T)$, we are in a position to apply Fatou's lemma (cf., e.g., [39]). A subsequent application of (4.58) gives:

$$\begin{aligned}
&\int_0^T \sup_{\substack{v \in W_0^{1,p}(\Omega)\\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \left| \left\langle \frac{\{u_\kappa(t)\}^-}{\kappa}, v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \right| \mathrm{d}t \leq \liminf_{k\to\infty} \int_0^T \left| \left\langle \frac{\{u_\kappa(t)\}^-}{\kappa}, v_k\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \right| \mathrm{d}t\\
&\leq \sup_{\substack{v \in W_0^{1,p}(\Omega)\\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \int_0^T \left| \left\langle \frac{\{u_\kappa(t)\}^-}{\kappa}, v\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \right| \mathrm{d}t,
\end{aligned}$$

and this proves (4.56). A a result of (4.56) and the properties of the Bochner integral (cf., e.g., Theorem 8.13 of [30]), we have that:

$$\int_0^T \left\| \frac{\{u_\kappa(t)\}^-}{\kappa} \right\|_{W^{-1,p'}(\Omega)} \mathrm{d}t = \int_0^T \sup_{\substack{v \in W_0^{1,p}(\Omega) \\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \left| \left\langle \frac{\{u_\kappa(t)\}^-}{\kappa}, v \right\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \right|$$

$$\leq \sup_{\substack{v \in W_0^{1,p}(\Omega) \\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \int_0^T \left| \left\langle \frac{\{u_\kappa(t)\}^-}{\kappa}, v \right\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \right| \mathrm{d}t$$

$$= \sup_{\substack{v \in W_0^{1,p}(\Omega) \\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \left| \left\langle \int_0^T \frac{\{u_\kappa(t)\}^-}{\kappa} \mathrm{d}t, v \right\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \right| = \left\| \int_0^T \frac{\{u_\kappa(t)\}^-}{\kappa} \mathrm{d}t \right\|_{W^{-1,p'}(\Omega)}.$$

In conclusion, we have shown that:

$$\left\| \frac{1}{\kappa} \int_0^T \{u_\kappa(t)\}^- \mathrm{d}t \right\|_{W^{-1,p'}(\Omega)} = \frac{1}{\kappa} \int_0^T \|\{u_\kappa(t)\}^-\|_{W^{-1,p'}(\Omega)} \mathrm{d}t. \tag{4.59}$$

Combining (4.55) and (4.59), we obtain that

$$\frac{1}{\kappa} \int_0^T \left\|\{u_\kappa(t)\}^-\right\|_{W^{-1,p'}(\Omega)} \mathrm{d}t \leq C, \qquad \text{for all } \kappa > 0,$$

for some $C > 0$ independent of κ or, equivalently, that

$$\left\{ \frac{\{u_\kappa\}^-}{\kappa} \right\}_{\kappa>0} \text{ is uniformly bounded in } L^1(0,T;W^{-1,p'}(\Omega)). \tag{4.60}$$

Therefore, we can derive the following estimate plugging (4.60) into (4.54):

$$\begin{aligned} &\sup_{\substack{v \in W_0^{1,p}(\Omega) \\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \left| \left\langle \frac{\mathrm{d}}{\mathrm{d}t} \left(|u_\kappa|^{\alpha-2} u_\kappa\right)(t), v \right\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \right| \\ &\leq \sup_{\substack{v \in W_0^{1,p}(\Omega) \\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \left| \left\langle \mu |\nabla u_\kappa(t)|^{p-2} \nabla u_\kappa(t), v \right\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \right| \\ &\quad + \frac{1}{\kappa} \|\{u_\kappa(t)\}^-\|_{W^{-1,p'}(\Omega)} + \|\tilde{a}(t)\|_{W^{-1,p'}(\Omega)}, \text{ for a.a. } t \in (0,T). \end{aligned} \tag{4.61}$$

Applying the boundedness of μ assumed in $(H2)$, the boundedness of $\{u_\kappa\}_{\kappa>0}$ established in (4.52), the boundedness of $\tilde{a}$ assumed in $(H4)$, the boundedness of the p-Laplacian, (4.60) and the monotonicity of the integral to (4.61) gives:

$$\int_0^T \sup_{\substack{v \in W_0^{1,p}(\Omega) \\ \|v\|_{W_0^{1,p}(\Omega)}=1}} \left| \left\langle \frac{\mathrm{d}}{\mathrm{d}t} \left(|u_\kappa|^{\alpha-2} u_\kappa\right)(t), v \right\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \right| \mathrm{d}t \leq C, \tag{4.62}$$

for some $C > 0$ independent of κ. This means that

$$\left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left(|u_\kappa|^{\alpha-2} u_\kappa\right) \right\}_{\kappa>0} \text{ is bounded in } L^1(0,T;W^{-1,p'}(\Omega)).$$

Therefore, there exists a vector-valued measure $\tilde{w}_t \in \mathcal{M}([0,T];W^{-1,p'}(\Omega))$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(|u_\kappa|^{\alpha-2} u_\kappa\right) \overset{*}{\rightharpoonup} \tilde{w}_t, \qquad \text{in } \mathcal{M}([0,T];W^{-1,p'}(\Omega)) \text{ as } \kappa \to 0^+. \tag{4.63}$$

The latter together with the fourth convergence in (4.52) allow us to apply Dubinskii's theorem (Theorem 3.9) with $A_0 = L^{(\alpha'-1)p}(\Omega)$, $A_1 = W^{-1,p'}(\Omega)$, $q_0 = (\alpha'-1)p > 1$ and $q_1 = 1$. The monotonicity of the mapping $\xi \mapsto |\xi|^{\alpha-2}\xi$ finally gives us:

$$|u_\kappa|^{\alpha-2}u_\kappa \to w = |u|^{\alpha-2}u, \quad \text{in } L^{(\alpha'-1)p}(0,T;L^{(\alpha'-1)p}(\Omega)), \tag{4.64}$$

and $w = |u|^{\alpha-2}u \in L^\infty(0,T;L^{\alpha'}(\Omega))$ by the fourth convergence in (4.52).

Thanks to Lemma 3.5 and the Dinculeanu-Zinger theorem (Theorem 3.11), we have that the following chain of embeddings holds:

$$L^1(0,T;W^{-1,p'}(\Omega)) \hookrightarrow \left(\mathcal{C}^0([0,T];W_0^{1,p}(\Omega))\right)^* \simeq \mathcal{M}([0,T];W^{-1,p'}(\Omega)).$$

Thanks to Lemma 3.5 and Lemma 3.8, we have that:

$$\begin{aligned}&\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(|u_\kappa(t)|^{\alpha-2}u_\kappa(t)\right), u_\kappa(t)\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \mathrm{d}t = \frac{\|u_\kappa(T)\|^\alpha_{L^\alpha(\Omega)}}{\alpha'} - \frac{\|u_0\|^\alpha_{L^\alpha(\Omega)}}{\alpha'}\\ &\to \frac{\|u(T)\|^\alpha_{L^\alpha(\Omega)}}{\alpha'} - \frac{\|u_0\|^\alpha_{L^\alpha(\Omega)}}{\alpha'}, \quad \text{as } \kappa \to 0^+.\end{aligned} \tag{4.65}$$

Observe that the first convergence of (4.52), namely $u_\kappa \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;W_0^{1,p}(\Omega))$, implies that

$$u_\kappa \rightharpoonup u, \quad \text{in } L^2(0,T;L^2(\Omega)) \simeq L^2((0,T)\times\Omega) \text{ as } \kappa \to 0^+.$$

The continuity of the negative part established in Lemma 3.2 and the third estimate in (4.14) give:

$$\|\{u\}^-\|_{L^2(0,T;L^2(\Omega))} \le \liminf_{\kappa\to 0} \|\{u_\kappa\}^-\|_{L^2(0,T;L^2(\Omega))} = 0,$$

which means that $u(t) \ge 0$ a.e. in Ω, for a.a. $t \in (0,T)$.

Specializing $v = u_\kappa - u$ in (4.1), we obtain:

$$\begin{aligned}&\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(|u_\kappa|^{\alpha-2}u_\kappa\right), u_\kappa - u\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \mathrm{d}t + \int_0^T\int_\Omega \mu(x)|\nabla u_\kappa(t)|^{p-2}\nabla u_\kappa(t)\cdot\nabla(u_\kappa(t)-u(t))\,\mathrm{d}x\,\mathrm{d}t\\ &-\frac{1}{\kappa}\int_0^T\int_\Omega \{u_\kappa(t)\}^-(u_\kappa(t)-u(t))\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \tilde{a}(t)(u_\kappa(t)-u(t))\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.66}$$

Since $u_\kappa \in \mathcal{C}^0([0,T];L^\alpha(\Omega))$, we define the extension $\tilde{u}_\kappa : [-T,2T] \to L^\alpha(\Omega)$ by:

$$\tilde{u}_\kappa(t) := \begin{cases} u_\kappa(-t), & \text{if } -T \le t < 0,\\ u_\kappa(t), & \text{if } 0 \le t < T,\\ u_\kappa(2T-t), & \text{if } T \le t \le 2T.\end{cases}$$

Since $u \in \mathcal{C}^0([0,T];L^\alpha(\Omega))$, we define the extension $\tilde{u} : [-T,2T] \to L^\alpha(\Omega)$ by:

$$\tilde{u}(t) := \begin{cases} u(-t), & \text{if } -T \le t < 0,\\ u(t), & \text{if } 0 \le t < T,\\ u(2T-t), & \text{if } T \le t \le 2T.\end{cases}$$

For each $-T < \tau < T$ and all $\kappa > 0$, define

$$Y_\tau^\kappa := \frac{1}{\tau}\int_0^T \langle |\tilde{u}_\kappa(t+\tau)|^{\alpha-2}\tilde{u}_\kappa(t+\tau) - |\tilde{u}_\kappa(t)|^{\alpha-2}\tilde{u}_\kappa(t), \tilde{u}(t)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,\mathrm{d}t.$$

By Lemma 3.7 with $\xi = \tilde{u}_\kappa(t)$, $\eta = \tilde{u}_\kappa(t+\tau)$ and $r = \alpha$ and (4.53), we have that:

$$\begin{aligned}Y_\tau^\kappa &\to \frac{1}{\tau}\int_0^T \langle |\tilde{u}(t+\tau)|^{\alpha-2}\tilde{u}(t+\tau) - |\tilde{u}(t)|^{\alpha-2}\tilde{u}(t), \tilde{u}(t)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\,\mathrm{d}t\\ &\le -\frac{1}{2\tau\alpha'}\int_{T-\tau}^{T+\tau} \|\tilde{u}(t)\|^\alpha_{L^\alpha(\Omega)}\,\mathrm{d}t + \frac{1}{2\tau\alpha'}\int_{-\tau}^{\tau} \|\tilde{u}(t)\|^\alpha_{L^\alpha(\Omega)}\,\mathrm{d}t,\end{aligned}$$

as $\kappa \to 0^+$. Letting $\tau \to 0$ in the latter term, we obtain that:

$$-\frac{1}{2\tau\alpha'}\int_{T-\tau}^{T+\tau} \|\tilde{u}(t)\|_{L^\alpha(\Omega)}^\alpha \,\mathrm{d}t + \frac{1}{2\tau\alpha'}\int_{-\tau}^{\tau} \|\tilde{u}(t)\|_{L^\alpha(\Omega)}^\alpha \,\mathrm{d}t \to -\frac{\|u(T)\|_{L^\alpha(\Omega)}^\alpha}{\alpha'} + \frac{\|u_0\|_{L^\alpha(\Omega)}^\alpha}{\alpha'}.$$

In the special case where $|u|^{\alpha-2}u$ is differentiable a.e. in $(0,T)$(this constitutes another *stability assumption*) then it results:

(4.67)
$$\lim_{\kappa\to 0^+}\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(|u_\kappa|^{\alpha-2}u_\kappa\right), u\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \mathrm{d}t = \lim_{\tau\to 0}\lim_{\kappa\to 0^+} Y_\tau^\kappa \le -\frac{\|u(T)\|_{L^\alpha(\Omega)}^\alpha}{\alpha'} + \frac{\|u_0\|_{L^\alpha(\Omega)}^\alpha}{\alpha'}.$$

Combining (4.65) and (4.67) and the convergence (4.53), we obtain that (4.66) gives:

$$\limsup_{\kappa\to 0^+}\int_0^T\int_\Omega \mu(x)|\nabla u_\kappa(t)|^{p-2}\nabla u_\kappa(t)\cdot\nabla(u_\kappa(t)-u(t))\,\mathrm{d}x\,\mathrm{d}t \le 0. \tag{4.68}$$

Since the p-Laplacian in divergence form is pseudo-monotone (cf., e.g., Proposition 2.5 of [31]), as it is hemi-continuous, strictly monotone and bounded, an application of (4.68) gives:

$$\begin{aligned}
&\int_0^T\int_\Omega \mu(x)|\nabla u(t)|^{p-2}\nabla u(t)\cdot\nabla(u(t)-v(t))\,\mathrm{d}x\,\mathrm{d}t\\
&\le \liminf_{\kappa\to 0^+}\int_0^T\int_\Omega \mu(x)|\nabla u_\kappa(t)|^{p-2}\nabla u_\kappa(t)\cdot\nabla(u_\kappa(t)-v(t))\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{4.69}$$

for all $v \in L^2(0,T;W_0^{1,p}(\Omega))$. Observe that the latter space is suitable for treating pseudo-monotonicity, as it is reflexive.

Let us now consider arbitrary functions $v \in \mathcal{C}^0([0,T];W_0^{1,p}(\Omega))$ such that $v(t) \ge 0$ in $\overline{\Omega}$ for all $t \in [0,T]$. Evaluate the variational equations (4.1) at $w_\kappa := v - u_\kappa$, obtaining:

$$\begin{aligned}
&\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(|u_\kappa(t)|^{\alpha-2}u_\kappa(t)\right), v(t)-u_\kappa(t)\right\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)} \mathrm{d}t\\
&\quad + \int_0^T\int_\Omega \mu(x)|\nabla u_\kappa(t)|^{p-2}\nabla u_\kappa(t)\cdot\nabla(v(t)-u_\kappa(t))\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \frac{1}{\kappa}\int_0^T\int_\Omega \{u_\kappa(t)\}^-(v(t)-u_\kappa(t))\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^T\int_\Omega \tilde{a}(t)(v(t)-u_\kappa(t))\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Let $\kappa \to 0^+$ and exploit the convergence process (4.52), (4.69) and the fact that the integral associated with the negative part is ≤ 0, being $u(t) \ge 0$ a.e. in $\overline{\Omega}$ for a.a. $t \in (0,T)$, gives:

$$\begin{aligned}
&\langle\langle \tilde{w}_t, v\rangle\rangle_{\mathcal{M}([0,T];W^{-1,p'}(\Omega)),\mathcal{C}^0([0,T];W_0^{1,p}(\Omega))} + \int_0^T\int_\Omega \mu(x)|\nabla u(t)|^{p-2}\nabla u(t)\cdot\nabla(v(t)-u(t))\,\mathrm{d}x\,\mathrm{d}t\\
&\ge \int_0^T\int_\Omega \tilde{a}(t)(v-u)\,\mathrm{d}x\,\mathrm{d}t + \frac{\|u(T)\|_{L^\alpha(\Omega)}^\alpha}{\alpha'} - \frac{\|u_0\|_{L^\alpha(\Omega)}^\alpha}{\alpha'}.
\end{aligned}$$

We are thus in a position to write down the model governing the evolution of the thickness of a shallow ice sheet, and to assert that this model admits at least one solution.

Theorem 4.5. *Let Ω be a domain in $\mathbb{R}^2$. Assume that (H1)–(H4) hold. Let $\{u_\kappa\}_{\kappa>0}$ be a family of solutions of Problem $\mathcal{P}_\kappa$. Then, the following convergences hold:*

$$
\begin{aligned}
u_\kappa &\overset{*}{\rightharpoonup} u, \text{ in } L^\infty(0,T;W_0^{1,p}(\Omega)),\\
|u_\kappa|^{\frac{\alpha-2}{2}}u_\kappa &\overset{*}{\rightharpoonup} |u|^{\frac{\alpha-2}{2}}u, \text{ in } L^\infty(0,T;L^2(\Omega)),\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(|u_\kappa|^{\frac{\alpha-2}{2}}u_\kappa\right) &\rightharpoonup \frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^{\frac{\alpha-2}{2}}u\right), \text{ in } L^\infty(0,T;L^2(\Omega)),\\
|u_\kappa|^{\alpha-2}u_\kappa &\overset{*}{\rightharpoonup} |u|^{\alpha-2}u, \text{ in } L^\infty(0,T;L^{\alpha'}(\Omega)),\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(|u_\kappa|^{\alpha-2}u_\kappa\right) &\overset{*}{\rightharpoonup} \frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^{\alpha-2}u\right), \text{ in } \mathcal{M}([0,T];W^{-1,p'}(\Omega)).
\end{aligned}
$$

Moreover, if the following stability condition holds

$$
|u|^{\alpha-2}u \text{ is differentiable a.e. in } (0,T), \tag{4.70}
$$

then u is a solution of the following variational problem:

Problem $\mathcal{P}$. Find $u \in \mathcal{K} := \{v \in L^\infty(0,T;W_0^{1,p}(\Omega)); v(t) \in K \text{ a.e. in } (0,T)\}$ such that

$$
\begin{aligned}
&u \in L^\infty(0,T;W_0^{1,p}(\Omega)),\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^{\frac{\alpha-2}{2}}u\right) \in L^2(0,T;L^2(\Omega)),\\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^{\alpha-2}u\right) \in \mathcal{M}([0,T];W^{-1,p'}(\Omega)),
\end{aligned}
$$

satisfying the following variational inequalities:

$$
\begin{aligned}
&\left\langle\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(|u|^{\alpha-2}u), v\right\rangle\right\rangle_{\mathcal{M}([0,T];W^{-1,p'}(\Omega)),\mathcal{C}^0([0,T];W_0^{1,p}(\Omega))} + \int_0^T \int_\Omega \mu(x)|\nabla u|^{p-2}\nabla \cdot \nabla(v-u)\,\mathrm{d}x\,\mathrm{d}t\\
&\geq \int_0^T \int_\Omega \tilde{a}(t)(v-u)\,\mathrm{d}x\,\mathrm{d}t + \frac{\|u(T)\|^\alpha_{L^\alpha(\Omega)}}{\alpha'} - \frac{\|u_0\|^\alpha_{L^\alpha(\Omega)}}{\alpha'},
\end{aligned}
$$

for all $v \in \mathcal{C}^0([0,T];W_0^{1,p}(\Omega))$ such that $v(t) \geq 0$ in $\overline{\Omega}$ for all $t \in [0,T]$, as well as the following initial condition

$$
u(0) = u_0,
$$

for some prescribed $u_0 \in K$. ■

□

We conclude the paper with a remark where we propose an alternative stability condition, which is more physically realistic than the abstract stability condition (4.70).

Remark 4.6. Observe that the following condition is sufficient to (4.70): There exists a constant $C>0$ independent of κ such that

$$
\frac{\|\{u_\kappa\}^-\|_{W^{-1,p'}}}{\kappa} \leq C. \tag{4.71}
$$

This stability condition, which is more physically realistic than (4.70), strengthens the convergence of the derivative of the nonlinear term and saves us the effort of resorting to vector-valued measures. ■

Conclusions

In this paper we formulated a time-dependent model governing the evolution of the thickness of a shallow ice sheet undergoing regimes of melting and ablation. The shallow ice sheet varying height, that is the unknown the model is described in terms of, is subjected to obey the constraint of being nonnegative. For this reason, the problem under consideration can be regarded as an obstacle problem.

First, we recovered the formal model, based on Glen's power law.

Second, we incorporated the constraint, in the form of a monotone term, in the model. By so doing it was possible to write down variational formulation of the model under consideration in terms of a set of variational equations posed over a vector space. The existence of solutions for the penalized model was established thanks to the Dubinskii's compactness theorem and a series of preliminary results, that we stated in the form of Lemmas.

Third, we let the penalty parameter approach zero and we recovered the actual model, which takes the form of a set of variational inequalities tested over a nonempty, closed, and convex subset of the space $\mathcal{C}^0([0,T];W_0^{1,p}(\Omega))$.

It is worth noticing that the proofs hinge on two *stability conditions*, i.e., (4.6) and (4.70) that, in the same spirit as [37], we were able to replace by the corresponding more realistic ones (4.21) and (4.71).

At a first glance, the time-dependent model we studied appears to be very challenging and it is not clear whether it is possible to establish the existence of solutions to it without resorting to the stability conditions (4.6) and (4.70) (or the more physically realistic stability conditions (4.21) and (4.71)).

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