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Affine Volterra processes with jumps^{*}

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Abstract

The theory of affine processes has been recently extended to continuous stochastic Volterra equations. These so-called affine Volterra processes overcome modeling shortcomings of affine processes by incorporating path-dependent features and trajectories with regularity different from the paths of Brownian motion. More specifically, singular kernels yield rough affine processes. This paper extends the theory by considering affine stochastic Volterra equations with jumps. This extension is not straightforward because the jump structure and possible singularities of the kernel may induce explosions of the trajectories. This study also provides exponential affine formulas for the conditional Fourier–Laplace transform of marked Hawkes processes.

Keywords: affine processes, affine Volterra processes, stochastic Volterra equations, Hawkes processes, Riccati–Volterra equations, rough volatility.

MSC2020 classifications: 60H20 (primary), 60G22, 45D05, 60G17 (secondary).

1. Introduction

Affine stochastic processes constitute unquestionably the most popular multi-factor framework to model rich and flexible stochastic dependence structures. Semi-explicit formulas for the Fourier–Laplace transform of affine processes make them numerically tractable since Fourier transform-based methods can be used to perform fast calculations.

We recall that a conservative regular affine process $X = (X_t)_{t \geq 0}$ with state space $E \subset \mathbb{R}^m$ is a stochastically continuous conservative Markov process having sufficiently regular Fourier–Laplace transforms given by exponential affine formulas in the initial state X_0 . Such a process X can also be seen as a special semimartingale whose semimartingale characteristics (B, C, ν) , with respect to the “truncation function” $h(\xi) = \xi$, are of the form

$$B_t = \int_0^t b(X_s) ds, \quad C_t = \int_0^t a(X_s) ds, \quad \nu(dt, d\xi) = \eta(X_t, d\xi) dt, \quad (1)$$

where

$$b(x) = b_0 + \sum_{k=1}^m x_k b_k, \quad a(x) = A_0 + \sum_{k=1}^m x_k A_k, \quad \eta(x, d\xi) = \nu_0(d\xi) + \sum_{k=1}^m x_k \nu_k(d\xi), \quad x \in E. \quad (2)$$

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In (2) we take $A_k \in \mathbb{R}^{m \times m}$, $b_k \in \mathbb{R}^m$, and $\nu_k(d\xi)$ signed measures on \mathbb{R}^m such that $\nu_k(\{0\}) = 0$ and $\int_{\mathbb{R}^m} |\xi|^2 |\nu_k|(d\xi) < \infty$, for every $k = 0, \dots, m$. Additional conditions on the parameters A_k , b_k and ν_k have to imposed in order to guarantee existence and invariance results depending on the state space E . See for instance [11] for $E = \mathbb{R}_+^k \times \mathbb{R}^l$ and [8] for E equal to the space of positive semidefinite matrices.

The conditional Fourier–Laplace transform of the affine process X is given by

$$\mathbb{E} \left[\exp \left(\int_0^T f(T-s)^\top X_s ds \right) \middle| \mathcal{F}_t \right] = \exp \left(\phi(T-t) + \int_0^t f(T-s)^\top X_s ds + \psi(T-t)^\top X_t \right), \quad (3)$$

with ψ a \mathbb{C}^m -valued function that solves the Riccati equation

$$\psi(t) = \int_0^t \mathcal{R}(s, \psi(s)) ds, \quad (4)$$

where

$$\mathcal{R}_k(s, z) = f_k(s) + \frac{1}{2} z^\top A_k z + z^\top b_k + \int_{\mathbb{R}^m} \left(e^{z^\top \xi} - 1 - z^\top \xi \right) \nu_k(d\xi), \quad k = 1, \dots, m, \quad (5)$$

and ϕ is the \mathbb{C} -valued function

$$\phi(t) = \int_0^t \left(\psi(s)^\top b_0 + \frac{1}{2} \psi(s)^\top A_0 \psi(s) + \int_{\mathbb{R}^m} \left(e^{\psi(s)^\top \xi} - 1 - \psi(s)^\top \xi \right) \nu_0(d\xi) \right) ds. \quad (6)$$

The identity (3) is only valid under additional hypotheses on the \mathbb{C}^m -valued function f and $t, T \geq 0$ that imply appropriate conditions on the functions ϕ and ψ .¹

The theory of affine processes was recently extended in [5, 14] to the framework of stochastic Volterra equations with continuous trajectories, where in general the semimartingale and Markov properties do not hold. These so-called affine Volterra processes overcome modeling shortcomings of affine processes because they may possess path-dependent features which introduce memory structures in the models. Furthermore, they can have trajectories whose Hölder’s regularity is different from the Hölder’s regularity of the paths of Brownian motion. More specifically, singular kernels yield rough processes in the spirit of [6, 12, 13]. The goal of this paper is to extend the results in [5, 14] by considering general affine stochastic Volterra equations with jumps. This extension is not straightforward because the jump structure together with possible singularities of the kernel may induce explosions of the trajectories.

Our study can be motivated by financial models for stock volatility, in particular by the observation in [24] that a complete description of volatility should take into account both path roughness and jumps. We also refer to [25] for an interesting discussion on the topic. In this paper, however, we concentrate on the mathematical properties of this family of processes and we address their possible applications in a separate article [7].

We summarize in this introduction the framework and the main results of our study. Suppose that X is a predictable solution of a stochastic Volterra equation of the form

$$X_t = g_0(t) + \int_0^t K(t-s) dZ_s, \quad \mathbb{P} \otimes dt\text{-a.e.} \quad (7)$$

It is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions, and has trajectories in $L_{\text{loc}}^1(\mathbb{R}_+; E)$, for some Borel-measurable state space $E \subset \mathbb{R}^m$. In (7) we take $g_0 \in L_{\text{loc}}^1(\mathbb{R}_+; \mathbb{R}^m)$, $K \in L_{\text{loc}}^2(\mathbb{R}_+; \mathbb{R}^{m \times d})$ a matrix-valued kernel, and Z a d -dimensional semimartingale whose characteristics depend on X . In order to have an affine structure we suppose that Z has characteristics of the form (1)-(2), with $A_k \in \mathbb{R}^{d \times d}$, $b_k \in \mathbb{R}^d$, and ν_k signed measures on \mathbb{R}^d such that $\nu_k(\{0\}) = 0$ and $\int_{\mathbb{R}^d} |\xi|^2 |\nu_k|(d\xi) < \infty$. In this case we call X an affine Volterra process. When $E = \mathbb{R}^m$,

¹One of these conditions could be for instance the boundedness of the right term in (3). On a related subject, we also refer to [18] where the authors analyze the possible explosions of the associated Riccati equation (4).

existence of weak solutions to (7) with trajectories in $L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^m)$ has been established in [2, Theorem 1.2]. For $E = \mathbb{R}_+$ and for a Volterra CIR-type of process with positive jumps results in this direction can be found in [1, Theorem 2.13].

Section 2 contains more details about this setting as well as important results regarding stochastic convolutions with respect to processes with jumps. These results, which play a crucial role in our arguments, extend and are inspired by those in [5] where the authors study stochastic convolutions with respect to processes with continuous trajectories.

Fix $T > 0$ and $f \in C(\mathbb{R}_+; \mathbb{C}^m)$. By analogy with (4), assume that $\psi \in C(\mathbb{R}_+; \mathbb{C}^d)$ solves the Riccati–Volterra equation

$$\psi(t)^\top = \int_0^t \mathcal{R}(s, \psi(s))^\top K(t-s) ds, \quad (8)$$

with \mathcal{R} as in (5), and let ϕ be given by (6).

In Section 3 we show our first main result, namely Theorem 5, which is a generalization of [5, Theorem 4.3] and provides a semi-explicit formula for the Fourier–Laplace transform of X . This theorem shows that, in the above-mentioned framework, if we define

$$M_t = \exp\left(\phi(T-t) + \int_0^t f(T-s)^\top X_s ds + \int_t^T \mathcal{R}(T-s, \psi(T-s))^\top g_t(s) ds\right), \quad (9)$$

where $(g_t(\cdot))_{t \geq 0}$ denotes the adjusted forward process²

$$g_t(s) = g_0(s) + \int_0^t K(s-r) dZ_r, \quad s > t, \quad (10)$$

then M is a local martingale, and if M is a martingale then one has the exponential affine formula

$$\mathbb{E}\left[\exp\left(\int_0^T f(T-s)^\top X_s ds\right) \middle| \mathcal{F}_t\right] = M_t. \quad (11)$$

As a consequence, under these conditions, uniqueness in law holds for the stochastic Volterra equation (7).

In Subsection 3.1, we discuss how the arguments in Section 3 can be adapted to obtain exponential affine formulas for the conditional Fourier–Laplace transforms of the marginal distributions of the solution process X and the semimartingale Z in (7). Notably, Theorem 6, which investigates the marginals of Z , can be applied to obtain original expressions for the conditional transforms of a multi-dimensional Hawkes process, see Corollary 7 and Remark 6.

Section 4 contains our second main result, which is Theorem 10. This theorem establishes, under the assumption $m = d$ and additional conditions on the kernel K , an alternative formula for the local martingale M_t in (9) in terms of $(X_s)_{s \leq t}$ and Z_t only, namely

$$\begin{aligned} \log(M_t) = \phi(T-t) + \int_0^t f(T-s)^\top X_s ds + \int_0^{T-t} \mathcal{R}(s, \psi(s))^\top g_0(T-s) ds \\ + \psi(T-t)^\top Z_t + (\pi_{T-t}^\top * (X - g_0))(t), \end{aligned} \quad (12)$$

with ϕ as in (6) and $\pi_h \in L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{C}^d)$, $h > 0$, a deterministic function that depends on K and ψ . This expression is a corollary of a similar formula for the adjusted forward process (10), shown in Lemma 9. The identity (12) can be used to show that (3) is a particular instance of (11) when g_0 is constant, and K is constant and equal to the identity matrix.

In Section 5, see Theorems 13 and 14, using our first two main results, we give a complete proof of the exponential affine formula (11) in the particular case $m = d = 1$, $E = \mathbb{R}_+$ and for a Volterra CIR-type

²This adjusted forward process was also used in [3] and [19] to elucidate the affine structure of affine Volterra processes with continuous trajectories.

process with positive jumps. The argument hinges on a novel comparison result between solutions of Riccati–Volterra equations, namely between a solution of (8) and a solution of an analogous equation in which the functions ψ and \mathcal{R} are substituted with the corresponding real parts. This comparison result, together with the *affine with respect to the past formula* (12) of Theorem 10, yields the desired conclusion because we can bound the complex-valued local martingale M (9) of Theorem 5 with a real-valued martingale.

It is important at this point to mention [9], where the authors construct infinite dimensional Markovian lifts of affine Volterra processes, possibly with jumps, and study affine transform formulas for these lifts. Such formulas are closely related to those of the present study within a one-dimensional setting, although in [9] the focus is on the marginal distributions of X (see Remark 7 in Subsection 3.1 for a precise comparison). The novelty of our work stems from the approach that we propose, which dispenses with the abstract infinite dimensional theory in [9]. In addition, we carry out a complete analysis of the associated Riccati–Volterra equations as in Section 5. Furthermore, we obtain the *affine with respect to the past formula* (12), as well as new formulas for the semimartingale Z which can be applied to derive novel expressions for the conditional Fourier–Laplace transforms of multi-dimensional marked Hawkes processes, see Theorem 6 and Corollary 7. These last two points also distinguish our paper from [1].

Appendix A contains some basic results regarding the classical forward process and Appendix B results regarding the 1-dimensional Riccati–Volterra equations appearing in Section 5.

Notation: Throughout the paper, elements of \mathbb{R}^k and \mathbb{C}^k are column vectors. Given a matrix $A \in \mathbb{C}^{k \times l}$, the element in row i and column j is A^{ij} , $A^\top \in \mathbb{C}^{l \times k}$ is its transpose matrix, and $|A|$ is the Frobenius norm. We also use the notation $\mathbb{R}_+^k = \{x \in \mathbb{R}^k : x_i \geq 0, i = 1, \dots, k\}$ and $\mathbb{C}_-^k = \{x \in \mathbb{C}^k : \operatorname{Re}(x_i) \leq 0, i = 1, \dots, k\}$, where for $z \in \mathbb{C}$, $\operatorname{Re}(z)$ denotes its real part. The imaginary part of a complex number z is $\operatorname{Im} z$. We use the convolution notation $(f * g)(t) = \int_0^t f(t-s)g(s)ds$ for functions f, g .

2. Preliminaries

Fix $d, m \in \mathbb{N}$. Let $g_0 \in L_{\text{loc}}^1(\mathbb{R}_+; \mathbb{R}^m)$, $K \in L_{\text{loc}}^2(\mathbb{R}_+; \mathbb{R}^{m \times d})$ be a matrix-valued kernel and $E \subset \mathbb{R}^m$ be a Borel-measurable subset which will be the state-space that we consider. We also introduce a *characteristic triplet* (b, a, η) consisting of the measurable maps $b: \mathbb{R}^m \rightarrow \mathbb{R}^d$, $a: \mathbb{R}^m \rightarrow \mathbb{R}^{d \times d}$ and the (nonnegative) transition kernel $\eta(x, d\xi)$ from \mathbb{R}^m to \mathbb{R}^d . We require this triplet to be affine on E , meaning that, for every $x \in E$,

$$b(x) = b_0 + \sum_{k=1}^m x_k b_k, \quad a(x) = A_0 + \sum_{k=1}^m x_k A_k, \quad \eta(x, d\xi) = \nu_0(d\xi) + \sum_{k=1}^m x_k \nu_k(d\xi). \quad (13)$$

Here $b_0, b_1, \dots, b_m \in \mathbb{R}^d$, $A_0, A_1, \dots, A_m \in \mathbb{R}^{d \times d}$, and $(\nu_k)_{k=0, \dots, m}$ are signed measures on \mathbb{R}^d such that $\int_{\mathbb{R}^d} |\xi|^2 |\nu_k| (d\xi) < \infty$, with $\nu_k(\{0\}) = 0$. Throughout the paper, we denote by $X = (X_t)_{t \geq 0}$ a predictable process with trajectories in $L_{\text{loc}}^1(\mathbb{R}_+; \mathbb{R}^m)$ and such that $X \in E$, $\mathbb{P} \otimes dt$ -a.e. It is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ where the filtration \mathbb{F} satisfies the usual conditions. Moreover, we assume that X solves the following affine stochastic Volterra equation of convolution type

$$X_t = g_0(t) + \int_0^t K(t-s) dZ_s, \quad \mathbb{P} - \text{a.s.}, \text{ for a.e. } t \in \mathbb{R}_+. \quad (14)$$

Here Z is a d -dimensional semimartingale starting at 0 whose differential characteristics with respect to the Lebesgue measure are $(b(X_t), a(X_t), \eta(X_t, d\xi))$, $t \geq 0$. These characteristics are taken with respect to the “truncation function” $h(\xi) = \xi$, $\xi \in \mathbb{R}^d$, which can be chosen because Z is a special semimartingale due to [16, Proposition 2.29, Chapter II] and the local integrability of the trajectories of X . In the sequel, we denote by $\mu(dt, d\xi)$ the measure associated with the jumps of Z and by $\nu(dt, d\xi) = \eta(X_t, d\xi) dt$ its compensator. We remark that $\nu(dt, d\xi)$ is nonnegative, even though ν_k , $k = 0, \dots, m$, is a signed measure and we do not impose any requirements on the sign of the components of $x \in E$. Indeed, from the definition of (nonnegative) transition kernel, $\eta(x, d\xi)$ is a nonnegative measure on \mathbb{R}^d for every $x \in \mathbb{R}^m$. This is

coherent with the fact that $\nu(dt, d\xi)$ should be nonnegative, as it is the compensator of the nonnegative jump–measure $\mu(dt, d\xi)$.

It is worth discussing the good definition of the stochastic integral in (14). Recalling that $X \in E$, $\mathbb{P} \otimes dt$ –a.e., the canonical representation theorem for semimartingales (see [16, Proposition 2.34, Chapter II]) shows that Z admits the decomposition

$$Z_t = \int_0^t b(X_s) ds + M_t^c + M_t^d = b_0 t + \sum_{k=1}^d b_k \int_0^t X_{k,s} ds + M_t^c + M_t^d, \quad t \geq 0, \mathbb{P}\text{--a.s.},$$

where $dM_t^d = \int_{\mathbb{R}^d} \xi (\mu - \nu) (dt, d\xi)$ is an \mathbb{R}^d –valued, purely discontinuous local martingale and M^c is a d –dimensional, continuous local martingale satisfying $d\langle M^c, M^c \rangle_t = a(X_t) dt$. Now if we introduce, for every $j = 1, \dots, d$, the increasing process $C_t^j = \int_0^t \int_{\mathbb{R}^d} |\xi_j|^2 \nu(ds, d\xi)$, $t \geq 0$, then we have

$$C_t^j = \left(\int_{\mathbb{R}^d} |\xi_j|^2 \nu_0(d\xi) \right) t + \sum_{k=1}^m \int_{\mathbb{R}^d} |\xi_j|^2 \nu_k(d\xi) \left(\int_0^t X_{k,s} ds \right), \quad t \geq 0, \mathbb{P}\text{--a.s.}$$

As a consequence of this expression, the local integrability of the paths of X implies that C^j is locally integrable. Hence [16, Theorem 1.33 (a), Chapter II] yields that M^d is a locally square–integrable martingale with

$$d\langle M^{d,j}, M^{d,j} \rangle_t = \left[\int_{\mathbb{R}^d} |\xi_j|^2 \nu_0(d\xi) + \sum_{k=1}^m \left(\int_{\mathbb{R}^d} |\xi_j|^2 \nu_k(d\xi) \right) X_{k,t} \right] dt, \quad (15)$$

where $M^{d,j}$ is the j –th component of M^d , $j = 1, \dots, d$. It is convenient to introduce the locally square–integrable martingale $\tilde{Z} = M^c + M^d$, which satisfies

$$\tilde{Z}_t = Z_t - \int_0^t b(X_s) ds = Z_t - b_0 t - \sum_{k=1}^d b_k \int_0^t X_{k,s} ds, \quad t \geq 0, \mathbb{P}\text{--a.s.} \quad (16)$$

Given an integer $l \in \mathbb{N}$ and $F \in L_{\text{loc}}^2(\mathbb{R}_+; \mathbb{R}^{l \times d})$, we define the l –dimensional random variable

$$(F * d\tilde{Z})_T = (F * dM^c)_T + (F * dM^d)_T = \int_0^T F(T-s) dM_s^c + \int_0^T F(T-s) dM_s^d.$$

This is well–defined for a.e. $T \in \mathbb{R}_+$. Indeed, consider the stopping times $\tau_n = \inf \{ t \geq 0 : \int_0^t |X_s| ds > n \}$ for all $n \in \mathbb{N}$. Since $X \cdot (\omega) \in L_{\text{loc}}^1(\mathbb{R}_+, \mathbb{R}^m)$, $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$ in Ω . Then for every $\bar{T} > 0$, we can apply the Young’s type inequality in [2, Lemma A.1] with $p = q = r = 1$ and Tonelli’s theorem to deduce that

$$\begin{aligned} \int_0^{\bar{T}} \left(\mathbb{E} \left[\int_0^{T \wedge \tau_n} |F(T-s)|^2 |X_{k,s}| ds \right] \right) dT &= \int_0^{\bar{T}} \left(\int_0^T |F(T-s)|^2 \mathbb{E} [1_{\{s \leq \tau_n\}} |X_{k,s}|] ds \right) dT \\ &\leq \|F\|_{L^2([0, \bar{T}]; \mathbb{R}^{l \times d})}^2 \mathbb{E} \left[\int_0^{\bar{T} \wedge \tau_n} |X_{k,s}| ds \right] \leq n \|F\|_{L^2([0, \bar{T}]; \mathbb{R}^{l \times d})}^2 < \infty, \quad k = 1, \dots, m. \end{aligned}$$

This ensures that $\mathbb{E} \left[\int_0^{T \wedge \tau_n} |F(T-s)|^2 |X_{k,s}| ds \right] < \infty$, $k = 1, \dots, m$, $n \in \mathbb{N}$, for a.e. $T \in \mathbb{R}_+$, say for every $T \in \mathbb{R}_+ \setminus N$, where $N \subset \mathbb{R}_+$ is a dt –null set. As a consequence, it is straightforward to conclude that the processes

$$\left(\int_0^t F(T-s) dM_s^c \right)_{t \in [0, T]}, \quad \left(\int_0^t F(T-s) dM_s^d \right)_{t \in [0, T]}, \quad (17)$$

are locally square–integrable martingales for every $T \in \mathbb{R}_+ \setminus N$. Indeed, denoting by $M^{c,j}$ the j –th component of M^c , $j = 1, \dots, d$, for every $n \in \mathbb{N}$ we can write

$$\sum_{j=1}^d \mathbb{E} \left[\int_0^{T \wedge \tau_n} |F(T-s)|^2 d\langle M^{c,j}, M^{c,j} \rangle_s \right] = \sum_{j=1}^d \mathbb{E} \left[\int_0^{T \wedge \tau_n} |F(T-s)|^2 \left(A_0^{jj} + \sum_{k=1}^m X_{k,s} A_k^{jj} \right) ds \right] < \infty,$$

and (by (15))

$$\begin{aligned} & \sum_{j=1}^d \mathbb{E} \left[\int_0^{T \wedge \tau_n} |F(T-s)|^2 d \langle M^{d,j}, M^{d,j} \rangle_s \right] \\ &= \sum_{j=1}^d \mathbb{E} \left[\int_0^{T \wedge \tau_n} |F(T-s)|^2 \left(\int_{\mathbb{R}^d} |\xi_j|^2 \nu_0(d\xi) + \sum_{k=1}^m X_{k,s} \int_{\mathbb{R}^d} |\xi_j|^2 \nu_k(d\xi) \right) ds \right] < \infty. \end{aligned}$$

We always work with a jointly measurable version of the stochastic convolution $F * d\tilde{Z}$ defined on $\Omega \times \mathbb{R}_+$ (such a modification exists, see, e.g., [23, Theorem 3.5]).

As for the convolution of F with the drift part of Z , using [15, Theorem 2.2 (i), Chapter 2] we compute

$$\begin{aligned} & \mathbb{E} \left[\int_0^T \left(\int_0^t 1_{\{t \leq \tau_n\}} 1_{\{s \leq t\}} |F(t-s)| \left(|b_0| + \sum_{k=1}^m |b_k| |X_{k,s}| \right) ds \right) dt \right] \\ &= \mathbb{E} \left[\int_0^{T \wedge \tau_n} \left(\int_0^t |F(t-s)| \left(|b_0| + \sum_{k=1}^m |b_k| |X_{k,s}| \right) ds \right) dt \right] \\ &\leq \|F\|_{L^1([0,T]; \mathbb{R}^l \times \mathbb{R}^d)} \left[|b_0|T + n \left(\sum_{k=1}^m |b_k| \right) \right] < \infty, \quad T > 0. \end{aligned}$$

This shows that there exists a $\mathbb{P} \otimes dt$ -null set $N_1 \subset \Omega \times \mathbb{R}_+$ such that the next expression is well-defined

$$1_{\{t \leq \tau_n(\omega)\}} \int_0^t F(t-s) \left(b_0 + \sum_{k=1}^m b_k X_{k,s}(\omega) \right) ds, \quad n \in \mathbb{N}, (\omega, t) \in (\Omega \times \mathbb{R}_+) \setminus N_1.$$

Moreover, by Fubini's theorem the resulting processes are jointly measurable in $(\Omega \times \mathbb{R}_+) \setminus N_1$, hence passing to the limit as $n \rightarrow \infty$, we obtain the jointly measurable process $\int_0^t F(t-s) (b_0 + \sum_{k=1}^m b_k X_{k,s}) ds$ (defined on the same set). Finally we introduce

$$(F * b(X))(\omega, t) = \begin{cases} \int_0^t F(t-s) (b_0 + \sum_{k=1}^m b_k X_{k,s}(\omega)) ds, & (\omega, t) \in (\Omega \times \mathbb{R}_+) \setminus N_1, \\ 0, & (\omega, t) \in N_1. \end{cases}$$

This is a jointly measurable process defined on the whole $\Omega \times \mathbb{R}_+$. This machinery for constructing jointly measurable modifications of given processes will be used several times in the sequel.

Overall, the previous argument proves that the integral on the right side of (14) is well-defined \mathbb{P} -a.s., for a.e. $t \in \mathbb{R}_+$. We denote by $(F * dZ) = (F * b(X)) + (F * d\tilde{Z})$; with this notation, Equation (14) can be written as follows

$$X = g_0 + (K * dZ) = g_0 + (K * b(X)) + (K * d\tilde{Z}), \quad \mathbb{P} \otimes dt - \text{a.e.} \quad (18)$$

The following lemma will be useful in the sequel.

Lemma 1. *For every $T > 0$,*

$$\mathbb{E} \left[\|X\|_{L^1([0,T]; \mathbb{R}^m)} \right] < \infty. \quad (19)$$

Proof. The proof follows the same steps as those in [2, Theorem 1.4]. The difference is that the affine structure of our model guaranteed by (13) is substituted for [2, Condition (1.5)], and makes the L^1_{loc} -integrability of the paths of X sufficient (instead of the L^p_{loc} -integrability, $p \geq 2$ required in [2]). \blacksquare

Thanks to the additional property in (19), the same argument as the one above (without stopping times) shows that the processes in (17) are indeed square-integrable martingales for a.e. $T \in \mathbb{R}_+$.

Remark 1. We refer to [2] for a general solution theory concerning equations of the type in (18) when $g_0 \in L_{loc}^p(\mathbb{R}_+; \mathbb{R}^m)$, $p \geq 2$, and $E = \mathbb{R}^m$.

In the case $m = d = 1$ and $E = \mathbb{R}_+$, if one defines $Y_t = \int_0^t X_s ds$, $t \geq 0$, then $Y = (Y_t)_{t \geq 0}$ is a nondecreasing process and an application of [2, Lemma 3.2] shows

$$Y_t = \int_0^t g_0(s) ds + \int_0^t K(t-s) Z_s ds = \int_0^t g_0(s) ds + (K * Z)_t, \quad t \geq 0, \mathbb{P} - a.s.$$

This type of stochastic Volterra equations is analyzed in [1] for locally integrable kernels $K \in L_{loc}^1(\mathbb{R}_+; \mathbb{R})$.

2.1. Stochastic convolution for processes with jumps

The goal of this subsection is to develop technical results concerning the stochastic convolution. In particular, we aim to make Lemma 2.1 and Lemma 2.6 in [5] feasible in our context, where we are dealing with discontinuities for Z and, more importantly, with a process X which a priori is not bounded. This requires to modify the statements and the proofs of the aforementioned results, which are crucial for the development of the theory. Such changes are important from a conceptual point of view and after every result we add a remark showing the parallel with the setting in [5].

We start with a preliminary claim.

Lemma 2. Fix $p \in \mathbb{N}$. Let $F, G \in L_{loc}^2(\mathbb{R}_+; \mathbb{R}^{p \times d})$ and $S \subset \mathbb{R}_+$ be such $\mathbb{R}_+ \setminus S$ is dt -null set. Suppose that $F = G$ a.e. in \mathbb{R}_+ . Then

$$\int_0^T F(T-s) dZ_s = \int_0^T 1_S(s) G(T-s) dZ_s, \quad \mathbb{P} - a.s., \text{ for a.e. } T \in \mathbb{R}_+. \quad (20)$$

In particular,

$$F * dZ = G * dZ, \quad \mathbb{P} \otimes dt - a.e. \quad (21)$$

Proof. It is sufficient to prove (20) replacing Z with \tilde{Z} , because trivially $F * b(X) = G * (1_S b(X))$, $\mathbb{P} \otimes dt$ -a.e. on $\Omega \times \mathbb{R}_+$. Moreover, we only work with the stochastic integral in dM^c , as by (15) we can repeat the next procedure (component-wise) for the convolution in dM^d recovering (20).

The argument above in the section implies the existence of a dt -null set $N \subset \mathbb{R}_+$ such that, for every $T \in \mathbb{R}_+ \setminus N$, we have

$$\int_0^T 1_S(s) G(T-s) dM_s^c - \int_0^T F(T-s) dM_s^c = \int_0^T (1_S(s) G(T-s) - F(T-s)) dM_s^c, \quad \mathbb{P} - a.s. \quad (22)$$

Consider the square-integrable, p -dimensional martingale $Q = (\int_0^t (1_S(s) G(T-s) - F(T-s)) dM_s^c)_{t \leq T}$, whose predictable quadratic covariation is, due to the hypotheses,

$$\langle Q, Q \rangle_t = \int_0^t (1_S(s) G(T-s) - F(T-s)) a(X_s) (1_S(s) G(T-s) - F(T-s))^\top ds = 0, \quad t \in [0, T], \mathbb{P} - a.s.$$

Since Q starts at 0, we can conclude that $Q = 0$ up to evanescence, hence (20) follows.

Regarding (21), it is an immediate consequence of (20) with $S = \mathbb{R}_+$ and the joint measurability of the stochastic convolutions, which allows to state an equality $\mathbb{P} \otimes dt$ -a.e. This completes the proof. \blacksquare

Remark 2. In [5], the authors consider the stochastic convolution of a function $F \in L_{loc}^2(\mathbb{R}_+; \mathbb{R}^{p \times d})$ with respect to a continuous local martingale M with predictable quadratic covariation $d\langle M, M \rangle_t = a_t dt$, where (a_t) is an adapted, locally bounded process. These assumptions allow to define $(F * dM)_t$ for every $t \in \mathbb{R}_+$. In particular, two jointly measurable versions of the stochastic convolution are equal \mathbb{P} -a.s., for every $t \geq 0$. This concept is stronger than the $\mathbb{P} \otimes dt$ -uniqueness that we have in our framework. As for (21) in Lemma 2, in the continuous case it can be stated as follows: for every $F, G \in L_{loc}^2(\mathbb{R}_+; \mathbb{R}^{p \times d})$, with $F = G$ a.e. in \mathbb{R}_+ , one has

$$(F * dM)_t = (G * dM)_t, \quad \mathbb{P} - a.s., t \geq 0.$$

Now we state a result concerning the associativity of the stochastic convolution.

Lemma 3. Fix $p, q \in \mathbb{N}$. Let $\rho \in L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{q \times p})$ and $F \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{p \times d})$. Then

$$((\rho * F) * dZ)_t = (\rho * (F * dZ))(t), \quad \mathbb{P} - \text{a.s.}, \text{ for a.e. } t \in \mathbb{R}_+. \quad (23)$$

Proof. Also in this case we just need to show the statement with $d\tilde{Z}$ in place of dZ , because an application of Fubini's theorem provides $(\rho * F) * b(X) = \rho * (F * b(X))$, $\mathbb{P} \otimes dt$ -a.e. on $\Omega \times \mathbb{R}_+$. In addition it is sufficient to focus only on the stochastic convolutions in dM^c , as discussed in the preceding proof. By linearity we can assume $d = p = q = 1$ without loss of generality, and we consider $\rho \geq 0$ to keep the notation simple, otherwise we should split it into positive and negative part.

First note that the function $\rho * F \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{R})$, hence for every $t \in \mathbb{R}_+ \setminus N_1$, being N_1 a dt -null set, we have

$$\begin{aligned} ((\rho * F) * dM^c)_t &= \int_0^t \left(1_{\{(t-u) \in S\}} \int_0^{t-u} F(t-u-s) \rho(s) ds \right) dM^c_u \\ &= \int_0^t \left(\int_0^t 1_{\{(t-u) \in S\}} 1_{\{s \leq t-u\}} F(t-u-s) \rho(s) ds \right) dM^c_u, \quad \mathbb{P} - \text{a.s.} \end{aligned} \quad (24)$$

where $S \subset \mathbb{R}_+$ is such that $\int_0^t F(t-s) \rho(s) ds$, $t \in S$, is well-defined. In particular, $\mathbb{R}_+ \setminus S$ is a dt -null set. Our goal is to apply the stochastic Fubini theorem (see, e.g., [22, Theorem 65, Chapter IV]), but before we can do that we need a preliminary step. For every $T > 0$, a change of variables, sequential applications of Tonelli's theorem and Young's inequality yield (in the whole Ω)

$$\begin{aligned} &\int_0^T \left[\int_0^t \left(\int_0^{t-u} 1_{\{(t-u) \in S\}} 1_{\{s \leq t-u\}} |F(t-s-u)|^2 \rho(s) ds \right) |X_u| du \right] dt \\ &\leq \int_0^T \left[\int_0^t \left(\int_0^{t-s} |F(t-s-u)|^2 |X_u| du \right) \rho(s) ds \right] dt = \int_0^T \left[\int_s^T (|F|^2 * |X|)(t-s) dt \right] \rho(s) ds \\ &= \int_0^T \left[\int_0^{T-s} (|F|^2 * |X|)(t) dt \right] \rho(s) ds \leq \|\rho\|_{L^1([0, T])} \|F\|_{L^2([0, T])}^2 \|X\|_{L^1([0, T])}. \end{aligned}$$

Taking expectation and recalling (19) we have

$$\begin{aligned} &\int_0^T \mathbb{E} \left[\int_0^t \left(\int_0^{t-u} 1_{\{(t-u) \in S\}} 1_{\{s \leq t-u\}} |F(t-s-u)|^2 \rho(s) ds \right) |X_u| du \right] dt \\ &\leq \|\rho\|_{L^1([0, T])} \|F\|_{L^2([0, T])}^2 \mathbb{E} \left[\|X\|_{L^1([0, T])} \right] < \infty. \end{aligned}$$

This proves that there exists $N_2 \subset \mathbb{R}_+$ such that

$$\mathbb{E} \left[\int_0^t \left(\int_0^{t-u} 1_{\{(t-u) \in S\}} 1_{\{s \leq t-u\}} |F(t-s-u)|^2 \rho(s) ds \right) |X_u| du \right] < \infty, \quad t \in \mathbb{R}_+ \setminus N_2. \quad (25)$$

Taking $t \in \mathbb{R}_+ \setminus (N_1 \cup N_2)$, thanks to (25) and Lemma 2 (see (20)) we can apply the stochastic Fubini theorem in (24) to deduce that

$$\begin{aligned} &((\rho * F) * dM^c)_t \\ &= \int_0^t \left(\int_0^t 1_{\{(t-u) \in S\}} 1_{\{s \leq t-u\}} F(t-u-s) \rho(s) ds \right) dM^c_u = \int_0^t \left(\int_0^{t-s} 1_{\{(t-u) \in S\}} F(t-s-u) dM^c_u \right) \rho(s) ds \\ &= \int_0^t (F * dM^c)_{t-s} \rho(s) ds = (\rho * (F * dM^c))(t), \quad \mathbb{P} - \text{a.s.}, \end{aligned}$$

and the proof is complete. ■

Remark 3. The previous result is the analogue of [5, Lemma 2.1], where the authors are able –in the framework described in Remark 2– to handle a generic signed measure of locally bounded variation L . Essentially they can do so because the convolution $F * dM$ is defined as a stochastic integral for every $t \in \mathbb{R}_+$. As a consequence, it is unique up to a $\mathbb{P} \otimes |L|$ -null set, being $|L|$ the total variation measure of L .

In contrast with this, notice that in our setting it is not possible to make sense of the right side of (23) for a fixed time $t > 0$ when ρ is replaced by L . Indeed, $F * dZ$ is only defined up to a $\mathbb{P} \otimes dt$ -null set, therefore the value of $(L * (F * dZ))(t)$ would depend on the modification one chooses. However, Lemma 3 can be slightly extended by replacing ρ in (23) with an $\mathbb{R}^{q \times p}$ -valued measure which is the sum of a locally integrable function and a point mass in 0 (this extension can be inferred directly from (23)). We are going to need this final comment in Section 4.

We are now ready to state an analogue of [5, Lemma 2.6].

Proposition 4. Assume that $m = d$, and that the kernel $K \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{d \times d})$ admits a resolvent of the first kind L^3 . Let $F \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{d \times d})$ be such that $F * L$ is locally absolutely continuous. Then

$$(F * dZ)_t = (F * L)(0)(X - g_0)(t) + ((F * L)' * (X - g_0))(t), \quad \text{for a.e. } t \in \mathbb{R}_+, \mathbb{P} - \text{a.s.} \quad (26)$$

Proof. By Lebesgue's fundamental theorem of calculus we can write (denoting by I the identity matrix in $\mathbb{R}^{d \times d}$)

$$(F * L)(t) = (F * L)(0) + \int_0^t (F * L)'(s) ds = (F * L)(0) + ((F * L)' * I)(t), \quad t \geq 0,$$

which implies, convolving with K , using [15, Theorem 6.1 (ix), Chapter 3] and a change of variables,

$$\int_0^t F(s) ds = (F * L)(0) \int_0^t K(s) ds + \int_0^t ((F * L)' * K)(s) ds, \quad t \geq 0.$$

We can differentiate both sides of the previous equation, as they are absolutely continuous functions, and we obtain

$$F(t) = (F * L)(0)K(t) + ((F * L)' * K)(t), \quad \text{for a.e. } t \in \mathbb{R}_+.$$

Then convolving with dZ yields

$$\begin{aligned} (F * dZ)_t &= (F * L)(0)(K * dZ)_t + (((F * L)' * K) * dZ)_t \\ &= (F * L)(0)(K * dZ)_t + ((F * L)' * (K * dZ))(t), \quad \mathbb{P} - \text{a.s. for a.e. } t \in \mathbb{R}_+, \end{aligned} \quad (27)$$

where in the first equality we use Lemma 2 (see (21)) and in the second Lemma 3 with $\rho = (F * L)'$. The crucial point here is to pass to the trajectories. In order to do so, observe that by (18) we have

$$X_t - g_0(t) = (K * dZ)_t, \quad \text{for a.e. } t \in \mathbb{R}_+, \mathbb{P} - \text{a.s.},$$

hence $((F * L)' * (K * dZ))(t) = ((F * L)' * (X - g_0))(t)$, \mathbb{P} -a.s., for a.e. $t \in \mathbb{R}_+$. Moreover we can consider a jointly measurable modification of the process $((F * L)' * (X - g_0))$ thanks to Fubini's theorem, which in turn can be applied as

$$\begin{aligned} &\mathbb{E} \left[\int_0^T \left(\int_0^T 1_{\{s \leq t\}} |(F * L)'(s)| |(X - g_0)(t - s)| ds \right) dt \right] \\ &= \mathbb{E} \left[\int_0^T \left(\int_0^t |(F * L)'(s)| |(X - g_0)(t - s)| ds \right) dt \right] \\ &\leq \|(F * L)'\|_{L^1([0, T]; \mathbb{R}^{d \times d})} \left(\mathbb{E} \left[\|X\|_{L^1([0, T]; \mathbb{R}^d)} \right] + \|g_0\|_{L^1([0, T]; \mathbb{R}^d)} \right) < \infty, \quad T > 0, \end{aligned} \quad (28)$$

³Given a kernel $K \in L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^{d \times d})$, an $\mathbb{R}^{d \times d}$ -valued measure L is called its (measure) resolvent of the first kind if $L * K = K * L = I$, where $I \in \mathbb{R}^{d \times d}$ is the identity matrix. L does not always exist, but if it does then it is unique (cfr. [15, Theorem 5.2, Chapter 5]).

by Tonelli's theorem, Equation (19) and [15, Theorem 2.2 (i), Chapter 2]. Substituting this term in (27) and recalling once again (18) we deduce that

$$(F * dZ)_t = (F * L)(0)(X - g_0)(t) + ((F * L)' * (X - g_0))(t), \quad \mathbb{P} - \text{a.s.}, \text{ for a.e. } t \in \mathbb{R}_+.$$

This equality can be understood up to a $\mathbb{P} \otimes dt$ -null set because it involves only jointly measurable processes. Therefore (26) holds and the proposition is completely proved. \blacksquare

Remark 4. In [5, Lemma 2.6] the authors require $F * L$ to be right-continuous and of locally bounded variation. The loss of generality in Proposition 4, where we assume the local absolute continuity for the same function, is triggered by Lemma 3 and Remark 3.

3. Towards the conditional Fourier–Laplace transform

In this section we are going to introduce processes $V^T = (V_t^T)_{t \in [0, T]}$ which will be used to find an *ansatz* for the conditional Fourier–Laplace transform of $(f^\top * X)(T)$, $T > 0$, where f is a suitable given function. The procedure that we employ can also be adapted to characterize the marginal distributions of the solution process X and of the semimartingale Z in (14), see Subsection 3.1.

We first introduce some notation. For a \mathbb{C} -valued function $g \in L^1(\nu_k)$, $k = 0, 1, \dots, m$, we denote

$$\begin{aligned} \langle \eta(x, d\xi), g(\xi) \rangle &= \int_{\mathbb{R}^d} g(\xi) \nu_0(d\xi) + \sum_{k=1}^m \left(\int_{\mathbb{R}^d} g(\xi) \nu_k(d\xi) \right) x_k, \quad x \in E; \\ \nu(g(\xi)) &= \left[\int_{\mathbb{R}^d} g(\xi) \nu_1(d\xi) \quad \int_{\mathbb{R}^d} g(\xi) \nu_2(d\xi) \quad \dots \quad \int_{\mathbb{R}^d} g(\xi) \nu_m(d\xi) \right]^\top \in \mathbb{C}^m. \end{aligned}$$

Note that $\langle \eta(x, d\xi), g(\xi) \rangle = \int_{\mathbb{R}^d} g(\xi) \nu_0(d\xi) + \nu(g(\xi))^\top x$ for every $x \in E$. In addition, we consider

$$\begin{aligned} B &= [b_1 \quad b_2 \quad \dots \quad b_m] \in \mathbb{R}^{d \times m}, \\ A(u) &= [u^\top A_1 u \quad u^\top A_2 u \quad \dots \quad u^\top A_m u]^\top \in \mathbb{C}^m, \quad u \in \mathbb{C}^d. \end{aligned}$$

Notice that $b(x) = b_0 + Bx$, and $u^\top a(x)u = u^\top A_0 u + A(u)^\top x$, for every $x \in E$, $u \in \mathbb{C}^d$.

Take $f \in C(\mathbb{R}_+; \mathbb{C}^m)$ and denote by

$$D_\nu = \left\{ u \in \mathbb{C}^d : (e^{u^\top \cdot} - 1 - u^\top \cdot) \in L^1(\nu_k), \text{ for every } k = 0, \dots, m \right\}. \quad (29)$$

Recalling that the signed measures ν_k , $k = 0, \dots, m$, have finite second moment, one can show that $i\mathbb{R}^d \subset D_\nu$, and that $D_\nu \cap \mathbb{R}^d$ is a star-shaped domain at 0.

Consider a subset $D \subset D_\nu$ where the \mathbb{C} -valued functions $u \mapsto \int_{\mathbb{R}^d} (e^{u^\top \xi} - 1 - u^\top \xi) \nu_k(d\xi)$ are locally bounded, for every $k = 0, \dots, m$. We define the map $\mathcal{R}: \mathbb{R}_+ \times D \rightarrow \mathbb{C}^m$ as follows:

$$\mathcal{R}(t, u) = f(t) + B^\top u + \frac{1}{2} A(u) + \nu \left(e^{u^\top \xi} - 1 - u^\top \xi \right), \quad (t, u) \in \mathbb{R}_+ \times D. \quad (30)$$

Notice that, by the definition of D , the mapping \mathcal{R} is locally bounded. The following hypothesis introduces the deterministic Riccati–Volterra equation that allows us to determine a semi-explicit exponential affine formula for $\mathbb{E}[\exp\{(f^\top * X)(T)\} | \mathcal{F}_t]$, with $T > 0$ and $t \in [0, T]$.

Hypothesis 1. *There exists a continuous global solution $\psi: \mathbb{R}_+ \rightarrow D$ to the deterministic Riccati–Volterra equation*

$$\psi(t)^\top = \int_0^t \mathcal{R}(s, \psi(s))^\top K(t-s) ds = \left(\mathcal{R}(\cdot, \psi(\cdot))^\top * K \right)(t), \quad t \geq 0. \quad (31)$$

Since ψ is continuous and takes values in D , the \mathbb{C}^m -valued map $s \mapsto \mathcal{R}(s, \psi(s))$ is locally bounded.

We introduce the \mathbb{C} -valued function $\phi: \mathbb{R}_+ \rightarrow \mathbb{C}$ given by

$$\phi(t) = \int_0^t \left(\psi(s)^\top b_0 + \frac{1}{2} \psi(s)^\top A_0 \psi(s) + \int_{\mathbb{R}^d} \left(e^{\psi(s)^\top \xi} - 1 - \psi(s)^\top \xi \right) \nu_0(d\xi) \right) ds, \quad t \geq 0. \quad (32)$$

For every $T > 0$ we define the following càdlàg, adapted, \mathbb{C} -valued semimartingale on $\Omega \times [0, T]$:

$$V_t^T = V_0^T - \int_0^t \left[\frac{1}{2} \psi(T-s)^\top a(X_s) \psi(T-s) + \left\langle \eta(X_s, d\xi), e^{\psi(T-s)^\top \xi} - 1 - \psi(T-s)^\top \xi \right\rangle \right] ds + \int_0^t \psi(T-s)^\top d\tilde{Z}_s, \quad (33)$$

$$V_0^T = \int_0^T \left(f(T-s) + B^\top \psi(T-s) + \frac{1}{2} A(\psi(T-s)) + \nu \left(e^{\psi(T-s)^\top \xi} - 1 - \psi(T-s)^\top \xi \right) \right)^\top g_0(s) ds + \phi(T). \quad (34)$$

Observe that V^T is left-continuous in T because $\psi(0) = 0$ by (31). This process is the natural extension of [5, Equations (4.4) – (4.5)] to the framework with jumps. Moreover, one can write

$$V_0^T = \phi(T) + \int_0^T \mathcal{R}(T-s, \psi(T-s))^\top g_0(s) ds. \quad (35)$$

Our aim is to find, using the stochastic Fubini theorem, an alternative expression for the random variables V_t^T by means of integrals in time of the trajectories of suitable processes.

In the case $b \equiv 0$, we are going to use the paths of the forward process. Precisely, for a fixed $t \in [0, T]$, by (A.1) in Appendix A we have

$$\mathbb{E}[X_s | \mathcal{F}_t] = g_0(s) + \int_0^t K(s-r) d\tilde{Z}_r, \quad \mathbb{P} - \text{a.s.}, \text{ for a.e. } s > t. \quad (36)$$

Hence requiring the kernel K to be continuous on $(0, \infty)$, the process on the right side of the previous equation has a jointly measurable version that we denote by $\tilde{g}_t(s)$, $s > t$. Note that it makes sense to integrate in time the trajectories of such $\tilde{g}_t(\cdot)$ since it is unique up to a $\mathbb{P} \otimes dt$ -null set.

In the case $b \neq 0$ we consider the paths of a process $g_t(\cdot)$ such that

$$g_t(s) = g_0(s) + \int_0^t K(s-r) dZ_r, \quad \mathbb{P} - \text{a.s.}, s > t. \quad (37)$$

Also in this case we assume K to be continuous on $(0, \infty)$, so that $g_t(\cdot)$ can be taken jointly measurable on $\Omega \times (t, \infty)$ and is uniquely defined up to a $\mathbb{P} \otimes dt$ -null set. An application of the stochastic Fubini theorem (see, e.g., [22, Theorem 65, Chapter IV]) shows that the trajectories of $g_t(\cdot)$ are locally integrable in (t, ∞) . Note that when $t = 0$ we have an abuse of notation, as g_0 represents both the initial input curve in (14) and the process just defined in (37). This, however, is not an issue as these two concepts coincide $\mathbb{P} \otimes dt$ -a.e. in $\Omega \times (0, \infty)$. In the following, we continue to consider g_0 as the initial input curve. Finally, notice that

$$g_t(s) = \mathbb{E} \left[X_s - \int_0^{s-t} K(s-t-r) b(X_{t+r}) dr \mid \mathcal{F}_t \right], \quad \mathbb{P} - \text{a.s.}, \text{ for a.e. } s > t. \quad (38)$$

For this reason $g_t(\cdot)$ is called *adjusted forward process*.

Theorem 5. *Assume Hypothesis 1. Let $K \in L_{\text{loc}}^2(\mathbb{R}_+; \mathbb{R}^{m \times d})$ be a continuous kernel on $(0, \infty)$ and define, for every $t \in [0, T]$,*

$$\tilde{V}_t^T = \phi(T-t) + \int_0^t f(T-s)^\top X_s ds + \int_t^T \mathcal{R}(T-s, \psi(T-s))^\top g_t(s) ds. \quad (39)$$

Then

$$V_t^T = \tilde{V}_t^T, \quad \mathbb{P} - \text{a.s.}, t \in [0, T]. \quad (40)$$

In addition, the process $(\exp \{V_t^T\})_{t \in [0, T]}$ is a \mathbb{C} -valued local martingale, and if it is a true martingale then

$$\mathbb{E} \left[\exp \{ (f^\top * X)(T) \} \middle| \mathcal{F}_t \right] = \exp \{ \tilde{V}_t^T \}, \quad \mathbb{P} - \text{a.s.}, t \in [0, T]. \quad (41)$$

Proof. It is straightforward to check that (40) holds for $t = 0$.

Focusing on the case $t \in (0, T]$, we rewrite the definition of \tilde{V}_t^T in (39) as follows

$$\begin{aligned} \tilde{V}_t^T &= \phi(T-t) + \int_0^t f(T-s)^\top X_s \, ds \\ &\quad + \int_t^T \mathcal{R}(T-s, \psi(T-s))^\top g_0(s) \, ds + \int_t^T \mathcal{R}(T-s, \psi(T-s))^\top (g_t - g_0)(s) \, ds. \end{aligned} \quad (42)$$

It is convenient to introduce the process

$$\bar{g}_t(s) = \begin{cases} X_s, & s \leq t \\ g_t(s), & s > t \end{cases}. \quad (43)$$

Recall that by (37) $g_t(s) = g_0(s) + \int_0^t K(s-r) \, dZ_r$, \mathbb{P} -a.s. for a.e. $s > t$, and that by (18) $X_s = g_0(s) + \int_0^s K(s-r) \, dZ_r$, \mathbb{P} -a.s., for a.e. $s \in [0, t]$. Therefore $\bar{g}_t(\cdot)$ is a jointly measurable modification of the process $g_0(\cdot) + \int_0^t \mathbf{1}_{\{r \leq \cdot\}} K(\cdot - r) \, dZ_r$. Invoking the stochastic Fubini theorem in [22, Theorem 65, Chapter IV] and recalling the Riccati–Volterra equation in (31), after a suitable change of variables we obtain

$$\begin{aligned} &\int_0^T \mathcal{R}(T-s, \psi(T-s))^\top (\bar{g}_t - g_0)(s) \, ds \\ &= \int_0^T \mathcal{R}(T-s, \psi(T-s))^\top \left[\int_0^t \mathbf{1}_{\{r \leq s\}} K(s-r) \, dZ_r \right] \, ds \\ &= \int_0^t \left[\int_r^T \mathcal{R}(T-s, \psi(T-s))^\top K(s-r) \, ds \right] \, dZ_r = \int_0^t \left[\int_0^{T-r} \mathcal{R}(s, \psi(s))^\top K(T-r-s) \, ds \right] \, dZ_r \\ &= \int_0^t \psi(T-r)^\top \, dZ_r, \quad \mathbb{P} - \text{a.s.} \end{aligned} \quad (44)$$

Such an application is legitimate, because the boundedness of $\mathcal{R}(\cdot, \psi(\cdot))$ in $[0, T]$ by a positive constant C_T and a change of variables yield, for every $k = 1, \dots, m$,

$$\begin{aligned} &\int_0^t \left[\int_0^T \mathbf{1}_{\{r \leq s\}} |\mathcal{R}(T-s, \psi(T-s))|^2 |K(s-r)|^2 \, ds \right] |X_{k,r}| \, dr \\ &= \int_0^t \left[\int_0^{T-r} |\mathcal{R}(s, \psi(s))|^2 |K(T-r-s)|^2 \, ds \right] |X_{k,r}| \, dr \leq C_T^2 \|K\|_{L^2([0, T]; \mathbb{R}^{m \times d})}^2 \|X\|_{L^1([0, t]; \mathbb{R}^m)}, \end{aligned} \quad (45)$$

so the expectation of the leftmost side is finite thanks to (19). As for the drift part,

$$\int_0^t \left(\int_0^T \mathbf{1}_{\{r \leq s\}} |\mathcal{R}(T-s, \psi(T-s))|^2 |K(s-r)|^2 \, ds \right)^{\frac{1}{2}} |X_{k,r}| \, dr \leq C_T \|K\|_{L^2([0, T]; \mathbb{R}^{m \times d})} \|X\|_{L^1([0, t]; \mathbb{R}^m)}. \quad (46)$$

Going back to (42) and recalling the definitions of V^T in (33)–(35) we obtain, \mathbb{P} -a.s.,

$$\begin{aligned}
\tilde{V}_t^T &= \phi(T-t) + \int_0^t f(T-s)^\top X_s \, ds + \int_t^T \mathcal{R}(T-s, \psi(T-s))^\top g_0(s) \, ds \\
&\quad + \int_0^t \psi(T-s)^\top dZ_s - \int_0^t \mathcal{R}(T-s, \psi(T-s))^\top (X_s - g_0(s)) \, ds \\
&= \phi(T-t) + \int_0^T \mathcal{R}(T-s, \psi(T-s))^\top g_0(s) \, ds + \int_0^t \psi(T-s)^\top dZ_s \\
&\quad - \int_0^t \left[B^\top \psi(T-s) + \frac{1}{2} A(\psi(T-s)) + \nu \left(e^{\psi(T-s)^\top \xi} - 1 - \psi(T-s)^\top \xi \right) \right]^\top X_s \, ds \\
&= \phi(T) + \int_0^T \mathcal{R}(T-s, \psi(T-s))^\top g_0(s) \, ds + \int_0^t \psi(T-s)^\top d\tilde{Z}_s \\
&\quad - \int_0^t \left[\frac{1}{2} \psi(T-s)^\top a(X_s) \psi(T-s) + \left\langle \eta(X_s, d\xi), e^{\psi(T-s)^\top \xi} - 1 - \psi(T-s)^\top \xi \right\rangle \right] ds \\
&= V_t^T, \tag{47}
\end{aligned}$$

where in the second-to-last equality we use (16). This proves (40).

Moving on to the next assertion, denote by $H^T = (H_t^T)_{t \in [0, T]} = (\exp\{V_t^T\})_{t \in [0, T]}$. By Itô's formula and the dynamics in (33) we have

$$\begin{aligned}
dH_t^T &= H_{t-}^T \left[- \left(\frac{1}{2} \psi(T-t)^\top a(X_t) \psi(T-t) + \left\langle \eta(X_t, d\xi), e^{\psi(T-t)^\top \xi} - 1 - \psi(T-t)^\top \xi \right\rangle \right) dt + \psi(T-t)^\top d\tilde{Z}_t \right] \\
&\quad + \frac{1}{2} H_{t-}^T \psi(T-t)^\top a(X_t) \psi(T-t) dt + H_{t-}^T \int_{\mathbb{R}^d} \left(e^{\psi(T-t)^\top \xi} - 1 - \psi(T-t)^\top \xi \right) \mu(dt, d\xi) \\
&= H_{t-}^T \left[\psi(T-t)^\top dM_t^c + \int_{\mathbb{R}^d} \left(e^{\psi(T-t)^\top \xi} - 1 \right) (\mu - \nu)(dt, d\xi) \right], \quad H_0^T = \exp(V_0^T).
\end{aligned}$$

We define $N^T = (N_t^T)_{t \in [0, T]}$ by $dN_t^T = \psi(T-t)^\top dM_t^c + \int_{\mathbb{R}^d} (e^{\psi(T-t)^\top \xi} - 1) (\mu - \nu)(dt, d\xi)$, $N_0^T = 0$. Then N^T is a local martingale and the previous computations show that $H^T = \exp\{V_0^T\} \mathcal{E}(N^T)$ up to evanescence, where \mathcal{E} denotes the Doléans–Dade exponential. Therefore H^T is a local martingale, as stated. Finally, in case it is a true martingale, (41) directly follows from (40), and the proof is complete. \blacksquare

Remark 5. Assuming $m = d$, it is possible to find an expression for V^T in terms of the true forward process even in the case $b \neq 0$. Indeed, by (A.4) in Appendix A,

$$\mathbb{E}[X_s | \mathcal{F}_t] = (g_0 - (R_B * g_0) + (E_B * b_0))(s) + \int_0^t E_B(s-r) d\tilde{Z}_r, \quad \mathbb{P} - a.s., \text{ for a.e. } s > t. \tag{48}$$

Here R_B is the resolvent of the second kind⁴ of $-KB$ and $E_B = K - R_B * K$. If K is continuous on $(0, \infty)$, then E_B is continuous on the same interval, as well. Thus, one can choose a jointly measurable version $f_t(s)$, $s > t$, of the process on the right side of (48), which is unique up to a $\mathbb{P} \otimes dt$ -null set. Arguing as in [5, Lemma 4.4], we obtain the variation of constants formula

$$\psi(t)^\top = \int_0^t \left[f(s) + \frac{1}{2} A(\psi(s)) + \nu \left(e^{\psi(s)^\top \xi} - 1 - \psi(s)^\top \xi \right) \right]^\top E_B(t-s) \, ds, \quad t \geq 0,$$

⁴Given $K \in L_{\text{loc}}^1(\mathbb{R}_+; \mathbb{R}^{d \times d})$, its resolvent of the second kind is the unique solution $R \in L_{\text{loc}}^1(\mathbb{R}_+; \mathbb{R}^{d \times d})$ of the two equations $K * R = R * K = K - R$ (cfr. [15, Theorem 3.1, Chapter 2] and the subsequent definition).

which combined with the strategy in the proof of Theorem 5 leads to

$$V_t^T = \int_0^t f(T-s)^\top X_s ds + \int_t^T \left[(\mathcal{R}(T-s, \psi(T-s)) - B^\top \psi(T-s))^\top f_t(s) + \frac{1}{2} \psi(T-s)^\top A_0 \psi(T-s) + \int_{\mathbb{R}^d} \left(e^{\psi(T-s)^\top \xi} - 1 - \psi(T-s)^\top \xi \right) \nu_0(d\xi) \right] ds, \quad \mathbb{P} - a.s.$$

However in the framework of jumps it is preferable to work with the adjusted forward process, because – as will become clear in the next section – certain properties can be assumed for the kernel K , but they can be neither required (i.e. it would not be a reasonable hypothesis) nor inferred for E_B .

3.1. The marginal distributions of X and Z

The procedure that we have used above to obtain a formula for $\mathbb{E}[\exp\{(f^\top * X)(T)\} | \mathcal{F}_t]$ (see (41) in Theorem 5) can also be followed to deduce an exponential affine expression for the conditional Fourier–Laplace transform of the marginal distributions of the solution process X and the semimartingale Z in (14), i.e.,

$$\mathbb{E}[\exp\{u_1^\top Z_T\} | \mathcal{F}_t], \quad \mathbb{E}[\exp\{u_2^\top X_T\} | \mathcal{F}_t],$$

for suitable $u_1 \in \mathbb{C}^d$ and $u_2 \in \mathbb{C}^m$.

We first show the formula for the conditional transform of the \mathbb{R}^d -valued càdlàg semimartingale Z .

Theorem 6. *Assume that $K \in L_{\text{loc}}^2(\mathbb{R}_+; \mathbb{R}^{m \times d})$ is continuous in $(0, \infty)$. Given $u_1 \in \mathbb{C}^d$ and $f \in C(\mathbb{R}_+; \mathbb{C}^m)$, suppose that there exists a continuous global solution $\psi_1: \mathbb{R}_+ \rightarrow D$ of the deterministic Riccati–Volterra equation*

$$\psi_1(t)^\top = u_1^\top + \int_0^t \mathcal{R}(s, \psi_1(s))^\top K(t-s) ds, \quad t \geq 0,$$

where $\mathcal{R}: \mathbb{R}_+ \times D \rightarrow \mathbb{C}^m$ is given in (30). Then, for every $T > 0$, defining $\tilde{V}^T = (\tilde{V}_t^T)_{t \in [0, T]}$ as in (39) with ψ_1 instead of ψ and $\tilde{V}_1^T = (\tilde{V}_{1,t}^T)_{t \in [0, T]}$ by

$$\tilde{V}_{1,t}^T = \tilde{V}_t^T + u_1^\top Z_t, \quad t \in [0, T],$$

the process $\exp\{\tilde{V}_1^T\}$ is a local martingale. In particular, if $\exp\{\tilde{V}_1^T\}$ is a true martingale, then

$$\mathbb{E} \left[\exp \left\{ u_1^\top Z_T + (f^\top * X)(T) \right\} \middle| \mathcal{F}_t \right] = \exp \left\{ \tilde{V}_{1,t}^T \right\}, \quad \mathbb{P} - a.s., \quad t \in [0, T].$$

Proof. We use the same argument as in the proof of Theorem 5. It can be split into two parts.

In the first step, we define the process $V_1^T = (V_{1,t}^T)_{t \in [0, T]}$ as in (33)–(35), replacing ψ with ψ_1 . Then, using the stochastic Fubini theorem, we prove that V_1^T and \tilde{V}_1^T are versions of each other.

In the second step, we use Itô’s formula to infer that $\exp\{V_1^T\}$ is a local martingale, since it can be seen as the Doléans–Dade exponential of a local martingale. \blacksquare

We now fix an i.i.d. sequence $(Y_{i,k})_k$ of \mathcal{F}_0 -measurable, \mathbb{R} -valued random variables distributed according to a probability measure θ_i on \mathbb{R} with finite first and second moments, for every $i = 1, \dots, d$. Denoting by $(e_i)_{i=1, \dots, d}$ the canonical basis of \mathbb{R}^d , we extend θ_i to a probability measure on \mathbb{R}^d by setting

$$\bar{\theta}_i(A) = \theta_i(\pi_i(A \cap \text{span}(e_i))), \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where $\pi_i: \mathbb{R}^d \rightarrow \mathbb{R}$ is defined by $\pi_i(x) = x^\top e_i$ and $\mathcal{B}(\mathbb{R}^d)$ are the Borel-measurable sets of \mathbb{R}^d . We also take an increasing sequence $(T_{i,k})_{k \in \mathbb{N}}$ of \mathbb{F} -stopping times independent from $(Y_{i,k})_k$ and such that $T_{i,k} \rightarrow \infty$ as $k \rightarrow \infty$, \mathbb{P} -a.s., for every $i = 1, \dots, d$.

We focus on a d -dimensional marked point process $N = (N_t)_{t \geq 0} = [N_{1,t}, \dots, N_{d,t}]_{t \geq 0}^\top$ with jump times $(T_{i,k})_{k \in \mathbb{N}}$ and jump sizes $(Y_{i,k})_{k \in \mathbb{N}}$, namely,

$$N_{i,t} = \sum_{t \geq T_{i,k}} Y_{i,k} = \sum_{k \in \mathbb{N}} Y_{i,k} 1_{\{t \geq T_{i,k}\}}, \quad t \geq 0, \quad i = 1, \dots, d.$$

Given a vector $\lambda_0 \in \mathbb{R}^m$ and two matrices $\bar{\Lambda}_0, \bar{\Lambda}_1 \in \mathbb{R}^{d \times m}$, we suppose that the conditional intensity of N is $\bar{\Lambda}_1 \lambda$, where $\lambda = (\lambda_t)_{t \geq 0} = [\lambda_{1,t}, \dots, \lambda_{m,t}]_{t \geq 0}^\top$ is an \mathbb{R}^m -valued process satisfying

$$\lambda_t = \lambda_0 + \int_0^t K(t-s) \bar{\Lambda}_0 \lambda_s ds + \sum_{i=1}^d \sum_{t > T_{i,k}} Y_{i,k} K(t-T_{i,k}) e_i, \quad t \geq 0. \quad (49)$$

This means that, denoting by $\mu^N(dt, d\xi)$ the measure on $\mathbb{R}_+ \times \mathbb{R}^d$ associated with the jumps of N and by $\nu^N(dt, d\xi)$ its compensator, then

$$\nu^N(dt, d\xi) = \sum_{i=1}^d (\bar{\Lambda}_1 \lambda_t)_i dt \otimes \bar{\theta}_i(d\xi). \quad (50)$$

We call N a *marked Hawkes process* in \mathbb{R}^d . In particular, note that

$$N_{i,t} - \int_0^t \int_{\mathbb{R}^d} \xi_i \nu^N(ds, d\xi) = N_{i,t} - \left(\int_{\mathbb{R}} \xi_i \theta_i(d\xi_i) \right) \int_0^t (\bar{\Lambda}_1 \lambda_s)_i ds, \quad t \geq 0, \quad i = 1, \dots, d,$$

are local martingales. Thus, for every $i = 1, \dots, d$, the i -th component N_i of N is a one-dimensional marked point process with intensity $(\bar{\Lambda}_1 \lambda)_i$ and marks $Y_{i,k} \sim \theta_i$, $k \in \mathbb{N}$.

We define the process

$$\bar{N} = N + \int_0^\cdot \bar{\Lambda}_0 \lambda_s ds$$

and the measures

$$\tilde{\theta}_i(d\xi) = \sum_{j=1}^d \bar{\Lambda}_1^{ji} \bar{\theta}_j(d\xi), \quad i = 1, \dots, m; \quad (51)$$

in the sequel, for a function $g \in L^1(\tilde{\theta}_i)$, $i = 1, \dots, m$, we write

$$\tilde{\theta}(g(\xi)) = \left[\int_{\mathbb{R}^d} g(\xi) \tilde{\theta}_1(d\xi) \quad \dots \quad \int_{\mathbb{R}^d} g(\xi) \tilde{\theta}_m(d\xi) \right]^\top \in \mathbb{C}^m.$$

By analogy with (29), we introduce the set

$$D_{\tilde{\theta}} = \left\{ u \in \mathbb{C}^d : (e^{u^\top \cdot} - 1 - u^\top \cdot) \in L^1(\tilde{\theta}_k), \text{ for every } k = 1, \dots, m \right\},$$

and we consider a subset $\bar{D} \subset D_{\tilde{\theta}}$ where the \mathbb{C} -valued functions $u \mapsto \int_{\mathbb{R}^d} (e^{u^\top \xi} - 1 - u^\top \xi) \tilde{\theta}_k(d\xi)$ are locally bounded, for every $k = 1, \dots, m$. In this framework, Theorem 6 is relevant because it provides a formula for the conditional Fourier-Laplace transform of N and \bar{N} . More precisely, the following corollary holds.

Corollary 7. *Assume that the kernel $K \in L_{\text{loc}}^2(\mathbb{R}_+; \mathbb{R}^{m \times d})$ is continuous in $(0, \infty)$. Given $u_1 \in \mathbb{C}^d$ and $f \in C(\mathbb{R}_+; \mathbb{C}^m)$, suppose that there exists a continuous global solution $\bar{\psi}_1: \mathbb{R}_+ \rightarrow \bar{D}$ of the deterministic Riccati-Volterra equation*

$$\bar{\psi}_1(t)^\top = u_1^\top + \int_0^t \bar{\mathcal{R}}_1(s, \bar{\psi}_1(s))^\top K(t-s) ds, \quad t \geq 0, \quad (52)$$

where $\bar{\mathcal{R}}_1: \mathbb{R}_+ \times \bar{D} \rightarrow \mathbb{C}^m$ is defined by

$$\bar{\mathcal{R}}_1(t, u) = f(t) + \bar{\Lambda}_0^\top u + \tilde{\theta} \left(e^{u^\top \xi} - 1 \right).$$

For every $T > 0$, consider the process $\bar{V}_1^T = (\bar{V}_{1,t}^T)_{t \in [0, T]}$ given by

$$\bar{V}_{1,t}^T = u_1^\top \bar{N}_t + \int_0^t f(T-s)^\top \lambda_s ds + \int_0^t \bar{\mathcal{R}}_1(T-s, \bar{\psi}_1(T-s))^\top \tilde{g}_t(s) ds, \quad t \in [0, T],$$

where $\tilde{g}_t(\cdot)$ is defined as in (37) with \bar{N} instead of Z . Then the process $\exp\{\bar{V}_1^T\}$ is a local martingale. In particular, if $\exp\{\bar{V}_1^T\}$ is a true martingale, then

$$\mathbb{E} \left[\exp \left\{ u_1^\top \bar{N}_T + (f^\top * \lambda)(T) \right\} \middle| \mathcal{F}_t \right] = \exp \left\{ \bar{V}_{1,t}^T \right\}, \quad \mathbb{P} - a.s., \quad t \in [0, T]. \quad (53)$$

Proof. We first observe that the differential characteristics of the special semimartingale \bar{N} with respect to the ‘‘truncation function’’ $h(\xi) = \xi$ are

$$\left(\left(\bar{\Lambda}_0 + \text{diag} \left[\int_{\mathbb{R}} \xi_1 \theta_1(d\xi_1), \dots, \int_{\mathbb{R}} \xi_d \theta_d(d\xi_d) \right] \bar{\Lambda}_1 \right) \lambda_t, \quad 0, \quad \sum_{i=1}^m \lambda_{i,t} \tilde{\theta}_i(d\xi) \right).$$

Indeed, by (51), the compensator $\nu^{\bar{N}}(dt, d\xi)$ of the jump–measure of \bar{N} , which coincides with $\nu^N(dt, d\xi)$ in (50), can be rewritten as

$$\nu^{\bar{N}}(dt, d\xi) = \sum_{i=1}^d (\bar{\Lambda}_1 \lambda_t)_i dt \otimes \tilde{\theta}_i(d\xi) = \sum_{i=1}^m \lambda_{i,t} dt \otimes \tilde{\theta}_i(d\xi).$$

From (49) we infer that λ is a càg and adapted, hence predictable process with locally (square–)integrable trajectories in \mathbb{R}^m . In addition, since

$$\sum_{t > T_{i,k}} K(t - T_{i,k}) = \sum_{t \geq T_{i,k}} K(t - T_{i,k}), \quad t \in \mathbb{R}_+ \setminus \{T_{i,k}, k \in \mathbb{N}\}, \quad \mathbb{P}\text{-a.s.}, \quad i = 1, \dots, d,$$

then λ satisfies the following stochastic Volterra equation of convolution type with jumps (cfr. (18)):

$$\lambda = \lambda_0 + (K * d\bar{N}), \quad \mathbb{P} \otimes dt - a.e.$$

Thus, the statements of the corollary are an immediate consequence of Theorem 6, obtained by replacing Z with \bar{N} and, respectively, B , A and $\nu(e^{\cdot \top \xi} - 1 - \cdot \top \xi)$ in the definition of \mathcal{R} in (30) with

$$\bar{\Lambda}_0 + \text{diag} \left[\int_{\mathbb{R}} \xi_1 \theta_1(d\xi_1), \dots, \int_{\mathbb{R}} \xi_d \theta_d(d\xi_d) \right] \bar{\Lambda}_1, \quad 0, \quad \tilde{\theta}(e^{\cdot \top \xi} - 1 - \cdot \top \xi).$$

This completes the proof. ■

Remark 6. *To the best of our knowledge, Corollary 7 contains an original expression for the conditional Fourier–Laplace transform of the Hawkes process N , see (53) with $f = -\bar{\Lambda}_0^\top u_1$. Comparing our findings to the existing literature, we note that in the one–dimensional case, results akin to Equation (53) can be found in [14]. To be precise, in [14, Theorem 3.1] the authors, albeit concentrating on other quantities, outline a procedure that can be adapted to derive an expression for $\mathbb{E}[\exp\{u_1 N_T\} | \mathcal{F}_t]$. Here, $N = (N_t)_{t \geq 0}$ is a Hawkes process with intensity $\lambda = \lambda_0 + K * d\tilde{N}$, where \tilde{N} represents the compensated process of N . Notably, this formula coincides with Equation (53) when we set $\bar{\Lambda}_0 = -\int_{\mathbb{R}} \xi \theta(d\xi)$, $\bar{\Lambda}_1 = 1$ and $f = u_1 \int_{\mathbb{R}} \xi \theta(d\xi)$, although the Riccati–Volterra equations in [14] have a different formulation from (53), see for instance [14, Equation (3.4)]. In particular, one retrieves a solution to (53) by convolving a solution of the corresponding equation in [14] with the kernel K . Thus, even within this restricted framework, Corollary 7 stands as an original contribution because it covers Hawkes processes with more general intensities, e.g., $\lambda = \lambda_0 + K * dN$.*

We now analyze the conditional distributions of the solution process X of (18). In particular, in Theorem 8 we obtain an expression for $\mathbb{E}[\exp\{u_2^\top X_T\}|\mathcal{F}_t]$, for some $u_2 \in \mathbb{C}^m$. However, since X is defined up to a $\mathbb{P} \otimes dt$ -null set, such a formula will be meaningful only for a.e. $T > 0$. This, in conjunction with the fact that we may be dealing with unbounded solutions of Riccati–Volterra equations in the new setting (cfr. [5, Appendix B]), makes the applicability of the arguments in Theorem 5 more delicate, as detailed in the proof of Theorem 8.

Theorem 8. *Assume that the kernel $K \in L_{\text{loc}}^2(\mathbb{R}_+; \mathbb{R}^{m \times d})$ is continuous in $(0, \infty)$. Given $u_2 \in \mathbb{C}^m$ and $f \in C(\mathbb{R}_+; \mathbb{C}^m)$, suppose that there exists a locally square-integrable function $\psi_2: \mathbb{R}_+ \rightarrow D_\nu$ such that the mappings $s \mapsto \int_{\mathbb{R}^d} (e^{\psi(s)^\top \xi} - 1 - \psi(s)^\top \xi) \nu_k(d\xi)$, $k = 0, \dots, m$, are locally integrable on \mathbb{R}_+ and that the following deterministic Riccati–Volterra equation is satisfied:*

$$\psi_2(t)^\top = u_2^\top K + \int_0^t \mathcal{R}(s, \psi_2(s))^\top K(t-s) ds, \quad \text{for a.e. } t \geq 0,$$

where D_ν is given in (29) and $\mathcal{R}: \mathbb{R}_+ \times D_\nu \rightarrow \mathbb{C}^m$ in (30). Then, for a.e. $T > 0$, defining $\tilde{V}^T = (\tilde{V}_t^T)_{t \in [0, T]}$ as in (39) with ψ_2 instead of ψ and $\tilde{V}_2^T = (\tilde{V}_{2,t}^T)_{t \in [0, T]}$ by

$$\tilde{V}_{2,t}^T = \tilde{V}_t^T + u_2^\top \int_0^t K(T-s) dZ_s, \quad t \in [0, T], \quad (54)$$

the process $\exp\{\tilde{V}_2^T\}$ is a local martingale. In particular, if $\exp\{\tilde{V}_2^T\}$ is a true martingale, then

$$\mathbb{E} \left[\exp \{ u_2^\top X_T + (f * X)(T) \} \middle| \mathcal{F}_t \right] = \exp \left\{ \tilde{V}_{2,t}^T \right\}, \quad \mathbb{P} - \text{a.s.}, \quad t \in [0, T]. \quad (55)$$

Proof. In this proof, we highlight the main changes that need to be implemented into the arguments employed in Theorem 5.

Notice that, contrary to before, we work with a solution ψ_2 of (8) which is supposed to be only locally square-integrable, with potential explosions precluding the boundedness on compact sets of $\mathcal{R}(\cdot, \psi_2(\cdot))$ and $\int_{\mathbb{R}^d} (e^{\psi_2(\cdot)^\top \xi} - 1 - \psi_2(\cdot)^\top \xi) \nu_k(d\xi)$, $k = 0, \dots, m$. As a consequence, recalling also the construction of the stochastic convolution in Section 2, we can define the process $V_2^T = (V_{2,t}^T)_{t \in [0, T]}$ as in (33)–(35) with ψ_2 instead of ψ only for a.e. $T > 0$.

We now discuss the good definition of the process $\tilde{V}^T = (\tilde{V}_t^T)_{t \in [0, T]}$ in (39) (with ψ replaced by ψ_2) for a.e. $T > 0$. By (38), the conditional Jensen’s inequality, the tower property and the change of variables $r' = r + t$, we observe that, for a.e. $s > t$,

$$\begin{aligned} 1_{\{s>t\}} \mathbb{E} [|g_t(s)|] &\leq c \mathbb{E} \left[|X_s| + 1_{\{s>t\}} \int_0^{s-t} |K(s-t-r)| |b(X_{t+r})| dr \right] \\ &\leq c \left(\mathbb{E} [|X_s|] + \int_0^s |K(s-r')| \mathbb{E} [|b(X_{r'})|] dr' \right) \leq c \left(\mathbb{E} [|X_s|] + \|K\|_{L^1([0, s]; \mathbb{R}^{m \times d})} + (|K| * \mathbb{E}[|X_\cdot|])(s) \right), \end{aligned} \quad (56)$$

for some constant $c = c(d, m, b_0, \dots, b_m) > 0$ allowed to change from line to line. This estimate enables us to show that $\int_t^T \mathcal{R}(T-s, \psi_2(T-s))^\top g_t(s) ds$, which appears in the definition of \tilde{V}_t^T in (39), exists \mathbb{P} -a.s. in \mathbb{C} , for every $t \in [0, T]$, for a.e. $T > 0$. Indeed, recalling Lemma 1, for every $\bar{T} > 0$ we have

$$\begin{aligned} &\int_0^{\bar{T}} \left(\int_0^T |\mathcal{R}(T-s, \psi_2(T-s))| (\mathbb{E} [|X_s|] + (|K| * \mathbb{E}[|X_\cdot|])(s)) ds \right) dT \\ &\leq \|\mathcal{R}(\cdot, \psi_2(\cdot))\|_{L^1([0, \bar{T}]; \mathbb{C}^m)} \mathbb{E} \left[\|X\|_{L^1([0, \bar{T}]; \mathbb{R}^m)} \right] \left(1 + \|K\|_{L^1([0, \bar{T}]; \mathbb{R}^{m \times d})} \right) < \infty, \end{aligned}$$

hence

$$\int_0^T |\mathcal{R}(T-s, \psi_2(T-s))| (\mathbb{E} [|X_s|] + (|K| * \mathbb{E}[|X_\cdot|])(s)) ds < \infty, \quad \text{for a.e. } T > 0.$$

Combining this equation with (56) we conclude that, for a.e. $T > 0$, for every $t \in [0, T]$,

$$\int_0^T 1_{\{s>t\}} |\mathcal{R}(T-s, \psi_2(T-s))| \mathbb{E}[|g_t(s)|] ds < \infty. \quad (57)$$

Thus, \tilde{V}^T is well defined for a.e. $T > 0$. Moreover, as $K \in L_{\text{loc}}^2(\mathbb{R}_+; \mathbb{R}^{m \times d})$, the stochastic convolution $(K * dZ)_T$ is well defined for a.e. $T > 0$, see Section 2. As a result, the process \tilde{V}_2^T in (54) is well defined, for a.e. $T > 0$.

Since, by the same computations as in the proof of Theorem 5, $\exp\{V_2^T\}$ is a local martingale, the final step required to conclude the current proof is to establish the equality $V_2^T = \tilde{V}_2^T$. To do this, since $\mathcal{R}(\cdot, \psi_2(\cdot))$ is neither locally bounded nor locally square-integrable anymore, we cannot apply the stochastic Fubini theorem as in (44). Indeed, the conditions in (45)-(46) are no longer satisfied. To overcome this problem, we employ an approximation argument.

For every $n \in \mathbb{N}$, define the truncation function $h_n(z) = z 1_{\{|z| \leq n\}}$, $z \in \mathbb{C}^m$. Recalling the process $\bar{g}_t(\cdot)$ in (43), the stochastic Fubini theorem, whose application is possible thanks to computations similar to (45)-(46), yields, \mathbb{P} -a.s.,

$$\int_0^T h_n(\mathcal{R}(T-s, \psi_2(T-s)))^\top (\bar{g}_t - g_0)(s) ds = \int_0^t \left[\int_r^T h_n(\mathcal{R}(T-s, \psi_2(T-s)))^\top K(s-r) ds \right] dZ_r. \quad (58)$$

Since, for every $z \in \mathbb{C}^m$, $h_n(z) \rightarrow z$ as $n \rightarrow \infty$, and, by (57), $\int_0^T |\mathcal{R}(T-s, \psi_2(T-s))| |\bar{g}_t - g_0|(s) ds < \infty$, \mathbb{P} -a.s., for every $t \in [0, T]$, for a.e. $T > 0$, by the dominated convergence theorem we infer that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^T h_n(\mathcal{R}(T-s, \psi_2(T-s)))^\top (\bar{g}_t - g_0)(s) ds \\ = \int_0^T \mathcal{R}(T-s, \psi_2(T-s))^\top (\bar{g}_t - g_0)(s) ds, \quad \mathbb{P} - \text{a.s.} \end{aligned} \quad (59)$$

As for the right side of (58), we observe that, again by dominated convergence, for a.e. $T > 0$,

$$\lim_{n \rightarrow \infty} \int_r^T h_n(\mathcal{R}(T-s, \psi_2(T-s)))^\top K(s-r) ds = \int_r^T \mathcal{R}(T-s, \psi_2(T-s))^\top K(s-r) ds, \quad \text{for a.e. } r \in [0, T].$$

Moreover, noticing that the map $|\mathcal{R}(\cdot, \psi_2(\cdot))| * |K| \in L_{\text{loc}}^2(\mathbb{R}_+)$, for $i = 1, 2$ we have

$$\begin{aligned} \mathbb{E} \left[\int_0^t \left(\int_r^T |\mathcal{R}(T-s, \psi_2(T-s))| |K(s-r)| ds \right)^i |X_r| dr \right] \\ = \int_0^t \left(\int_0^{T-r} |\mathcal{R}(T-r-s, \psi_2(T-r-s))| |K(s)| ds \right)^i \mathbb{E}[|X_r|] dr \\ \leq \int_0^T ((|\mathcal{R}(\cdot, \psi_2(\cdot))| * |K|)(T-r))^i \mathbb{E}[|X_r|] dr < \infty, \quad \text{for every } t \in [0, T], \end{aligned}$$

which holds for a.e. $T > 0$. Thus, we can apply the dominated convergence theorem for stochastic integrals (see, for instance, [22, Theorem 32, Chapter IV]) to claim that, for a.e. $T > 0$ and for every $t \in [0, T]$,

$$\begin{aligned} \mathbb{P} - \lim_{n \rightarrow \infty} \int_0^t \left[\int_r^T h_n(\mathcal{R}(T-s, \psi_2(T-s)))^\top K(s-r) ds \right] dZ_r \\ = \int_0^t \left[\int_r^T \mathcal{R}(T-s, \psi_2(T-s))^\top K(s-r) ds \right] dZ_r, \end{aligned} \quad (60)$$

where \mathbb{P} -lim denotes the limit in probability. Combining (59)-(60) in (58), for a.e. $T > 0$ and $t \in [0, T]$ we obtain

$$\begin{aligned} \int_0^T \mathcal{R}(T-s, \psi_2(T-s))^\top (\bar{g}_t - g_0)(s) ds &= \int_0^t \left[\int_r^T \mathcal{R}(T-s, \psi_2(T-s))^\top K(s-r) ds \right] dZ_r \\ &= \int_0^t \left(\psi_2(T-r)^\top - u_2^\top K(T-r) \right) dZ_r, \quad \mathbb{P} - \text{a.s.} \end{aligned}$$

Therefore, an analogue of the equality (44) holds. Proceeding as in (47), this allows to show that V_2^T and \tilde{V}_2^T are versions of each other, which completes the proof. \blacksquare

Remark 7. When $m = d = 1$, $u_2 \in \mathbb{R}_-$, $\phi \equiv 0$, $f \equiv 0$ and g_0 is the Laplace transform of a finite signed measure on \mathbb{R}_+ , Equation (55) reduces to the expression in [9, Theorem 5.7(v)] for the (conditional) Laplace transform of the marginal distributions of X . To deduce such a result, however, in [9] the authors employ an abstract approach based on infinite dimensional Markovian lifts of stochastic affine Volterra processes, which is completely different from our procedure inspired by [5].

4. An expression for V^T affine in the past trajectory of X

In this section we consider $m = d$ and aim to find an alternative formula for V^T (see (33)-(34)) which is affine in the past trajectory of X . This new expression can be used to prove the martingale property of the complex-valued process $\exp\{V^T\}$ in particular cases (see Section 5). Similar formulas might also be obtained for the processes introduced in Subsection 3.1 to study the marginal distributions of X and Z , see Theorems 6-8.

Due to the lack of regularity of the trajectories of both X and the stochastic convolution in dZ , we are going to require mild, additional conditions on the kernel K , in particular on the shifted kernels $\Delta_h K$ for $h > 0$. We start with a preliminary result providing an alternative expression for the adjusted forward process $g_t(\cdot)$.

Lemma 9. Assume that $K \in L_{\text{loc}}^2(\mathbb{R}_+; \mathbb{R}^{d \times d})$ is continuous on $(0, \infty)$ and that it admits a resolvent of the first kind L with no point masses in $(0, \infty)$. In addition, suppose that for every $h > 0$ the shifted kernel $\Delta_h K$ is differentiable, with derivative $(\Delta_h K)' \in C(\mathbb{R}_+; \mathbb{R}^{d \times d})$. Then, for every $T > 0$, for every $t \in [0, T]$

$$g_t(T) = g_0(T) + K(T-t)Z_t + \left(((\Delta_{T-t}K)' * L) * (X - g_0) \right)(t), \quad \mathbb{P} - \text{a.s.} \quad (61)$$

Proof. Fix $h > 0$. We first show that the stochastic convolution $\Delta_h K * dZ$ has a càdlàg version. Indeed, for every $t \in \mathbb{R}_+$, $(\Delta_h K * dZ)_t = \int_0^t f_{t,h}(s) dZ_s$, \mathbb{P} -a.s., with $f_{t,h}(s) = K(t+h-s)$, $s \in [0, t]$. Integration by parts yields

$$\begin{aligned} \int_0^t f_{t,h}(s) dZ_s &= f_{t,h}(t)Z_t - f_{t,h}(0)Z_0 - \int_0^t (f_{t,h})'(s)Z_{s-} ds \\ &= K(h)Z_t + \int_0^t (\Delta_h K)'(t-s)Z_s ds, \quad \mathbb{P} - \text{a.s.}, \end{aligned}$$

where we also note that $Z_{t-} = Z_t$ for a.e. $t > 0$, \mathbb{P} -a.s. Since the rightmost side of the previous equality is a càdlàg process we obtain the desired claim. Hence in what follows we consider $\Delta_h K * dZ$ to be right-continuous. In particular, the process $(\Delta_h K - K(h)) * dZ$ is continuous.

Thanks to the assumptions on the kernel, we apply [15, Corollary 7.3, Chapter 3] to claim that the function $(\Delta_h K - K(h)) * L$ is locally absolutely continuous in \mathbb{R}_+ , with

$$((\Delta_h K - K(h)) * L)' = (\Delta_h K)' * L, \quad \text{a.e. in } \mathbb{R}_+.$$

In particular, the function $(\Delta_h K)' * L \in C(\mathbb{R}_+; \mathbb{R}^{d \times d})$ by [15, Corollary 6.2 (iii), Chapter 3], the absence of point masses of L in $(0, \infty)$ and the continuity of $(\Delta_h K)'$. Therefore we invoke Proposition 4 to obtain

$$\begin{aligned} ((\Delta_h K - K(h)) * dZ)_t &= ((\Delta_h K - K(h)) * L)(0)(X - g_0)(t) + (((\Delta_h K)' * L) * (X - g_0))(t) \\ &= (((\Delta_h K)' * L) * (X - g_0))(t), \quad \text{for a.e. } t \in \mathbb{R}_+, \mathbb{P}\text{-a.s.} \end{aligned}$$

Note that the last equality involves continuous processes, so it is indeed true for every $t \geq 0$ up to a \mathbb{P} -null set. Thus,

$$(\Delta_h K * dZ)_t = K(h) Z_t + (((\Delta_h K)' * L) * (X - g_0))(t), \quad t \geq 0, \mathbb{P}\text{-a.s.} \quad (62)$$

At this point, take $t < T$ and recall that, by (37),

$$\begin{aligned} g_t(T) &= g_0(T) + \int_0^t K(T-s) dZ_s = g_0(T) + \int_0^t (\Delta_{T-t} K)(t-s) dZ_s \\ &= g_0(T) + (\Delta_{T-t} K * dZ)_t, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

It suffices to take $h = T - t$ in (62) to deduce that

$$(\Delta_{T-t} K * dZ)_t = K(T-t) Z_t + (((\Delta_{T-t} K)' * L) * (X - g_0))(t), \quad \mathbb{P}\text{-a.s.}$$

Hence, combining the two previous equations, we conclude

$$g_t(T) = g_0(T) + K(T-t) Z_t + (((\Delta_{T-t} K)' * L) * (X - g_0))(t), \quad \mathbb{P}\text{-a.s.},$$

completing the proof. ■

Fix a generic $T > 0$. By Equation (61) we can write, for every $t \in [0, T)$,

$$g_t(s) = g_0(s) + K(s-t) Z_t + (((\Delta_{s-t} K)' * L) * (X - g_0))(t), \quad \mathbb{P}\text{-a.s.}, s \in (t, T). \quad (63)$$

Intuitively speaking, we want to plug this expression in (39), so that we obtain an alternative formulation for V_t^T which is an affine function on the past trajectory $\{X_s, s \leq t\}$. This is done in the next theorem, which extends [5, Theorem 4.5] under further conditions on the kernel K . These additional assumptions hold for instance in the one-dimensional case if K is completely monotone (recall that a function f is called completely monotone on $(0, \infty)$ if it is infinitely differentiable there with $(-1)^k f^{(k)}(t) \geq 0$ for all $t > 0$ and $k = 0, 1, \dots$).

Theorem 10. *Assume that $K \in L_{\text{loc}}^2(\mathbb{R}_+; \mathbb{R}^{d \times d})$ is continuous on $(0, \infty)$ and that it admits a resolvent of the first kind L with no point masses in $(0, \infty)$. In addition, suppose that for every $h > 0$ the shifted kernel $\Delta_h K$ is differentiable, with $(\Delta_h K)'$ continuous on \mathbb{R}_+ . Under Hypothesis 1, if the total variation bound*

$$\sup_{h \in (0, \bar{T}]} \|\Delta_h K * L\|_{TV([0, \bar{T}])} < \infty, \quad \text{for all } \bar{T} > 0, \quad (64)$$

holds, then for every $h > 0$ the \mathbb{C}^d -valued function

$$\pi_h(r) = \left(\mathcal{R}(\cdot, \psi(\cdot))^\top * ((\Delta_h K)' * L)(r) \right) (h)^\top \quad (65)$$

is well-defined for a.e. $r \in \mathbb{R}_+$ and belongs to $L_{\text{loc}}^1(\mathbb{R}_+; \mathbb{C}^d)$. Moreover, \mathbb{P} -a.s., for a.e. $t \in (0, T)$,

$$\begin{aligned} V_t^T &= \phi(T-t) + \int_0^t f(T-s)^\top X_s ds + \int_0^{T-t} \mathcal{R}(s, \psi(s))^\top g_0(T-s) ds \\ &\quad + \psi(T-t)^\top Z_t + (\pi_{T-t}^\top * (X - g_0))(t), \quad (66) \end{aligned}$$

where ϕ is defined in (32).

Proof. Fix $h > 0$; expanding the notation in (65) for π_h we have

$$\pi_h(r)^\top = \int_0^h \mathcal{R}(s, \psi(s))^\top \left[\int_0^r (\Delta_{h-s}K)'(r-u) L(du) \right] ds.$$

In order to see that it is well-defined a.e. on \mathbb{R}_+ and belongs to $L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{C}^d)$, first note that for every positive s , the continuity of $(\Delta_s K)'$ and the absence of point masses for L in $(0, \infty)$ allow to apply [15, Corollary 6.2 (iii), Chapter 3], which ensures the continuity on \mathbb{R}_+ of $(\Delta_s K)' * L$. As a consequence, we can define the \mathbb{C}^d -valued measurable function

$$\left[\mathcal{R}(s, \psi(s))^\top ((\Delta_{h-s}K)' * L)(r) \right]^\top, \quad (s, r) \in (0, h) \times \mathbb{R}_+.$$

Recalling the previous proof, we see that $(\Delta_h K)' * L$ is, almost everywhere, the derivative of the locally absolutely continuous function $(\Delta_h K - K(h)) * L$. The boundedness of $\mathcal{R}(\cdot, \psi(\cdot))$ on $[0, h]$ by a constant $C_h > 0$ (see Hypothesis 1 and the subsequent comment) coupled with Condition (64), Tonelli's theorem and [15, Theorem 6.1 (v), Chapter 3] yields, for a generic $\bar{T} > h$,

$$\begin{aligned} & \int_0^{\bar{T}} \left[\int_0^h |\mathcal{R}(s, \psi(s))| |((\Delta_{h-s}K)' * L)(r)| ds \right] dr = \int_0^h |\mathcal{R}(s, \psi(s))| \left[\int_0^{\bar{T}} |((\Delta_{h-s}K)' * L)(r)| dr \right] ds \\ & \leq d^2 \int_0^h |\mathcal{R}(s, \psi(s))| \left[\|(\Delta_{h-s}K - K(h-s)) * L\|_{\text{TV}([0, \bar{T}])} \right] ds \\ & \leq d^2 \left[\int_0^h |\mathcal{R}(s, \psi(s))| \|\Delta_{h-s}K * L\|_{\text{TV}([0, \bar{T}])} ds + |L|([0, \bar{T}]) \int_0^h |\mathcal{R}(s, \psi(s))| |K(h-s)| ds \right] \\ & \leq d^2 \left[\sup_{s \in (0, \bar{T})} \|\Delta_s K * L\|_{\text{TV}([0, \bar{T}])} C_h h + |L|([0, \bar{T}]) \int_0^h |\mathcal{R}(s, \psi(s))| |K(h-s)| ds \right] < \infty. \end{aligned}$$

Hence the conclusion on π_h follows. Furthermore, by Lebesgue's fundamental theorem of calculus, the \mathbb{C}^d -valued function $\Pi_h(r) = \int_0^r \pi_h(u) du$, $r \in \mathbb{R}_+$, is locally absolutely continuous on \mathbb{R}_+ , with $\Pi'_h = \pi_h$ a.e. Using Fubini's theorem we can obtain the following explicit expression for such Π_h

$$\Pi_h(r)^\top = \int_0^h \mathcal{R}(s, \psi(s))^\top ((\Delta_{h-s}K - K(h-s)) * L)(r) ds, \quad r \in \mathbb{R}_+. \quad (67)$$

At this point we observe that for every function $g \in L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^d)$ we have, reasoning as before and using the boundedness of $\mathcal{R}(\cdot, \psi(\cdot))$ on $[0, T]$ by a positive constant C_T ,

$$\begin{aligned} & \int_0^T \left[\int_0^t |g(t-u)| \left(\int_0^{T-t} |\mathcal{R}(s, \psi(s))| |((\Delta_{T-t-s}K)' * L)(u)| ds \right) du \right] dt \\ & = \int_0^T \left[\int_0^t |g(t-u)| \left(\int_0^{T-t} |\mathcal{R}(T-t-s, \psi(T-t-s))| |((\Delta_s K)' * L)(u)| ds \right) du \right] dt \\ & = \int_0^T \left[\int_0^{T-t} |\mathcal{R}(T-t-s, \psi(T-t-s))| \left(\int_0^t |g(t-u)| |((\Delta_s K)' * L)(u)| du \right) ds \right] dt \\ & = \int_0^T \left[\int_0^{T-s} |\mathcal{R}(T-s-t, \psi(T-s-t))| \left(\int_0^t |g(t-u)| |((\Delta_s K)' * L)(u)| du \right) dt \right] ds \\ & \leq C_T d^2 \|g\|_{L^1([0, T]; \mathbb{R}^d)} \int_0^T \|(\Delta_s K - K(s)) * L\|_{\text{TV}([0, T])} ds \\ & \leq C_T d^2 \|g\|_{L^1([0, T]; \mathbb{R}^d)} \left[T \sup_{s \in (0, T)} \|\Delta_s K * L\|_{\text{TV}([0, T])} + |L|([0, T]) \int_0^T |K(s)| ds \right] < \infty, \end{aligned} \quad (68)$$

where we apply Tonelli's theorem, together with [15, Theorem 2.2 (i), Chapter 2] and a change of variables. Consequently, for almost every $t \in (0, T)$ we can apply Fubini's theorem to obtain

$$\int_0^t \pi_{T-t}(u)^\top g(t-u) du = \int_0^{T-t} \mathcal{R}(s, \psi(s))^\top \left[\int_0^t ((\Delta_{T-t-s}K)' * L)(u) g(t-u) du \right] ds. \quad (69)$$

Computations analogous to those in (68) (with g [resp., $|\mathcal{R}(\cdot, \psi(\cdot))|$] substituted by $X - g_0$ [resp., 1]) enable us to conclude, by Fubini's theorem and Equation (19), that there is a jointly measurable modification of the process $((\Delta_{\cdot-t}K)' * L) * (X - g_0)(t)$ on $\Omega \times (t, T)$ for a.e. $t \in (0, T)$. Therefore we interpret (63) pathwise, namely the equality holds almost everywhere in (t, T) up to a \mathbb{P} -null set.

Now we focus on \tilde{V}_t^T . The previous analysis together with (39) and a suitable change of variables yields

$$\begin{aligned} \tilde{V}_t^T &= \phi(T-t) + \int_0^t f(T-s)^\top X_s ds + \int_0^{T-t} \mathcal{R}(s, \psi(s))^\top g_t(T-s) ds \\ &= \left\{ \phi(T-t) + \int_0^t f(T-s)^\top X_s ds + \int_0^{T-t} \mathcal{R}(s, \psi(s))^\top g_0(T-s) ds \right\} \\ &\quad + \left\{ \left(\int_0^{T-t} \mathcal{R}(s, \psi(s))^\top K(T-t-s) ds \right) Z_t \right\} \\ &\quad + \left\{ \int_0^{T-t} \mathcal{R}(s, \psi(s))^\top ((\Delta_{T-t-s}K)' * L) * (X - g_0)(t) ds \right\} \\ &= \mathbf{I}_t + \mathbf{II}_t + \mathbf{III}_t, \quad \mathbb{P}\text{-a.s.}, t \in (0, T). \end{aligned}$$

The idea is to analyze separately the addends that we have singled out in the previous computations. Note that \mathbf{III}_t is finite because $\tilde{V}_t^T, \mathbf{I}_t, \mathbf{II}_t$ are so, and that we can consider a jointly measurable modification of this process in $\Omega \times (0, T)$, again by Fubini's theorem and Equation (19) (see (68)). Taking into account (40) we have

$$V_t^T = \mathbf{I}_t + \mathbf{II}_t + \mathbf{III}_t, \quad \text{for a.e. } t \in (0, T), \mathbb{P}\text{-a.s.}, \quad (70)$$

where the equality can be understood pathwise as it involves jointly measurable processes.

Regarding \mathbf{II}_t , since ψ solves the Riccati-Volterra equation in (31) we have

$$\mathbf{II}_t = \psi(T-t)^\top Z_t.$$

As for \mathbf{III}_t , by (69) we have

$$\mathbf{III}_t = \int_0^t \pi_{T-t}(u)^\top (X - g_0)(t-u) du = (\pi_{T-t}^\top * (X - g_0))(t), \quad \text{for a.e. } t \in (0, T), \mathbb{P}\text{-a.s.}$$

Substituting the two previous equations in (70) we conclude

$$\begin{aligned} V_t^T &= \phi(T-t) + \int_0^t f(T-s)^\top X_s ds + \int_0^{T-t} \mathcal{R}(s, \psi(s))^\top g_0(T-s) ds \\ &\quad + \psi(T-t)^\top Z_t + (\pi_{T-t}^\top * (X - g_0))(t), \end{aligned}$$

for almost every $t \in (0, T)$, \mathbb{P} -a.s. The proof is now complete. \blacksquare

If the resolvent of the first kind L is the sum of a locally integrable function and a point mass in 0, then recalling (18) we can apply Lemma 3 (see also the final comment in Remark 3) and argue as in (28) to see that $Z_t = (L * (X - g_0))(t)$, for a.e. $t > 0$, \mathbb{P} -a.s. In addition, for every $h > 0$ we define the \mathbb{C}^d -valued function

$$\tilde{\Pi}_h(r)^\top = \Pi_h(r)^\top + \psi(h)^\top L(\{0\}) + \psi(h)^\top L((0, r]) = \int_0^h \mathcal{R}(s, \psi(s))^\top (\Delta_{h-s}K * L)(r) ds, \quad r \in \mathbb{R}_+, \quad (71)$$

where the second equality is due to (67). Note that $\tilde{\Pi}_h$ is locally absolutely continuous on \mathbb{R}_+ , and that

$$\begin{aligned} (\pi_{T-t}^\top * (X - g_0))(t) &= \left(d\Pi_{T-t}^\top * (X - g_0) \right)(t) \\ &= \left(d\tilde{\Pi}_{T-t}^\top * (X - g_0) \right)(t) - \psi(T-t)^\top (L * (X - g_0))(t) + \psi(T-t)^\top L(\{0\})(X - g_0)(t), \end{aligned}$$

which holds for a.e. $t \in (0, T)$, \mathbb{P} -a.s. Substituting in (66) we immediately deduce the following result.

Corollary 11. *Under the same hypotheses of Theorem 10, if the resolvent of the first kind L is the sum of a locally integrable function and a point mass in 0, then \mathbb{P} -a.s., for a.e. $t \in (0, T)$*

$$\begin{aligned} V_t^T &= \phi(T-t) + \int_0^t f(T-s)^\top X_s ds + \int_0^{T-t} \mathcal{R}(s, \psi(s))^\top g_0(T-s) ds \\ &\quad + \psi(T-t)^\top L(\{0\})(X - g_0)(t) - \left(d\tilde{\Pi}_{T-t}^\top * g_0 \right)(t) + \left(d\tilde{\Pi}_{T-t}^\top * X \right)(t). \end{aligned} \quad (72)$$

5. The 1-dimensional Volterra square root diffusion process with jumps

In this section we discuss a one-dimensional example ($m = d = 1$) where not only are we able to infer the assumptions made in the previous arguments, such as the existence of solutions to the stochastic Volterra equation (14) and the Riccati-Volterra equation (31) (i.e., Hypothesis 1), but also we can prove the martingale property of the process $\exp\{V^T\}$. In order to develop the theory we need to require more properties for the kernel K . In particular, we consider a hypothesis which is standard in the theory of stochastic Volterra equations, that is (see [1, Condition (2.10)], and also [5, Condition (3.4)] and [4, Assumption B.2])

Condition 1. *The kernel K is nonnegative, nonincreasing, not identically zero and continuously differentiable on $(0, \infty)$, and its resolvent of the first kind L is nonnegative and nonincreasing, i.e., $s \mapsto L([s, s+t])$ is nonincreasing for every $t \geq 0$.*

Notice that, under Condition 1, the map $s \mapsto L(\{s\})$ is nonincreasing, where L is the resolvent of the first kind of K . Combining this fact with Lebesgue's decomposition theorem, which ensures that L has at most a countable number of point masses in $(0, \infty)$, we deduce that L has no point masses in $(0, \infty)$.

In the sequel, we suppose that K and the shifted kernels $\Delta_{1/n}K$, $n \in \mathbb{N}$, satisfy Condition 1. This is the case, for example, when K is a completely monotone function not identically equal to 0.

We focus on the following stochastic Volterra equation of convolution type:

$$X = g_0 + (K * dZ), \quad \mathbb{P} \otimes dt - \text{a.e.}, \quad (73)$$

where Z is a real-valued semimartingale with differential characteristics (with respect to $h(\xi) = \xi$, $\xi \in \mathbb{R}$) given by $(b(X_t), a(X_t), \eta(X_t, d\xi))$, $t \geq 0$, with

$$b(x) = bx, \quad a(x) = cx, \quad \eta(x, d\xi) = x\nu(d\xi), \quad x \geq 0.$$

Here $b \in \mathbb{R}$, $c \geq 0$ and ν is a nonnegative measure on \mathbb{R}_+ such that $\int_{\mathbb{R}_+} |\xi|^2 \nu(d\xi) < \infty$. The function $g_0: \mathbb{R}_+ \rightarrow \mathbb{R}$ is an admissible input curve in either one of the following two forms

- i. g_0 is continuous and non-decreasing, with $g_0(0) \geq 0$;
- ii. $g_0(t) = x_0 + \int_0^t K(t-s)\theta(s)ds$, $t \geq 0$, where $x_0 \geq 0$ and $\theta: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is locally bounded.

Notice that (73) describes a 1-dimensional Volterra square root diffusion with jumps. In this framework, we can invoke [1, Theorem 2.13] to claim the existence of a weak, predictable solution $X = (X_t)_{t \geq 0}$ of (73) with trajectories in $L_{\text{loc}}^1(\mathbb{R}_+)$ such that $X \geq 0$, $\mathbb{P} \otimes dt$ -a.e. Actually, if $g_0 \in L_{\text{loc}}^2(\mathbb{R}_+)$, the paths of this solution X are in $L_{\text{loc}}^2(\mathbb{R}_+)$, \mathbb{P} -a.s., as the next result shows.

Lemma 12. *Suppose that $g_0 \in L^2_{\text{loc}}(\mathbb{R}_+)$ and let X be a solution of (73) with trajectories in $L^1_{\text{loc}}(\mathbb{R}_+)$ such that $X \geq 0$, $\mathbb{P} \otimes dt$ -a.e. Then, for every $T > 0$, $\mathbb{E}[(\int_0^T |X_t|^2 dt)^{1/2}] < \infty$.*

Proof. The convolution equation (73) enables us to write, \mathbb{P} -a.s.,

$$|X_t|^2 \leq 4 \left(|g_0(t)|^2 + |b|^2 |(K * X)(t)|^2 + |(K * dM^c)_t|^2 + |(K * dM^d)_t|^2 \right), \quad \text{for a.e. } t \geq 0.$$

Integrating over the interval $(0, T)$, $T > 0$, we have

$$\begin{aligned} \left(\int_0^T |X_t|^2 dt \right)^{\frac{1}{2}} &\leq 2 \left(2 + \|g_0\|_{L^2([0, T])} + |b| \|K * X\|_{L^2([0, T])} \right. \\ &\quad \left. + \int_0^T |(K * dM^c)_t|^2 dt + \int_0^T |(K * dM^d)_t|^2 dt \right), \quad \mathbb{P} - \text{a.s.}, \end{aligned}$$

where we also use that $\sqrt{x} \leq 1 + x$, $x \in \mathbb{R}_+$. By [15, Theorem 2.2 (i), Chapter 2] we have $\|K * X\|_{L^2([0, T])} \leq \|K\|_{L^2([0, T])} \|X\|_{L^1([0, T])}$, hence taking expectation in the previous inequality we obtain, using Tonelli's theorem,

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^T |X_t|^2 dt \right)^{\frac{1}{2}} \right] &\leq 2 \left(2 + \|g_0\|_{L^2([0, T])} + |b| \|K\|_{L^2([0, T])} \mathbb{E} \left[\|X\|_{L^1([0, T])} \right] \right. \\ &\quad \left. + \int_0^T \mathbb{E} \left[|(K * dM^c)_t|^2 \right] dt + \int_0^T \mathbb{E} \left[|(K * dM^d)_t|^2 \right] dt \right). \quad (74) \end{aligned}$$

Recall that $(K * dM^c)_t = \int_0^t K(t-s) dM_s^c$, \mathbb{P} -a.s. for a.e. $t \geq 0$; therefore we use the Burkholder–Davis–Gundy inequality and the Young's type inequality in [2, Lemma A.1] to write (always bearing in mind Tonelli's theorem)

$$\begin{aligned} \int_0^T \mathbb{E} \left[|(K * dM^c)_t|^2 \right] dt &\leq c \cdot c_1 \mathbb{E} \left[\int_0^T \left(\int_0^t |K(t-s)|^2 X_s ds \right) dt \right] \\ &\leq c \cdot c_1 \|K\|_{L^2([0, T])}^2 \mathbb{E} \left[\|X\|_{L^1([0, T])} \right], \quad \text{for some } c_1 > 0. \end{aligned}$$

Analogously, we invoke [21, Theorem 3.2] to assert

$$\int_0^T \mathbb{E} \left[|(K * dM^d)_t|^2 \right] dt \leq 2 \left(\int_{\mathbb{R}_+} |\xi|^2 \nu(d\xi) \right) c_2 \|K\|_{L^2([0, T])}^2 \mathbb{E} \left[\|X\|_{L^1([0, T])} \right], \quad \text{for some } c_2 > 0.$$

Now substituting the previous two bounds in (74) we see that the right side is finite by (19). This concludes the proof. \blacksquare

Fix $f \in C(\mathbb{R}_+; \mathbb{C}_-)$. Since $\nu(d\xi)$ is a nonnegative measure on \mathbb{R}_+ with finite second moment and, for every $z \in \mathbb{C}_-$,

$$\begin{aligned} |\operatorname{Re}(e^z - 1 - z)| &\leq |e^{\operatorname{Re} z} - 1 - \operatorname{Re} z| + e^{\operatorname{Re} z} (1 - \cos(\operatorname{Im} z)) \leq |z|^2, \\ |\operatorname{Im}(e^z - 1 - z)| &= |e^{\operatorname{Re} z} \sin(\operatorname{Im} z) - \operatorname{Im} z| \leq |\sin(\operatorname{Im} z) - \operatorname{Im} z| + |\operatorname{Im} z| (1 - e^{\operatorname{Re} z}) \leq 2|z|^2, \end{aligned}$$

we have $\mathbb{C}_- \subset D_\nu$, see (29). We then set $D = \mathbb{C}_-$ and define the map $\mathcal{R}: \mathbb{R}_+ \times D \rightarrow \mathbb{C}$ as in (30). Notice that, by the dominated convergence theorem, \mathcal{R} is continuous in its domain $\mathbb{R}_+ \times \mathbb{C}_-$, hence it is locally bounded. In addition to Equation (31), we consider the deterministic Riccati–Volterra equation

$$\bar{\psi}(t) = \int_0^t K(t-s) \bar{\mathcal{R}}(s, \bar{\psi}(s)) ds, \quad t \geq 0, \quad (75)$$

where $\bar{\mathcal{R}}: \mathbb{R}_+ \times \mathbb{C}_- \rightarrow \mathbb{C}$ is defined by

$$\bar{\mathcal{R}}(t, u) = \operatorname{Re} f(t) + bu + \frac{c}{2}u^2 + \int_{\mathbb{R}_+} (e^{u\xi} - 1 - u\xi) \nu(d\xi), \quad (t, u) \in \mathbb{R}_+ \times \mathbb{C}_- \quad (76)$$

and $\operatorname{Re} f: \mathbb{R}_+ \rightarrow \mathbb{R}_-$ denotes the real part of f . We observe that also $\bar{\mathcal{R}}$ is continuous in $\mathbb{R}_+ \times \mathbb{C}_-$. The next theorem shows the existence of global solutions to (31) and (75) (in particular, Hypothesis 1 is verified), as well as a comparison result between them which is crucial for the subsequent argument on the martingale property.

Theorem 13. *Let $f \in C(\mathbb{R}_+; \mathbb{C}_-)$ and assume Condition 1.*

- (i) *There exist a continuous global solution $\psi \in C(\mathbb{R}_+; \mathbb{C}_-)$ of (31) and a real-valued, continuous global solution $\bar{\psi} \in C(\mathbb{R}_+; \mathbb{R}_-)$ of (75).*
- (ii) *Given $\psi \in C(\mathbb{R}_+; \mathbb{C}_-)$ and $\bar{\psi} \in C(\mathbb{R}_+; \mathbb{R}_-)$ satisfying (31) and (75), respectively, the following inequality holds:*

$$\operatorname{Re} \psi(t) \leq \bar{\psi}(t), \quad t \geq 0. \quad (77)$$

Proof. The proof of (i) is in Appendix B.1, and the one of (ii) is in Appendix B.2. ■

In what follows, we take two continuous functions $\psi, \bar{\psi}$ as in Theorem 13 (i) and fix $T > 0$. We aim to prove the martingale property of the process $\exp\{V^T\}$, where V^T is given by (33)–(34). For this purpose, we define the process \bar{V}^T as in (33)–(34), substituting $\operatorname{Re} f$ [resp., $\bar{\psi}$] for f [resp., ψ]. Theorem 5 shows that $\bar{V}_t^T = \tilde{\bar{V}}_t^T$, \mathbb{P} -a.s., for every $t \in [0, T]$, where of course we define $\tilde{\bar{V}}^T$ as in (39) with the same substitution as before. It is known that $\exp\{\bar{V}^T\}$ is a true, real-valued martingale. This is due to [1, Lemma 6.1], which in turn is an interesting application of the Novikov-type condition in [20, Theorem IV.3]. The idea of the present section consists in using the expression (72) in order to prove the bound $|\exp\{V^T\}| \leq C \exp\{\bar{V}^T\}$ up to indistinguishability for some $C > 0$, so that we can conclude that $\exp\{V^T\}$ is a martingale, too.

Direct computations based on the Riccati–Volterra equation (31) yield, for every $h > 0$,

$$\Delta_h \psi(r) = (\Delta_h(\mathcal{R}(\cdot, \psi(\cdot))) * K)(r) + (\mathcal{R}(\cdot, \psi(\cdot)) * \Delta_r K)(h), \quad r \geq 0.$$

Focusing on the second addend on the right side, if we convolve it with L then

$$\begin{aligned} ((\mathcal{R}(\diamond, \psi(\diamond)) * \Delta_r K)(h) * L)(r) &= \int_0^r \left[\int_0^h \mathcal{R}(s, \psi(s)) (\Delta_{r-u} K)(h-s) ds \right] L(du) \\ &= \int_0^h \mathcal{R}(s, \psi(s)) ((\Delta_{h-s} K) * L)(r) ds, \quad r \geq 0, \end{aligned}$$

where the application of Fubini’s theorem is justified because K is nonnegative and nonincreasing and L is a nonnegative measure. Whence, since $\Delta_h(\psi * L)(r) = (\mathcal{R}(\cdot, \psi(\cdot)) * 1)(r+h)$, recalling (71) we can write

$$\begin{aligned} \tilde{\Pi}_h(r) &= (\Delta_h \psi * L)(r) - \Delta_h(\psi * L)(r) + \int_0^h \mathcal{R}(s, \psi(s)) ds \\ &= - \int_{(0, h]} \psi(h-s) L(r+ds) + \int_0^h \mathcal{R}(s, \psi(s)) ds, \quad r \geq 0, \end{aligned}$$

and in particular

$$\operatorname{Re}(\tilde{\Pi}_h(r)) = - \int_{(0, h]} \operatorname{Re}(\psi(h-s)) L(r+ds) + \int_0^h \operatorname{Re}(\mathcal{R}(s, \psi(s))) ds, \quad r \geq 0.$$

Repeating the same argument for $\bar{\psi}$ we also obtain

$$\tilde{\bar{\Pi}}_h(r) = - \int_{(0,h]} \bar{\psi}(h-s) L(r+ds) + \int_0^h \bar{\mathcal{R}}(s, \bar{\psi}(s)) ds, \quad r \geq 0.$$

Taking the difference between the two previous equations we infer, for every $r \geq 0$,

$$\begin{aligned} \tilde{\bar{\Pi}}_h(r) - \operatorname{Re}(\tilde{\Pi}_h(r)) &= - \int_{(0,h]} [\bar{\psi} - \operatorname{Re} \psi](h-s) L(r+ds) + \int_0^h [\bar{\mathcal{R}}(\cdot, \bar{\psi}(\cdot)) - \operatorname{Re}(\mathcal{R}(\cdot, \psi(\cdot)))](s) ds \\ &= (\bar{\psi} - \operatorname{Re} \psi)(h) L(\{r\}) - \int_{(0,h]} [\bar{\psi} - \operatorname{Re} \psi](h-s) L(r+ds) + \int_0^h [\bar{\mathcal{R}}(\cdot, \bar{\psi}(\cdot)) - \operatorname{Re}(\mathcal{R}(\cdot, \psi(\cdot)))](s) ds. \end{aligned} \quad (78)$$

Hence, we see that this function is increasing on the interval $(0, \infty)$ by (77) in Theorem 13 (ii) and Condition 1 (see also the subsequent comment). We are now in position to prove the next, important result.

Theorem 14. *Assume that the kernel $K \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{R})$ satisfies the requirements of Corollary 11 together with Condition 1. Then there exists a constant $C > 0$ such that*

$$|\exp\{V_t^T\}| \leq C \exp\{\bar{V}_t^T\}, \quad t \in [0, T], \mathbb{P} - a.s. \quad (79)$$

In particular, $(\exp\{V_t^T\})_{t \in [0, T]}$ is a complex-valued martingale.

Proof. First of all note that $|\exp\{V^T\}| = \exp\{\operatorname{Re}(V^T)\}$. For the reader's convenience, we write the expression for $\operatorname{Re}(V^T)$ provided by (72)

$$\begin{aligned} \operatorname{Re}(V_t^T) &= \int_0^t \operatorname{Re} f(T-s) X_s ds + \int_0^{T-t} \operatorname{Re}(\mathcal{R}(s, \psi(s))) g_0(T-s) ds + \operatorname{Re} \psi(T-t) L(\{0\})(X - g_0)(t) \\ &\quad - \left(d \left(\operatorname{Re}(\tilde{\Pi}_{T-t}) \right) * g_0 \right)(t) + \left(d \left(\operatorname{Re}(\tilde{\Pi}_{T-t}) \right) * X \right)(t), \quad \text{for a.e. } t \in (0, T), \mathbb{P} - a.s. \end{aligned}$$

The idea of the proof is simply to compare, term by term, the addends of this sum with the corresponding ones in the expansion of \bar{V}^T according to (72). We are going to consider a common set $\Omega_0 \subset \Omega$, with $\mathbb{P}(\Omega_0) = 1$, such that both the expressions for $\operatorname{Re}(V_t^T)$ and \bar{V}_t^T are valid on $(0, T) \setminus N^\omega$, being $N^\omega \subset (0, T)$ a dt -null set for every $\omega \in \Omega_0$.

Regarding the random terms, recall that $X \geq 0$, $\mathbb{P} \otimes dt$ -a.e. Therefore, without loss of generality, we can assume that for every $\omega \in \Omega_0$ and $t \in (0, T) \setminus N^\omega$ we have $X_t(\omega) \geq 0$. As a consequence (by (78))

$$\left(d \left(\tilde{\bar{\Pi}}_{T-t} - \operatorname{Re}(\tilde{\Pi}_{T-t}) \right) * X(\omega) \right)(t) \geq 0 \implies \left(d \left(\operatorname{Re}(\tilde{\Pi}_{T-t}) \right) * X(\omega) \right)(t) \leq \left(d \tilde{\bar{\Pi}}_{T-t} * X(\omega) \right)(t).$$

It is important to stress the fact that such an inequality can be stated because the measure L is absolutely continuous with respect to the Lebesgue measure on the interval $(0, \infty)$. Summing up,

$$\left(d \left(\operatorname{Re}(\tilde{\Pi}_{T-t}) \right) * X \right)(t) \leq \left(d \tilde{\bar{\Pi}}_{T-t} * X \right)(t), \quad t \in (0, T) \setminus N^\omega, \omega \in \Omega_0.$$

Moreover, since $L(\{0\}) \geq 0$, by (77) we immediately have

$$\operatorname{Re} \psi(T-t) L(\{0\}) X_t \leq \bar{\psi}(T-t) L(\{0\}) X_t, \quad t \in (0, T) \setminus N^\omega, \omega \in \Omega_0.$$

The other random addend $\int_0^t \operatorname{Re} f(T-s) X_s ds$ appears in both the expressions for $\operatorname{Re}(V_t^T)$ and \bar{V}_t^T , so it does not need to be discussed.

As for the deterministic terms, we observe that, by Hölder's inequality,

$$\left| \int_0^{T-t} g_0(T-s) (\operatorname{Re}(\mathcal{R}(s, \psi(s))) - \bar{\mathcal{R}}(s, \bar{\psi}(s))) ds \right| \leq \|g_0\|_{L^2([0, T])} \|\operatorname{Re}(\mathcal{R}(\cdot, \psi(\cdot))) - \bar{\mathcal{R}}(\cdot, \bar{\psi}(\cdot))\|_{L^2([0, T])},$$

for any $t \in (0, T)$. Hence, calling $C_1 = \|g_0\|_{L^2([0, T])} \|\operatorname{Re}(\mathcal{R}(\cdot, \psi)) - \bar{\mathcal{R}}(\cdot, \psi)\|_{L^2([0, T])}$, we have

$$\int_0^{T-t} g_0(T-s) \operatorname{Re}(\mathcal{R}(s, \psi(s))) \, ds \leq C_1 + \int_0^{T-t} g_0(T-s) \bar{\mathcal{R}}(s, \bar{\psi}(s)) \, ds, \quad t \in (0, T).$$

Furthermore, recalling the continuity of ψ , $\bar{\psi}$ and g_0 , we call $C_2 = \max_{t \in [0, T]} \{|\bar{\psi} - \operatorname{Re} \psi|(T-t) g_0(t)\}$, so that we have

$$-\operatorname{Re} \psi(T-t) L(\{0\}) g_0(t) \leq L(\{0\}) C_2 - \bar{\psi}(T-t) L(\{0\}) g_0(t), \quad t \in (0, T).$$

Finally, looking at (78) we compute

$$\begin{aligned} \left(d \left(\tilde{\Pi}_{T-t} - \operatorname{Re} \left(\tilde{\Pi}_{T-t} \right) \right) * 1 \right) (t) &\leq - \int_{[0, T-t]} [\bar{\psi} - \operatorname{Re} \psi] (T-t-s) [L(t+ds) - L(ds)] \\ &\leq 2 \max_{t \in [0, T]} |\bar{\psi}(t) - \operatorname{Re} \psi(t)| L([0, T]) = C_3, \quad t \in (0, T). \end{aligned}$$

Hence exploiting the continuity of the input curve we conclude that

$$\left| \left(d \left(\tilde{\Pi}_{T-t} - \operatorname{Re} \left(\tilde{\Pi}_{T-t} \right) \right) * g_0 \right) (t) \right| \leq C_3 \max_{t \in [0, T]} |g_0(t)|,$$

which in turn implies

$$- \left(d \left(\operatorname{Re} \left(\tilde{\Pi}_{T-t} \right) \right) * g_0 \right) (t) \leq C_3 \max_{t \in [0, T]} |g_0(t)| - \left(d \tilde{\Pi}_{T-t} * g_0 \right) (t), \quad t \in (0, T).$$

Combining all these results we deduce that

$$\operatorname{Re} (V_t^T(\omega)) \leq C_1 + L(\{0\}) C_2 + C_3 \max_{t \in [0, T]} |g_0(t)| + \bar{V}_t^T(\omega), \quad t \in (0, T) \setminus N^\omega, \omega \in \Omega_0. \quad (80)$$

Since N^ω is a null set, its complementary $(N^\omega)^c = (0, T) \setminus N^\omega$ is dense in $[0, T]$. Recalling the regularity for the trajectories of the processes $\operatorname{Re}(V^T)$ and \bar{V}^T , we can assume that for every $\omega \in \Omega_0$ both the functions $\operatorname{Re}(V^T(\omega))$ and $\bar{V}^T(\omega)$ are càdlàg in $[0, T]$ and left-continuous in T . Accordingly, we pass to the limit –from the right in $[0, T)$ and from the left in T^- – to deduce, from (80), that

$$\operatorname{Re} (V_t^T(\omega)) \leq C_1 + L(\{0\}) C_2 + C_3 \max_{t \in [0, T]} |g_0(t)| + \bar{V}_t^T(\omega), \quad t \in [0, T], \omega \in \Omega_0,$$

i.e., (79) holds choosing $C = \exp \{C_1 + L(\{0\}) C_2 + C_3 \max_{t \in [0, T]} |g_0(t)|\}$.

The second statement of the theorem immediately follows from [17, Lemma 1.4], as $(\exp \{\bar{V}^T\})_{t \in [0, T]}$ is a real-valued martingale. Thus, the proof is complete. \blacksquare

Combining Theorem 14 with Theorem 5 (see (41)) we deduce the following result about weak uniqueness for (73).

Corollary 15. *The weak solution X of (73) is unique in law in $L^2_{\text{loc}}(\mathbb{R}_+)$, that is: if $Y = (Y_t)_{t \geq 0}$ is another predictable process (defined on a possibly different stochastic basis) such that $Y \geq 0$, $\mathbb{P} \otimes dt$ -a.e., which satisfies (73), then the laws of X and Y on the spaces $L^2([0, T])$, $T > 0$, are the same.*

Proof. Fix $T > 0$ and consider another weak solution Y of (73). We assume that X and Y are defined on the same stochastic basis to keep notation simple. The paths of Y are in $L^2([0, T])$, \mathbb{P} -a.s., by Lemma 12. We want to show that

$$\mathbb{E} \left[\exp \left\{ i \int_0^T f(s) X_s \, ds \right\} \right] = \mathbb{E} \left[\exp \left\{ i \int_0^T f(s) Y_s \, ds \right\} \right], \quad f \in L^2([0, T]). \quad (81)$$

First, we verify the previous equation for $f \in C([0, T])$. Denoting by $\tilde{f}(s) = if(T - s)$, $s \in [0, T]$, by Theorem 5 and Theorem 14 we have

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ i \int_0^T f(s) X_s ds \right\} \right] &= \mathbb{E} \left[\exp \left\{ \int_0^T \tilde{f}(T - s) X_s ds \right\} \right] = \mathbb{E} [\exp \{V_0^T\}] \\ &= \mathbb{E} \left[\exp \left\{ \int_0^T \tilde{f}(T - s) Y_s ds \right\} \right] = \mathbb{E} \left[\exp \left\{ i \int_0^T f(s) Y_s ds \right\} \right], \end{aligned}$$

where we use the fact that V_0^T in (34) does not depend on the solution process, but only on the solution of the Riccati–Volterra equation. Therefore (81) holds for continuous functions. Since $C([0, T])$ is dense in $L^2([0, T])$, Hölder’s inequality allows to carry out a dominated convergence argument that yields (81) for all $f \in \tilde{L}^2([0, T])$. Hence, the laws of X and Y are the same on the space $L^2([0, T])$ by, for instance, [10, Proposition 2.5, Chapter 2]. This completes the proof. \blacksquare

Appendix A. The forward process

Given a kernel $K \in L_{\text{loc}}^2(\mathbb{R}_+; \mathbb{R}^{m \times d})$, we want to find an expression for the forward process

$$\mathbb{E} [X_T | \mathcal{F}_t], \quad 0 \leq t \leq T,$$

for almost every $T \in \mathbb{R}_+$.

If $b \equiv 0$, then (18) implies

$$X_T = g_0(T) + (K * d\tilde{Z})_T = g_0(T) + \int_0^T K(T - s) d\tilde{Z}_s, \quad \mathbb{P} - \text{a.s.}, \text{ for a.e. } T \in \mathbb{R}_+.$$

By the martingale property ensured by (19) we immediately infer that, for almost every $T \in \mathbb{R}_+$,

$$\mathbb{E} [X_T | \mathcal{F}_t] = g_0(T) + \int_0^t K(T - s) d\tilde{Z}_s, \quad \mathbb{P} - \text{a.s.}, t \in [0, T]. \quad (\text{A.1})$$

If $b \neq 0$, then we consider $m = d$ and introduce the resolvent of the second kind R_B associated with $-KB$. Note that $R_B \in L_{\text{loc}}^2(\mathbb{R}_+; \mathbb{R}^{d \times d})$ by [15, Theorem 3.5, Chapter 2]. Convolving (18) with R_B and [15, Theorem 2.2 (viii), Chapter 2] yield

$$(R_B * X)(T) = (R_B * g_0)(T) + ((R_B * K) * b(X))(T) + (R_B * (K * d\tilde{Z}))(T), \quad \text{for a.e. } T \in \mathbb{R}_+, \mathbb{P} - \text{a.s.}$$

The associativity of the stochastic convolution proved in Lemma 3 (with $\rho = R_B$) and the joint measurability of the processes involved enable us to rewrite this equality as follows:

$$\begin{aligned} (R_B * X)(T) &= (R_B * g_0)(T) + ((R_B * K) * b_0)(T) + ((R_B * KB) * X)(T) \\ &\quad + ((R_B * K) * d\tilde{Z})_T, \quad \mathbb{P} - \text{a.s.}, \text{ for a.e. } T \in \mathbb{R}_+. \end{aligned} \quad (\text{A.2})$$

From the resolvent identity (see the footnote ⁴) we have $R_B * KB = KB + R_B$ a.e. in \mathbb{R}_+ , so we rewrite Equation (A.2) as follows

$$0 = (R_B * g_0)(T) + ((R_B * K) * b_0)(T) + (KB * X)(T) + ((R_B * K) * d\tilde{Z})_T, \quad \mathbb{P} - \text{a.s.}, \text{ for a.e. } T \in \mathbb{R}_+. \quad (\text{A.3})$$

Consider the *canonical resolvent* $E_B = K - R_B * K$; subtracting (A.3) from (18) we have

$$X_T = (g_0 - (R_B * g_0))(T) + (E_B * b_0)(T) + (E_B * d\tilde{Z})_T, \quad \mathbb{P} - \text{a.s.}, \text{ for a.e. } T \in \mathbb{R}_+.$$

Hence by the martingale property guaranteed by (19) we are able to find an expression for the forward process $\mathbb{E}[X_T|\mathcal{F}_t]$, namely for almost every $T \in \mathbb{R}_+$, for every $t \in [0, T]$ it holds

$$\mathbb{E}[X_T|\mathcal{F}_t] = (g_0(T) - (R_B * g_0)(T)) + (E_B * b_0)(T) + \int_0^t E_B(T-s) d\tilde{Z}_s, \quad \mathbb{P} - \text{a.s.} \quad (\text{A.4})$$

Finally, notice that (A.4) reduces to (A.1) as $b \equiv 0$. Indeed, since $E_0 = K$ a.e. in \mathbb{R}_+ as $R_0 = 0$ ($\in \mathbb{R}^{d \times d}$), combining (18) with Lemma 2 (see (21)) we have

$$X_T = g_0(T) + (E_0 * d\tilde{Z})_T, \quad \mathbb{P} - \text{a.s.}, \text{ for a.e. } T \in \mathbb{R}_+, \quad (\text{A.5})$$

and the assertion follows by the martingale property.

Remark 8. Equation (A.4) with $t = 0$ implies that $\mathbb{E}[X_T] = (g_0 - (R_B * g_0))(T) + (E_B * b_0)(T)$ for a.e. $T \in \mathbb{R}_+$. This result can be confirmed with a direct method. Specifically, by (19) and Tonelli's theorem the function $\mathbb{E}[|X.|] \in L^1_{\text{loc}}(\mathbb{R}_+; \mathbb{R})$. Hence taking expectations in (14) we obtain, by Fubini's theorem,

$$\mathbb{E}[X_T] = (g_0 + K * b_0)(T) + (KB * \mathbb{E}[X.])(T), \quad \text{for a.e. } T \in \mathbb{R}_+,$$

i.e., $\mathbb{E}[X.] + ((-KB) * \mathbb{E}[X.]) = g_0 + K * b_0$ a.e. in \mathbb{R}_+ . By the variation of constants formula [15, Theorem 3.5, Chapter 2] we conclude

$$\mathbb{E}[X_T] = (g_0 - (R_B * g_0) + (E_B * b_0))(T), \quad \text{for a.e. } T \in \mathbb{R}_+,$$

as desired.

Appendix B. On the 1-dimensional, deterministic Riccati–Volterra equation

Here we focus on the Riccati–Volterra equation used in Section 5, i.e., (31) with

$$\mathcal{R}(t, u) = f(t) + bu + \frac{c}{2}u^2 + \int_{\mathbb{R}_+} (e^{u\xi} - 1 - u\xi) \nu(d\xi), \quad (t, u) \in \mathbb{R}_+ \times \mathbb{C}_-, \quad (\text{B.1})$$

where $f \in C(\mathbb{R}_+; \mathbb{C}_-)$. Throughout the section, we require Condition 1 on the kernel K .

Appendix B.1. Existence of a global solution

It is easy to argue that (31) admits a continuous, noncontinuable solution ψ , with $\text{Re } \psi \leq 0$, defined on the maximal interval $[0, T_{\max})$ (see [1, Theorem 2.5, Step 1]). We are concerned with showing that $T_{\max} = \infty$, i.e., that ψ does not explode in finite time (cfr. [15, Theorem 1.1, Chapter 12]).

Fix a generic $T \in (0, T_{\max})$; taking real and imaginary parts in (31) and (B.1) we have, on the interval $[0, T]$,

$$\text{Re } \psi = K * \left[\text{Re } f + b \text{Re } \psi + \frac{c}{2} (|\text{Re } \psi|^2 - |\text{Im } \psi|^2) + \int_{\mathbb{R}_+} (\cos(\text{Im } \psi \cdot \xi) e^{\text{Re } \psi \cdot \xi} - 1 - \text{Re } \psi \cdot \xi) \nu(d\xi) \right], \quad (\text{B.2})$$

$$\text{Im } \psi = K * \left[\text{Im } f + b \text{Im } \psi + c \text{Re } \psi \text{Im } \psi + \int_{\mathbb{R}_+} (\sin(\text{Im } \psi \cdot \xi) e^{\text{Re } \psi \cdot \xi} - \text{Im } \psi \cdot \xi) \nu(d\xi) \right]. \quad (\text{B.3})$$

First we study the imaginary part. In particular, we consider the function $h: \mathbb{R}_- \times \mathbb{R} \rightarrow \mathbb{R}$ defined as follows

$$h(x, y) = \begin{cases} \frac{1}{y} \int_{\mathbb{R}_+} (\sin(y\xi) - y\xi) e^{x \cdot \xi} \nu(d\xi), & y \neq 0 \\ 0, & y = 0 \end{cases}, \quad x \leq 0.$$

Note that h is continuous and nonpositive in its domain. By construction

$$y \cdot h(x, y) = \int_{\mathbb{R}_+} (\sin(y\xi) - y\xi) e^{x \cdot \xi} \nu(d\xi), \quad (x, y) \in \mathbb{R}_- \times \mathbb{R}.$$

Hence we can use this function to rewrite (B.3) in the following form

$$\operatorname{Im} \psi = K * \left[\operatorname{Im} f + b \operatorname{Im} \psi + c \operatorname{Re} \psi \operatorname{Im} \psi + \left(\int_{\mathbb{R}_+} \xi (e^{\operatorname{Re} \psi \cdot \xi} - 1) \nu(d\xi) \right) \operatorname{Im} \psi + h(\operatorname{Re} \psi, \operatorname{Im} \psi) \operatorname{Im} \psi \right],$$

which holds on $[0, T]$. Consider the unique, continuous, non-negative solution on $[0, T]$ of the linear equation

$$g = K * \left[|\operatorname{Im} f| + b g + \left(c \operatorname{Re} \psi + \int_{\mathbb{R}_+} \xi (e^{\operatorname{Re} \psi \cdot \xi} - 1) \nu(d\xi) + h(\operatorname{Re} \psi, \operatorname{Im} \psi) \right) g \right].$$

By Condition 1, we can invoke [4, Theorem C.1] to deduce that $|\operatorname{Im} \psi| \leq g$ on $[0, T]$. Next we introduce u , the unique, continuous solution of the linear equation

$$u = K * [|\operatorname{Im} f| + b u].$$

Notice that u is defined on \mathbb{R}_+ , and that $g \leq u$ on $[0, T]$ (again by [4, Theorem C.1]), as in this interval one has

$$c \operatorname{Re} \psi + \int_{\mathbb{R}_+} \xi (e^{\operatorname{Re} \psi \cdot \xi} - 1) \nu(d\xi) + h(\operatorname{Re} \psi, \operatorname{Im} \psi) \leq 0.$$

Therefore we have obtained the bound

$$|\operatorname{Im} \psi(t)| \leq u(t), \quad 0 \leq t \leq T. \quad (\text{B.4})$$

Secondly, Equation (B.2) ensures that $\operatorname{Re} \psi$ satisfies

$$\begin{aligned} \operatorname{Re} \psi = K * \left[\operatorname{Re} f + b \operatorname{Re} \psi + \frac{c}{2} (|\operatorname{Re} \psi|^2 - |\operatorname{Im} \psi|^2) + \int_{\mathbb{R}_+} (e^{\operatorname{Re} \psi \cdot \xi} - 1 - \operatorname{Re} \psi \cdot \xi) \nu(d\xi) \right. \\ \left. - \left| \int_{\mathbb{R}_+} e^{\operatorname{Re} \psi \cdot \xi} (\cos(\operatorname{Im} \psi \cdot \xi) - 1) \nu(d\xi) \right| \right] \end{aligned}$$

on $[0, T]$. Since $|\cos(x) - 1| = 1 - \cos(x) \leq x^2/2$, $x \in \mathbb{R}$, we have (also recalling (B.4))

$$\left| \int_{\mathbb{R}_+} e^{\operatorname{Re} \psi \cdot \xi} (\cos(\operatorname{Im} \psi \cdot \xi) - 1) \nu(d\xi) \right| \leq \frac{1}{2} \left(\int_{\mathbb{R}_+} |\xi|^2 \nu(d\xi) \right) |\operatorname{Im} \psi|^2 \leq \frac{1}{2} \left(\int_{\mathbb{R}_+} |\xi|^2 \nu(d\xi) \right) u^2, \quad (\text{B.5})$$

which holds on $[0, T]$. This suggests to introduce the linear equation

$$l = K * \left[\operatorname{Re} f + b l - \left(\frac{c}{2} + \frac{1}{2} \int_{\mathbb{R}_+} |\xi|^2 \nu(d\xi) \right) u^2 \right],$$

which has a unique, continuous, nonpositive solution l defined on the whole \mathbb{R}_+ . At this point, observe that the difference $\operatorname{Re} \psi - l$ satisfies the linear equation

$$\begin{aligned} \chi = K * \left[b \chi + \frac{c}{2} |\operatorname{Re} \psi|^2 + \frac{c}{2} (u^2 - |\operatorname{Im} \psi|^2) + \int_{\mathbb{R}_+} (e^{\operatorname{Re} \psi \cdot \xi} - 1 - \operatorname{Re} \psi \cdot \xi) \nu(d\xi) \right. \\ \left. + \left(\frac{1}{2} \left(\int_{\mathbb{R}_+} |\xi|^2 \nu(d\xi) \right) u^2 - \left| \int_{\mathbb{R}_+} e^{\operatorname{Re} \psi \cdot \xi} (\cos(\operatorname{Im} \psi \cdot \xi) - 1) \nu(d\xi) \right| \right) \right]. \end{aligned}$$

It admits a unique, continuous solution on $[0, T]$ which is non-negative by (B.4), (B.5) and the fact that $e^x - 1 - x \geq 0$, $x \in \mathbb{R}$. Since $T \in (0, T_{\max})$ was chosen arbitrarily, we conclude that

$$l(t) \leq \operatorname{Re} \psi(t) \leq 0 \quad \text{and} \quad |\operatorname{Im} \psi(t)| \leq u(t), \quad 0 \leq t < T_{\max}.$$

Now recalling that l and u are continuous in \mathbb{R}_+ , so they are bounded in every compact interval, we conclude that $T_{\max} = \infty$, as desired.

Finally we notice that if f takes values in \mathbb{R}_- , then from (B.4) we deduce that any solution of (31) is real-valued, as well. In particular, $\bar{\psi}$ in (75) is \mathbb{R}_- -valued.

Appendix B.2. A comparison result

The goal of this appendix is to prove the inequality (77) in Theorem 13 (ii), which is of utmost importance for the argument in Section 5. Precisely, we want to show that

$$\operatorname{Re} \psi(t) \leq \bar{\psi}(t), \quad t \geq 0,$$

where $\psi \in C(\mathbb{R}_+; \mathbb{C}_-)$ and $\bar{\psi} \in C(\mathbb{R}_+; \mathbb{R}_-)$ satisfy (31) and (75), respectively. Direct computations based on the definitions in (B.1) and (76) show that, for every $u \in \mathbb{C}_-$ and $t \geq 0$,

$$\begin{aligned} \operatorname{Re}(\mathcal{R}(t, u)) &= \operatorname{Re} f(t) + b \operatorname{Re}(u) + \frac{c}{2} (|\operatorname{Re}(u)|^2 - |\operatorname{Im}(u)|^2) + \int_{\mathbb{R}_+} (\cos(\operatorname{Im}(u)\xi) e^{\operatorname{Re}(u)\xi} - 1 - \operatorname{Re}(u)\xi) \nu(d\xi) \\ &\leq \operatorname{Re} f(t) + b \operatorname{Re}(u) + \frac{c}{2} |\operatorname{Re}(u)|^2 + \int_{\mathbb{R}_+} (e^{\operatorname{Re}(u)\xi} - 1 - \operatorname{Re}(u)\xi) \nu(d\xi) = \bar{\mathcal{R}}(t, \operatorname{Re}(u)). \end{aligned}$$

Summarizing,

$$\operatorname{Re}(\mathcal{R}(t, u)) \leq \bar{\mathcal{R}}(t, \operatorname{Re}(u)), \quad u \in \mathbb{C}_-, t \geq 0. \quad (\text{B.6})$$

Then taking the real parts in (31) and recalling that –under Condition 1– the kernel K is nonnegative on $(0, \infty)$, we obtain

$$\operatorname{Re}(\psi(t)) \leq \int_0^t K(t-s) \bar{\mathcal{R}}(s, \operatorname{Re}(\psi(s))) ds, \quad t \geq 0.$$

Therefore we can introduce a nonnegative function $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by the relation

$$\operatorname{Re}(\psi(t)) = -\gamma(t) + \int_0^t K(t-s) \bar{\mathcal{R}}(s, \operatorname{Re}(\psi(s))) ds, \quad t \geq 0; \quad (\text{B.7})$$

we immediately note that, using (31), one can rewrite γ as follows

$$\gamma(t) = \int_0^t K(t-s) (\bar{\mathcal{R}}(s, \operatorname{Re}(\psi(s))) - \operatorname{Re}(\mathcal{R}(s, \psi(s)))) ds, \quad t \geq 0. \quad (\text{B.8})$$

For a generic map $g: \mathbb{R}_+ \rightarrow \mathbb{R}$ consider the condition

$$\Delta_h g - (\Delta_h K * L)(0)g - d(\Delta_h K * L) * g \geq 0, \quad h \geq 0; \quad (\text{B.9})$$

we denote by $\mathcal{G}_K = \{g: \mathbb{R}_+ \rightarrow \mathbb{R} \text{ s.t. } g \text{ is continuous, satisfies (B.9) and } g(0) \geq 0\}$ the set of admissible curves, see [4, Condition B.3], and also [1, Equations (2.14)-(2.15)]. By [4, Remark B.6] and (B.8) we infer that $\gamma \in \mathcal{G}_K$.

At this point we subtract (B.7) from (75) to deduce, calling $\delta = \bar{\psi} - \operatorname{Re} \psi$, that

$$\delta(t) = \gamma(t) + \int_0^t K(t-s) (\bar{\mathcal{R}}(s, \bar{\psi}(s)) - \bar{\mathcal{R}}(s, \operatorname{Re}(\psi(s)))) ds, \quad t \geq 0. \quad (\text{B.10})$$

We then need to study the increments of $\bar{\mathcal{R}}$ in the second variable. Namely, fix $u_1, u_2 \in \mathbb{R}_-$ and use the definition (76) to write

$$\begin{aligned} \bar{\mathcal{R}}(t, u_1) - \bar{\mathcal{R}}(t, u_2) &= b(u_1 - u_2) + \frac{c}{2} (u_1^2 - u_2^2) + \int_{\mathbb{R}_+} (e^{u_1\xi} - e^{u_2\xi} - (u_1 - u_2)\xi) \nu(d\xi) \\ &= \left[b + \frac{c}{2} (u_1 + u_2) \right] (u_1 - u_2) + \int_{\mathbb{R}_+} (e^{u_1\xi} - e^{u_2\xi} - (u_1 - u_2)\xi) \nu(d\xi), \quad t \geq 0. \end{aligned}$$

Hence substituting $\bar{\psi}$ and $\operatorname{Re} \psi$ to u_1 and u_2 , respectively, we have

$$\bar{\mathcal{R}}(t, \bar{\psi}(t)) - \bar{\mathcal{R}}(t, \operatorname{Re}(\psi(t))) = \underbrace{\left[b + \frac{c}{2} (\bar{\psi}(t) + \operatorname{Re}(\psi(t))) \right]}_{=z(t)} \delta(t) + \underbrace{\int_{\mathbb{R}_+} (e^{\bar{\psi}(t)\xi} - e^{\operatorname{Re}(\psi(t))\xi} - \delta(t)\xi) \nu(d\xi)}_{=w(t)}$$

for $t \geq 0$. Going back to (B.10),

$$\delta(t) = \gamma(t) + \int_0^t K(t-s)(z(s)\delta(s) + w(s)) ds, \quad t \geq 0. \quad (\text{B.11})$$

We aim to apply [4, Theorem C.1] in order to conclude $\delta \geq 0$ in \mathbb{R}_+ .

- In the continuous case the integral in $\nu(d\xi)$, i.e., the function w , simply disappears, hence the application of [4, Theorem C.1] is straightforward.
- In the jump case we need to deal with such an integral. Observe that the function w has opposite sign with respect to δ , so there is no hope of applying [4, Theorem C.1] unless we modify its expression. Fortunately this can be done using the mean value theorem, in combination with simple real-analysis arguments.

First, for every $\xi > 0$ we define $f_\xi(u) = e^{\xi u}$, $u \in \mathbb{R}$, so $f'_\xi(u) = \xi e^{\xi u}$. Observe that the derivative f'_ξ is continuous and strictly increasing in \mathbb{R} , hence its inverse $h_\xi = (f'_\xi)^{-1}$ is continuous on $(0, \infty)$, as well. By the mean value theorem, for every $u_1, u_2 \in \mathbb{R}$ there exists $c_\xi \in [u_1 \wedge u_2, u_1 \vee u_2]$ such that

$$f_\xi(u_2) - f_\xi(u_1) = f'_\xi(c_\xi)(u_2 - u_1).$$

In particular $c_\xi \in (u_1 \wedge u_2, u_1 \vee u_2)$ when $u_1 \neq u_2$.

Secondly, we consider the functions $\bar{\psi}$ and $\text{Re } \psi$, and we can say that for every $t \in \mathbb{R}_+$ there exists $c_\xi(t) \in [\bar{\psi}(t) \wedge \text{Re } \psi(t), \bar{\psi}(t) \vee \text{Re } \psi(t)]$ (in the interior of such interval whenever $\bar{\psi} \neq \text{Re } \psi$, i.e., whenever $\delta \neq 0$) such that

$$e^{\xi \bar{\psi}(t)} - e^{\xi \text{Re } \psi(t)} = f'_\xi(c_\xi(t))(\bar{\psi}(t) - \text{Re } \psi(t)) = \xi e^{\xi c_\xi(t)} \delta(t). \quad (\text{B.12})$$

By the axiom of choice we construct the function $c: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_-$ defined by

$$c(\xi, t) = c_\xi(t), \quad \xi > 0, t \geq 0,$$

and $c(0, t) = 0$, $t \geq 0$. Note that the codomain of $c(\cdot, \cdot)$ is \mathbb{R}_- since both $\bar{\psi}$ and $\text{Re } \psi$ take values there. Recalling the definition of w and using (B.12) we can write

$$w(t) = \int_{\mathbb{R}_+} \left[e^{\bar{\psi}(t)\xi} - e^{\text{Re } \psi(t)\xi} - \delta(t)\xi \right] \nu(d\xi) = \underbrace{\left(\int_{\mathbb{R}_+} \xi \left[e^{\xi c(\xi, t)} - 1 \right] \nu(d\xi) \right)}_{=\bar{w}(t)} \delta(t), \quad t \geq 0.$$

Now we have to prove that \bar{w} is continuous on \mathbb{R}_+ . The first step is to show the continuity of the function $c(\xi, \cdot)$ in \mathbb{R}_+ for every fixed $\xi \in \mathbb{R}_+$. It is of course trivial for $\xi = 0$, so we just focus on $\xi > 0$. If $\bar{t} \in (0, \infty)$ such that $\delta(\bar{t}) > 0$, then we can find $\epsilon > 0$ such that $\delta > 0$ in $(\bar{t} - \epsilon, \bar{t} + \epsilon)$. Hence we use (B.12) to prove that

$$c(\xi, t) = h_\xi \left(\frac{e^{\xi \bar{\psi}(t)} - e^{\xi \text{Re } \psi(t)}}{\delta(t)} \right), \quad t \in (\bar{t} - \epsilon, \bar{t} + \epsilon),$$

recalling that $h_\xi = (f'_\xi)^{-1}$. So $c(\xi, \cdot)$ is continuous in the points $\bar{t} \in (0, \infty)$ where $\delta(\bar{t}) > 0$. An analogous reasoning shows the continuity in the points where $\delta < 0$. Consider now $\bar{t} \in \mathbb{R}_+$ a zero for δ , i.e., $\delta(\bar{t}) = 0$. For every sequence $(t_n)_n \subset \mathbb{R}_+$ such that $t_n \rightarrow \bar{t}$ as $n \rightarrow \infty$ one has, by construction,

$$\bar{\psi}(t_n) \wedge \text{Re } \psi(t_n) \leq c(\xi, t_n) \leq \bar{\psi}(t_n) \vee \text{Re } \psi(t_n), \quad n \in \mathbb{N}.$$

Therefore an application of the squeeze theorem gives

$$\lim_{n \rightarrow \infty} c(\xi, t_n) = \text{Re } \psi(\bar{t}) = \bar{\psi}(\bar{t}) = c(\xi, \bar{t}).$$

At this point we deduce the continuity of the function \bar{w} using the dominated convergence theorem. Indeed, take $t \in \mathbb{R}_+$, a sequence $t_n \rightarrow t$, and define $g_{(n)}(\xi) = \xi \left[e^{\xi c(\xi, t_n)} - 1 \right]$, $\xi \in \mathbb{R}_+$. Then $g_n \rightarrow g$ pointwise in \mathbb{R}_+ by the continuity of $c(\xi, \cdot)$ and, for a certain $C > 0$ s.t. $t_n \leq C$, $n \in \mathbb{N}$ (which exists since $(t_n)_n$ is bounded), we have

$$\begin{aligned} \left| \xi \left[e^{\xi c(\xi, t_n)} - 1 \right] \right| &= \xi \left[1 - e^{\xi c(\xi, t_n)} \right] \leq \xi \left[1 - e^{\xi \min_{0 \leq s \leq C} c(\xi, s)} \right] \\ &\leq \xi \left[1 - e^{\xi \min_{0 \leq s \leq C} \{(\operatorname{Re} \psi \wedge \bar{\psi})(s)\}} \right] \in L^1(d\nu), \quad n \in \mathbb{N}. \end{aligned}$$

Therefore we can rewrite (B.11) as follows

$$\delta(t) = \gamma(t) + \int_0^t K(t-s)(z(s) + \bar{w}(s))\delta(s)ds, \quad t \geq 0,$$

and we invoke [4, Theorem C.1] to assert that $\delta \geq 0$, i.e., that (77) holds.

References

- [1] ABI JABER, E. (2021). Weak existence and uniqueness for affine stochastic Volterra equations with L^1 -kernels. *Bernoulli*, 27(3), 1583–1615.
- [2] ABI JABER, E., CUCHIERO, C., LARSSON, M., & PULIDO, S. (2019). A weak solution theory for stochastic Volterra equations of convolution type. *The Annals of Applied Probability*, 2021, 31(6), 2924–2952.
- [3] ABI JABER, E., & EL EUCH, O. (2019). Markovian structure of the Volterra Heston model. *Statistics & Probability Letters*, 149, 63–72.
- [4] ABI JABER, E., & EL EUCH, O. (2019). Multifactor approximation of rough volatility models. *SIAM Journal on Financial Mathematics*, 10(2), 309–349.
- [5] ABI JABER, E., LARSSON, M., & PULIDO, S. (2019). Affine Volterra processes. *The Annals of Applied Probability*, 29(5), 3155–3200.
- [6] BAYER, C., FRIZ, P., & GATHERAL, J. (2016). Pricing under rough volatility. *Quantitative Finance*, 16(6), 887–904.
- [7] BONDI, A., PULIDO, S., & SCOTTI, S. (2022). The rough Hawkes Heston stochastic volatility model. *arXiv preprint arXiv:2210.12393*.
- [8] CUCHIERO, C., FILIPOVIĆ, D., MAYERHOFER, E., & TEICHMANN, J. (2011). Affine processes on positive semidefinite matrices. *The Annals of Applied Probability*, 21(2), 397–463.
- [9] CUCHIERO, C., & TEICHMANN, J. (2020). Generalized Feller processes and Markovian lifts of stochastic Volterra processes: the affine case. *Journal of evolution equations*, 20(4), 1301–1348.
- [10] DA PRATO, G., & ZABCZYK, J. (2014). *Stochastic equations in infinite dimensions*. Cambridge University Press.
- [11] DUFFIE, D., FILIPOVIĆ, D., & SCHACHERMAYER, W. (2003). Affine processes and applications in finance. *The Annals of Applied Probability*, 13(3), 984–1053.
- [12] EL EUCH, O., & ROSENBAUM, M. (2019). The characteristic function of rough Heston models. *Mathematical Finance*, 29(1), 3–38.
- [13] GATHERAL, J., JAISSON, T., & ROSENBAUM, M. (2018). Volatility is rough. *Quantitative Finance*, 18(6), 933–949.
- [14] GATHERAL, J., & KELLER-RESSEL, M. (2019). Affine forward variance models. *Finance and Stochastics*, 23(3), 501–533.
- [15] GRIPENBERG, G., LONDEN, S. O., & STAFFANS, O. (1990). *Volterra integral and functional equations* (No. 34). Cambridge University Press.
- [16] JACOD, J., & SHIRYAEV, A. (2013). *Limit theorems for stochastic processes* (Vol. 288). Springer Science & Business Media.
- [17] JARROW, R. A. (2018). *Continuous-time asset pricing theory*. Springer.
- [18] KELLER-RESSEL, M. (2011). Moment explosions and long-term behavior of affine stochastic volatility models. *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, 21(1), 73–98.
- [19] KELLER-RESSEL, M., LARSSON, M., & PULIDO, S. (2018). Affine rough models. *arXiv preprint arXiv:1812.08486*.
- [20] LÉPINGLE, D., & MÉMIN, J. (1978). Sur l'intégrabilité uniforme des martingales exponentielles. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 42(3), 175–203.
- [21] MARINELLI, C., & RÖCKNER, M. (2014). On maximal inequalities for purely discontinuous martingales in infinite dimensions. In *Séminaire de Probabilités XLVI* (pp. 293–315). Springer, Cham.
- [22] PROTTER, P. E. (2005). *Stochastic integration and differential equations*. Springer, Berlin, Heidelberg.
- [23] ROSESTOLATO, M. (2018). A note on stochastic Fubini's theorem and stochastic convolution. *arXiv preprint arXiv:1606.06340v2*.
- [24] TODOROV, V., & TAUCHEN, G. (2011). Volatility jumps. *Journal of Business & Economic Statistics*, 29(3), 356–371.
- [25] XIA, W. (2020). Power-type derivatives for rough volatility with jumps. *arXiv preprint arXiv:2008.10184*.