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A Data-Driven Trajectory Representation for Nonlinear Systems under quasi-Linear Parameter Varying Embeddings

Marcelo M. Morato^{1,2}, Julio E. Normey-Rico¹ and Olivier Sename²

Abstract—Recent literature has shown how linear time-invariant (LTI) systems can be represented through trajectory-based features, relying on a single measured input-output (IO) trajectory dictionary, as long as the input is persistently exciting. The so-called behavioural framework is a promising alternative for controller synthesis without the necessity of system identification. In this paper, we extend and translate previous results to a wide class of nonlinear systems, using quasi-Linear Parameter Varying (qLPV) embeddings along suitable IO coordinates. Accordingly, we show how nonlinear data-driven simulation and predictions can be made based on the proposed qLPV approach. A parameter-dependent dissipativity analysis verification setup is also given. Realistic results are included to demonstrate the effectiveness of the tools. [Paper under corrections, a new version will be uploaded soon.]

I. INTRODUCTION

Modern control theory relies on the availability of trustworthy process models, and thus system identification has been an active field of research. Yet, fostering accurate models is costly, ponderous, and hindered by uncertainties. Thus, over the last decades, developing controllers directly from data has received a considerable amount of attention, specially due to reinforcement learning techniques [1] and virtual reference feedback tuning approaches [2]. Withal, as argued extensively in [1], these methods consistently require large data sets, while lacking formal guarantees on stability and performance of the resulting closed-loop.

More recently, concrete results were presented using behavioural theory as an unified approach to data-driven control [3]–[6]. This framework enables to characterise all possible trajectories of an unknown system using a single measured input–output dictionary of a fixed length, as long as the input is persistently exciting. This representation structure has been thoroughly exploited in the context of linear time-invariant (LTI), as well as for Hammerstein-Wiener plants, offering a well-suited set of tools for the development of data-driven control with inherent formal guarantees, such as dissipativity and corresponding stability properties. Data-driven simulation and prediction have also been assessed, including the case of Linear Parameter Varying (LPV) systems [7].

In parallel to these results, the LPV toolkit has been shown capable to describe a wide range of time-varying

behaviours under linear structures, with experimental examples registered in [8]. Under differential inclusion properties, quasi-LPV (qLPV) embeddings are a viable way to encompass nonlinearities into bounded scheduling parameters, thus maintaining linearity along suitable input-output (IO) channels [9]. W.r.t. this context, our main contributions are:

- A data-driven trajectory representation is proposed for nonlinear systems, benefiting from qLPV embeddings. The framework is an extension of the results from [5] to a much wider class of nonlinear plants.
- In consonance with [5], [7], we present data-driven simulation and prediction algorithms for nonlinear systems using input-output data and a scheduling function.
- A parameter-dependent dissipativity analysis framework is conceived for nonlinear systems, enabled through a direct verification test, as in [3], [6], yet constrained by the available scheduling variable space.
- Realistic simulation results of a rotational pendulum benchmark are presented in order to demonstrate the effectiveness and accurateness of the proposed tools.

Paper organisation. Sec. II provides preliminaries. Sec. III gives the main result: the trajectory representation for nonlinear systems via qLPV embeddings, and also data-driven simulation and prediction algorithms.

Notation. The identity matrix of size j is denoted as I_j . The orthogonal complement of a matrix A is denoted A^\perp . For a discrete-time signal $v : \mathbb{N} \rightarrow \mathbb{R}^{n_v}$, we denote $v(k) \in \mathbb{R}^{n_v}$ each of its entries and $\{v(k)\}_{k=0}^{N-1}$ the corresponding sequence of N data entries, or just v in short. We use $\text{col}\{v\} := [v(0)^T \dots v(N-1)^T]^T$ to denote the column vectorisation, and $\text{diag}\{v\}$ as the block-diagonal matrix formed with $\text{col}\{v\}$. The Kronecker product is represented by \otimes ; the corresponding block-diagonal operator is denoted \oplus , implying that $(v \otimes I_\xi) = \text{diag}\{v(0) \otimes I_\xi \dots v(N-1) \otimes I_\xi\}$. For a sequence $\{v(k)\}_{k=0}^{N-1}$, we have the corresponding Hankel matrix, for a window of L entries, given by:

$$H_L(v) := \begin{bmatrix} v(0) & v(1) & \dots & v(N-L) \\ v(1) & v(2) & \dots & v(N-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ v(L-1) & v(L) & \dots & v(N-1) \end{bmatrix}.$$

(\star) denotes the corresponding symmetrical transpose. For two sets \mathbb{W} and \mathbb{T} , $\mathbb{W}^\mathbb{T}$ marks all maps from \mathbb{T} to \mathbb{W} .

II. PRELIMINARIES AND SETTING

In this Section, we briefly recall key concepts on behavioural theory and trajectory representation for LTI systems, as well as main the arguments used to generate qLPV embeddings.

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A. Behavioural Theory

Definition 1 (System Behaviour [10]): A dynamic system is given by $\mathcal{G} := (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T} \subset \mathbb{R}$ is called the time dimension, \mathbb{W} the signal space, and $\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}}$ the system behaviour, which represents all possible trajectories of \mathcal{G} .

Definition 2 (Manifest Behaviour [10]): The manifest behaviour of a system $\mathcal{G} := (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ with inputs $u \in \mathbb{R}^{n_u}$ and outputs $y \in \mathbb{R}^{n_y}$ is given by $\mathfrak{B}_M := \{\text{col}\{u, y\} \in \mathfrak{B} \mid \exists x \in (\mathbb{R}^{n_x})^{\mathbb{N}} \text{ s.t. Eq. (1) holds}\}$.

$$\begin{cases} x(k+1) &= f_x(x(k), u(k)), \\ y(k) &= f_y(x(k), u(k)). \end{cases} \quad (1)$$

Thus, we say that Eq. (1) is a state-space representation of \mathcal{G} if $\mathfrak{B}_M = \mathfrak{B}$, i.e. all possible trajectories of \mathcal{G} are mapped.

Definition 3 (Persistent excitation [10]): A signal $\{u(k)\}_{k=0}^{N-1}$, with $u(k) \in \mathbb{R}^{n_u}, \forall k \geq 0$, is persistently exciting of order L if the rank of $H_L(u) = n_u L$.

The condition of persistent excitation is widely used in system identification theory. Def. 3 implies that $N \geq (n_u + 1)L - 1$. Based on the assumption of a persistently exciting input u , Willem's Lemma [10] is exploited in control theory:

Theorem 1 (Trajectory Representation [5]): Consider an LTI system \mathcal{G} with inputs $u(k) \in \mathbb{R}^{n_u}$ and outputs $y(k) \in \mathbb{R}^{n_y}$, whose behaviour is given by the set of all trajectories $\text{col}\{u, y\}$, s.t. $\exists x \in (\mathbb{R}^{n_x})^{\mathbb{N}}$ that validates Eq. (2).

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) + Du(k). \end{cases} \quad (2)$$

Consider $\{u(k), y(k)\}_{k=0}^{N-1}$ as a trajectory of \mathcal{G} , with u persistently exciting of order $L + n_x$. Then, any sequence $\{\bar{u}(k), \bar{y}(k)\}_{k=0}^{L-1}$ is also a trajectory of \mathcal{G} iff $\exists \alpha \in \mathbb{R}^{N-L+1}$ s.t.:

$$\begin{bmatrix} H_L(u) \\ H_L(y) \end{bmatrix} \alpha = \begin{bmatrix} \text{col}\{\bar{u}\} \\ \text{col}\{\bar{y}\} \end{bmatrix}. \quad (3)$$

Proof: Follows from the linearity that the set of all trajectories of an LTI system is a vector space. Thus, a direct application of [10, Theo. 1] yields Eq. (3); details in [5]. ■

Remark 1: Theo. 1 uses the LTI model from Eq. (2) as a vector space that maps all corresponding trajectories. Moreover, it shows how time-shifts of a single measured trajectory can serve as a basis for this LTI vector space, as long as if the input is persistently exciting of sufficient order. This theorem exploits the well-known property of the existence of minimal (controllable, observable) realisations of LTI systems. The particular choice of the specific realisation is not relevant, but rather the fact that a fixed window IO trajectory $\{u(k), y(k)\}_{k=a}^b$ imposes an unique LTI state trajectory $\{x(k)\}_{k=a}^b$, whenever $b - a \geq n_x - 1$.

Definition 4 (Dissipativity [3]): A system \mathcal{G} is said dissipative w.r.t. a supply rate $\Pi \in \mathbb{R}^{(n_u+n_y) \times (n_u+n_y)}$ if the following inequality holds for all input-output trajectories of \mathcal{G} , i.e. $\{u(k), y(k)\}_{k=0}^{\infty}$, for null initial conditions:

$$\sum_k^r \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}^T \overbrace{\begin{bmatrix} Q & S \\ \star & R \end{bmatrix}}^{\Pi} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \geq 0 \forall r \geq 0, \quad (4)$$

where $Q = Q^T$, $R = R^T$ and S are supply weights.

Definition 5 (L-Dissipativity [3]): A system \mathcal{G} is said L -dissipative w.r.t. a supply rate Π if the following inequality holds for all L -sized input-output trajectories of \mathcal{G} , i.e. $\{u(k), y(k)\}_{k=0}^{L-1}$, for null initial conditions:

$$\sum (k)^r \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}^T \Pi \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \geq 0 \forall r \in \mathbb{N}_{[0, L-1]} \quad (5)$$

Theorem 2 (Dissipativity from Data [6]): Suppose that $\{u(k), y(k)\}_{k=0}^{N-1}$ is a data-dictionary of an LTI system \mathcal{G} . Then, the following statements are equivalent:

- 1) \mathcal{G} is L -dissipative w.r.t. a given supply rate Π .
- 2) Data $\{u(k)\}_{k=0}^{N-1}$ is persistently exciting of order $L + n_x$ and Ineq. (6) holds for any ν s.t. $n_x \leq \nu < L$.

$$(\star) \Pi_L \left(\begin{bmatrix} H_L(u) \\ H_L(y) \end{bmatrix} V_{L,\nu}(u, y) \right) \geq 0, \quad (6)$$

$$\Pi_L := \begin{bmatrix} I_L \otimes Q & I_L \otimes S \\ \star & I_L \otimes R \end{bmatrix},$$

$$V_{L,\nu}(u, y) := \left(T_{L,\nu} \begin{bmatrix} H_L(u) \\ H_L(y) \end{bmatrix} \right)^\perp,$$

$$T_{L,\nu} := [I_{(n_u+n_y)\nu} \ 0_{((n_u+n_y)\nu) \times ((n_u+n_y)(L-\nu))}].$$

Proof: Full proof given in [3], [6]. ■

B. qLPV Embeddings

Definition 6 (Differential Inclusion [9]): Consider that the following difference equation gives the discrete-time input-output nonlinear manifest behaviour of a system \mathcal{G} :

$$y(k) = f(y(k-1), \dots, y(k-n_a), u(k), \dots, u(k-n_b)),$$

being $u \in \mathbb{R}^{n_u}$ the vector of inputs, and $y \in \mathbb{R}^{n_y}$ the vector of outputs. We say that \mathcal{G} satisfies the differential inclusion property if there exists a map $D(y(k-1), \dots, y(k-n_a), u(k), \dots, u(k-n_b)) \subseteq \mathbb{R}^{n_y \times (n_y n_a + n_u (n_b + 1))}$ such that $f(\cdot) := D(\cdot)[y(k-1)^T \dots y(k-n_a)^T u(k)^T \dots u(k-n_b)^T]^T$. Then, the manifest behaviour of \mathcal{G} can be stated as:

$$y(k) + \sum_{i=1}^{n_a} a_i(\rho(k-i))y(k-i) = \sum_{j=1}^{n_b} b_j(\rho(k-j))u(k-j), \quad (7)$$

where $n_a, n_b \geq 0$, and $a_i \in \mathbb{R}^{n_y \times n_y}$ and $b_i \in \mathbb{R}^{n_y \times n_u}$ are coefficient functions.

Remark 2: The model in Eq. (7) is quasi-LPV, with an endogenous nonlinear scheduling function $f_\rho(\cdot)$. For simplicity, we use $\rho(k) = f_\rho(y(k-1))$ in the sequel; all the presented procedures can be applied to the more generic case without loss of generality.

Proposition 1: Consider a nonlinear system \mathcal{G} which satisfies differential inclusion, being states as in Eq. (7) with $\rho(k) = f_\rho(y(k-1))$. Assume that the following compact, convex constraints are respected: $y(k) \in \mathcal{Y} \subset \mathbb{R}^{n_y}$ and $u(k) \in \mathcal{U} \subset \mathbb{R}^{n_u}, \forall k \geq 0$. Thus, $\rho(k) \in \mathcal{P} \subset \mathbb{R}^{n_\rho}, \forall k \geq 0$.

Remark 3: Using an IO LPV realisation, as gives Eq. (7), is rather common in LPV identification, as seen in many application results [8], [11], [12]. Consider a behaviour $\mathfrak{B}_{\text{qLPV}} := \{\text{col}(u, \rho, y) \in (\mathcal{U} \times \mathcal{P} \times \mathcal{Y}) \mid \text{s.t. Eq. (7) holds}\}$. Note that $\mathfrak{B}_{\text{qLPV}}$ is linear along the (u, y) IO channels, in the sense that for any $(u, \rho, y), (\tilde{u}, \rho, \tilde{y}) \in \mathfrak{B}_{\text{qLPV}}$ and $\lambda, \tilde{\lambda} \in \mathbb{R}$,

it follows that $(\lambda u + \tilde{\lambda}\tilde{u}, \rho, \lambda y + \tilde{\lambda}\tilde{y}) \in \mathfrak{B}_{\text{qLPV}}$. Moreover, $\mathfrak{B}_{\text{qLPV}}$ is time-invariant and well-defined, with a direct state-space (SS) realisation (e.g. Proposition 2).

Proposition 2: Assume that a nonlinear system \mathcal{G} satisfies differential inclusion, and that there exists a scheduling proxy $f_\rho(\cdot)$ s.t. Propo. 1 holds. Then, the corresponding non-minimal SS realisation of Eq. (7) is:

$$\begin{aligned} x(k+1) &= A(\rho(k))x(k) + B(\rho(k))u(k), \\ y(k) &= C(\rho(k))x(k) + D(\rho(k))u(k), \end{aligned} \quad (8)$$

where $x(k) = [y(k-1)', \dots, y(k-n_a)']', u(k-1)', \dots, u(k-n_b)']' \in \mathbb{R}^{n_x}$ defines the state vector with $n_x = (n_a n_y + n_b n_u)$. Matrices $\begin{bmatrix} A(\cdot) & B(\cdot) \\ C(\cdot) & D(\cdot) \end{bmatrix}$ are:

$$\begin{bmatrix} -a_1(\cdot) & \dots & -a_{n_a}(\cdot) & b_1(\cdot) & \dots & b_{n_b}(\cdot) & b_0 \\ I_{n_y} & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & \dots & 0 & I_{n_u} & \dots & 0 & I_{n_y} \\ 0 & \dots & 0 & I_{n_u} & \dots & 0 & 0 \\ \vdots & & & & & & \vdots \\ \hline -a_1(\cdot) & \dots & -a_{n_a}(\cdot) & b_1(\cdot) & \dots & b_{n_b}(\cdot) & b_0(\cdot) \end{bmatrix}.$$

Remark 4: The system order is n_x ; n_u and n_y argue the number of inputs and outputs, respectively. Next, only a rough upper bound over n_x is required to quantify the persistent excitation.

III. MAIN RESULT: A TRAJECTORY REPRESENTATION FOR NONLINEAR SYSTEMS UNDER QLPV EMBEDDING

The remainder of this paper is under corrections. A suitable version will be uploaded to HAL soon.

REFERENCES

- [1] B. Recht, "A tour of reinforcement learning: The view from continuous control," *Annual Review of Control, Robotics, and Autonomous Systems*, vol. 2, pp. 253–279, 2019.
- [2] S. Formentin, D. Piga, R. Tóth, and S. M. Savaresi, "Direct learning of LPV controllers from data," *Automatica*, vol. 65, pp. 98–110, 2016.
- [3] A. Romer, J. Berberich, J. Köhler, and F. Allgöwer, "One-shot verification of dissipativity properties from input–output data," *IEEE Control Systems Letters*, vol. 3, no. 3, pp. 709–714, 2019.
- [4] C. De Persis and P. Tesi, "Formulas for data-driven control: Stabilization, optimality, and robustness," *IEEE Trans. on Automatic Control*, vol. 65, no. 3, pp. 909–924, 2019.
- [5] J. Berberich and F. Allgöwer, "A trajectory-based framework for data-driven system analysis and control," in *2020 European Control Conference (ECC)*. IEEE, 2020, pp. 1365–1370.
- [6] A. Koch, J. Berberich, J. Köhler, and F. Allgöwer, "Determining optimal input–output properties: A data-driven approach," *Automatica*, vol. 134, p. 109906, 2021.
- [7] C. Verhoek, H. S. Abbas, R. Tóth, and S. Haesaert, "Data-driven predictive control for linear parameter-varying systems," *IFAC-PapersOnLine*, vol. 54, no. 8, pp. 101–108, 2021.
- [8] C. Hoffmann and H. Werner, "A survey of linear parameter-varying control applications validated by experiments or high-fidelity simulations," *IEEE Trans. on Control Systems Technology*, vol. 23, no. 2, pp. 416–433, 2014.
- [9] M. M. Morato, J. E. Normey-Rico, and O. Sename, "Model predictive control design for linear parameter varying systems: A survey," *Annual Reviews in Control*, vol. 49, pp. 64 – 80, 2020.
- [10] J. C. Willems, P. Rapisarda, I. Markovskiy, and B. L. De Moor, "A note on persistency of excitation," *Systems & Control Letters*, vol. 54, no. 4, pp. 325–329, 2005.

- [11] R. Tóth, J. C. Willems, P. S. Heuberger, and P. M. Van den Hof, "The behavioral approach to linear parameter-varying systems," *IEEE Trans. on Automatic Control*, vol. 56, no. 11, pp. 2499–2514, 2011.
- [12] R. Tóth, H. S. Abbas, and H. Werner, "On the state-space realization of LPV input-output models: Practical approaches," *IEEE Trans. on Control Systems Technology*, vol. 20, no. 1, pp. 139–153, 2011.