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# A Data-Driven Trajectory Representation for Nonlinear Systems under quasi-Linear Parameter Varying Embeddings

Marcelo M. Morato<sup>1,2</sup>, Julio E. Normey-Rico<sup>1</sup> and Olivier Sename<sup>2</sup>

**Abstract**—Recent literature has shown how linear time-invariant (LTI) systems can be represented through trajectory-based features, relying on a single measured input-output (IO) trajectory dictionary, as long as the input is persistently exciting. The so-called behavioural framework is a promising alternative for controller synthesis without the necessity of system identification. In this paper, we extend and translate previous results to a wide class of nonlinear systems, using quasi-Linear Parameter Varying (qLPV) embeddings along suitable IO coordinates. Accordingly, we show how nonlinear data-driven simulation and predictions can be made based on the proposed qLPV approach. A parameter-dependent dissipativity analysis verification setup is also given. Realistic results are included to demonstrate the effectiveness of the tools.

## I. INTRODUCTION

Modern control theory relies on the availability of trustworthy process models, and thus system identification has been an active field of research. Yet, fostering accurate models is costly, ponderous, and hindered by uncertainties. Thus, over the last decades, developing controllers directly from data has received a considerable amount of attention, specially due to reinforcement learning techniques [1] and virtual reference feedback tuning approaches [2]. Withal, as argued extensively in [1], these methods consistently require large data sets, while lacking formal guarantees on stability and performance of the resulting closed-loop.

More recently, concrete results were presented using behavioural theory as an unified approach to data-driven control [3]–[6]. This framework enables to characterise all possible trajectories of an unknown system using a single measured input–output dictionary of a fixed length, as long as the input is persistently exciting. This representation structure has been thoroughly exploited in the context of linear time-invariant (LTI), as well as for Hammerstein-Wiener plants, offering a well-suited set of tools for the development of data-driven control with inherent formal guarantees, such as dissipativity and corresponding stability properties. Data-driven simulation and prediction have also been assessed, including the case of Linear Parameter Varying (LPV) systems [7].

In parallel to these results, the LPV toolkit has been shown capable to describe a wide range of time-varying behaviours under linear structures, with experimental examples registered in [8]. Under differential inclusion properties,

quasi-LPV (qLPV) embeddings are a viable way to encompass nonlinearities into bounded scheduling parameters, thus maintaining linearity along suitable input-output (IO) channels [9]. W.r.t. this context, our main contributions are:

- A data-driven trajectory representation is proposed for nonlinear systems, benefiting from qLPV embeddings. The framework is an extension of the results from [5] to a much wider class of nonlinear plants.
- In consonance with [5], [7], we present data-driven simulation and prediction algorithms for nonlinear systems using input-output data and a scheduling function.
- A parameter-dependent dissipativity analysis framework is conceived for nonlinear systems, enabled through a direct verification test, as in [3], [6], yet constrained by the available scheduling variable space.
- Realistic simulation results of a rotational pendulum benchmark are presented in order to demonstrate the effectiveness and accurateness of the proposed tools.

**Paper organisation.** Sec. II provides preliminaries. Sec. III gives the main result: the trajectory representation for nonlinear systems via qLPV embeddings, and also data-driven simulation and prediction algorithms. Sec. IV details the parameter-dependent dissipativity analysis tool. Sec. IV shows some results. Conclusions are drawn in Sec. VI.

**Notation.** The identity matrix of size  $j$  is denoted as  $I_j$ . The orthogonal complement of a matrix  $A$  is denoted  $A^\perp$ . For a discrete-time signal  $v : \mathbb{N} \rightarrow \mathbb{R}^{n_v}$ , we denote  $v(k) \in \mathbb{R}^{n_v}$  each of its entries and  $\{v(k)\}_{k=0}^{N-1}$  the corresponding sequence of  $N$  data entries, or just  $v$  in short. We use  $\text{col}\{v\} := [v(0)^T \dots v(N-1)^T]^T$  to denote the column vectorisation, and  $\text{diag}\{v\}$  as the block-diagonal matrix formed with  $\text{col}\{v\}$ . The Kronecker product is represented by  $\otimes$ ; the corresponding block-diagonal operator is denoted  $\oplus$ , implying that  $(v \oplus I_\xi) = \text{diag}\{v(0) \otimes I_\xi \dots v(N-1) \otimes I_\xi\}$ . For a sequence  $\{v(k)\}_{k=0}^{N-1}$ , we have the corresponding Hankel matrix, for a window of  $L$  entries, given by:

$$H_L(v) := \begin{bmatrix} v(0) & v(1) & \dots & v(N-L) \\ v(1) & v(2) & \dots & v(N-L+1) \\ \vdots & \vdots & \ddots & \vdots \\ v(L-1) & v(L) & \dots & v(N-1) \end{bmatrix}.$$

( $\star$ ) denotes the corresponding symmetrical transpose. For two sets  $\mathbb{W}$  and  $\mathbb{T}$ ,  $\mathbb{W}^\mathbb{T}$  marks all maps from  $\mathbb{T}$  to  $\mathbb{W}$ .

## II. PRELIMINARIES AND SETTING

In this Section, we briefly recall key concepts on behavioural theory and trajectory representation for LTI systems, as well as main the arguments used to generate qLPV embeddings.

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## A. Behavioural Theory

*Definition 1 (System Behaviour [10]):* A dynamic system is given by  $\mathcal{G} := (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ , where  $\mathbb{T} \subset \mathbb{R}$  is called the time dimension,  $\mathbb{W}$  the signal space, and  $\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}}$  the system behaviour, which represents all possible trajectories of  $\mathcal{G}$ .

*Definition 2 (Manifest Behaviour [10]):* The manifest behaviour of a system  $\mathcal{G} := (\mathbb{T}, \mathbb{W}, \mathfrak{B})$  with inputs  $u \in \mathbb{R}^{n_u}$  and outputs  $y \in \mathbb{R}^{n_y}$  is given by  $\mathfrak{B}_M := \{\text{col}\{u, y\} \in \mathfrak{B} \mid \exists x \in (\mathbb{R}^{n_x})^{\mathbb{N}} \text{ s.t. Eq. (1) holds}\}$ .

$$\begin{cases} x(k+1) &= f_x(x(k), u(k)), \\ y(k) &= f_y(x(k), u(k)). \end{cases} \quad (1)$$

Thus, we say that Eq. (1) is a state-space representation of  $\mathcal{G}$  if  $\mathfrak{B}_M = \mathfrak{B}$ , i.e. all possible trajectories of  $\mathcal{G}$  are mapped.

*Definition 3 (Persistent excitation [10]):* A signal  $\{u(k)\}_{k=0}^{N-1}$ , with  $u(k) \in \mathbb{R}^{n_u}, \forall k \geq 0$ , is persistently exciting of order  $L$  if the rank of  $H_L(u) = n_u L$ .

The condition of persistent excitation is widely used in system identification theory. Def. 3 implies that  $N \geq (n_u + 1)L - 1$ . Based on the assumption of a persistently exciting input  $u$ , Willem's Lemma [10] is exploited in control theory:

*Theorem 1 (Trajectory Representation [5]):* Consider an LTI system  $\mathcal{G}$  with inputs  $u(k) \in \mathbb{R}^{n_u}$  and outputs  $y(k) \in \mathbb{R}^{n_y}$ , whose behaviour is given by the set of all trajectories  $\text{col}\{u, y\}$ , s.t.  $\exists x \in (\mathbb{R}^{n_x})^{\mathbb{N}}$  that validates Eq. (2).

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) + Du(k). \end{cases} \quad (2)$$

Consider  $\{u(k), y(k)\}_{k=0}^{N-1}$  as a trajectory of  $\mathcal{G}$ , with  $u$  persistently exciting of order  $L + n_x$ . Then, any sequence  $\{\bar{u}(k), \bar{y}(k)\}_{k=0}^{L-1}$  is also a trajectory of  $\mathcal{G}$  iff  $\exists \alpha \in \mathbb{R}^{N-L+1}$  s.t.:

$$\begin{bmatrix} H_L(u) \\ H_L(y) \end{bmatrix} \alpha = \begin{bmatrix} \text{col}\{\bar{u}\} \\ \text{col}\{\bar{y}\} \end{bmatrix}. \quad (3)$$

*Proof:* Follows from the linearity that the set of all trajectories of an LTI system is a vector space. Thus, a direct application of [10, Theo. 1] yields Eq. (3); details in [5]. ■

*Remark 1:* Theo. 1 uses the LTI model from Eq. (2) as a vector space that maps all corresponding trajectories. Moreover, it shows how time-shifts of a single measured trajectory can serve as a basis for this LTI vector space, as long as if the input is persistently exciting of sufficient order. This theorem exploits the well-known property of the existence of minimal (controllable, observable) realisations of LTI systems. The particular choice of the specific realisation is not relevant, but rather the fact that a fixed window IO trajectory  $\{u(k), y(k)\}_{k=a}^b$  imposes an unique LTI state trajectory  $\{x(k)\}_{k=a}^b$ , whenever  $b - a \geq n_x - 1$ .

*Definition 4 (Dissipativity [3]):* A system  $\mathcal{G}$  is said dissipative w.r.t. a supply rate  $\Pi \in \mathbb{R}^{(n_u+n_y) \times (n_u+n_y)}$  if the following inequality holds for all input-output trajectories of  $\mathcal{G}$ , i.e.  $\{u(k), y(k)\}_{k=0}^{\infty}$ , for null initial conditions:

$$\sum_k^r \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}^T \overbrace{\begin{bmatrix} Q & S \\ \star & R \end{bmatrix}}^{\Pi} \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \geq 0 \forall r \geq 0, \quad (4)$$

where  $Q = Q^T$ ,  $R = R^T$  and  $S$  are supply weights.

*Definition 5 (L-Dissipativity [3]):* A system  $\mathcal{G}$  is said  $L$ -dissipative w.r.t. a supply rate  $\Pi$  if the following inequality holds for all  $L$ -sized input-output trajectories of  $\mathcal{G}$ , i.e.  $\{u(k), y(k)\}_{k=0}^{L-1}$ , for null initial conditions:

$$\sum (k)^r \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}^T \Pi \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \geq 0 \forall r \in \mathbb{N}_{[0, L-1]} \quad (5)$$

*Theorem 2 (Dissipativity from Data [6]):* Suppose that  $\{u(k), y(k)\}_{k=0}^{N-1}$  is a data-dictionary of an LTI system  $\mathcal{G}$ . Then, the following statements are equivalent:

- 1)  $\mathcal{G}$  is  $L$ -dissipative w.r.t. a given supply rate  $\Pi$ .
- 2) Data  $\{u(k)\}_{k=0}^{N-1}$  is persistently exciting of order  $L + n_x$  and Ineq. (6) holds for any  $\nu$  s.t.  $n_x \leq \nu < L$ .

$$(\star) \Pi_L \left( \begin{bmatrix} H_L(u) \\ H_L(y) \end{bmatrix} V_{L,\nu}(u, y) \right) \geq 0, \quad (6)$$

$$\Pi_L := \begin{bmatrix} I_L \otimes Q & I_L \otimes S \\ \star & I_L \otimes R \end{bmatrix},$$

$$V_{L,\nu}(u, y) := \left( T_{L,\nu} \begin{bmatrix} H_L(u) \\ H_L(y) \end{bmatrix} \right)^\perp,$$

$$T_{L,\nu} := [I_{(n_u+n_y)\nu} \ 0_{((n_u+n_y)\nu) \times ((n_u+n_y)(L-\nu))}].$$

*Proof:* Full proof given in [3], [6]. ■

## B. qLPV Embeddings

*Definition 6 (Differential Inclusion [9]):* Consider that the following difference equation gives the discrete-time input-output nonlinear manifest behaviour of a system  $\mathcal{G}$ :

$$y(k) = f(y(k-1), \dots, y(k-n_a), u(k), \dots, u(k-n_b)),$$

being  $u \in \mathbb{R}^{n_u}$  the vector of inputs, and  $y \in \mathbb{R}^{n_y}$  the vector of outputs. We say that  $\mathcal{G}$  satisfies the differential inclusion property if there exists a map  $D(y(k-1), \dots, y(k-n_a), u(k), \dots, u(k-n_b)) \subseteq \mathbb{R}^{n_y \times (n_y n_a + n_u (n_b + 1))}$  such that  $f(\cdot) := D(\cdot)[y(k-1)^T \dots y(k-n_a)^T u(k)^T \dots u(k-n_b)^T]^T$ . Then, the manifest behaviour of  $\mathcal{G}$  can be stated as:

$$y(k) + \sum_{i=1}^{n_a} a_i(\rho(k-i))y(k-i) = \sum_{j=1}^{n_b} b_j(\rho(k-j))u(k-j), \quad (7)$$

where  $n_a, n_b \geq 0$ , and  $a_i \in \mathbb{R}^{n_y \times n_y}$  and  $b_i \in \mathbb{R}^{n_y \times n_u}$  are coefficient functions.

*Remark 2:* The model in Eq. (7) is quasi-LPV, with an endogenous nonlinear scheduling function  $f_\rho(\cdot)$ . For simplicity, we use  $\rho(k) = f_\rho(y(k-1))$  in the sequel; all the presented procedures can be applied to the more generic case without loss of generality.

*Proposition 1:* Consider a nonlinear system  $\mathcal{G}$  which satisfies differential inclusion, being states as in Eq. (7) with  $\rho(k) = f_\rho(y(k-1))$ . Assume that the following compact, convex constraints are respected:  $y(k) \in \mathcal{Y} \subset \mathbb{R}^{n_y}$  and  $u(k) \in \mathcal{U} \subset \mathbb{R}^{n_u}, \forall k \geq 0$ . Thus,  $\rho(k) \in \mathcal{P} \subset \mathbb{R}^{n_\rho}, \forall k \geq 0$ .

*Remark 3:* Using an IO LPV realisation, as gives Eq. (7), is rather common in LPV identification, as seen in many application results [8], [11], [12]. Consider a behaviour  $\mathfrak{B}_{\text{qLPV}} := \{\text{col}(u, \rho, y) \in (\mathcal{U} \times \mathcal{P} \times \mathcal{Y}) \mid \text{s.t. Eq. (7) holds}\}$ . Note that  $\mathfrak{B}_{\text{qLPV}}$  is linear along the  $(u, y)$  IO channels, in the sense that for any  $(u, \rho, y), (\tilde{u}, \rho, \tilde{y}) \in \mathfrak{B}_{\text{qLPV}}$  and  $\lambda, \tilde{\lambda} \in \mathbb{R}$ ,

it follows that  $(\lambda u + \tilde{\lambda}\tilde{u}, \rho, \lambda y + \tilde{\lambda}\tilde{y}) \in \mathfrak{B}_{\text{qLPV}}$ . Moreover,  $\mathfrak{B}_{\text{qLPV}}$  is time-invariant and well-defined, with a direct state-space (SS) realisation (e.g. Proposition 2).

*Proposition 2:* Assume that a nonlinear system  $\mathcal{G}$  satisfies differential inclusion, and that there exists a scheduling proxy  $f_\rho(\cdot)$  s.t. Propo. 1 holds. Then, the corresponding non-minimal SS realisation of Eq. (7) is:

$$\begin{aligned} x(k+1) &= A(\rho(k))x(k) + B(\rho(k))u(k), \\ y(k) &= C(\rho(k))x(k) + D(\rho(k))u(k), \end{aligned} \quad (8)$$

where  $x(k) = [y(k-1)', \dots, y(k-n_a)', u(k-1)', \dots, u(k-n_b)'] \in \mathbb{R}^{n_x}$  defines the state vector with  $n_x = (n_a n_y + n_b n_u)$ . Matrices  $\begin{bmatrix} A(\cdot) & B(\cdot) \\ C(\cdot) & D(\cdot) \end{bmatrix}$  are:

$$\left[ \begin{array}{cccccc|c} -a_1(\cdot) & \dots & -a_{n_a}(\cdot) & b_1(\cdot) & \dots & b_{n_b}(\cdot) & b_0 \\ I_{n_y} & \dots & 0 & 0 & \dots & 0 & 0 \\ & & \vdots & & & & \vdots \\ 0 & \dots & 0 & I_{n_u} & \dots & 0 & I_{n_y} \\ 0 & \dots & 0 & I_{n_u} & \dots & 0 & 0 \\ & & \vdots & & & & \vdots \\ \hline -a_1(\cdot) & \dots & -a_{n_a}(\cdot) & b_1(\cdot) & \dots & b_{n_b}(\cdot) & b_0(\cdot) \end{array} \right].$$

*Remark 4:* The system order is  $n_x$ ;  $n_u$  and  $n_y$  argue the number of inputs and outputs, respectively. Next, only a rough upper bound over  $n_x$  is required to quantify the persistent excitation.

### III. MAIN RESULT: A TRAJECTORY REPRESENTATION FOR NONLINEAR SYSTEMS UNDER QLPV EMBEDDING

In this Section, we extend and translate the main result of [5] to the case of nonlinear systems that allow differential inclusion (Def. 6). Specifically, we adapt the framework from [10], which describes an unknown system through the trajectory space made from measured data dictionaries, to the case of nonlinear manifest behaviours, enabled by exploiting the linearity property that is retained along the IO channels via differential inclusion. The result is an elegant trajectory representation proposition for a wide variety nonlinear systems; in [5], only Hammerstein-Wiener nonlinearities were treated, whereas herein any nonlinearity can be considered using an appropriate differential inclusion realisation, under the assumption that the scheduling proxy  $f_\rho(\cdot)$  is known. The following theorem synthesises the main result.

*Theorem 3 (Nonlinear Trajectory Representation):*

Consider a nonlinear system  $\mathcal{G}$  with inputs  $u(k) \in \mathbb{R}^{n_u}$  and outputs  $y(k) \in \mathbb{R}^{n_y}$ , which satisfies the differential inclusion property in such a way that its behaviour is given by the set of all trajectories  $\text{col}\{u, y\}$  that validate Eq. (7) with  $\rho(k) = f_\rho(y(k-1))$  and scheduling coefficients  $a_i$  and  $b_i$  affine on  $\rho$ . Equivalently, assume that these trajectories are s.t.  $\exists x \in (\mathbb{R}^{n_x})^{\mathbb{N}}$  that validates Eq. (8). Assume that the scheduling proxy  $f_\rho(\cdot)$  is a known function. Suppose that  $\{u(k), \rho(k), y(k)\}_{k=0}^{N-1}$  defines a trajectory of  $\mathcal{G}$ , being  $u$  persistently exciting of order  $L + n_x$ . Denote  $u_\rho = \{\rho(k) \otimes u(k)\}_{k=0}^{N-1}$  and  $y_\rho = \{\rho(k) \otimes y(k)\}_{k=0}^{N-1}$ . Assume that  $u_\rho$  is also persistently exciting of order  $L + n_x$ .

Then,  $\{\bar{u}(k), \bar{\rho}(k), \bar{y}(k)\}_{k=0}^{L-1}$  is also a trajectory of  $\mathcal{G}$  iff there exists some constant vector  $\alpha \in \mathbb{R}^{N-L+1}$  s.t.:

$$\begin{bmatrix} H_L(u) \\ H_L(u_\rho) \\ H_L(y) \\ H_L(y_\rho) \end{bmatrix} \alpha = \begin{bmatrix} \text{col}\{\bar{u}\} \\ \text{col}\{\bar{u}_\rho\} \\ \text{col}\{\bar{y}\} \\ \text{col}\{\bar{y}_\rho\} \end{bmatrix}. \quad (9)$$

*Proof:* Consider that the manifest behaviour of  $\mathcal{G}$  can indeed be given by Eq. (7). Consider that the input-output data dictionary  $\{u(k), y(k)\}_{k=0}^{N-1}$  is available and thus build  $\{u(k), \rho(k), y(k)\}_{k=0}^{N-1}$  using the scheduling proxy  $\rho(k) = f_\rho(y(k-1))$ . Consider that the IO coefficients  $a_i$  and  $b_i$  are affine on  $\rho \in \mathbb{R}^{n_\rho}$  and thus write  $a_i(\rho) = a_{i,0} + \sum_{j=1}^{n_\rho} a_{i,j} \rho_j$ , and  $b_i(\rho) = b_{i,0} + \sum_{j=1}^{n_\rho} b_{i,j} \rho_j$ . Append these expressions to Eq. (7) in order to obtain:  $y(k) + \sum_{i=0}^{n_a} a_{i,0} y(k-i) + \sum_{i=1}^{n_a} \tilde{a}_i(\rho(k-i) \otimes y(k-i)) = \sum_{i=0}^{n_b} b_{i,0} u(k-i) + \sum_{i=1}^{n_b} \tilde{b}_i(\rho(k-i) \otimes u(k-i))$ , where  $\tilde{a}_i := [a_{i,1} \dots a_{i,n_\rho}]$  and  $\tilde{b}_i := [b_{i,1} \dots b_{i,n_\rho}]$ . Then, let us define:

$$U_\rho(k) := \begin{bmatrix} u(k) \\ \rho(k) \otimes u(k) \end{bmatrix}, Y_\rho(k) := \begin{bmatrix} y(k) \\ \rho(k) \otimes y(k) \end{bmatrix},$$

which generate the following implicit LTI realisation:

$$\begin{aligned} \begin{bmatrix} I_{n_y} & 0 \end{bmatrix} Y_\rho(k) &= - \sum_{i=1}^{n_a} \begin{bmatrix} a_{i,0} & \tilde{a}_i \end{bmatrix} Y_\rho(k) \\ &+ \sum_{i=1}^{n_b} \begin{bmatrix} b_{i,0} & \tilde{b}_i \end{bmatrix} U_\rho(k). \end{aligned} \quad (10)$$

Assume that  $u$  and  $u_\rho$  are both persistently exciting of order  $L + n_x$ ; thus,  $U_\rho$  is also persistently exciting of order  $L + n_x$ . Then, apply Theo. (1) to Eq. (10), which implies that there exists a vector  $\alpha$  such that Eq. (9) holds. ■

*Remark 5:* From Eq. (9), it follows that  $\text{col}\{\bar{u}\} = \alpha H_L(u) = \sum_{i=0}^{N-L} \alpha_i \{u(k)\}_{k=i}^{L-1+i}$ ,  $\text{col}\{\bar{u}_\rho\} = \alpha H_L(u_\rho) = \sum_{i=0}^{N-L} \alpha_i \{\rho(k) \otimes u(k)\}_{k=i}^{L-1+i}$ ,  $\text{col}\{\bar{y}\} = \alpha H_L(y) = \sum_{i=0}^{N-L} \alpha_i \{y(k)\}_{k=i}^{L-1+i}$ , and  $\text{col}\{\bar{y}_\rho\} = \alpha H_L(y_\rho) = \sum_{i=0}^{N-L} \alpha_i \{\rho(k) \otimes y(k)\}_{k=i}^{L-1+i}$ . This means that trajectory space determined by the manifest IO LPV behaviour of  $\mathcal{G}$  is spanned by time-shifts of the measured input-scheduling-variable-output data trajectory, where the scheduling variable trajectory is given by the application of the scheduling proxy  $f_\rho(\cdot)$  over the corresponding output trajectory.

*Remark 6:* We stress that Eq. (9) is equivalent to  $\{\bar{u}(k)\}_{k=0}^{L-1} = \sum_{i=0}^{N-L} \alpha_i \{u(k)\}_{k=i}^{L-1+i}$ ,  $\{\bar{u}_\rho(k)\}_{k=0}^{L-1} = \sum_{i=0}^{N-L} \alpha_i \{u_\rho(k)\}_{k=i}^{L-1+i}$ ,  $\{\bar{y}(k)\}_{k=0}^{L-1} = \sum_{i=0}^{N-L} \alpha_i \{y(k)\}_{k=i}^{L-1+i}$ , and  $\{\bar{y}_\rho(k)\}_{k=0}^{L-1} = \sum_{i=0}^{N-L} \alpha_i \{y_\rho(k)\}_{k=i}^{L-1+i}$ . These expressions implicitly define the trajectory space spanned by time-shifts of the measured input-scheduling-data-output trajectory. Thus, the vector space defined by Eq. (10) maps some minimal state realisation (as gives Propo. 2) s.t.  $\{\bar{x}(k)\}_{k=0}^{L-1} = \sum_{i=0}^{N-L} \alpha_i \{x(k)\}_{k=i}^{L-1+i}$ , where  $\bar{x}$  is related to  $(\bar{u}, \bar{\rho}, \bar{y})$  and  $x$  relates to  $(u, \rho, y)$ .

*Remark 7:* Theo. 3 requires the persistent excitation condition over the input sequence  $u$  and input-scheduling sequence  $u_\rho$  (and not on  $u$  and  $\rho$  as in global LPV identification). Nevertheless, since we use a qLPV representation, this

requirement often boils down to having only  $u$  as persistently exciting, since  $\rho$  is a function of the inputs and outputs and thus  $\{\rho(k) \otimes u(k)\}_{k=0}^{N-1}$  also becomes persistently exciting.

The main result of this paper, synthesised through Theorem 3, shows that we can exploit the linearity property along the IO channels of Eq. (7) to span all trajectories of an unknown nonlinear system, as long as if it verifies differential inclusion, the scheduling proxy is known and at hand, and that the input is persistently exciting. In this way, one can use a single input-output data dictionary of a system  $\mathcal{G}$  to construct a larger input-scheduling-variable-output dictionary and, then, represent the trajectories of  $\mathcal{G}$  without using an explicit model identification technique. The only prior knowledge necessary is the system order size  $n_x$  (or an upper bound  $\nu \geq n_x$ ) and the scheduling proxy  $f_\rho(\cdot)$ .

The condition of a persistently exciting input signal of order  $L + n_x$  imposes a practical limit to Theo. 3: in order to construct the qLPV-embedded trajectories of length  $L$ , the window size must be  $N \geq (n_u + 1)(L + n_x) - 1$ . Thus, the major drawback is that the length of the spanned trajectory  $L$  becomes upper-bounded by  $\frac{N+1}{n_u+1} - n_x$ . This issue can be attenuated by using multiple measured trajectories to construct an overall larger trajectory, aligning the internal non-minimal states at the intersections (e.g. [5, Propo. 4]).

*Remark 8:* The scheduling proxy description  $f_\rho(\cdot)$  can be generated by phenomenological insights of the system. In many cases, the designer can derive an approximate description of  $f_\rho(\cdot)$  and consider the biases as noises upon the output measurements. In future works, we will discuss the issues of noise and non-exact knowledge of  $f_\rho(\cdot)$  in details. We mention that [13, Theos. 1 and 2] provide exact conditions for qLPV embeddings of nonlinear systems, with corresponding discussions on unique choices of the scheduling proxy, altogether with assessments on developing projecting scheduling variables that may simplify the dependency. A wide variety of examples is available in [8].

*Proposition 3 (Data-driven Simulation):* Suppose that  $\{u(k), \rho(k), y(k)\}_{k=0}^{N-1}$  is a trajectory of a discrete-time nonlinear system  $\mathcal{G}$  which verifies differential inclusion, being  $u$  persistently exciting of order  $L + n_x$ . Let  $\{\tilde{u}(k), \tilde{\rho}(k), \tilde{y}(k)\}_{k=0}^{L-1}$  be an arbitrary input-scheduling-variable-output dictionary of  $\mathcal{G}$ . If  $\nu \geq n_x$ , then  $\exists \alpha \in \mathbb{R}^{N-L+1}$  s.t. for  $\xi = N - L + \nu - 1$ :

$$\begin{bmatrix} H_L(u) \\ H_L(u_\rho) \\ H_\nu(\{y(k)\}_{k=0}^\xi) \\ H_\nu(\{y_\rho(k)\}_{k=0}^\xi) \end{bmatrix} \alpha = \begin{bmatrix} \text{col}\{\tilde{u}\} \\ \text{col}\{\tilde{u}_{\tilde{\rho}}\} \\ \text{col}\{\{\tilde{y}(k)\}_{k=0}^{\nu-1}\} \\ \text{col}\{\{\tilde{y}_{\tilde{\rho}}(k)\}_{k=0}^{\nu-1}\} \end{bmatrix},$$

while  $[\text{col}\{\tilde{y}\}^T \text{col}\{\tilde{y}_{\tilde{\rho}}\}^T]^T = [H_L(y)^T H_L(y_\rho)]^T \alpha$ .

*Proof:* Apply [5, Propo. 8] to Eq. (10) via Theo. 3. ■

*Remark 9:* The concept behind Proposition 3 is that the persistently input trajectory  $\{\tilde{u}(k)\}_{k=0}^{N-1}$  together with an initial dictionary  $\{\tilde{u}(k), \tilde{\rho}(k), \tilde{y}(k)\}_{k=0}^{\nu-1}$  define a vector  $\alpha$  which can be used to map the remaining elements of  $\tilde{y}$ . Notably, one uses an initial data dictionary  $\{u(k), \rho(k), y(k)\}_{k=0}^{N-1}$  and the new input  $\{\tilde{u}(k)\}_{k=0}^{N-1}$  and initial conditions  $\{\tilde{u}(k), \tilde{\rho}(k), \tilde{y}(k)\}_{k=0}^{\nu-1}$  to determine  $\alpha$  and,

then, the remaining elements can be directly given through  $\text{col}\{\tilde{y}\} = H_L(y)\alpha$ . For this data-driven simulation tool, the corresponding scheduling variable data  $\text{col}\{\tilde{\rho}\}$  is computed using  $\text{col}\{f_\rho^{-1}(\tilde{y})\}$ , as long as if  $f_\rho(\cdot)$  is bijective.

*Proposition 4 (Data-driven Predictor):* Suppose that  $\{u(k), \rho(k), y(k)\}_{k=0}^{N-1}$  is a sequence of data from a nonlinear discrete-time system  $\mathcal{G}$  which verifies differential inclusion, being  $u$  persistently exciting of order  $L + n_x$ . Let  $\{\tilde{u}(k), \tilde{\rho}(k), \tilde{y}(k)\}_{k=0}^{\nu-1}$  be another input-scheduling-variable-output trajectory dictionary of  $\mathcal{G}$ . Assume that the scheduling proxy  $f_\rho(\cdot)$  is not known. Let  $\{\bar{u}(k), \bar{\rho}(k)\}_{k=\nu}^{\nu+L-1}$  be the future input-scheduling-variable data of  $\mathcal{G}$ . If  $\nu \geq n_x$ , then,  $\exists \alpha \in \mathbb{R}^{N-L+1}$  for  $\xi = N - L + \nu - 1$  s.t.:

$$\begin{bmatrix} H_\nu(\{u(k)\}_{k=0}^\xi) \\ H_\nu(\{u_\rho(k)\}_{k=0}^\xi) \\ H_\nu(\{y(k)\}_{k=0}^\xi) \\ H_\nu(\{y_\rho(k)\}_{k=0}^\xi) \\ H_L(u) \\ H_L(u_\rho) - (\{\bar{\rho}(k)\} \otimes I_{n_u})H_L(u) \\ H_L(y_\rho) - (\{\bar{\rho}(k)\} \otimes I_{n_y})H_L(y) \end{bmatrix} \alpha = \begin{bmatrix} \text{col}\{\{\tilde{u}\}\} \\ \text{col}\{\{\tilde{u}_{\tilde{\rho}}\}\} \\ \text{col}\{\{\tilde{y}\}\} \\ \text{col}\{\{\tilde{y}_{\tilde{\rho}}(k)\}\} \\ \text{col}\{\{\bar{u}\}\} \\ 0 \\ 0 \end{bmatrix}.$$

Thus, the future output behaviour is  $\text{col}\{\bar{y}\} = H_L(y)\alpha$ .

*Proof:* In order to demonstrate this proposition, we make use of the reduced nominal form in Theo. 3. From Eq. (9), use  $H_L(y)\alpha = \text{col}\{\bar{y}\}$  and  $H_L(y_\rho)\alpha = \text{col}\{\bar{y}_{\tilde{\rho}}\}$ . Combining these expressions yields  $H_L(y_\rho)\alpha = \text{col}\{\bar{y}_{\tilde{\rho}}\} = (\{\bar{\rho}\} \otimes I_{n_y})\text{col}\{\bar{y}\} = (\{\bar{\rho}\} \otimes I_{n_y})H_L(y)\alpha$ , i.e.  $[H_L(y_\rho) - (\{\bar{\rho}\} \otimes I_{n_y})H_L(y)]\alpha = 0$ . Thus, we can expand (9):

$$\begin{bmatrix} H_L(u) \\ H_L(u_\rho) - (\{\bar{\rho}\} \otimes I_{n_u})H_L(u) \\ H_L(y) \\ H_L(y_\rho) - (\{\bar{\rho}\} \otimes I_{n_y})H_L(y) \end{bmatrix} \alpha = \begin{bmatrix} \text{col}\{\{\bar{u}\}\} \\ 0 \\ \text{col}\{\{\bar{y}\}\} \\ 0 \end{bmatrix}.$$

Then, by using this expression, we are able to generate a data-driven predictor, considering that future input and scheduling variable trajectories are known. ■

*Remark 10:* Propo. 4 is similar to the LPV data-driven predictor from [7, Eqs. (17)-(18)], whereas here we exploit the qLPV embedding for the nonlinear system description is a broader sense. We stress that the differences from the data-driven simulation (Propo. 3) and predictor (Propo. 4) stand for the knowledge of the scheduling parameter data along the simulation horizon.

#### IV. PARAMETER-DEPENDENT DISSIPATIVITY

Using the trajectory representation framework for nonlinear systems enabled through Theo. 3, the second main result of this work is a data-driven dissipativity analysis tool for nonlinear system. Specifically, we note that Theo. 3 constrains the representation analysis of the nonlinearities by the means of the differential inclusion, thus only describing behaviours for the measured scheduling variable space, i.e. the representation depends on  $\{\rho(k)\}_{k=0}^{N-1}$ . Thus, the following theorem adapts the LTI dissipativity analysis from data from [3] for a partial description of arbitrary nonlinear systems, valid within the measured scheduling variable space.

**Theorem 4 (Nonlinear Dissipativity Analysis from Data):** Consider an arbitrary nonlinear system  $\mathcal{G}$  for which Propo. 1 holds. Furthermore, assume that the corresponding qLPV embedding is such that coefficients  $a_i(\rho)$  and  $b_i(\rho)$  in Eq. (7) are affine on  $\rho$ . Suppose that  $\{u(k), \rho(k), y(k)\}_{k=0}^{N-1}$  is a measured input-scheduling-variable-output data dictionary of  $\mathcal{G}$  and that  $\rho(k) \in \mathcal{P}_{\text{mes}} \subseteq \mathcal{P}, \forall k \in \mathbb{N}_{[0, N-1]}$ , being  $\mathcal{P}_{\text{mes}} := \{\rho \in \mathbb{R}^{n_\rho} \mid \rho_{\min}^{\text{mes}} \leq \rho \leq \rho_{\max}^{\text{mes}}\}$ . Then, the following statements are equivalent:

- 1)  $\mathcal{G}$  is  $L$ -dissipative for any  $\{\rho(k)\}$  s.t.  $\rho(k) = f_\rho(y(k-1)) \in \mathcal{P}_{\text{mes}}$  w.r.t. to a given supply rate  $\Pi$ .
- 2) The input  $\{u(k)\}_{k=0}^{N-1}$  and the input-scheduling  $\{u_\rho(k)\}_{k=0}^{N-1}$  are persistently exciting of order  $L + n_x$  and Ineq. (11) holds for any  $n_x \leq \nu \leq L$ .

$$(\star) \Pi_L^\rho \left( \begin{bmatrix} H_L(u) \\ H_L(u_\rho) \\ H_L(y) \\ H_L(y_\rho) \end{bmatrix} V_{L,\nu} \left( \begin{bmatrix} u \\ u_\rho \end{bmatrix}, \begin{bmatrix} y \\ y_\rho \end{bmatrix} \right) \right) \geq 0, \quad (11)$$

$$\Pi_L^\rho := (\star) \Pi_L \begin{bmatrix} I & I & 0 & 0 \\ 0 & 0 & I & I \end{bmatrix}.$$

*Proof:* Follows directly from the application of Finsler's Lemma and Theo. 2 to Eq. (10), assuming Theo. 3 holds. ■

**Remark 11:** We say that Theo. 4 provides a “parameter-dependent” dissipativity condition since Ineq. (11) is related to the scheduling variable space defined implicitly by  $\mathcal{P}_{\text{mes}}$ . Through Theo. 4, dissipativity cannot be assured for all trajectories of the nonlinear system  $\mathcal{G}$ ; yet enough data can be collected s.t.  $\mathcal{P}_{\text{mes}} \approx \mathcal{P}$  and thus the result can be extended to  $\mathcal{G}$  entirely. In practice, dissipativity can be checked for a known operational region  $\mathcal{Y}_{\text{op}} \subset \mathcal{Y}$  s.t.  $\mathcal{P}_{\text{mes}} = f_\rho(\mathcal{Y}_{\text{op}})$ .

**Remark 12:** The supply rate  $\Pi$  can indicate different qualities of the system, such as the induced  $L_2$  gain of the system or its shortage of passivity. In the presence of measurement noise, Ineq. (11) can be relaxed by a sufficient and necessary condition, for which its right-side,  $(\cdot) \geq 0$ , is replaced by  $(\cdot) \geq \delta I$ , where  $\delta < 0$  is a small constant which indicates the signal-to-noise ratio, e.g. [3, Algo. 1].

**Remark 13:** All prior developments can be directly translated to the regular LPV case, with no loss of generality; the difference is that  $\rho$  is not computed via  $f_\rho(\cdot)$ , but measured.

## V. DATA-DRIVEN RESULTS

Next, we provide realistic simulation results in order to validate proposed set of tools, demonstrating their capabilities in capturing nonlinear behaviour using the trajectory data-driven framework via qLPV embeddings. For such, we use data from a nonlinear benchmark: a rotational pendulum system (Fig. 1). Accordingly, we show how Theo. 3 can be applied: we use Propo. 3 to compare the data-driven simulation results to the true outputs of the nonlinear system, for which we also compute the dissipativity verification (Theo. 4); further, we illustrate how predictions can be generated via Propo. 4, enabling the application of predictive controllers.

**Setup:** Consider the following rotational pendulum discrete-time dynamics, as detailed in [14], for which  $y$  gives the

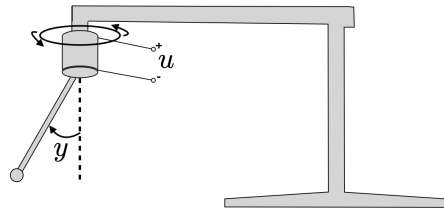


Fig. 1. Rotational pendulum system setup.

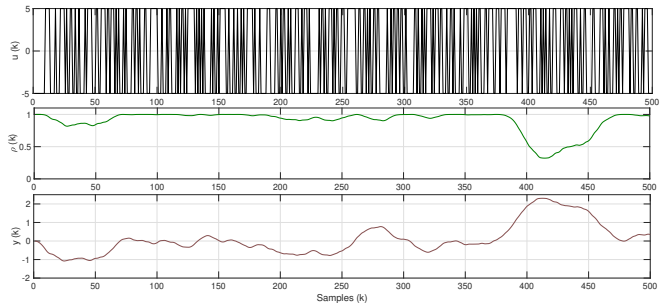


Fig. 2. Initial input-scheduling-variable-output data dictionary.

angular position, and  $u$  is an input voltage signal:

$$y(k) + \left( \frac{T_s}{\tau} - 2 \right) y(k-1) + \left( 1 - \frac{T_s}{\tau} \right) y(k-2) \quad (12)$$

$$+ T_s^2 K_y \sin(y(k-2)) y(k-2) = \left( \frac{T_s^2 K_u}{\tau} \right) u(k-2),$$

where  $\tau = 0.6$  s is a settling time parameter,  $K_y = 1.31$  m and  $K_u = 0.76$  V<sup>-1</sup> are static gains, and  $T_s = 50$  ms is the sampling period of the system. Insight into the rotational dynamics of this process leads us to the choice of the following scheduling variable  $\rho(k) = \text{sinc}(y(k)) = \frac{\sin(y(k))}{y(k)}$ , which embeds the nonlinear dynamics of Eq. (12) into a qLPV IO model in the form of Eq. (7), with affine coefficients:

$$\begin{cases} a_1(\rho(k-1)) &= \left( \frac{T_s}{\tau} - 2 \right), \\ a_2(\rho(k-2)) &= \left( 1 - \frac{T_s}{\tau} + T_s^2 K_y \rho(k-2) \right), \\ b_2(\rho(k-2)) &= \left( \frac{T_s^2 K_u}{\tau} \right). \end{cases}$$

The choice<sup>1</sup> of a cardinal sinusoidal scheduling map is natural, being coherent with the rotational dynamics of the process. Consider henceforth that this system abides to the following process constraints:  $u(k) \in \mathcal{U} = [-5, 5]$  V, and  $y(k) \in \mathcal{Y} = [-2.5, 2.5]$  rad,  $\forall k \geq 0$ , thus implying  $\rho(k) \in \mathcal{P} = [-0.22, 1], \forall k \geq 0$ ; the system order  $n_x = 4$  is known.

**Data collection:** In order to demonstrate the proposed trajectory representation tools, we first collect an initial data dictionary  $\{u(k), \rho(k), y(k)\}_{k=0}^{N-1}$  for  $N = 500$  samples. The used input is a persistently exciting frequency-rich pseudo-random signal. In Fig. 2, we show these system trajectories.

**Parameter-dependent dissipativity:** First, we test the dissipativity of the rotational pendulum system, enabled through

<sup>1</sup>In references [8], [13], one can find a wide variety of nonlinear system with corresponding scheduling parameter maps for coherent qLPV embeddings, with experimental validation results.

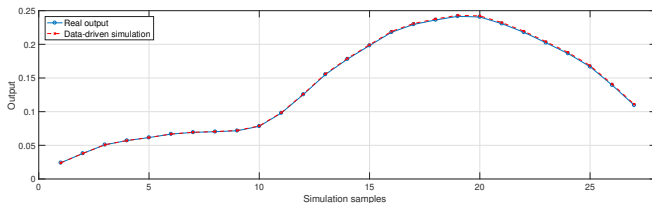


Fig. 3. Data-driven simulation: true and estimated output, using Propo. 3.

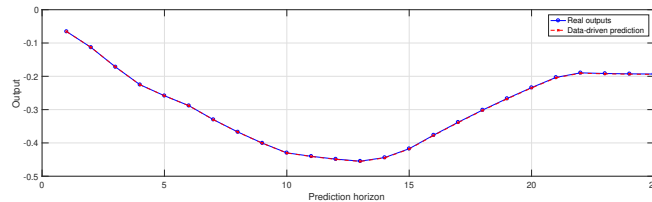


Fig. 4. Data-driven prediction: true and estimated output, using Propo. 4.

the inequality check from Theo. 4. For such, we consider the validity of the dissipativity analysis only within the scheduling parameter space defined implicitly by  $\{\rho(k)\}_{k=0}^{N-1}$ , where  $\rho(k) \in \mathcal{P}_{\text{mes}}$ , with  $\rho_{\min}^{\text{mes}} = 0.32$  and  $\rho_{\max}^{\text{mes}} = 1$  (see Fig. 2). As done in [3], we estimate the shortage of input- and output-strict passivity<sup>2</sup> by finding the minimal  $\theta$  such that the system is dissipative w.r.t. the following supply rate  $\Pi = \begin{bmatrix} \theta & \frac{1}{2} \\ \star & \theta \end{bmatrix}$ . The trajectory representation from Theo. 3 is enabled using  $L - n_x = 30$ . Using a simple bisection search, taking  $\nu = n_x = 4$ , we determine through Theo. 4 that the minimal shortage of passivity is given by  $\theta^* = 0.49$ , for any input-output sequence  $\{u(k), y(k)\}$  such that  $\rho(k) = f_\rho(y(k)) \in \mathcal{P}_{\text{mes}}$ . The knowledge of  $\theta^*$  can be exploited to design robust controller via standard tools [15].

**Data-driven simulation:** In order to demonstrate the fidelity of the proposed data-driven representation for nonlinear systems, we exploit Propo. 3 as follows: taking  $\nu = n_x = 4$  (known system order) and  $L = 30$ , we simulate the system behaviour using the data-driven tool, taking the simulated outputs as  $\text{col}\{\tilde{y}\} = H_L(y)\alpha$ , and comparing them to the real system outputs. Using a simulation horizon of 28 steps, we show in Fig. 3 the real outputs and the estimated ones. The average estimation error is of 0.09%. Clearly, the representation is trustworthy and can be extended to control synthesis.

**Data-driven predictor:** Finally, we use Propo. 4 to demonstrate the prediction qualities of the developed framework. Now, we disregard the knowledge of  $f_\rho(\cdot)$ , thus predicting the system behaviour for a future horizon with future inputs and scheduling parameters at hand. Using a prediction horizon of 25 steps, we present in Fig. 4 the real outputs and the predicted ones. The average estimation error is of 0.06%, again indicating the quality of the qLPV representation.

<sup>2</sup>Dissipativity w.r.t. measurement noise can be exploited using the relaxed inequality from [3, Algo. 1], or through the IQC features debated in [6]. We are unable to discuss such phenomena due to lack of space, leaving them as topic of future studies.

## VI. CONCLUDING REMARKS

In this brief paper, we propose a novel data-based characterisation framework for nonlinear systems. For such, we extend previous results on behavioural theory to the context of nonlinear systems, using qLPV embeddings to ensure linearity along suitable IO coordinates. The proposed tool requires only the knowledge of an appropriate scheduling function and the input trajectory to be persistently exciting over a sufficient length. Then, using a single input-output data dictionary, a larger input-scheduling-variable-output dictionary is generated, from which the data-driven trajectory framework is conceived. Data-driven simulation and prediction algorithms are also formulated, and a simple dissipativity test tool is given. Through realistic simulation of a nonlinear benchmark (rotational pendulum), we demonstrate the accurateness and fidelity of the proposed set of tools. In future works, we will discuss the issue of noisy measurements and the use of biased scheduling functions.

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