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To cite this version:
Marcelo Menezes Morato, Julio Normey-Rico, Olivier Sename. A Data-Driven Trajectory Representation for Nonlinear Systems under quasi-Linear Parameter Varying Embeddings. 2022. hal-03613735v2

HAL Id: hal-03613735
https://hal.science/hal-03613735v2
Preprint submitted on 11 Apr 2022

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A Data-Driven Trajectory Representation for Nonlinear Systems under quasi-Linear Parameter Varying Embeddings

Marcelo M. Morato¹,², Julio E. Normey-Rico¹ and Olivier Sename²

Abstract—Recent literature has shown how linear time-invariant (LTI) systems can be represented through trajectory-based features, relying on a single measured input-output (IO) trajectory dictionary, as long as the input is persistently exciting. The so-called behavioural framework is a promising alternative for controller synthesis without the necessity of system identification. In this paper, we extend and translate previous results to a wide class of nonlinear systems, using quasi-Linear Parameter Varying (qLPV) embeddings along suitable IO coordinates. Accordingly, we show how nonlinear data-driven simulation and predictions can be made based on the proposed qLPV approach. A parameter-dependent dissipativity analysis verification setup is also given. Realistic results are included to demonstrate the effectiveness of the tools. [Paper under corrections, a new version will be uploaded soon.]

I. INTRODUCTION

Modern control theory relies on the availability of trustworthy process models, and thus system identification has been an active field of research. Yet, fostering accurate models is costly, ponderous, and hindered by uncertainties. Thus, over the last decades, developing controllers directly from data has received a considerable amount of attention, specially due to reinforcement learning techniques [1] and virtual reference feedback tuning approaches [2]. Withal, as argued extensively in [1], these methods consistently require large data sets, while lacking formal guarantees on stability and performance of the resulting closed-loop.

More recently, concrete results were presented using behavioural theory as an unified approach to data-driven control [3]–[6]. This framework enables to characterise all possible trajectories of an unknown system using a single measured input–output dictionary of a fixed length, as long as the input is persistently exciting. This representation structure has been thoroughly exploited in the context of linear time-invariant (LTI), as well as for Hammerstein-Wiener plants, offering a thoroughly exploited in the context of linear time-invariant (LTI), as well as for Hammerstein-Wiener plants, offering a

behaviours under linear structures, with experimental examples registered in [8]. Under differential inclusion properties, quasi-LPV (qLPV) embeddings are a viable way to encompass nonlinearities into bounded scheduling parameters, thus maintaining linearity along suitable input-output (IO) channels [9]. W.r.t. this context, our main contributions are:

- A data-driven trajectory representation is proposed for nonlinear systems, benefiting from qLPV embeddings. The framework is an extension of the results from [5] to a much wider class of nonlinear plants.
- In consonance with [5], [7], we present data-driven simulation and prediction algorithms for nonlinear systems using input-output data and a scheduling function.
- A parameter-dependent dissipativity analysis framework is conceived for nonlinear systems, enabled through a direct verification test, as in [3], [6], yet constrained by the available scheduling variable space.
- Realistic simulation results of a rotational pendulum benchmark are presented in order to demonstrate the effectiveness and accurateness of the proposed tools.

Paper organisation. Sec. II provides preliminaries. Sec. III gives the main result: the trajectory representation for nonlinear systems via qLPV embeddings, and also data-driven simulation and prediction algorithms.

Notation. The identity matrix of size \( j \) is denoted as \( I_j \). The orthogonal complement of a matrix \( A \) is denoted \( A^\perp \). For a discrete-time signal \( v : \mathbb{N} \to \mathbb{R}^n \), we denote \( v(k) \in \mathbb{R}^n \) each of its entries and \( \{v(k)\}_{k=0}^{N-1} \) the corresponding sequence of \( N \) data entries, or just \( v \) in short. We use \( \text{col}\{v\} := [v(0)^T \ldots v(N-1)^T]^T \) to denote the column vectorisation, and \( \text{diag}\{v\} \) as the block-diagonal matrix formed with \( \text{col}\{v\} \). The Kronecker product is represented by \( \otimes \); the corresponding block-diagonal operator is denoted \( \mathcal{O} \), implying that \( (v \otimes I_\xi) = \text{diag}\{v(0) \otimes I_\xi \ldots v(N-1) \otimes I_\xi\} \). For a sequence \( \{v(k)\}_{k=0}^{N-1} \), we have the corresponding Hankel matrix, for a window of \( L \) entries, given by:

\[
H_L(v) := \begin{bmatrix}
v(0) & v(1) & \ldots & v(N-L) 
v(1) & v(2) & \ldots & v(N-L+1) 
\vdots & \vdots & \ddots & \vdots 
v(L-1) & v(L) & \ldots & v(N-1)
\end{bmatrix}
\]

\((\ast)\) denotes the corresponding symmetrical transpose. For two sets \( \mathcal{W} \) and \( T \), \( \mathcal{W}^T \) marks all maps from \( T \) to \( \mathcal{W} \).

II. PRELIMINARIES AND SETTING

In this Section, we briefly recall key concepts on behavioural theory and trajectory representation for LTI systems, as well as main the arguments used to generate qLPV embeddings.
A. Behavioural Theory

Definition 1 (System Behaviour [10]): A dynamic system is given by \( G := (T, \mathbb{W}, \mathbb{B}) \), where \( T \subset \mathbb{R} \) is called the time dimension, \( \mathbb{W} \) the signal space, and \( \mathbb{B} \subset \mathbb{W}^T \) the system behaviour, which represents all possible trajectories of \( G \).

Definition 2 (Manifest Behaviour [10]): The manifest behaviour of a system \( G := (T, \mathbb{W}, \mathbb{B}) \) with inputs \( u \in \mathbb{R}^{n_u} \) and outputs \( y \in \mathbb{R}^{n_y} \) is given by
\[
\mathcal{B}_M := \{ \text{col}\{u, y\} \mid \exists x \in (\mathbb{R}^{n_x})^T \text{s.t. Eq. (1) holds} \}
\]

Thus, we say that Eq. (1) is a state-space representation of \( G \) if \( \mathcal{B}_M = \mathcal{B} \), i.e. all possible trajectories of \( G \) are mapped.

Definition 3 (Persistent excitation [10]): A signal \( \{u(k)\}_{k=0}^{N-1} \), with \( u(k) \in \mathbb{R}^{n_u}, \forall k \geq 0 \), is persistently exciting of order \( L \) if the rank of \( H_L(u) = n_x L \).

The condition of persistent excitation is widely used in system identification theory. Def. 3 implies that \( N \geq (n_u + 1)L - 1 \). Based on the assumption of a persistently exciting input \( u \), Willem’s Lemma [10] is exploited in control theory:

Theorem 1 (Trajectory Representation [5]): Consider an LTI system \( G \) with inputs \( u(k) \in \mathbb{R}^{n_u} \) and outputs \( y(k) \in \mathbb{R}^{n_y} \), whose behaviour is given by the set of all trajectories \( \text{col}\{u, y\} \) s.t. \( \exists x \in (\mathbb{R}^{n_x})^T \) that validates Eq. (2).
\[
\begin{align*}
  x(k+1) &= Ax(k) + Bu(k), \\
  y(k) &= Cx(k) + Du(k).
\end{align*}
\]

Consider \( \{u(k), y(k)\}_{k=0}^{N-1} \) as a trajectory of \( G \), with \( u(k) \) persistently exciting of order \( L + n_x \). Then, any sequence \( \{x(k), y(k)\}_{k=0}^{L-1} \) is also a trajectory of \( G \) iff \( \exists \alpha \in \mathbb{R}^{N-L+1} \) s.t.:
\[
\begin{bmatrix}
  H_L(u) \\
  H_L(y)
\end{bmatrix}
\begin{bmatrix}
  \text{col}\{\pi\} \\
  \text{col}\{\gamma\}
\end{bmatrix}
= 0
\]

Proof: Follows from the linearity that the set of all trajectories of an LTI system is a vector space. Thus, a direct application of [10, Theo. 1] yields Eq. (3); details in [5].

Remark 1: Theo. 1 uses the LTI model from Eq. (2) as a vector space that maps all corresponding trajectories. Moreover, it shows how time-shifts of a single measured trajectory can serve as a basis for this LTI vector space, as long as if the input is persistently exciting of sufficient order. This theorem exploits the well-known property of the existence of minimal (controllable, observable) realisations of LTI systems. The particular choice of the specific realisation is not relevant, but rather the fact that a fixed window IO trajectory \( \{u(k), y(k)\}_{k=0}^{b} \) imposes an unique LTI state trajectory \( \{x(k)\}_{k=0}^{b} \) whenever \( b - a \geq n_x - 1 \).

Definition 4 (Dissipativity [3]): A system \( G \) is said dissipative w.r.t. a supply rate \( \Pi \in \mathbb{R}^{(n_u+n_y) \times (n_u+n_y)} \) if the following inequality holds for all input-output trajectories of \( G \), i.e. \( \{u(k), y(k)\}_{k=0}^{\infty} \), for null initial conditions:
\[
\sum_{k} \begin{bmatrix}
  u(k) \\
  y(k)
\end{bmatrix}^T \begin{bmatrix}
  Q & S \\
  * & R
\end{bmatrix} \begin{bmatrix}
  u(k) \\
  y(k)
\end{bmatrix} \geq 0 \forall r \geq 0 ,
\]

where \( Q = Q^T, R = R^T \) and \( S \) are supply matrices.

Definition 5 (L-Dissipativity [3]): A system \( G \) is said L-dissipative w.r.t. a supply rate \( \Pi \) if the following inequality holds for all \( L \)-sized input-output trajectories of \( G \), i.e. \( \{u(k), y(k)\}_{k=0}^{L-1} \), for null initial conditions:
\[
\sum_{k} \begin{bmatrix}
  u(k) \\
  y(k)
\end{bmatrix}^T \Pi \begin{bmatrix}
  u(k) \\
  y(k)
\end{bmatrix} \geq 0 \forall r \in \mathbb{N}_{[0,L-1]} \]

Theorem 2 (Dissipativity from Data [6]): Suppose that \( \{u(k), y(k)\}_{k=0}^{N-1} \) is a data-dictionary of an LTI system \( G \). Then, the following statements are equivalent:
1) \( G \) is L-dissipative w.r.t. a given supply rate \( \Pi \).
2) Data \( \{u(k)\}_{k=0}^{N-1} \) is persistently exciting of order \( L + n_x \) and Ineq. (6) holds for any \( \nu \) s.t. \( n_x < \nu < L \).
it follows that \((\lambda u + \tilde{\lambda} \tilde{u}, \rho, \lambda y + \tilde{\lambda} \tilde{y}) \in \mathcal{B}_{\text{qLPV}}\). Moreover, \(\mathcal{B}_{\text{qLPV}}\) is time-invariant and well-defined, with a direct state-space (SS) realisation (e.g. Proposition 2).

**Proposition 2:** Assume that a nonlinear system \(\mathcal{G}\) satisfies differential inclusion, and that there exists a scheduling proxy \(f_\rho(\cdot)\) s.t. Prop. 1 holds. Then, the corresponding non-minimal SS realisation of Eq. (7) is:

\[
\begin{align*}
    x(k+1) &= A(\rho(k))x(k) + B(\rho(k))u(k), \\
    y(k) &= C(\rho(k))x(k) + D(\rho(k))u(k),
\end{align*}
\]

where \(x(k) = [y(k-1)', \ldots, y(k-n_y)', u(k-1)', \ldots, u(k-n_u)'] \in \mathbb{R}^{n_x}\) defines the state vector with \(n_x = (n_uy + nb_uy)\). Matrices \(\begin{bmatrix} A(\cdot) & B(\cdot) \\ C(\cdot) & D(\cdot) \end{bmatrix}\) are:

\[
\begin{bmatrix}
    -a_1(\cdot) & \ldots & -a_{n_u}(\cdot) & b_1(\cdot) & \ldots & b_{n_u}(\cdot) & b_0(\cdot) \\
    I_{n_y} & \ldots & 0 & 0 & \ldots & 0 & 0 \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & \ldots & 0 & I_{n_u} & \ldots & 0 & I_{n_y} \\
    0 & \ldots & 0 & I_{n_u} & \ldots & 0 & 0 \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    -a_1(\cdot) & \ldots & -a_{n_u}(\cdot) & b_1(\cdot) & \ldots & b_{n_u}(\cdot) & b_0(\cdot)
\end{bmatrix}
\]

**Remark 4:** The system order is \(n_x\); \(n_u\) and \(n_y\) argue the number of inputs and outputs, respectively. Next, only a rough upper bound over \(n_x\) is required to quantify the persistent excitation.

### III. Main Result: A Trajectory Representation for Nonlinear Systems under qLPV Embedding

The remainder of this paper is under corrections. A suitable version will be uploaded to HAL soon.

### References


