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# Geometric optics expansion for weakly well-posed hyperbolic boundary value problem: the glancing degeneracy

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## Abstract

This article aims to finalize the classification of weakly well-posed hyperbolic boundary value problems in the half-space. Such problems with loss of derivatives are rather classical in the literature and appear for example in [19] or [2]. It is known that depending on the kind of the area of the boundary of the frequency space on which the uniform Kreiss-Lopatinskii condition degenerates then the energy estimate can include different losses. The three first possible area of degeneracy have been studied in [9] and [3] by the use of geometric optics expansions. In this article we reiterate the same kind of tools in order to deal with the last remaining case, namely a degeneracy in the glancing area. In comparison to the first cases studied we will see that the equation giving the amplitude of the leading order in the expansion, and thus initializing the whole construction of the expansion, is not a transport equation any more but it is given by some Fourier multiplier. This multiplier need to be invert in order to recover the first amplitude. As an application we discuss the existing estimates of [11]-[10] for the wave equation with Neumann boundary condition.

**AMS subject classification:** 35L04 78A05

**Key words:** hyperbolic boundary value problems, geometric optics expansions, weak well-posedness, glancing modes.

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## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Assumptions and main result</b>	<b>5</b>
2.1	Assumptions and notations . . . . .	5
2.2	Main results . . . . .	11
<b>3</b>	<b>Determination of the WKB expansion</b>	<b>12</b>
3.1	Construction of the leading order terms . . . . .	14
3.1.1	Determination of the hyperbolic leading order term . . . . .	16

3.1.2	Determination of the evanescent leading order term . . . . .	16
3.1.3	Determination of the glancing leading order term . . . . .	17
3.2	Construction of terms of order one . . . . .	22
3.2.1	Determination of the hyperbolic amplitudes of order one . . . . .	24
3.2.2	Determination of the evanescent amplitudes of order one . . . . .	25
3.2.3	Determination of the glancing leading order term . . . . .	25
3.3	Higher order terms . . . . .	28
<b>4</b>	<b>Discussion about the energy estimates</b>	<b>31</b>
<b>5</b>	<b>Some examples</b>	<b>34</b>
5.1	The $2d$ -wave equation . . . . .	34
5.2	Linearisation of Euler equation . . . . .	35

# 1 Introduction

This article deals with geometric optics expansion for hyperbolic initial boundary value problems and more precisely to the question of the loss of derivatives of such problems when the boundary condition leads to weak well-posedness. The considered problems read under the form: for fixed  $T > 0$

$$\begin{cases} L(\partial)u := \partial_t u + \sum_{j=1}^d A_j \partial_{x_j} u = f & \text{in } \Omega_T, \\ Bu|_{x_d=0} = g & \text{on } \omega_T, \\ u|_{t \leq 0} = 0, & \text{on } \mathbb{R}_+^d, \end{cases} \quad (1)$$

where the space domain is the half-space  $\mathbb{R}_+^d := \{x = (x', x_d) \in \mathbb{R}^d, x_d > 0\}$  and where we defined the sets  $\Omega_T := ]-\infty, T] \times \mathbb{R}_+^d$  and  $\omega_T := ]-\infty, T] \times \mathbb{R}^{d-1}$ .

In (1) the coefficient matrices  $A_1, \dots, A_d$  belong to  $\mathcal{M}_N(\mathbb{R})$  ( $N$  is a strictly positive integer), the matrix  $B$  belongs to  $\mathcal{M}_{p,N}(\mathbb{R})$  ( $p$  is made more precise in Assumption 2.2), and the unknown  $u$  takes its values in  $\mathbb{R}^N$ .

The strong well-posedness of the problem (1) is well established from the seminal work of [14] in which the author characterizes all the boundary matrices  $B$  leading to strong well-posedness. By strong well-posedness we mean that for all choices of the sources  $(f, g) \in L^2(\Omega_T) \times L^2(\omega_T)$  the problem (1) admits a unique solution  $u$  (with trace on  $\omega_T$  in  $L^2(\omega_T)$ ) that satisfies the following energy estimate : there exists  $C_T > 0$  such we have the inequality

$$\|u\|_{L^2(\Omega_T)}^2 + \|u|_{x_d=0}\|_{L^2(\omega_T)}^2 \leq C_T \left( \|f\|_{L^2(\Omega_T)}^2 + \|g\|_{L^2(\omega_T)}^2 \right). \quad (2)$$

In particular in the equation (2) we have a control of the solution in the same functional space than the one of the data, this optimal control is referred as strong well-posedness. Without enter into technical details the full characterization given in [14], the so-called Kreiss-Lopatinskii condition states that in the normal mode analysis of the problem (1) there is no stable modes solution of the homogeneous boundary condition. Indeed if we denote by  $\mathbb{E}^s$  the stable space, in the sense of dynamical systems, associated to the problem (1) in the frequency space then the existence of some non trivial element in  $\ker B \cap \mathbb{E}^s$  gives rise to a non trivial solution to a homogeneous linear problem and thus leads to a contradiction.

However as firstly remarked in [17] and [1] on formal geometric optics expansions the uniform Kreiss-Lopatinskii condition can degenerate without that the problem generates a Hadamard instability and that it is ill-posed. Indeed when the uniform Kreiss-Lopatinskii condition fails

then one can expect the problem to admit a unique solution  $u$  but in a less regular space than the one of the sources. Such phenomenon is referred as a loss of derivative in the following and the associated concept of well-posedness is called the weak well-posedness.

Historically the first result establishing such weak well-posedness behaviour is due to [19] and is related to elastodynamics. In this paper the author shows the existence and the uniqueness of the solution  $u$  but with an energy estimate reading

$$\|u\|_{L^2(\Omega_T)}^2 + \|u|_{x_d=0}\|_{H^{-1/2}(\omega_T)}^2 \leq C_T \left( \|f\|_{L^2(\Omega_T)}^2 + \|g\|_{H^{1/2}(\omega_T)}^2 \right), \quad (3)$$

that is to say that the solution exhibits a loss of one derivative on the boundary but no loss of derivative in the interior. Then other kinds of estimates have also been demonstrated in [8] and [7] and the associated energy estimate both shows a loss of one derivative on the boundary coupled with a loss of one half derivative respectively one derivative in the interior. Without enter into technical details we know from the so-called block structure condition that there are four kinds of degeneracy of the uniform Kreiss-Lopatinskii condition depending on the area of the frequency space where it fails. Namely the degeneracy can occur in the elliptic, mixed, hyperbolic or glancing area. Consequently there are four possible estimates, the ones described above correspond to the three first ones. The aim of the following article is to investigate the last case that is a degeneracy of the uniform Kreiss-Lopatinskii condition in the so-called glancing area.

Note that once an estimate of the form (3) has been established then a natural question is the one of the sharpness of the losses of derivatives. In order to investigate what can be the optimal losses then geometric optics expansions are commonly used. Indeed they are used for example in [9] and [3] in order to show that the estimates obtained in [19], [8] and [7] are effectively optimal.

The whole idea of geometric optics expansions is to consider the highly oscillating problem

$$\begin{cases} L(\partial)u^\varepsilon = f^\varepsilon & \text{in } \Omega_T, \\ Bu^\varepsilon|_{x_d=0} = g^\varepsilon & \text{on } \omega_T, \\ u^\varepsilon|_{t \leq 0} = 0, & \text{on } \mathbb{R}_+^d, \end{cases}$$

where the (small) parameter  $0 < \varepsilon \ll 1$  represents the typical wavelength of the highly oscillatory source terms  $f^\varepsilon$  and  $g^\varepsilon$  and to look for an approximate solution to  $u^\varepsilon$  as a sum of waves packets with amplitudes that are written as asymptotic expansion with respect to the small parameter  $\varepsilon$ . Of course such approximate solution have an interest for their own but moreover if ones is able to construct sufficiently enough terms in the geometric optics expansions then he can expect that this expansion is an approximation of  $u^\varepsilon$  with high accuracy and to use the approximate solution to exhibit that some qualitative phenomenon (such that losses of derivatives) occur on the approximate solution and so they do on the exact one.

As already mentioned this technic has been successfully used firstly in [9] and then in [3] to show the optimality of the energy estimates with the different possible losses of derivatives for the degeneracies in the hyperbolic, the mixed and the elliptic area. In this paper we use the same kind of method to investigate what can be the loss of derivatives for the last remaining case namely the degeneracy in the so-called glancing area.

A main difference between the construction of geometric optics expansion for a failure of the uniform Kreiss-Lopatinskii condition in the glancing area compared to the other degeneracies is the nature of the propagation along the boundary which is the keystone that need to be understood in order to start the resolution of the WKB cascade. To explain this let us sketch

the main ideas in the construction of the leading order amplitude as it is done in [9] and [3]. Let  $u_0$  denotes the leading order amplitude in the geometric optics expansion and let  $u_1$  be the first corrector. Then because some losses of derivatives are expected, we expect to have some amplification in the expansion compared with the source terms so that for a boundary source term of scale  $\varepsilon^\alpha$ ,  $\alpha > 0$ , we expect  $u_0$  to be of scale  $\varepsilon^0$ .

The leading order term  $u_0$  satisfies in a classical setting in geometric optics expansion some kind of polarization condition meaning essentially (and to simplify the exposition) that  $u_0 \in \mathbb{E}^s$ , the stable subspace of the problem, and that to determine the whole  $u_0$  then only the value of its trace on  $\{x_d = 0\}$  is required.

However because of the scale on the boundary source then this leading order should also satisfied the homogeneous boundary condition  $Bu_{0|_{x_d=0}} = 0$ , equation that can not be used in order to determine  $u_{0|_{x_d=0}}$  but that implies that  $u_{0|_{x_d=0}} \in \ker B \cap \mathbb{E}^s := \text{span}\{e\}$  because of the failure of the uniform Kreiss-Lopatinskii condition. So that we can write

$$u_{0|_{x_d=0}}(t, x') := \alpha_0(t, x')e,$$

and the question is to determine  $\alpha_0$ . In order to do this following [9] and [3] we should have a look to the only boundary condition involving  $u_{0|_{x_d=0}}$  that is the boundary condition for  $u_1$  in which  $u_{0|_{x_d=0}}$  appears via the unpolarized part of  $u_1$ . This boundary condition essentially read

$$B\bar{u}_{1|_{x_d=0}} = g - B\mathcal{L}u_{0|_{x_d=0}}, \quad (4)$$

where  $\bar{u}_1$  stands for the polarized part of  $u_1$ ,  $g$  is the amplitude of the boundary source and  $\mathcal{L}$  is some operator. Then the common point in [9] and [3] is to show that the operator  $\mathcal{L}$  is in fact a transport operator (with respect to the boundary variables  $(t, x')$ ) so that (4) can be solved explicitly in order to determine  $\alpha_0$ .

For the failure of the uniform Kreiss-Lopatinskii condition in the glancing area we will follow the same approach but the operator  $\mathcal{L}$  will not be a transport operator any more but some Fourier multiplier. However equation (4) will still be used in order to determine  $\alpha_0$  just by reversing the Fourier multiplier  $\mathcal{L}$ . Once that  $\alpha_0$  is determined we can initialize the resolution of the cascades of equations of the geometric optics expansion and then the order of resolution is rather classical and essentially follows [20] (see also [3]).

Let us point that in contrary to the other degeneracy then we do not have (in general) an energy estimate for such a degeneracy. Indeed it seems rather difficult to adapt the construction of the so-called Kreiss symmetrizer (the classical tool used to show *a priori* energy estimates, see [14]) as it has been done for example in [19] in order to show an *a priori* estimate. That is why in our result about losses of derivatives we will only show that losses of one half a derivative in the interior and/or on the boundary should happened (we refer to Theorem 2.3 for a precise statement). However because we do not have an energy estimate with such losses we can not conclude that it is sharp.

But let us say that in some particular setting such estimates can be found in the litterature (see [11] and [10]). More precisely in [11] the author obtains a weak well-posedness result when the so-called Kreiss-Sakamoto condition with power  $s$  holds. Even if a full characterization of the fulfilment of the Kreiss-Sakamoto condition in terms of the area of degeneracy of the uniform Kreiss-Loaptinskii condition as not achieved yet we can use this estimate for the very interesting (in view of the applications) 2d wave equation with Neumann boundary condition.

The paper is organize as follows Section 2 contains some classical definitions and notations used for the construction of geometric optics expansion for boundary value problems. The main

results of the article namely Theorems 2.2 and 2.3 state respectively, the existence of a solution to the WKB expansion when the uniform Kreiss-Lopatinskii condition fails in the glancing area, and some results about what can be the losses of derivatives in such a framework can be found in Paragraph 2.2. Section 3 gives the construction of the WKB expansion and thus the proof of Theorem 2.2. It is the technical part of the article. Then the proof of Theorem 2.3 is given in Section 4. Finally some examples namely the  $2d$ -waves equation with Neumann boundary condition and a linearisation of Euler equation are discussed in Section 5.

## 2 Assumptions and main result

In all the article  $C > 0$  stands for a constant which can change from one line to the other without changing of notation. For two vectors of same size  $u$  and  $v$  we will denote  $u \cdot v$  the euclidian scalar product of  $u$  and  $v$ . Finally for  $p, q \geq 1$  and for a matrix  $A \in \mathcal{M}_{p,q}$  we will denote  $A^T \in \mathcal{M}_{q,p}$  for the transpose matrix of  $A$ .

We introduce the following frequency space  $\Xi$  and its boundary  $\Xi_0$ :

$$\begin{aligned}\Xi &:= \{\zeta = (\gamma + i\tau, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} \setminus \{(0,0)\}, \gamma \geq 0\}, \\ \Xi_0 &:= \Xi \cap \{\gamma = 0\}.\end{aligned}$$

### 2.1 Assumptions and notations

In the following we consider the boundary value problem:

$$\begin{cases} L(\partial)u := \partial_t u + \sum_{j=1}^d A_j \partial_{x_j} u = f & \text{in } \Omega_T, \\ Bu|_{x_d=0} = g & \text{on } \omega_T, \\ u|_{t \leq 0} = 0 & \text{on } \mathbb{R}_+^d. \end{cases} \quad (5)$$

We assume that the constant coefficients  $A_j \in \mathcal{M}_N(\mathbb{R})$  give rise to an operator  $L(\partial)$  of hyperbolic type and more precisely a constantly hyperbolic operator. With more details we assume that:

**Assumption 2.1.** *There exists an integer  $q \geq 1$  and analytic, homogeneous of degree one functions  $\lambda_1, \dots, \lambda_q$  on  $\mathbb{R}^d \setminus \{0\}$  and integers  $\mu_1, \dots, \mu_q$  such that*

$$\forall \xi := (\xi_1, \dots, \xi_d) \in \mathbb{S}^{d-1}, \det \left( \tau I + \sum_{j=1}^d \xi_j A_j \right) = \prod_{k=1}^q (\tau - \lambda_k(\xi))^{\mu_k}.$$

*The eigenvalues  $\lambda_1(\xi), \dots, \lambda_q(\xi)$  are semi-simple and satisfy  $\lambda_1(\xi) < \dots < \lambda_q(\xi)$  for all  $\xi \in \mathbb{R}^d \setminus \{0\}$ .*

We restrict our analysis to a non characteristic boundary and therefore make the following assumption:

**Assumption 2.2.** *The matrix  $A_d$  is invertible and the matrix  $B$  has maximal rank, its rank  $p$  being equal to the number of positive eigenvalues of  $A_d$  (counted with their multiplicity). Moreover, the integer  $p$  satisfies  $1 \leq p \leq N - 1$ .*

Let us assume that the problem (5) is well-posed in the sense that it admits a unique solution  $u \in L^2(\Omega_T)$  then we can perform a Laplace transform in the time variable  $t$  and a Fourier transform in the tangential space variable  $x'$ . Let  $\sigma := \gamma + i\tau \in \mathbb{C}_+ := \{z \in \mathbb{C} \text{ s.t. } \text{Re}(z) > 0\}$

and  $\eta \in \mathbb{R}^{d-1}$  denote the dual variables of  $t$  and  $x'$  and let  $\widehat{\cdot}$  be the Fourier-Laplace transform. So, thanks to Assumption 2.2 the problem (5) reads in the resolvent form:

$$\begin{cases} \frac{d}{dx_d} \widehat{u}(\zeta, x_d) = \mathcal{A}(\zeta) \widehat{u}(\zeta, x_d) + A_d^{-1} \widehat{f} & \text{for } x_d \in \mathbb{R}_+, \\ B \widehat{u}(\zeta, 0) = \widehat{g}(\zeta), \end{cases} \quad (6)$$

in which  $\zeta \in \Xi$  acts like a parameter and where the so-called resolvent matrix is defined by:

$$\mathcal{A}(\zeta) := -A_d^{-1} \left( \sigma I + i \sum_{j=1}^{d-1} \eta_j A_j \right), \text{ where } \zeta := (\sigma, \eta) \in \mathbb{C}_+ \times \mathbb{R}^{d-1}.$$

So, the spectrum of  $\mathcal{A}(\zeta)$  encodes the behaviour of the solution to (6) and so the one to (5). The spectrum of  $\mathcal{A}(\zeta)$  is known and given by the following lemma due to Hersh (see [12]).

**Lemma 2.1** (Hersh). *Under Assumptions 2.1 and 2.2, for all frequency parameter  $\zeta \in \Xi \setminus \Xi_0$ , the resolvent matrix  $\mathcal{A}(\zeta)$  only admits eigenvalues with non-zero real part, and thus does not have purely imaginary eigenvalues. We denote by  $\mathbb{E}^s(\zeta)$  (resp.  $\mathbb{E}^u(\zeta)$ ) the stable (resp. unstable) space of  $\mathcal{A}(\zeta)$  that is the eigenspace associated with the negative (resp. positive) real part eigenvalues. Then, independently of  $\zeta \in \Xi \setminus \Xi_0$ ,  $\dim \mathbb{E}^s(\zeta) = p$  and  $\dim \mathbb{E}^u(\zeta) = N - p$  and we have the following decomposition:*

$$\mathbb{C}^N = \mathbb{E}^s(\zeta) \oplus \mathbb{E}^u(\zeta). \quad (7)$$

However, this lemma only gives information as long as the frequency parameter lives away from the boundary  $\Xi_0$ . The study of hyperbolic boundary value problems needs to have a look to the frequency parameters  $\zeta \in \Xi_0$  for which Lemma 2.1 does not apply anymore. Indeed in the limit  $\gamma \downarrow 0$  then the real parts of the eigenvalues may (and they do) vanish. For glancing modes in which we are especially interested in, such a degeneracy of the eigenvalue occur at the order at least two meaning that in the limit  $\gamma \downarrow 0$  some eigenvalues associated to  $\mathbb{E}^s$  and some eigenvalues associated to  $\mathbb{E}^u$  coincide. More precisely the theorem describing the behaviour of the eigenvalues of  $\mathcal{A}(\zeta)$  for  $\zeta \in \Xi_0$  is the so-called block structure theorem firstly shown in [14] for strictly hyperbolic systems (that is to say that Assumption 2.1 is satisfied with  $\mu_1 = \dots = \mu_q = 1$ ) and then extended by [18] for constantly hyperbolic operators.

**Theorem 2.1** (Block structure). *Under Assumptions 2.1 and 2.2, for all  $\underline{\zeta} \in \Xi$ , there exists a neighborhood  $\mathcal{V}$  in  $\Xi$  of  $\underline{\zeta}$ , an integer  $L \geq 1$ , a partition  $N = \mu_1 + \dots + \mu_L$ , with  $\mu_1, \dots, \mu_L \geq 1$  and an invertible matrix  $T := T(\underline{\zeta})$ , regular on  $\mathcal{V}$  such that:*

$$\forall \zeta \in \mathcal{V}, \quad T^{-1}(\zeta) \mathcal{A}(\zeta) T(\zeta) = \text{diag}(\mathcal{A}_1(\zeta), \dots, \mathcal{A}_L(\zeta))$$

where the blocks  $\mathcal{A}_j \in \mathcal{M}_{\mu_j}(\mathbb{C})$  satisfy one of the following alternatives:

1. all the elements in the spectrum of  $\mathcal{A}_j(\underline{\zeta})$  have positive real part.
2. all the elements in the spectrum of  $\mathcal{A}_j(\underline{\zeta})$  have negative real part.
3.  $\mu_j = 1$ ,  $\mathcal{A}_j(\underline{\zeta}) \in i\mathbb{R}$ ,  $\partial_\gamma \mathcal{A}_j(\underline{\zeta}) \in \mathbb{R} \setminus \{0\}$  and  $\mathcal{A}_j(\underline{\zeta}) \in i\mathbb{R}$  for all  $\zeta \in \Xi_0 \cap \mathcal{V}$ .
4.  $\mu_j > 1$  and there exists  $k_j \in i\mathbb{R}$  such that

$$\mathcal{A}_j(\underline{\zeta}) = \begin{pmatrix} k_j & i & 0 \\ & \ddots & i \\ 0 & & k_j \end{pmatrix},$$

the coefficient in the lower left corner of  $\partial_\gamma \mathcal{A}_j(\underline{\zeta}) \in \mathbb{R} \setminus \{0\}$  and for all  $\zeta \in \Xi_0 \cap \mathcal{V}$ ,  $\mathcal{A}_j(\underline{\zeta}) \in i\mathcal{M}_{\mu_j}(\mathbb{R})$ .

Such theorem motivates the following definition clarifying the terminology of glancing modes used in the introduction.

**Definition 2.1.** For  $\underline{\zeta} \in \Xi_0$ , we define:

- $\mathbb{E}$  the elliptic area which is the set of  $\underline{\zeta}$  such that Theorem 2.1 is satisfied with blocks of type 1 and 2 only.
- $\mathbb{EH}$  the mixed area which is the set of  $\underline{\zeta}$  such that Theorem 2.1 is satisfied with blocks of type 1, 2 and at least one block of type 3.
- $\mathbb{H}$  the hyperbolic area which is the set of  $\underline{\zeta}$  such that Theorem 2.1 is satisfied with blocks of type 3 only.
- $\mathbb{G}$  the glancing area which is the set of  $\underline{\zeta}$  such that Theorem 2.1 is satisfied with at least one block of type 4.

Thanks to Theorem 2.1, we thus have the following partition of  $\Xi_0$ :

$$\Xi_0 = \mathbb{E} \cup \mathbb{EH} \cup \mathbb{H} \cup \mathbb{G},$$

and in the following we will be particularly interested in a boundary frequency  $\underline{\zeta} \in \mathbb{G}$ .

However let us say that when  $\underline{\zeta} \in \Xi_0 \setminus \mathbb{G}$ , then the decomposition (7) still holds and we write:

$$\mathbb{C}^N = \mathbb{E}^s(\underline{\zeta}) \oplus \mathbb{E}^u(\underline{\zeta}), \quad (8)$$

where  $\mathbb{E}^s(\underline{\zeta})$  (resp.  $\mathbb{E}^u(\underline{\zeta})$ ) is the extension by continuity of  $\mathbb{E}^s(\zeta)$  (resp.  $\mathbb{E}^u(\zeta)$ ) up to the boundary  $\Xi_0$ .

These spaces admit the following decompositions:

$$\mathbb{E}^s(\underline{\zeta}) = \mathbb{E}_e^s(\underline{\zeta}) \oplus \mathbb{E}_h^s(\underline{\zeta}) \text{ and } \mathbb{E}^u(\underline{\zeta}) = \mathbb{E}_e^u(\underline{\zeta}) \oplus \mathbb{E}_h^u(\underline{\zeta}),$$

where  $\mathbb{E}_e^s(\underline{\zeta})$  (resp.  $\mathbb{E}_e^u(\underline{\zeta})$ ) is the generalized eigenspace associated to eigenvalues of  $\mathcal{A}(\underline{\zeta})$  with negative (resp. positive) real part and where the  $\mathbb{E}_h^s(\underline{\zeta})$ ,  $\mathbb{E}_h^u(\underline{\zeta})$  are sums of eigenspaces associated to some purely imaginary eigenvalues of  $\mathcal{A}(\underline{\zeta})$ .

But for  $\underline{\zeta} \in \mathbb{G}$ , the decomposition (8) does not hold any more because at a glancing frequency, we have  $\mathbb{E}^s(\underline{\zeta}) \cap \mathbb{E}^u(\underline{\zeta}) \neq \{0\}$ . In this setting, we introduce the following decomposition of the stable and unstable spaces:

$$\mathbb{E}^s(\underline{\zeta}) = \mathbb{E}_e^s(\underline{\zeta}) \oplus \mathbb{E}_h^s(\underline{\zeta}) \oplus \mathbb{E}_g^s(\underline{\zeta}) \text{ and } \mathbb{E}^u(\underline{\zeta}) = \mathbb{E}_e^u(\underline{\zeta}) \oplus \mathbb{E}_h^u(\underline{\zeta}) \oplus \mathbb{E}_g^u(\underline{\zeta}) \quad (9)$$

where  $\mathbb{E}_g^s(\underline{\zeta})$ ,  $\mathbb{E}_g^u(\underline{\zeta})$  are sums of eigenspaces associated to the Jordan block of type 4 of  $\mathcal{A}(\underline{\zeta})$  in Theorem 2.1 and consequently satisfying  $\mathbb{E}_g^s(\underline{\zeta}) \cap \mathbb{E}_g^u(\underline{\zeta}) \neq \{0\}$ .

In this article we will make the rather classical assumption in geometric optics expansion construction for glancing modes that these modes are of order two (see [21]). Indeed without this restriction the construction of geometric optics expansions is a rather open question. What it is known is that the associated glancing boundary layer may blow up in  $L^\infty$ -norm (see [[20]-Part III]) We will also make the simplifying (and probably not necessary assumption) that there is only one block of type 4 in Theorem 2.1.



**Assumption 2.3.** Let  $\zeta \in \mathbb{G}$  then Theorem 2.1 is satisfied with one block of type 4 only and moreover this block is of size two. In this setting, there exists  $\tilde{e} \in \mathbb{C}^N \setminus \{0\}$  such that:

$$\mathbb{E}_g^s(\zeta) = \mathbb{E}_g^u(\zeta) = \text{span}(\tilde{e}).$$

In order to perform the geometric optics expansion we need to be more precise about the hyperbolic and the glancing modes of  $\mathcal{A}(\zeta)$  that is the one associated with purely imaginary eigenvalues. Let  $i\underline{\xi}_m \in i\mathbb{R}$  be a purely imaginary eigenvalue of  $\mathcal{A}(\zeta)$  then

$$\det\left(\tau I + \sum_{j=1}^{d-1} \eta_j A_j + \underline{\xi}_m A_d\right) = 0.$$

From the hyperbolicity Assumption 2.1, there exists an index  $k_m$  such that

$$\tau + \lambda_{k_m}(\underline{\eta}, \underline{\xi}_m) = 0$$

where  $\lambda_{k_m}$  is smooth in both variables so that we can introduce

**Definition 2.2.** The set of incoming (resp. outgoing) phases for the side  $\omega_T$  denoted by  $\mathcal{I}$  (resp.  $\mathcal{O}$ ) is the set of indices  $m$  such that the group velocity  $v_m := \nabla \lambda_{k_m}(\underline{\eta}, \underline{\xi}_m)$  satisfies  $v_{m,d} > 0$  (resp.  $v_{m,d} < 0$ ).

The set of glancing phases for the side  $\omega_T$  denoted by  $\mathcal{G}$  is the set of indices  $m$  such that the group velocity  $v_m := \nabla \lambda_{k_m}(\underline{\eta}, \underline{\xi}_m)$  satisfies  $v_{m,d} = 0$ .

With this definition in hand, we can give the following description of the spaces  $\mathbb{E}_h^s(\zeta)$ ,  $\mathbb{E}_h^u(\zeta)$ ,  $\mathbb{E}_g^s(\zeta)$  and  $\mathbb{E}_g^u(\zeta)$  (see for example [4]).

**Lemma 2.2.** Under Assumptions 2.1 and 2.2; for all  $\zeta \in \Xi_0$ , we have:

$$\mathbb{E}_h^s(\zeta) = \bigoplus_{k \in \mathcal{I}} \ker \mathcal{L}(\tau, \underline{\eta}, \underline{\xi}_k), \quad \mathbb{E}_h^u(\zeta) = \bigoplus_{k \in \mathcal{O}} \ker \mathcal{L}(\tau, \underline{\eta}, \underline{\xi}_k), \quad \text{and} \quad \mathbb{E}_g^s(\zeta) = \mathbb{E}_g^u(\zeta) = \bigoplus_{k \in \mathcal{G}} \ker \mathcal{L}(\tau, \underline{\eta}, \underline{\xi}_k)$$

where  $\mathcal{L}$  stands for the symbol of  $L(\partial)$  defined by

$$\forall \omega := (\omega_0, \dots, \omega_d) \in \mathbb{R}^{d+1}, \quad \mathcal{L}(\omega) := \omega_0 I + \sum_{j=1}^d \omega_j A_j.$$

Consequently, for  $\zeta \in \mathbb{G}$ , (9) reads:

$$\mathbb{E}^s(\zeta) = \bigoplus_{k \in \mathcal{I}} \ker \mathcal{L}(\tau, \underline{\eta}, \underline{\xi}_k) \oplus \bigoplus_{k \in \mathcal{G}} \ker \mathcal{L}(\tau, \underline{\eta}, \underline{\xi}_k) \oplus \mathbb{E}_e^s(\zeta), \quad (10)$$

$$\mathbb{E}^u(\zeta) = \bigoplus_{k \in \mathcal{O}} \ker \mathcal{L}(\tau, \underline{\eta}, \underline{\xi}_k) \oplus \bigoplus_{k \in \mathcal{G}} \ker \mathcal{L}(\tau, \underline{\eta}, \underline{\xi}_k) \oplus \mathbb{E}_e^u(\zeta). \quad (11)$$

**Remark 2.1.** In fact, under Assumption 2.3, we can be more precise, the decompositions (10) and (11) are:

$$\begin{aligned} \mathbb{E}^s(\zeta) &= \bigoplus_{k \in \mathcal{I}} \ker \mathcal{L}(\tau, \underline{\eta}, \underline{\xi}_k) \oplus \text{span}(\tilde{e}) \oplus \mathbb{E}_e^s(\zeta), \\ \mathbb{E}^u(\zeta) &= \bigoplus_{k \in \mathcal{O}} \ker \mathcal{L}(\tau, \underline{\eta}, \underline{\xi}_k) \oplus \text{span}(\tilde{e}) \oplus \mathbb{E}_e^u(\zeta), \end{aligned}$$

where  $\tilde{e}$  has been introduced in Assumption 2.3 can now be chosen as a generator of  $\ker \mathcal{L}(\tau, \underline{\eta}, \underline{\xi}_{k_g})$ , where  $k_g \in \mathcal{G}$  stands for the only glancing index.

As mentioned in the introduction, the strong well posedness of the boundary value problem (5) is totally characterised in [14] by the so-called uniform Kreiss-Lopatinskii condition (that is recalled below for the reader convenience). Because we are interested in weakly well-posed problems this condition will not be satisfied in our study and we describe in Assumption 2.4 how the uniform Kreiss-Lopatinskii condition degenerates.

**Definition 2.3** (Uniform Kreiss-Lopatinskii condition). *Under Assumptions 2.1 and 2.2, let  $\zeta \in \Xi$ , and as previously we still denote by  $\mathbb{E}^s(\zeta)$  the extension by continuity of  $\mathbb{E}^s(\zeta)$  up to  $\xi \in \Xi_0$  of the well-defined (for  $\zeta \in \Xi \setminus \Xi_0$ ) stable subspace of  $\mathcal{A}(\zeta)$ . Then the boundary condition  $B$  satisfies the uniform Kreiss-Lopatinskii condition (UKL) if:*

$$\forall \zeta \in \Xi, \ker B \cap \mathbb{E}^s(\zeta) = \{0\}.$$

*In other words, the restriction of  $B$  to  $\mathbb{E}^s(\zeta)$  is invertible and we denote its inverse by  $\Phi(\zeta) := B^{-1}|_{\mathbb{E}^s(\zeta)}$ .*

**Assumption 2.4.** *Under Assumptions 2.1, 2.2 we assume that the boundary value problem (5) satisfies:*

1. *For all  $\zeta \in \Xi \setminus \mathbb{G}$ ,  $\ker B \cap \mathbb{E}^s(\zeta) = \{0\}$*
2. *There exists  $\zeta \in \mathbb{G}$  such that  $\ker B \cap \mathbb{E}^s(\zeta) = \ker B \cap \mathbb{E}_g^s(\zeta) \neq \{0\}$ . Moreover, we suppose that  $\ker B \cap \mathbb{E}^s(\zeta)$  is one-dimensional. So that there exist a vector  $e \in \mathbb{C}^N \setminus \{0\}$ ,  $\ker B \cap \mathbb{E}^s(\zeta) = \ker B \cap \mathbb{E}_g^s(\zeta) = \text{span}(e)$ .*

**Remark 2.2.** 1. *The first point of Assumption 2.4 implies in particular the so-called (weak) Kreiss-Lopatinskii condition that is*

$$\forall \zeta \in \Xi \setminus \Xi_0, \ker B \cap \mathbb{E}^s(\zeta) = \{0\}.$$

*Indeed it is known (see for example [[6] Section 4.2]) that if this condition fails then the boundary value problem (5) will develop an Hadamard instability meaning that we have an infinite number of losses of derivatives and thus we can not expected weak well-posedness. So that the uniform Kreiss-Lopatinskii condition can only degenerate for  $\zeta \in \Xi_0$ .*

2. *The second point of Assumption 2.4 states that the uniform Kreiss-Lopatinskii condition is not satisfied at one frequency  $\zeta \in \mathbb{G}$  (so that it is a glancing frequency) and that the failure of the uniform Kreiss-Lopatinskii condition occurs on the component of the stable subspace which is associated to the glancing eigenvalue (that is the block of type 4 in Theorem 2.1). Moreover we assume in a classical setting (see [9] and [3]) that this non trivial intersection becomes one dimensional only.*
3. *We have  $\tilde{e} = \lambda e$  for  $\lambda \in \mathbb{C}$  where  $\tilde{e}$  and  $e$  are defined respectively in Assumption 2.3 and 2.4. Hence, we have  $\mathbb{E}_g^s(\zeta) = \mathbb{E}_g^u(\zeta) = \text{span}(\tilde{e}) = \text{span}(e)$ . Indeed, since  $\dim \mathbb{E}_g^s = 1$ , we have  $\ker B \cap \mathbb{E}_g^s(\zeta) = \text{span}(\tilde{e})$  and combining this with the fact that  $\ker B \cap \mathbb{E}_g^u(\zeta) = \text{span}(e)$ , we obtain the result. So that in the following we will assume without loss of generality that  $\tilde{e} = e$*

We conclude this preliminary section with the introduction of some projectors that are commonly used in the construction of geometric optics expansions.

**Definition 2.4.** *Under Assumptions 2.1 and 2.2, for  $\zeta = (i\underline{\tau}, \underline{\eta}) \in \Xi_0$ , we define:*

- $\Pi_e^s := \Pi_e^s(\zeta)$  (resp.  $\Pi_e^u = \Pi_e^u(\zeta)$ ) *the spectral projector on  $\mathbb{E}_e^s(\zeta)$  (resp.  $\mathbb{E}_e^u(\zeta)$ ).*

- For  $k \in \mathcal{I} \cup \mathcal{O} \cup \mathcal{G}$ ,  $\Pi^k := \Pi^k(\underline{\zeta})$  the orthogonal projector on  $\ker \mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k)$ .
- For  $k \in \mathcal{I} \cup \mathcal{O} \cup \mathcal{G}$ ,  $\Upsilon^k := \Upsilon^k(\underline{\zeta})$  the partial inverse of  $\mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k)$  characterized by the relations:

$$\begin{cases} \Upsilon^k \mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k) = \mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k) \Upsilon^k = I - \Pi^k, \\ \text{Im } \Pi^k = \ker \mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k) = \ker \Upsilon^k, \\ \ker \Pi^k = \text{Im } \mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k) = \text{Im } \Upsilon^k. \end{cases} \quad (12)$$

**Remark 2.3.** Since we have only one glancing phase, we denote  $\Pi^g := \Pi^k$  and  $\Upsilon^g := \Upsilon^k$  for the glancing phase  $k \in \mathcal{G}$ .

We now introduce some material which are commonly used for geometric optics expansions without the uniform Kreiss-Lopatinskii condition (see [9]). From Assumption 2.4 the vector space  $B\mathbb{E}^s(\underline{\zeta})$  is  $(p-1)$ -dimensional. We can therefore write it as the kernel of a complex linear form

$$B\mathbb{E}^s(\underline{\zeta}) = \{X \in \mathbb{C}^p, b \cdot X = 0\} \quad (13)$$

for a suitable vector  $b \in \mathbb{C}^p \setminus \{0\}$ .

Then, we can choose a supplementary vector space  $\widetilde{\mathbb{E}}^s(\underline{\zeta})$  of  $\text{span}(e)$  in  $\mathbb{E}^s(\underline{\zeta})$ :

$$\mathbb{E}^s(\underline{\zeta}) = \text{span}(e) \oplus \widetilde{\mathbb{E}}^s(\underline{\zeta}) \quad (14)$$

The matrix  $B$  induces an isomorphism from  $\widetilde{\mathbb{E}}^s(\underline{\zeta})$  to the hyperplane  $B\mathbb{E}^s(\underline{\zeta})$  and we denote its inverse  $\Phi := \Phi(\underline{\zeta}) := B|_{\widetilde{\mathbb{E}}^s(\underline{\zeta})}^{-1}$ .

**Remark 2.4.** In practice, we can take  $\widetilde{\mathbb{E}}^s(\underline{\zeta}) = \mathbb{E}_h^s(\underline{\zeta}) \oplus \mathbb{E}_e^s(\underline{\zeta})$ , due to Remark 2.2. Thus, we have the decomposition:

$$\mathbb{E}^s(\underline{\zeta}) = \mathbb{E}_h^s(\underline{\zeta}) \oplus \text{span}(e) \oplus \mathbb{E}_e^s(\underline{\zeta}) = \bigoplus_{k \in \mathcal{I}} \ker \mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k) \oplus \text{span}(e) \oplus \mathbb{E}_e^s(\underline{\zeta}). \quad (15)$$

In the following, to determine the hyperbolic and elliptic amplitudes of high order (and more precisely their traces) we will need the following projectors<sup>1</sup>.

**Definition 2.5.** Under Assumptions 2.1, 2.2 and 2.3 for  $\underline{\zeta} \in \Xi_0$ , we define:

- for  $k \in \mathcal{I}$ ,  $P^k := P^k(\underline{\zeta})$  the projector on  $\ker \mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}_k)$  with respect to the decomposition (15).
- $P^e := P^e(\underline{\zeta})$  the projector on  $\mathbb{E}_e^s(\underline{\zeta})$  with respect to the decomposition (15).

For technical reasons that will be made precise during the construction of the WKB expansion, we assume that the following holds.

**Assumption 2.5.** Under Assumptions 2.1, 2.2, 2.3 and 2.4, we assume that for the index  $g \in \mathcal{G}$ ,  $b \cdot B\Upsilon^g A_d e \neq 0$ .

<sup>1</sup>Note in particular that we do not use or require the boundedness of the projector on  $\mathbb{E}_g^s$  in the decomposition (15). So for this special point we do not require the glancing mode to be on size two (see [20] and [13]).

**Remark 2.5.** If we have  $\mathbb{E}^s(\underline{\zeta}) = \ker \mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\xi}) = \text{span } e$  which is automatic for  $2 \times 2$  systems, then we have  $b \cdot B\Upsilon^g A_d e \neq 0$ .

Indeed proceed by contradiction and assume that  $\Upsilon^g A_d e \in \mathbb{E}^s(\underline{\zeta}) = \text{Im } \Pi^g$ . However,  $\Upsilon^g A_d e \in \text{Im } \Upsilon^g = \ker \Pi^g$  and  $\ker \Pi^g \cap \text{Im } \Pi^g = \{0\}$ , because  $\Pi^g$  is a projection. So, we deduce that  $\Upsilon^g A_d e = 0$  and so, applying the symbol  $\mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\xi})$ ,

$$\Pi^g A_d e = A_d e.$$

Next, since  $e \in \mathbb{E}^s(\underline{\zeta}) = \text{Im } \Pi^g$ , we can write  $e = \Pi^g f$  where  $f \in \mathbb{C}^N \setminus \{0\}$ . And, using Lemma 3.1, we obtain:

$$A_d e = \Pi^g A_d e = \Pi^g A_d \Pi^g f = 0.$$

We deduce that  $e = 0$  which is a contradiction and so  $b \cdot B\Upsilon^g A_d e \neq 0$ .

□

## 2.2 Main results

In this section we state the two main results of the article. The first one deals with the existence of a solution to the geometric optic expansion cascade of equations at any order. The amplitudes of the profiles are rather similar to the one introduced in [21] in order to treat glancing phases. In particular they involve two boundary layers one in  $\frac{1}{\varepsilon}$ , it is associated to the elliptic modes, and one in  $\frac{1}{\sqrt{\varepsilon}}$  which is proper to the glancing mode.

**Theorem 2.2.** Under Assumptions 2.1-2.2-2.3-2.4 and 2.5 then for all  $n \in \mathbb{N}$ , the cascades of equations (21)-(22) and (23) admit a unique solution  $(u_{n,k}, u_n^g, U_{ev,n})_{n \in \mathbb{N}, k \in \mathcal{I} \cup \mathcal{O}, g \in \mathcal{G}} \subset H^\infty(\Omega_T) \times \mathbf{P}_g \times \mathbf{P}_{ev}$  where the evanescent and glancing profile sets,  $\mathbf{P}_g$  and  $\mathbf{P}_{ev}$ , are defined in Definition 3.1.

The construction of the geometric optics expansion is given in Section 3.

Note that in a general setting because there is no general well-posedness theory for hyperbolic boundary value problems when the uniform Kreiss-Lopatinskii condition fails in the glancing area then we can not justify that the approximated solution given by the geometric optics expansion is effectively an approximate solution to the boundary value problem (5) for highly oscillating source terms.

However if such a weak well-posedness theory is available then we can show (see Section 4) that the solution to the geometric optics expansion is effectively an approximate solution. Moreover for some examples like the interesting case of the wave equation for Neumann boundary condition such a weak well-posedness theory exists (see [11] and [10]) so, on some examples, we can also conclude that the geometric optics expansion is an approximate solution.

Our second result follows closely [9] and [3] by using the geometric optics expansion in order to understand what can be (we can not say are here because of the lack of a weak well-posedness theory) the losses of derivatives in the energy estimate. The result is the following and its proof is given in Section 4.

**Theorem 2.3.** Let Assumptions 2.1, 2.2, 2.3, 2.4 and 2.5 be satisfied and let  $s_1, s_2 \geq 0$  and  $T > 0$ . Assume that for all sources  $f \in L^2(\mathbb{R}_{x_d}^+; H_{t,x'}^{s_1}(\omega_T))$  and  $g \in H^{s_2}(\omega_T)$  that vanish for

$t \leq 0$ , there exists a unique  $u \in L^2(\Omega_T)$  vanishing for  $t \leq 0$  that is a weak solution to the problem (5), and that satisfies an energy estimate of the form:

$$\|u\|_{L^2(\Omega_T)} \leq C_T \left( \|f\|_{L^2(\mathbb{R}_{x_d}^+; H_{t,x'}^{s_1}(\omega_T))} + \|g\|_{H^{s_2}(\omega_T)} \right). \quad (16)$$

Then we have the following possible alternatives

1. If the extra Assumption 4.1 about the existence of some suitable for amplification outgoing mode holds then we have  $s_1 \geq \frac{1}{2}$ .
2. If the system (5) admits an energy estimate with no loss in the interior i.e.  $s_1 = 0$  then  $s_2 \geq \frac{1}{2}$ .
3. In the general framework we have a loss of derivative on the interior or on the boundary. More precisely  $s_1 \geq \frac{1}{2}$  or  $s_2 \geq \frac{1}{2}$ .

### 3 Determination of the WKB expansion

This section is devoted to the construction of the geometric optics expansion that is the proof of Theorem 2.2. We consider the initial boundary value problem (5) with highly oscillating interior and boundary source terms. More precisely

$$\begin{cases} L(\partial)u^\varepsilon = f^\varepsilon & \text{in } \Omega_T, \\ Bu^\varepsilon|_{x_d=0} = g^\varepsilon & \text{on } \omega_T, \\ u^\varepsilon|_{t \leq 0} = 0. \end{cases} \quad (17)$$

In all that follows we fix  $\zeta := (\underline{i}\tau, \underline{\eta}) \in \mathbb{G}$  such that Assumptions 2.3 and 2.4 are satisfied. In order to define the sources in (17), we introduce the phases functions:

$$\psi(t, x') = \underline{i}\tau t + \underline{\eta} \cdot x' \quad \text{and} \quad \begin{cases} \varphi_k(t, x) = \psi(t, x') + \underline{\xi}_k x_d & \text{for } k \in \mathcal{I} \cup \mathcal{O} \\ \varphi^g(t, x) = \psi(t, x') + \underline{\xi}^g x_d & \text{for } k \in \mathcal{G} := \{g\}, \end{cases}$$

where the  $\underline{\xi}_k$  stand for the roots in the  $\xi$  variable of the dispersion relation  $\det \mathcal{L}(\underline{i}\tau, \underline{\eta}, \xi) = 0$ .

We define the sources  $f^\varepsilon$  and  $g^\varepsilon$  under the form

$$f^\varepsilon := \varepsilon^{\frac{1}{2}} \left( \sum_{k \in \mathcal{I} \cup \mathcal{O}} e^{\frac{i}{\varepsilon} \varphi_k} f_k \right) + e^{\frac{i}{\varepsilon} \varphi^g} f^g + e^{\frac{i}{\varepsilon} \psi} f_{ev}, \quad (18)$$

$$g^\varepsilon := \varepsilon^{\frac{1}{2}} e^{\frac{i}{\varepsilon} \psi} g, \quad (19)$$

where for all  $k \in \mathcal{I} \cup \mathcal{O}$ , the amplitudes  $f_k \in H^\infty(\Omega_T)$ ,  $f^g \in \mathbf{P}_g$  (see below) and  $g \in H_\gamma^\infty(\omega_T)$ . All these terms vanish for negative times and for  $X \subset \Omega_T$  the space  $H_\gamma^\infty(X)$  being defined by

$$H_\gamma^\infty(X) := \{u \in L^2(X) \mid e^{-\gamma t} u \in H^\infty(X)\}.$$

The elliptic source term  $f_{ev}$  lies in  $\mathbf{P}_{ev}$  the space of elliptic profiles introduced in [16], note that this treatment of the elliptic modes differs a little from the one of [21] because it is done in a monoblock framework and in particular no diagonalization property is required on the elliptic block. Also remark that compared to [16] our elliptic profiles depend on  $x_d$  only through the fast variable  $\frac{x_d}{\varepsilon}$ . This will simplify some points of the proofs compared to [3] where some lifting of double traces on  $\{x_d = \chi = 0\}$  was required (see [16] or [3] for more details).

**Definition 3.1.** [Boundary layer profiles] The set of elliptic (or evanescent) profiles  $\mathbf{P}_{ev}$  is defined by

$$\mathbf{P}_{ev} := \{F = F(t, x', X_d) \in H^\infty(\omega_T \times \mathbb{R}_+) \setminus \exists \delta > 0, e^{\delta X_d} F(t, x', X_d) \in H^\infty(\omega_T \times \mathbb{R}_+)\}.$$

We define similarly the set of glancing profiles  $\mathbf{P}_g$ :

$$\mathbf{P}_g := \mathbf{P}_{ev} = \{F = F(t, x', \chi) \in H^\infty(\omega_T \times \mathbb{R}_+) \setminus \exists \delta > 0, e^{\delta \chi} F(t, x', \chi) \in H^\infty(\omega_T \times \mathbb{R}_+)\}.$$

We postulate for ansatz

$$\begin{aligned} u^\varepsilon(t, x) \sim & \sum_{n \geq 0} \varepsilon^{\frac{n}{2}} \sum_{k \in \mathcal{I} \cup \mathcal{O}} e^{\frac{i}{\varepsilon} \varphi_k(t, x)} u_{n,k}(t, x) + \sum_{n \geq 0} \varepsilon^{\frac{n}{2}} e^{\frac{i}{\varepsilon} \varphi^g(t, x)} u_n^g(t, x', \frac{x_d}{\sqrt{\varepsilon}}) \\ & + \sum_{n \geq 0} \varepsilon^{\frac{n}{2}} e^{\frac{i}{\varepsilon} \psi(t, x')} U_{ev,n}(t, x', \frac{x_d}{\varepsilon}), \end{aligned} \quad (20)$$

where for all  $n \geq 0$  the evanescent profile  $U_{ev,n} \in \mathbf{P}_{ev}$ , the hyperbolic and glancing amplitudes  $(u_{n,k})_{k \in \mathcal{I} \cup \mathcal{O}} \in H^\infty(\Omega_T)$  and  $u_n^g \in \mathbf{P}_g$ .

We denote  $\chi := \frac{x_d}{\sqrt{\varepsilon}}$  the fast variable for the glancing boundary layer and  $X_d := \frac{x_d}{\varepsilon}$  the one of the elliptic boundary layer.

Remark in particular that the amplitudes  $u_n^g$  for glancing mode and  $U_{ev,n}$  for the elliptic modes are functions of  $(t, x', \chi)$  and  $(t, x', X_d)$  so that their only dependency with respect to  $x_d$  is made in the fast variable. It differs from the hyperbolic amplitudes  $u_{n,k}$ .

Plugging the ansatz (20) in the evolution equation of (17) leads, after identification on the  $\varepsilon^{\frac{n}{2}}$  and using the linear independence of the phases functions, to the following cascade of equations:

$$\begin{cases} \mathbb{I}_{-1} : \begin{cases} \mathcal{L}(d\varphi^g)u_0^g = 0, \\ \mathcal{L}(d\varphi_k)u_{0,k} = 0, \quad \forall k \in \mathcal{I} \cup \mathcal{O}, \\ L(\partial_{X_d})U_{ev,0} = 0, \end{cases} \\ \mathbb{I}_{-\frac{1}{2}} : \begin{cases} i\mathcal{L}(d\varphi^g)u_1^g + A_d \partial_\chi u_0^g = 0, \\ \mathcal{L}(d\varphi_k)u_{1,k} = 0, \quad \forall k \in \mathcal{I} \cup \mathcal{O}, \\ L(\partial_{X_d})U_{ev,1} = 0, \end{cases} \\ \text{for } n \geq 0, \mathbb{I}_{\frac{n}{2}} : \begin{cases} i\mathcal{L}(d\varphi^g)u_{n+2}^g + A_d \partial_\chi u_{n+1}^g + L'(\partial)u_n^g = \delta_{n,0} f^g, \\ i\mathcal{L}(d\varphi_k)u_{n+2,k} + L(\partial)u_{n,k} = \delta_{n,1} f_k \quad k \in \mathcal{I} \cup \mathcal{O}, \\ L(\partial_{X_d})U_{ev,n+2} + L'(\partial)U_{ev,n} = \delta_{n,0} f_{ev}, \end{cases} \end{cases} \quad (21)$$

where  $L(\partial_{X_d}) := A_d(\partial_{X_d} - \mathcal{A}(\zeta))$  is the operator of derivation with respect to the fast variable  $X_d$  and  $L'(\partial) := \partial_t + \sum_{j=1}^{d-1} A_j \partial_{x_j}$  is the operator of differentiation with respect to the tangential (slow) variables. In (21),  $\delta_{\cdot, \cdot}$  stands for the Kronecker symbol.

Plugging the ansatz (20) in the boundary condition of (17) gives:

$$\mathbb{B}_{\frac{n}{2}} : B \left( \sum_{k \in \mathcal{I} \cup \mathcal{O}} u_{n,k}|_{x_d=0} + u_n^g|_{\chi=0} + U_{ev,n}|_{x_d=0} \right) = \delta_{n,1} g. \quad (22)$$

Finally, plugging the ansatz (20) in the initial condition (17) gives:

$$\forall n \in \mathbb{N}, \forall k \in \mathcal{I} \cup \mathcal{O}, \quad u_{n,t < 0}^g = u_{n,k}|_{t < 0} = U_{ev,n}|_{t < 0} = 0. \quad (23)$$

In the following, we describe how to construct the amplitudes  $u_n^g$ ,  $u_{n,k}$  and  $U_{ev,n}$  solving the cascades of equation (21), (22), (23). The construction is really classical for hyperbolic modes. Indeed because of polarization conditions and Lax lemma [15] the hyperbolic amplitudes solve transport equations. If the mode is incoming (resp. outgoing) there is (resp. there is not) a boundary condition on  $\{x_d = 0\}$  to solve. However because the uniform Kreiss-Lopatinskii condition holds for hyperbolic modes this condition can be solved by inverting the boundary matrix  $B$ . The construction for such modes is performed in Paragraph 3.1.1.

The construction for elliptic modes is also rather classical. In the space  $P_{ev}$  the solution of the evolution equation in the fast variable  $X_d$  can be written via Duhamel formula as the evolution of the trace on  $\{X_d = 0\}$  plus the contribution to the interior source (which are supposed to be known). Once again because the uniform Kreiss-Lopatinskii condition holds for elliptic mode we have an explicit formula for the trace on  $\{X_d = 0\}$ . The determination of elliptic modes occupies Paragraph 3.1.2.

The main difficulty in the construction is the determination of glancing mode. Indeed on the one hand Lax lemma applies so that it should solve the tangential (note that for a glancing mode the group velocity  $v_g := (v'_g, 0)$ ) transport equation

$$\partial_t u_0^g + v'_g \cdot \nabla_{x'} u_0^g = 0, \quad (24)$$

equation that does not require any boundary condition on  $\omega_T$ . But on the other hand we require a boundary condition in (22) to have a good error term on the boundary. Boundary condition that overdetermined (24). This difficulty has been first encounter in [20] and then overcome in [21] by the introduction of the large boundary layer of size  $\sqrt{\varepsilon}$ .

One extra difficulty compared to [20] is that to determine the value of the boundary layer trace on  $\{\chi = 0\}$  which is required for its determination we can not use the uniform Kreiss-Lopatinskii condition any more. Indeed at some step we will obtain that  $u_{0|\chi=0}^g \in \ker B \cap \mathbb{E}_g^s(\underline{\zeta})$  so that  $u_{0|\chi=0}^g$  reads  $u_{0|\chi=0}^g(t, x') := \alpha_0(t, x')e$  and the whole question is to find a way to determine the good value amplitude  $\alpha_0$ . To do so we will follow the method of [9] (see also [3]) consisting in considering the higher order boundary condition in order to derive some compatibility condition on the trace. Compared to [9] and [3] the equation determining  $\alpha_0$  will not be a simple transport equation any more but it will involve some Fourier multiplier (see equation (47) for more details). The construction of the glancing boundary layer is made in Paragraph 3.1.3.

### 3.1 Construction of the leading order terms

In this paragraph, we first use the pseudo-inverses introduced in Definition 2.4 in order to recover the usual so-called polarization condition for hyperbolic and glancing modes.

We next use Lax lemma [15] to obtain transport equation with respect to the group velocity for the hyperbolic and glancing leading amplitudes which are classical in geometric optics expansions. Finally, we will determine the hyperbolic and elliptic amplitudes first because they are mandatory to determine the glancing one.

From  $\mathbb{I}_{-1}$ , we see that  $u_0^g \in \ker \mathcal{L}(d\varphi^g)$  and for all  $k \in \mathcal{I} \cup \mathcal{O}$  we also have  $u_{0,k} \in \ker \mathcal{L}(d\varphi^k)$ . So that it turns out that we have the usual polarization conditions

$$\Pi^g u_0^g = u_0^g, \quad (25)$$

$$\Pi^k u_{0,k} = u_{0,k}, \quad \forall k \in \mathcal{I} \cup \mathcal{O}. \quad (26)$$

Next, applying the pseudo-inverse  $\Upsilon^g$  to the equation for glancing mode of  $\mathbb{I}_{-\frac{1}{2}}$  gives:

$$(I - \Pi^g)u_1^g = i\Upsilon^g A_d \partial_\chi u_0^g. \quad (27)$$

Now, we apply respectively the projector  $\Pi^g$  and  $\Pi^k$  to the equations for glancing and hyperbolic modes of  $\mathbb{I}_0$  to obtain:

$$\begin{cases} \Pi^g L'(\partial)u_0^g + \Pi^g A_d \partial_\chi u_1^g = \Pi^g f^g, \\ \Pi^k L(\partial)\Pi^k u_{0,k} = 0 \end{cases} \quad k \in \mathcal{I} \cup \mathcal{O}. \quad (28)$$

Decomposing  $u_1^g$  in terms of polarized and unpolarized parts, gives:

$$\Pi^g L(\partial)u_0^g + \Pi^g A_d \Pi^g \partial_\chi u_1^g + \Pi^g A_d \partial_\chi (I - \Pi^g)u_1^g = \Pi^g f^g. \quad (29)$$

Using equations (25), (27) and (29), we thus obtain:

$$\Pi^g L(\partial)\Pi^g u_0^g + \Pi^g A_d \Pi^g \partial_\chi u_1^g + i\Pi^g A_d \Upsilon^g A_d \Pi^g \partial_\chi^2 u_0^g = \Pi^g f^g. \quad (30)$$

We now need some lemmas to recover a transport equation due to Williams [21] and Lax [15].

**Lemma 3.1** ([21], Lemma 8.1). *Under Assumptions 2.1, 2.2 and 2.3 we have:*

$$\Pi^g A_d \Upsilon^g A_d \Pi^g = \frac{1}{c} \Pi^g \text{ and } \Pi^g A_d \Pi^g = 0,$$

where  $c \in \mathbb{R} \setminus \{0\}$ .

**Lemma 3.2** (Lax lemma [15]). *Under Assumption 2.1, we have:*

$$\forall k \in \mathcal{I} \cup \mathcal{O} \cup \mathcal{G}, \Pi^k L(\partial)\Pi^k = (\partial_t + v_k \cdot \nabla_x) \Pi^k$$

where the velocity  $v_k$  is the so-called group velocity associated to  $k$  introduced in Definition 2.2.

We recall that we have only one glancing frequency so that  $\mathcal{G} = \{g\}$  and the associated group velocity  $v_g := (v'_g, 0)$  for some  $v'_g \in \mathbb{R}^{d-1}$ .

Using these two lemmas and equations (25),(26) permit to rewrite equations (30) and (28) as:

$$-\partial_\chi^2 u_0^g + ic (\partial_t + v'_g \cdot \nabla_{x'}) u_0^g = ic \Pi^g f^g, \quad (31)$$

$$(\partial_t + v_k \cdot \nabla_x) u_{0,k} = 0, \quad k \in \mathcal{I} \cup \mathcal{O}. \quad (32)$$

Combined with the equation for the first elliptic mode

$$L(\partial_{X_d})U_{ev,0} = 0. \quad (33)$$

Then we consider the first order boundary condition, namely  $\mathbb{B}_0$ . It reads:

$$B \left( \sum_{k \in \mathcal{I} \cup \mathcal{O}} u_{0,k}|_{x_d=0} + u_{0|_{x=0}}^g + U_{ev,0}|_{x_d=0} \right) = 0,$$

so that we can decouple the boundary condition like:

$$Bu_{0,k}|_{x_d=0} = 0 \quad \forall k \in \mathcal{I} \cup \mathcal{O}, \quad (34)$$

$$Bu_{0|_{x=0}}^g = 0, \quad (35)$$

$$BU_{ev,0}|_{x_d=0} = 0. \quad (36)$$



Finally for the initial condition we will solve

$$\forall k \in \mathcal{I} \cup \mathcal{O}, \quad u_{0|t \leq 0}^g = u_{0,k|t \leq 0} = U_{ev,0|t \leq 0} = 0.$$

To determine the leading order amplitudes, we determine in the following the hyperbolic one then the elliptic one and finally the glancing one. We can determine the hyperbolic and evanescent ones but for the glancing one, since the uniform Kreiss-Lopatinskii condition does not hold, we follow the analysis of Coulombel-Guès [9] which requires to consider the boundary condition  $\mathbb{B}_{\frac{1}{2}}$  which reveals the hyperbolic and evanescent amplitudes of order one.

### 3.1.1 Determination of the hyperbolic leading order term

The transport for the outgoing phases goes from the interior of the domain  $\Omega_T$  to the boundary  $\omega_T$  and consequently in the resolution of (32) no boundary condition has to be imposed on the boundary  $\omega_T$ . Using (32) we thus determine the outgoing amplitudes by resolving the homogeneous transport equation:

$$\forall k \in \mathcal{O}, \quad \begin{cases} (\partial_t + v_k \cdot \nabla_x) u_{0,k} = 0, \\ u_{0,k|t \leq 0} = 0, \end{cases}$$

and thus  $u_{0,k} \equiv 0$  for all  $k \in \mathcal{O}$ .

To determine the glancing leading order term, we will also require to know the  $u_{1,k} = \Pi^k u_{1,k}$  for  $k \in \mathcal{O}$ . Proceeding as it has been done for the leading order term, it is easy to see that these amplitudes satisfy the transport equations

$$\forall k \in \mathcal{O}, \quad \begin{cases} (\partial_t + v_k \cdot \nabla_x) \Pi^k u_{1,k} = \Pi^k f_k, \\ \Pi^k u_{1,k|t \leq 0} = 0, \end{cases}$$

so that integrating the equation along the characteristics, we determine the  $u_{1,k}$  for all outgoing  $k$  in terms of the given sources  $f_k$ . Because they are solution to a linear transport equation with a source in  $H^\infty(\Omega_T)$  the  $u_{1,k} \in H^\infty(\Omega_T)$ .

We now turn to the construction of the incoming amplitudes. In that case the transport goes from the boundary  $\omega_T$  to the interior of  $\Omega_T$  and it is thus needed to know the value of  $u_{0,k}$  in order to solve the transport equation. For these phases, due to the Remark 2.4, we see that for  $k \in \mathcal{I}$ ,  $\Pi^k u_{0,k} \in \widehat{\mathbb{E}}^s(\zeta)$  and so we deduce that  $\Pi^k u_{0,k|_{x_d=0}} = \Phi(0) = 0$ , for all  $k \in \mathcal{I}$ . Hence, the amplitudes  $u_{0,k}$  for incoming modes solve the transport equations

$$\begin{cases} (\partial_t + v_k \cdot \nabla_x) u_{0,k} = 0, \\ u_{0,k|_{x_d=0}} = 0, \\ u_{0,k|t \leq 0} = 0, \end{cases}$$

which is a homogeneous linear transport equation so the  $u_{0,k}$  for  $k \in \mathcal{I}$  are zero.

### 3.1.2 Determination of the evanescent leading order term

From the cascade of equations  $\mathbb{I}_{-1}$  and the boundary condition (36), we have to solve for elliptic modes:

$$\begin{cases} L(\partial_{X_d}) U_{ev,0} = 0 & \text{for } (t, x) \in \Omega_T, X_d \in \mathbb{R}_+ \\ BU_{ev,0}|_{X_d=0} = 0, \\ U_{ev,0|t \leq 0}, \end{cases}$$

which is a standard ordinary differential equation with respect to the fast variable  $X_d$  in which the variables  $(t, x')$  act as parameters. In order to solve this equation we will use the following lemma (see [16] or [3]):

**Lemma 3.3** (Lescarret). *We define for  $X_d \geq 0$ :*

$$(P_{ev}U)(X_d) := e^{X_d A(\zeta)} \Pi_e^s U(0), \quad (37)$$

$$(Q_{ev}F)(X_d) := \int_0^{X_d} e^{(X_d-s)A(\zeta)} \Pi_e^s A_d^{-1} F(s) ds - \int_{X_d}^{\infty} e^{(X_d-s)A(\zeta)} \Pi_e^u A_d^{-1} F(s) ds. \quad (38)$$

Then, for all  $F \in \mathbf{P}_{ev}$ , the equation  $L(\partial_\chi)U = F$  for  $X_d \geq 0$ , admits a unique solution  $U \in \mathbf{P}_{ev}$  reading  $U = P_{ev}U + Q_{ev}F$ .

Using this lemma, we have the polarization condition for elliptic modes  $U_{ev,0} = P_{ev}U_{ev,0}$ . So it is sufficient to determine  $U_{ev,0}|_{X_d=0}$  in order to determine  $U_{ev,0}$ .

Since  $U_{ev,0} = P_{ev}U_{ev,0}$ , we deduce that  $U_{ev,0} \in \mathbb{E}_e^s(\zeta)$ . Hence, we can invert  $B$  and the boundary condition (36) gives  $U_{ev,0}|_{X_d=0} = 0$ .

Hence, we deduce that  $U_{ev,0} = 0$  as the free evolution of the trivial trace  $U_{ev,0}|_{X_d=0}$ .

Next, we will have to be more precise about the form of  $U_{ev,1}$  to determine the glancing leading order term. Since  $U_{ev,0} = 0$ , we have from the cascade of equations  $\mathbb{I}_{-\frac{1}{2}}$ ,  $L(\partial_{X_d})U_{ev,1} = 0$ , so that from Lemma 3.3 we have,  $U_{ev,1} = P_{ev}U_{ev,1}$ , polarization condition which will be sufficient for the determination of the leading order glancing term.

### 3.1.3 Determination of the glancing leading order term

For  $\gamma > 0$ , let  $v_0^g = e^{-\gamma t} u_0^g$ . We observe that  $\partial_t v_0^g = -\gamma v_0^g + e^{-\gamma t} \partial_t u_0^g$ . Hence, in the new unknown, the equation for the glancing leading order term (31) becomes:

$$-\partial_\chi^2 v_0^g + ic(\gamma + \partial_t + v'_g \cdot \nabla_{x'}) v_0^g = ice^{-\gamma t} \Pi^g f^g. \quad (39)$$

We now perform a Fourier transform in the time variable  $t$  and in the tangential space variable  $x'$  to this equation. Let  $\tau \in \mathbb{R}$  and  $\eta \in \mathbb{R}^{d-1}$  denote the dual variables of  $t$  and  $x'$  and  $\widehat{\cdot}$  denotes the Fourier transform. We thus have:

$$-\partial_\chi^2 \widehat{v}_0^g + ic\left(\gamma + i\tau + i\sum_{j=1}^{d-1} v'_{g,j} \eta_j\right) \widehat{v}_0^g = ic \Pi^g \widehat{e^{-\gamma t} f^g},$$

Let  $X = X(\sigma, \eta) := ic\left(\gamma + i\tau + i\sum_{j=1}^{d-1} v'_{g,j} \eta_j\right) = c\left(-\tau - \sum_{j=1}^{d-1} v'_{g,j} \eta_j + i\gamma\right)$ , we obtain the following equation:

$$-\partial_\chi^2 \widehat{v}_0^g + X \widehat{v}_0^g = ic \Pi^g \widehat{e^{-\gamma t} f^g}. \quad (40)$$

So, if  $f^g$  has exponential decay with respect to  $\chi$  for example if there exists  $\widetilde{f}^g \in L^2(\mathbb{R}^d)$  and  $C, \delta > 0$  such that

$$|f^g(t, x', \chi)| \leq C |\widetilde{f}^g(t, x')| e^{-\delta \chi},$$

then we have a unique solution  $\widehat{v}_0^g$  which is in  $L^2_{\tau, \eta}(\mathbb{R}^d)$  with exponential decay with respect to  $\chi$  (see [21], equation 8.40):

$$\widehat{v}_0^g(\tau, \eta, \chi) = e^{\sqrt{X}\chi} \widehat{v}_0^g|_{\chi=0}(\tau, \eta) + ic \int_0^\chi e^{(x-x')\sqrt{X}} \int_{x'}^\infty e^{-(x'-x'')\sqrt{X}} \Pi^g \widehat{e^{-\gamma t} f^g}(\tau, \eta, x'') dx'' d\chi'. \quad (41)$$

where  $\sqrt{X}$  stands for the root of  $X$  with negative real part in order that  $v_0^g \in L^2_{\chi}(\mathbb{R}_+)$ . Reversing the Fourier transform, we deduce that:

$$\begin{aligned} v_0^g(t, x', \chi) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t, x') \cdot (\tau, \eta)} e^{\sqrt{X}\chi} \widehat{v_0^g}|_{\chi=0}(\tau, \eta) d\tau d\eta \\ &\quad + \frac{ic}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t, x') \cdot (\tau, \eta)} \int_0^X e^{(x-x')\sqrt{X}} \int_{\chi'}^{\infty} e^{-(x'-x'')\sqrt{X}} \widehat{\Pi^g e^{-\gamma t} f^g}(\tau, \eta, \chi'') d\chi'' d\chi' d\tau d\eta. \end{aligned} \quad (42)$$

So, to determine  $v_0^g$  (and so  $u_0^g$ ), we have to determine  $\widehat{v_0^g}|_{\chi=0}$ . Indeed the second term in (42) is a known function depending on the source  $f^g$ . Moreover following [[20]-Proposition 9.6] we have that  $v_0^g$  vanishes for negative times if  $v_0^g|_{\chi=0}$  and  $e^{-\gamma t} f^g$  do so if and only if  $v_0^g|_{\chi=0}$  does.

If we first consider the boundary condition  $\mathbb{B}_0$  then (25) shows us that  $u_0^g$  is polarized so  $u_0^g \in \mathbb{E}^s$  and the boundary condition shows us that the trace  $u_0^g|_{\chi=0} \in \ker B$ . Thus, we deduce that  $u_0^g|_{\chi=0} \in \mathbb{E}_g^s \cap \ker(B)$  so, using Assumption 2.4 there exists a scalar function  $\alpha_0$  defined on  $\omega_T$  such that

$$u_0^g|_{\chi=0}(t, x') = \alpha_0(t, x')e \text{ and thus } v_0^g|_{\chi=0}(t, x') = e^{-\gamma t} \alpha_0(t, x')e, \quad (43)$$

and for compatibility reasons we should keep in mind that we should have  $\alpha_0|_{t \leq 0} = 0$ .

In order to determine  $\alpha_0$ , we look at the boundary condition  $\mathbb{B}_{\frac{1}{2}}$  which is the only other equation involving  $u_0^g|_{\chi=0}$  (via the unpolarized part). More precisely, we have:

$$B \left( \sum_{k \in \mathcal{I} \cup \mathcal{O}} u_{1,k}|_{x_d=0} + u_1^g|_{\chi=0} + U_{ev,1}|_{x_d=0} \right) = g$$

We have already justified that for all  $k \in \mathcal{I} \cup \mathcal{O}$ , we have  $\Pi^k u_{1,k} = u_{1,k}$  so that we can replace  $u_{1,k}$  by  $\Pi^k u_{1,k}$  in the previous equation. Similarly we have that  $U_{1,ev} = P_{ev} U_{1,ev}$  so that the boundary condition can be written as

$$B \left( \sum_{k \in \mathcal{I}} \Pi^k u_{1,k}|_{x_d=0} + B u_1^g|_{\chi=0} + B(P_{ev} U_{ev,1})|_{x_d=0} \right) = g - B \sum_{k \in \mathcal{O}} \Pi^k u_{1,k}|_{x_d=0}.$$

So, decomposing  $u_1^g$  in terms of polarized and unpolarized parts and applying  $b$  defined in (13), note that for all  $k \in \mathcal{I}$ ,  $\Pi^k u_{1,k} \in \mathbb{E}_h^s(\zeta)$  and that  $P_{ev} U_{1,ev} \in \mathbb{E}_e^s(\zeta)$ , we obtain:

$$b \cdot B(I - \Pi^g) u_1^g|_{\chi=0} = b \cdot g - b \cdot B \sum_{k \in \mathcal{O}} \Pi^k u_{1,k}|_{x_d=0}$$

Using (27), we can rewrite this equation as:

$$ib \cdot B \Upsilon^g A_d (\partial_{\chi} u_0^g)|_{\chi=0} = b \cdot g - b \cdot B \sum_{k \in \mathcal{O}} \Pi^k u_{1,k}|_{x_d=0}. \quad (44)$$

Multiplying it by  $e^{-\gamma t}$ , we obtain:

$$ib \cdot B \Upsilon^g A_d (\partial_{\chi} v_0^g)|_{\chi=0} = e^{-\gamma t} \left( b \cdot g - b \cdot B \sum_{k \in \mathcal{O}} \Pi^k u_{1,k}|_{x_d=0} \right). \quad (45)$$

Using (42) and (43), we see that

$$\begin{aligned} (\partial_{\chi} v_0^g)|_{\chi=0} &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t,x') \cdot (\tau,\eta)} \sqrt{X} \widehat{e^{-\gamma t} \alpha_0}(\tau, \eta) d\tau d\eta e \\ &+ \frac{ic}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t,x') \cdot (\tau,\eta)} \int_0^\infty e^{\chi' \sqrt{X}} \Pi^g \widehat{e^{-\gamma t} f^g}(\tau, \eta, \chi') d\chi' d\tau d\eta. \end{aligned} \quad (46)$$

This motivates the following definition: let  $T$  be the Fourier multiplier defined from  $H_\gamma^\infty(\mathbb{R}^d)$  into  $H^\infty(\mathbb{R}^d)$  given by

$$(Tf)(t, x') := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t,x') \cdot \xi} \sqrt{X} \widehat{e^{-\gamma t} f}(\xi) d\xi, \quad (47)$$

where  $\xi$  is a short hand notation for  $(\tau, \eta)$ . We also introduce  $\lambda := b \cdot B \Upsilon^g A_d e \neq 0$  from Assumption 2.5. In this new notations using (46) in (45) thus gives the following condition on the unknown trace  $\alpha_0$  :

$$\begin{aligned} i\lambda T(\alpha_0) &= b \cdot e^{-\gamma t} g - b \cdot B \sum_{k \in \mathcal{O}} e^{-\gamma t} \Pi^k u_{1,k}|_{x_d=0} \\ &+ \frac{c}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t,x') \cdot (\tau,\eta)} \int_0^\infty e^{\chi' \sqrt{X}} b \cdot B \Upsilon^g A_d \Pi^g \widehat{e^{-\gamma t} f^g}(\tau, \eta, \chi') d\chi' d\tau d\eta \\ &:= G, \end{aligned} \quad (48)$$

where  $G$  is explicit in terms of  $g$ ,  $f^k$ ,  $k \in \mathcal{O}$  and  $f^g$ . We also remark for later purposes that all the terms in the right hand side of (48) vanish for negative times if  $g$ ,  $f^g$  and the  $f^k$  do.

**Proposition 3.1.** *The operator  $T$  defined by*

$$T(f)(t, x') := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t,x') \cdot \xi} \sqrt{X} \widehat{e^{-\gamma t} f}(\xi) d\xi$$

maps  $H_\gamma^\infty(\mathbb{R}^d)$  into  $H^\infty(\mathbb{R}^d)$  where  $H_\gamma^\infty(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) \text{ s.t. } e^{-\gamma t} u \in H^\infty(\mathbb{R}^d)\}$ .

Moreover,  $T$  is invertible and  $T^{-1}$  is defined by

$$T^{-1}(f)(t, x') = \frac{e^{\gamma t}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t,x') \cdot \xi} \frac{1}{\sqrt{X}} \widehat{f}(\xi) d\xi;$$

it maps  $H^\infty(\mathbb{R}^d)$  into  $H_\gamma^\infty(\mathbb{R}^d)$ .

Finally if the function  $f$  vanishes for negative times then so do  $T(f)$  and  $T^{-1}(f)$ .

Proof: In a first time, we show that  $T$  maps  $H_\gamma^\infty(\mathbb{R}^d)$  into  $H^\infty(\mathbb{R}^d)$ .

Let  $f \in H_\gamma^\infty(\mathbb{R}^d)$ . We want to show that  $T(f) \in H^\infty(\mathbb{R}^d)$  or equivalently that

$$\forall m \in \mathbb{N}, \exists C_m > 0, \int_{\mathbb{R}^d} |\widehat{T(f)}(\xi)|^2 (1 + |\xi|^2)^m d\xi < C_m.$$

Then using the definition of  $X$  and Cauchy-Schwarz inequality, we have:

$$\begin{aligned} |\widehat{T(f)}(\xi)|^2 &= |c| \sqrt{\gamma^2 + (\tau + v' \cdot \eta)^2} |\widehat{e^{-\gamma t} f}(\xi)|^2, \\ &\leq \begin{cases} |c| (\gamma + |(1, v')|) |\widehat{e^{-\gamma t} f}(\xi)|^2 & \text{if } |\xi| \leq 1, \\ |c| (\gamma + (1 + |\xi|^2) |(1, v')|) |\widehat{e^{-\gamma t} f}(\xi)|^2 & \text{if } |\xi| > 1. \end{cases} \end{aligned}$$

Hence, denoting  $v'_1 := (1, v')$ :

$$\begin{aligned}
\int_{\mathbb{R}^d} |\widehat{T(f)}(\xi)|^2 (1 + |\xi|^2)^m d\xi &= \int_{|\xi| \leq 1} |\widehat{T(f)}(\xi)|^2 (1 + |\xi|^2)^m d\xi + \int_{|\xi| > 1} |\widehat{T(f)}(\xi)|^2 (1 + |\xi|^2)^m d\xi \\
&\leq |c|(\gamma + |v'_1|) \int_{|\xi| \leq 1} |\widehat{e^{-\gamma t} f}(\xi)|^2 (1 + |\xi|^2)^m d\xi \\
&\quad + |c|\gamma \int_{|\xi| > 1} |\widehat{e^{-\gamma t} f}(\xi)|^2 (1 + |\xi|^2)^m d\xi \\
&\quad + |c||v'_1| \int_{|\xi| > 1} |\widehat{e^{-\gamma t} f}(\xi)|^2 (1 + |\xi|^2)^{m+1} d\xi, \\
&\leq |c|(\gamma + |v'_1|) \int_{\mathbb{R}^d} |\widehat{e^{-\gamma t} f}(\xi)|^2 (1 + |\xi|^2)^m d\xi \\
&\quad + |c|\gamma \int_{\mathbb{R}^d} |\widehat{e^{-\gamma t} f}(\xi)|^2 (1 + |\xi|^2)^m d\xi \\
&\quad + |c||v'_1| \int_{\mathbb{R}^d} |\widehat{e^{-\gamma t} f}(\xi)|^2 (1 + |\xi|^2)^{m+1} d\xi,
\end{aligned}$$

Since  $f \in H_\gamma^\infty(\mathbb{R}^d)$ , we have  $\int_{\mathbb{R}^d} |\widehat{T(f)}(\xi)|^2 (1 + |\xi|^2)^m d\xi < +\infty$ ,  $\forall m \in \mathbb{N}$ , so that  $T(f) \in H^\infty(\mathbb{R}^d)$ . We have shown that  $\|T(f)\|_{H^m(\mathbb{R}^d)} \leq C\|e^{-\gamma t} f\|_{H^{m+1}(\mathbb{R}^d)}$  from which we deduce that if  $f$  vanishes for negative times then so do  $T(f)$ .

Next, we show that the expression given for  $T^{-1}$  is really the inverse.

We rewrite  $T(f) = \mathcal{F}^{-1}(\sqrt{X}\widehat{e^{-\gamma t} f})$  and  $T^{-1}(f) = e^{\gamma t}\mathcal{F}^{-1}(\frac{1}{\sqrt{X}}\widehat{f})$ , where  $\mathcal{F}^{-1}$  stands for the inverse of the Fourier transform on  $L^2(\mathbb{R}^d)$ .

By easy computations, we can see that  $T \circ T^{-1} = T^{-1} \circ T = I$ .

Finally, we show that  $T^{-1}$  maps  $H^\infty(\mathbb{R}^d)$  into  $H_\gamma^\infty(\mathbb{R}^d)$ . Let  $f \in H^\infty(\mathbb{R}^d)$ . We have:

$$e^{-\gamma t}T^{-1}(f) = \mathcal{F}^{-1}\left(\frac{1}{\sqrt{X}}\widehat{f}\right).$$

And so:

$$|e^{-\gamma t}\widehat{T^{-1}(f)}(\xi)|^2 = \frac{1}{|X(\xi)|}|\widehat{f}(\xi)|^2.$$

But,  $|X(\xi)| = |c|\sqrt{\gamma^2 + (\tau + v' \cdot \eta)^2} \geq |c|\gamma$  and  $c, \gamma \neq 0$  by definition of  $\gamma$  and Lemma 3.1. Hence

$$|e^{-\gamma t}\widehat{T^{-1}(f)}(\xi)|^2 \leq \frac{1}{|c|\gamma}|\widehat{f}(\xi)|^2,$$

and consequently

$$\int_{\mathbb{R}^d} |e^{-\gamma t}\widehat{T^{-1}(f)}(\xi)|^2 (1 + |\xi|^2)^m d\xi \leq \frac{1}{|c|\gamma} \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 (1 + |\xi|^2)^m d\xi,$$

that is to say that  $\|e^{-\gamma t}T^{-1}(f)\|_{H^m(\mathbb{R}^d)} \leq C\|f\|_{H^m(\mathbb{R}^d)}$ , since  $f \in H^\infty(\mathbb{R}^d)$ , we deduce that  $T^{-1}(f) \in H_\gamma^\infty(\mathbb{R}^d)$  and that it vanishes for negative times if  $f$  does.  $\square$

Then with Proposition 3.1 in hand we can easily determine the unknown trace  $\alpha_0$  from (48) by applying  $T^{-1}$  on each side to obtain

$$\alpha_0(t, x') = \frac{1}{i\lambda} T^{-1} G.$$

In order to do so we shall justify that  $G$  defined in (49) is in  $H^\infty(\omega_T)$  so that we can effectively apply  $T^{-1}$ . We recall that

$$\begin{aligned} G := & b \cdot e^{-\gamma t} g - b \cdot \sum_{k \in \mathcal{O}} e^{-\gamma t} \Pi^k u_{1,k}|_{x_d=0} \\ & + \frac{c}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t,x') \cdot (\tau,\eta)} \int_0^\infty e^{x' \sqrt{X}} b \cdot B \Upsilon^g A_d \Pi^g \widehat{e^{-\gamma t} f^g}(\tau, \eta, \chi') d\chi' d\tau d\eta, \end{aligned}$$

the first term is clearly in  $H^\infty(\omega_T)$  because  $g \in H^\infty(\omega_T)$ . Similarly if we choose  $f_k \in H^\infty(\Omega_T)$  then as a solution of a transport equation  $\Pi^k u_{1,k} \in H^\infty(\Omega_T) \subset H_\gamma^\infty(\Omega_T)$  and thus the second term is in  $H_\gamma^\infty(\omega_T)$ . The last term reads under the form

$$\mathcal{F}^{-1} \int_0^\infty e^{x' \sqrt{X}} \widehat{\mathfrak{F}} d\chi' d\tau d\eta$$

where  $\mathfrak{F} \in H^\infty(\omega_T \times \mathbb{R}_+)$ . And thus in order to show that this term is in  $H^\infty(\Omega_T)$  we shall consider the integrals for a multi-index  $\delta \in \mathbb{N}^d$ .

$$\int_{\mathbb{R}^d} \xi^\delta \left| \int_0^\infty e^{x' \sqrt{X}} \widehat{\mathfrak{F}}(\xi, \chi') d\chi' \right|^2 d\xi \leq \frac{C}{\gamma} \int_{\mathbb{R}^d} \xi^\delta \sup_{\chi' \in \mathbb{R}_+} |\widehat{\mathfrak{F}}(\xi, \chi')|^2 d\xi,$$

which are finite for all  $\delta$  because  $\mathfrak{F} \in H^\infty(\omega_T \times \mathbb{R}_+)$ .

To conclude, we have determine  $u_0^g$  as

$$u_0^g(t, x', \chi) = e^{\gamma t} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t,x') \cdot (\tau,\eta)} e^{\sqrt{X} \chi} \widehat{e^{-\gamma t} \alpha_0}(\tau, \eta) d\tau d\eta e, \quad (50)$$

where  $\alpha_0 = \frac{1}{i\lambda} T^{-1}(G)$  vanishes for negative times and so do  $u_0^g$ .

For later purpose let us remark that we can read on (50) that  $u_0^g$  and its derivatives in the fast variable  $\chi$  decay exponentially fast with respect to this variable. More precisely we give the following proposition establishing the regularity of  $u_0^g$ .

**Proposition 3.2.** *We have  $u_0^g \in \mathbf{P}_g$  and  $u_0^g$  vanishes for negative times.*

*Proof:* Let us remark that in view of its expression  $u_0^g$  has exponential decay with respect to  $\chi$ .

In the following we consider  $\alpha \in \mathbb{N}^{d-1}$  a multi-index associated to  $x'$  and we use classical notations for multi-index.

From Leibniz formula to show that  $u_0^g \in H_{t,x',\chi}^\infty(\Omega_T \times \mathbb{R}_+)$  it is sufficient to show that the  $e^{\gamma t} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t,x') \cdot (\tau,\eta)} e^{\sqrt{X} \chi} \tau^k \eta^\alpha \sqrt{X}^m \widehat{e^{-\gamma t} \alpha_0}(\tau, \eta) d\tau d\eta$  are in  $L_{t,x',\chi}^2(\Omega_T \times \mathbb{R}_+)$  for all  $k, m \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^{d-1}$ .

Because we are in finite time the factor  $e^{\gamma t}$  can be replaced by  $e^{\gamma T}$  and we can integrate on the whole line for  $t$ .

Thus, using Plancherel theorem:

$$\|\mathcal{F}_{(t,x') \leftrightarrow (\tau,\eta)}^{-1} \left( e^{\sqrt{X} \chi} \tau^k \eta^\alpha \sqrt{X}^m \widehat{e^{-\gamma t} \alpha_0} \right)\|_{L_{t,x',\chi}^2(\mathbb{R}^d \times \mathbb{R}_+)} = \|e^{\sqrt{X} \chi} \tau^k \eta^\alpha \sqrt{X}^m \widehat{e^{-\gamma t} \alpha_0}\|_{L_{\tau,\eta,\chi}^2(\mathbb{R}^d \times \mathbb{R}_+)}.$$

Integrating with respect to  $\chi$  first then gives:

$$\begin{aligned} \|e^{\sqrt{X}\chi}\tau^k\eta^\alpha\sqrt{X^m}e^{-\widehat{\gamma t}\alpha_0}\|_{L^2_{\tau,\eta,\chi}(\mathbb{R}^d\times\mathbb{R}_+)}^2 &= \int_{\mathbb{R}_+} |e^{\sqrt{X}\chi}\tau^k\eta^\alpha\sqrt{X^m}e^{-\widehat{\gamma t}\alpha_0}|^2 d\chi, \\ &= |\tau^k\eta^\alpha\sqrt{X^m}e^{-\widehat{\gamma t}\alpha_0}|^2 \int_{\mathbb{R}_+} |e^{\sqrt{X}\chi}|^2 d\chi, \\ &= -|\tau^k\eta^\alpha\sqrt{X^m}e^{-\widehat{\gamma t}\alpha_0}|^2 \times \frac{1}{2\Re(\sqrt{X})}. \end{aligned}$$

But if  $\Re(X) > 0$  we write  $-\Re(\sqrt{X}) = \sqrt{\frac{1}{2}(\Re(X) + |X|)} \geq \frac{1}{\sqrt{2}}\sqrt{|c|\gamma}$  and if  $\Re(X) < 0$  we write  $\Re(\sqrt{X}) = -\frac{1}{\sqrt{2}}\frac{|\Im(X)|}{\sqrt{|X|-\Re(X)}}$  so that in both cases  $-\frac{1}{2\Re(\sqrt{X})} \leq \frac{1}{\sqrt{2}|c|\gamma}$ . Consequently

$$\|e^{\sqrt{X}\chi}\tau^k\eta^\alpha\sqrt{X^m}e^{-\widehat{\gamma t}\alpha_0}\|_{L^2_{\tau,\eta,\chi}(\mathbb{R}^d\times\mathbb{R}_+)}^2 \leq \frac{1}{\sqrt{2}|c|\gamma} \|\tau^k\eta^\alpha\sqrt{X^m}e^{-\widehat{\gamma t}\alpha_0}\|_{L^2_{\tau,\eta}(\mathbb{R}^d)}^2$$

We decompose like in Proposition 3.1

$$\begin{aligned} \|\tau^k\eta^\alpha\sqrt{X^m}e^{-\widehat{\gamma t}\alpha_0}\|_{L^2_{\tau,\eta}(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} |\tau^k\eta^\alpha\sqrt{X^m}e^{-\widehat{\gamma t}\alpha_0}|^2 d\tau d\eta \\ &\leq |c|^m(\gamma + |v'_1|)^m \int_{|\xi|\leq 1} |e^{-\widehat{\gamma t}\alpha_0}|^2 d\tau d\eta \\ &\quad + \int_{|\xi|>1} |\tau^k\eta^\alpha\sqrt{X^m}e^{-\widehat{\gamma t}\alpha_0}|^2 d\tau d\eta \end{aligned}$$

The first right hand side term is finite because  $e^{-\widehat{\gamma t}\alpha_0} \in H^\infty(\omega_T)$  and the second term can also be bounded by using Holder inequality for  $\sqrt{X^m}$  by

$$\int_{|\xi|>1} |\tau^k\eta^\alpha\sqrt{X^m}e^{-\widehat{\gamma t}\alpha_0}|^2 d\tau d\eta \leq \int_{\mathbb{R}^d} (1 + |\xi|^2)^{k+|\alpha|+m} |e^{-\widehat{\gamma t}\alpha_0}|^2 d\tau d\eta,$$

which is finite because  $e^{-\widehat{\gamma t}\alpha_0} \in H^\infty(\omega_T)$ . This prove that  $u_0^g \in H^\infty(\Omega_T \times \mathbb{R}_+)$  and so  $u_0^g \in \mathbf{P}_g$  because of the exponential decay with respect to  $\chi$ .  $\square$

So we sum up the construction of the Paragraphs 3.1.1, 3.1.2 and 3.1.3 in the following proposition

**Proposition 3.3.** *Under Assumptions 2.1, 2.2, 2.3, 2.4 and 2.5, for all  $k \in \mathcal{I} \cup \mathcal{O}$ , there exist  $u_{0,k} \in H^\infty(\Omega_T)$ ,  $u_0^g \in \mathbf{P}_g$  and  $U_{ev,0} \in \mathbf{P}_{ev}$  satisfying the cascades of equations (21), (22) and (23) at order zero. In fact we have that for all  $k \in \mathcal{I} \cup \mathcal{O}$ ,  $u_{0,k} = U_{ev,0} = 0$ .*

## 3.2 Construction of terms of order one

In this paragraph, we determine the amplitudes of order one. For this, we find transport equations for the hyperbolic and glancing amplitudes too. We follow the same steps as the precedent paragraphs but the determination of the traces is a little more technical than for the leading amplitudes because the amplitudes are not polarized any more.

Using (12), we apply the pseudo-inverse  $\Upsilon^g$  (resp.  $\Upsilon^k$ ) to the equation for the glancing mode of  $\mathbb{I}_0$  (resp. to the equation for the hyperbolic modes of  $\mathbb{I}_0$ ) to derive:

$$(I - \Pi^g)u_2^g = -i\Upsilon^g f^g + i\Upsilon^g A_d \partial_\chi u_1^g + i\Upsilon^g L(\partial)u_0^g, \quad (51)$$

$$(I - \Pi^k)u_{2,k} = i\Upsilon^k L(\partial)u_{0,k}, \quad k \in \mathcal{I} \cup \mathcal{O}. \quad (52)$$

We remark that (52) gives the unpolarized part of  $u_{2,k}$  in terms of the constructed term  $u_{0,k}$ . Similarly in the right hand side of (51) the last term is now a known function.

Now, we apply the projector  $\Pi^g$  (resp.  $\Pi^k$ ) to the glancing (resp. hyperbolic) equation of  $\mathbb{I}_{\frac{1}{2}}$  to obtain (recall that  $u_{1,k}$  is polarized):

$$\begin{aligned}\Pi^g L'(\partial)u_1^g + \Pi^g A_d \partial_\chi u_2^g &= 0, \\ \Pi^k L(\partial)\Pi^k u_{1,k} &= \Pi^k f_k, \quad k \in \mathcal{I} \cup \mathcal{O}.\end{aligned}$$

Decomposing  $u_1^g$  and  $u_2^g$  in terms of polarized and unpolarized parts, we have:

$$\begin{aligned}\Pi^g L'(\partial)\Pi^g u_1^g + \Pi^g A_d \Pi^g \partial_\chi u_2^g + \Pi^g A_d \partial_\chi (I - \Pi^g)u_2^g &= -\Pi^g L'(\partial)(I - \Pi^g)u_1^g, \\ \Pi^k L(\partial)\Pi^k u_{1,k} &= \Pi^k f_k,\end{aligned}\tag{53}$$

where  $(I - \Pi^g)u_1^g$  is known from equation (27) and thus the right hand side of (53) is.

Using equations (51) and (53) and decomposing again  $u_1^g$  in terms of polarized and unpolarized parts, we thus obtain:

$$\begin{aligned}\Pi^g L'(\partial)\Pi^g u_1^g + \Pi^g A_d \Pi^g \partial_\chi u_2^g + i\Pi^g A_d \Upsilon^g A_d \partial_\chi^2 \Pi^g u_1^g & \\ = -\Pi^g L'(\partial)(I - \Pi^g)u_1^g + i\Pi^g A_d \Upsilon^g \partial_\chi f^g & \\ - i\Pi^g A_d \Upsilon^g L'(\partial)\partial_\chi u_0^g - i\Pi^g A_d \Upsilon^g A_d \partial_\chi^2 (I - \Pi^g)u_1^g, & \\ := F_1. &\end{aligned}\tag{54}$$

$$\Pi^k L(\partial)\Pi^k u_{1,k} = \Pi^k f_k.\tag{55}$$

Using Lemmas 3.1 and 3.2, equations (54) and (55) can be rewritten as:

$$-\partial_\chi^2 \Pi^g u_1^g + ic(\partial_t + v'_g \cdot \nabla_{x'}) \Pi^g u_1^g = icF_1,\tag{56}$$

$$(\partial_t + v_k \cdot \nabla_x) \Pi^k u_{1,k} = \Pi^k f_k \quad k \in \mathcal{I} \cup \mathcal{O}.\tag{57}$$

Then we look at the boundary condition: the equation  $\mathbb{B}_{\frac{1}{2}}$  reads

$$B \left( \sum_{k \in \mathcal{I} \cup \mathcal{O}} u_{1,k}|_{x_d=0} + u_{1|_{x=0}}^g + U_{ev,1}|_{x_d=0} \right) = g.\tag{58}$$

In the following, we need to determine the  $\Pi^k u_{1,k}|_{x_d=0}$  for  $k \in \mathcal{I}$  and  $P_{ev} U_{ev,1}|_{x_d=0}$  so we will do this from this boundary condition. We recall that the  $u_{1,k}$  for  $k \in \mathcal{I} \cup \mathcal{O}$  are polarized and that we know  $\Pi^k u_{1,k}$ ,  $k \in \mathcal{O}$  from Paragraph 3.1.1. And so:

$$B \left( \sum_{k \in \mathcal{I}} \Pi^k u_{1,k}|_{x_d=0} + u_{1|_{x=0}}^g + U_{ev,1}|_{x_d=0} \right) = g - B \sum_{k \in \mathcal{O}} \Pi^k u_{1,k}|_{x_d=0}.\tag{59}$$

Equation (27) gives the unpolarized part of  $u_1^g$ . Remember that we have  $\text{Im } \Pi^g = \ker \mathcal{L}(\underline{\zeta}, \underline{\xi}_k) = E_g^s(\underline{\zeta})$ , so we can write, using Assumption 2.3 and Remark 2.2:

$$\Pi^g u_{1|_{x=0}}^g = \alpha_1 e \text{ where } \alpha_1 \text{ is a scalar function defined on } \omega_T.\tag{60}$$

Hence,  $B \Pi^g u_{1|_{x=0}}^g = \alpha_1 B e = 0$  and we can rewrite equation (59) as:

$$B \left( \sum_{k \in \mathcal{I}} \Pi^k u_{1,k}|_{x_d=0} + U_{ev,1}|_{x_d=0} \right) = g - B \sum_{k \in \mathcal{O}} \Pi^k u_{1,k}|_{x_d=0} - B(I - \Pi^g)u_{1|_{x=0}}^g.$$



Finally using that  $U_{ev,1} = P_{ev}U_{ev,1}$ , it turns out

$$\begin{aligned} B \left( \sum_{k \in \mathcal{I}} \Pi^k u_{1,k|_{x_d=0}} + P_{ev}U_{ev,1|_{x_d=0}} \right) &= g - B \sum_{k \in \mathcal{O}} \Pi^k u_{1,k|_{x_d=0}} - B(I - \Pi^g)u_{1|_{x=0}}^g, \\ &:= G_0. \end{aligned}$$

Since  $\Pi^k u_{1,k|_{x_d=0}} \in \mathbb{E}_h^s(\zeta)$  and  $P_{ev}U_{ev,1|_{x_d=0}} \in \mathbb{E}_e^s(\zeta)$ , from (14), we can inverse  $B$  and so applying  $\Phi$ , we deduce that:

$$\sum_{k \in \mathcal{I}} \Pi^k u_{1,k|_{x_d=0}} + P_{ev}U_{ev,1|_{x_d=0}} = \Phi G_0.$$

Next, applying the projectors of Definition 2.5, we thus obtain:

$$\begin{aligned} \forall k \in \mathcal{I}, P^k \Pi^k u_{1,k|_{x_d=0}} &= P^k \Phi G_0, \\ P^e P_{ev}U_{ev,1|_{x_d=0}} &= P^e \Phi G_0. \end{aligned}$$

Since  $\Pi^k u_{1,k|_{x_d=0}} \in \mathbb{E}_h^s(\zeta)$  and  $P_{ev}U_{ev,1|_{x_d=0}} \in \mathbb{E}_e^s(\zeta)$ , we see that  $P^k \Pi^k u_{1,k|_{x_d=0}} = \Pi^k u_{1,k|_{x_d=0}}$  and  $P^e P_{ev}U_{ev,1|_{x_d=0}} = P_{ev}U_{ev,1|_{x_d=0}}$ . Therefore,

$$\forall k \in \mathcal{I}, \Pi^k u_{1,k|_{x_d=0}} = P^k \Phi G_0, \quad (61)$$

$$P_{ev}U_{ev,1|_{x_d=0}} = P^e \Phi G_0. \quad (62)$$

In the following paragraphs we describe the construction of the higher order hyperbolic and elliptic terms. We then use this construction for the determination of the glancing amplitude of order one in Paragraph 3.2.3.

### 3.2.1 Determination of the hyperbolic amplitudes of order one

The outgoing amplitudes of order one namely the  $u_{1,k}$  for  $k \in \mathcal{O}$  have been determined in Paragraph 3.1.1. However like the leading order, the determination of the first order glancing amplitude will require the knowledge of one order more on the outgoing modes, more precisely we will need to know  $u_{2,k}$  for  $k \in \mathcal{O}$ .

Reiterating the same analysis as the one performed in Paragraph 3.1.1, these terms can be determined independently of the other by solving an outgoing transport equation.

Indeed, applying the projector  $\Pi^k$  on the hyperbolic equation of  $\mathbb{I}_1$ , we have:

$$\Pi^k L(\partial)u_{2,k} = 0.$$

Equation (52) gives the unpolarized part of  $u_{2,k}$ . So decomposing  $u_{2,k}$  in terms of polarized and unpolarized parts and using Lemma 3.2, we deduce that:

$$(\partial_t + v_k \cdot \nabla_x) \Pi^k u_{2,k} = -\Pi^k L(\partial)(I - \Pi^k)u_{2,k} := f_{k,2}.$$

Hence, we determine  $\Pi^k u_{2,k}$  and so  $u_{2,k}$  by solving the transport equation:

$$\begin{cases} (\partial_t + v_k \cdot \nabla_x) \Pi^k u_{2,k} = f_{k,2}, \\ \Pi^k u_{2,k|_{t \leq 0}} = 0, \end{cases}$$

by integration along the characteristics. The obtained solution is explicit in terms of  $f_{k,2}$  that is to say explicit in terms of  $u_{0,k}$  from (52). Moreover the solution  $u_{2,k}$  lies in the same functional space than the source  $f_{2,k}$ .

For the incoming phases, we have to consider the boundary equation (61). For these phases, as in Paragraph 3.1.1 for the outgoing phases of order one, we have  $(\partial_t + v_k \cdot \nabla_x) \Pi^k u_{1,k} = \Pi^k f_k$ . Hence, each of the incoming amplitude solves the transport equation:

$$\begin{cases} (\partial_t + v_k \cdot \nabla_x) \Pi^k u_{1,k} = \Pi^k f_k, \\ \Pi^k u_{1,k}|_{x_d=0} = P^k \Phi G_0, \\ \Pi^k u_{1,k}|_{t \leq 0} = 0, \end{cases}$$

equation that can be explicitly solved in terms of  $f_k$ ,  $k \in \mathcal{I}$  and  $G_0$  (which depends on  $g$  and on the  $f_k$  for  $k \in \mathcal{O}$ ) so that  $u_{1,k}$  depends on all the sources of the problem. We have that  $u_{1,k}$  lies in  $H^\infty(\Omega_T)$  vanishes for negative times if the source does.

### 3.2.2 Determination of the evanescent amplitudes of order one

We recall that from Paragraph 3.1.2 that  $U_{ev,1} = P_{ev} U_{ev,1}$ . So that in order to determine the whole elliptic amplitude it is sufficient, from the definition of  $P_{ev}$  to determine  $U_{ev,1}|_{x_d=0}$ .

We use the boundary equation (62) to recover that

$$U_{ev,1}(t, x', X_d) = (P_{ev} U_{ev,1})(t, x', X_d) = e^{X_d A(\zeta)} P^e \Phi G_0(t, x'),$$

equation that determines  $U_{ev,1}$  in terms of  $G_0$ . Moreover we readily see that  $U_{ev,1} \in \mathbf{P}_{ev}$ . Note that as mentioned in the introduction of Section 3 the fact that the elliptic profile is independent of the normal variable  $x_d$  permit us to avoid the step of the lifting of some double trace as simple one as it is done in [4] or [16].

Like for the leading order glancing term, the determination of the term of order one requires the knowledge of the elliptic amplitude of order plus one. More precisely we will need to know the form of  $U_{ev,2}$  to determine the glancing leading term of order one.

From the cascade of equations  $\mathbb{I}_0$  for elliptic modes we have

$$L(\partial_\chi) U_{ev,2} = - \underbrace{L'(\partial) U_{ev,0}}_{=0} + f_{ev}.$$

From Lemma 3.3, we deduce that:

$$U_{ev,2} = P_{ev} U_{ev,2} + Q_{ev} f_{ev}, \tag{63}$$

equation that does not fully determine  $U_{ev,2}$  because the polarized part  $P_{ev} U_{ev,2}$  has not been determined yet (it is however easy to determine this term by determining the trace  $U_{ev,2}|_{x_d=0}$  like it has been done for  $U_{ev,1}|_{x_d=0}$ ) but which will be sufficient to determine the glancing amplitude of order one.

### 3.2.3 Determination of the glancing leading order term

For  $\gamma > 0$ , let  $v_1^g = e^{-\gamma t} \Pi^g u_1^g$ . As for  $u_0^g$ , equation (56) becomes:

$$-\partial_\chi^2 v_1^g + ic \left( \gamma + \partial_t + v' \cdot \nabla_{x'} \right) v_1^g = ice^{-\gamma t} F_1, \tag{64}$$

where we recall that the right hand side  $F_1$  depends on  $u_0^g$ , the unpolarized part of  $u_1^g$  and  $f^g$  and eventually on their derivatives is defined in (54). In particular this term has exponential decay with respect to  $\chi$ .

We now perform a Fourier transform with respect to  $t$  and  $x'$ , we thus have the same ordinary differential equation as the one determining the leading order glancing amplitude:

$$-\partial_x^2 \widehat{v}_1^g + X \widehat{v}_1^g = ic e^{-\gamma t} \widehat{F}_1, \quad (65)$$

where we recall that  $X = X(\sigma, \eta) = ic \left( \gamma + i\tau + i \sum_{j=1}^{d-1} v'_{g,j} \eta_j \right)$ .

So, because  $\widehat{F}_1$  has exponential decay with respect to  $\chi$  we have a unique solution with exponential decay with respect to  $\chi$  (see [21], equation 8.40):

$$\begin{aligned} \widehat{v}_1^g(\tau, \eta, \chi) &= ic \int_0^\chi e^{(x-x')\sqrt{X}} \int_{x'}^\infty e^{-(x'-x'')\sqrt{X}} e^{-\gamma t} \widehat{F}_1(\tau, \eta, \chi'') d\chi'' d\chi' + e^{x\sqrt{X}} \widehat{v}_1^g|_{\chi=0}(\tau, \eta), \\ &:= F_2(\tau, \eta, \chi) + e^{x\sqrt{X}} \widehat{v}_1^g|_{\chi=0}(\tau, \eta). \end{aligned} \quad (66)$$

Reversing the Fourier transform, we deduce that:

$$\begin{aligned} v_1^g(t, x', \chi) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t, x') \cdot (\tau, \eta)} F_2(\tau, \eta, \chi) d\tau d\eta + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t, x') \cdot (\tau, \eta)} e^{x\sqrt{X}} \widehat{v}_1^g|_{\chi=0}(\tau, \eta) d\tau d\eta \\ &:= F_3(t, x', \chi) + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t, x') \cdot (\tau, \eta)} e^{x\sqrt{X}} \widehat{v}_1^g|_{\chi=0}(\tau, \eta) d\tau d\eta \end{aligned} \quad (67)$$

So, to determine  $v_1^g$  (and so  $\Pi^g u_1^g$ ), it is sufficient to determine  $\widehat{v}_1^g|_{\chi=0}$  because  $F_3$  is known from (54) and (66).

Recall that from (60) we have:

$$v_1^g|_{\chi=0} = e^{-\gamma t} \alpha_1 e. \quad (68)$$

Thus, it remains to determine  $\alpha_1$ . For this, we look at the boundary condition  $\mathbb{B}_1$  that is:

$$B \left( \sum_{k \in \mathcal{I} \cup \mathcal{O}} u_{2,k}|_{x_d=0} + u_{2^g}|_{\chi=0} + U_{ev,2}|_{x_d=0} \right) = 0.$$

The  $u_{2,k}$ , for  $k \in \mathcal{O}$  are known because the polarized part of them are determined from Paragraph 3.2.1 and the unpolarized part from (52). From (52) we know  $(I - \Pi^k)u_{2,k}$ ,  $k \in \mathcal{I}$

Hence, the precedent equation can be rewritten as

$$B \sum_{k \in \mathcal{I}} \Pi^k u_{2,k}|_{x_d=0} + B u_{2^g}|_{\chi=0} + B U_{ev,2}|_{x_d=0} = -B \sum_{k \in \mathcal{O}} u_{2,k}|_{x_d=0} - B \sum_{k \in \mathcal{I}} (I - \Pi^k) u_{2,k}|_{x_d=0}.$$

Furthermore, from (63), we see that:

$$U_{ev,2}|_{x_d=0} = \Pi_e^s U_{ev,2}|_{x_d=0} - \int_0^{+\infty} e^{-s\mathcal{A}(\underline{\zeta})} \Pi_e^u A_d^{-1} f_{ev}(t, x', s) ds.$$

The first right hand side term is in  $E_e^s(\underline{\zeta})$  so  $b \cdot B \Pi_e^s U_{ev,2}|_{x_d=0} = 0$  and the second term is known.

So, decomposing  $u_2^g$  in terms of polarized and unpolarized parts and testing against the vector  $b$  defined in (13), we obtain:

$$\begin{aligned} b \cdot B(I - \Pi^g)u_{2|\chi=0}^g &= -b \cdot B \sum_{k \in \mathcal{O}} u_{2,k|_{x_d=0}} - b \cdot B \sum_{k \in \mathcal{I}} (I - \Pi^k)u_{2,k|_{x_d=0}} \\ &\quad + b \cdot B \int_0^{+\infty} e^{-sA}(\underline{s}) \Pi_e^u A_d^{-1} f_{ev}(t, x', s) ds, \\ &:= \widetilde{G}_1; \end{aligned} \tag{69}$$

clearly as the sum of the solutions of transport equations and the free evolution of the regular source  $f_{ev}$  then the new source  $\widetilde{G}_1$  lies in  $H_{t,x'}^\infty(\omega_T)$ .

Using (51), we can rewrite the left hand side of (69) as:

$$ib \cdot B\Upsilon^g A_d (\partial_\chi u_1^g)|_{\chi=0} = \widetilde{G}_1 + ib \cdot B\Upsilon^g f_{|\chi=0}^g - ib \cdot B\Upsilon^g L'(\partial)u_{0|\chi=0}^g. \tag{70}$$

We stress that the right hand side is known from (50). Multiplying by  $e^{-\gamma t}$ , we obtain:

$$ib \cdot B\Upsilon^g A_d (\partial_\chi v_1^g)|_{\chi=0} = e^{-\gamma t} \widetilde{G}_1 + ie^{-\gamma t} b \cdot B\Upsilon^g f_{|\chi=0}^g - ie^{-\gamma t} b \cdot B\Upsilon^g L'(\partial)u_{0|\chi=0}^g. \tag{71}$$

Using (67) and (68), we see that:

$$\begin{aligned} (\partial_\chi v_1^g)|_{\chi=0} &= (\partial_\chi F_3)(t, x', 0) + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t,x') \cdot (\tau, \eta)} \sqrt{X} e^{-\gamma t} \widehat{\alpha}_1(\tau, \eta) d\tau d\eta e, \\ &= \frac{ic}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t,x') \cdot (\tau, \eta)} \int_0^\infty e^{\chi' \sqrt{X}} e^{-\gamma t} \widehat{F}_1(\tau, \eta, \chi') d\chi' d\tau d\eta + T(\alpha_1)e. \end{aligned}$$

Combining (71) and the precedent equation, we deduce that:

$$\begin{aligned} i\lambda T(\alpha_1) &= e^{-\gamma t} \widetilde{G}_1 + ie^{-\gamma t} b \cdot B\Upsilon^g f_{|\chi=0}^g - ie^{-\gamma t} b \cdot B\Upsilon^g L'(\partial)u_{0|\chi=0}^g \\ &\quad + \frac{c}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t,x') \cdot (\tau, \eta)} \int_0^\infty e^{\chi' \sqrt{X}} b \cdot B\Upsilon^g A_d e^{-\gamma t} \widehat{F}_1(\tau, \eta, \chi') d\chi' d\tau d\eta, \\ &:= G_1, \end{aligned}$$

where we recall that  $\lambda := b \cdot B\Upsilon^g A_d e \neq 0$ .

The first terms of  $G_1$  read  $e^{-\gamma t} f$  with  $f \in H_{t,x'}^\infty(\omega_T)$  vanishes for negative times so that they are in  $H_{t,x'}^\infty(\omega_T)$  and hence in  $H_\gamma^\infty(\omega_T)$ . The integral term in  $G_1$  is shown to be in  $H_\gamma^\infty(\omega_T)$  by the same arguments as for the integral term in the source term for the leader order glancing amplitude (see (49) and Proposition ( ) ). Reversing the operator  $T$ , we find  $\alpha_1$  as

$$\alpha_1(t, x') = \frac{1}{i\lambda} (T^{-1} G_1)(t, x').$$

Arguing like for the first amplitude  $\alpha_0$  we can show that such  $\alpha_1 \in H^\infty(\omega_T)$  vanishes for negative times. To conclude, we have determined  $\Pi^g u_1^g \in H_\gamma^\infty(\omega_T)$  as

$$\Pi^g u_1^g(t, x', \chi) = e^{\gamma t} F_3(t, x', \chi) + e^{\gamma t} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t,x') \cdot (\tau, \eta)} e^{\sqrt{X}\chi} e^{-\gamma t} \widehat{\alpha}_1(\tau, \eta) d\tau d\eta e,$$

where  $F_3$  is given by (54), (66) and (67). Reiterating the same kind of arguments as in the proof of Proposition ( ) we have fully determined  $u_1^g \in \mathbf{P}_g$  such that  $u_{1|t \leq 0}^g = 0$ .

In this paragraph, we thus have shown:

**Proposition 3.4.** *Under Assumptions 2.1, 2.2, 2.3, 2.4 and 2.5, for all  $k \in \mathcal{I} \cup \mathcal{O}$ , there exist  $u_{1,k} \in H^\infty(\Omega_T)$ ,  $u_1^g \in \mathbf{P}_g$  and  $U_{ev,1} \in \mathbf{P}_{ev}$  satisfying the cascades of equations (21), (22) and (23) at order one.*

### 3.3 Higher order terms

In this paragraph, we describe briefly how to construct terms of order  $n$  for  $n \geq 2$ . The construction is very close to the construction of the terms of order one.

We suppose that we know the terms of order  $n-1, n-2, \dots$ . So, applying the pseudo-inverses on the equation for the glancing mode of  $\mathbb{I}_{\frac{n-2}{2}}$  and the equation for the hyperbolic mode of  $\mathbb{I}_{\frac{n-2}{2}}$ , we know the unpolarized parts of the glancing and hyperbolic amplitudes of order  $n$  as :

$$\begin{aligned} (I - \Pi^g) u_n^g &= -i\delta_{n-2,0} \Upsilon^g f^g + i\Upsilon^g A_d \partial_\chi u_{n-1}^g + i\Upsilon^g L'(\partial) u_{n-2}^g \\ (I - \Pi^k) u_{n,k} &= -i\delta_{n-2,1} \Upsilon^k f_k + i\Upsilon^k L(\partial) u_{n-2,k} \end{aligned}$$

Thus, we know the unpolarized parts of the glancing and the hyperbolic amplitudes of order  $n$  and it remains to determine their polarized terms. For this, we will need to know the unpolarized parts of the terms of order  $n+1$  which are obtain applying the pseudo-inverses to  $\mathbb{I}_{\frac{n-1}{2}}$ :

$$(I - \Pi^g) u_{n+1}^g = i\Upsilon^g A_d \partial_\chi u_n^g + i\Upsilon^g L'(\partial) u_{n-1}^g, \quad (72)$$

$$(I - \Pi^k) u_{n+1,k} = -i\delta_{n-1,1} \Upsilon^k f_k + i\Upsilon^k L(\partial) u_{n-1,k}. \quad (73)$$

Next, applying the projectors  $\Pi^k$  and  $\Pi^g$  on  $\mathbb{I}_{\frac{n}{2}}$  and decomposing the amplitudes in terms of their polarized and unpolarized parts, we deduce that:

$$\begin{aligned} \Pi^g A_d \Pi^g \partial_\chi u_{n+1}^g + i\Pi^g A_d \Upsilon^g A_d \Pi^g \partial_\chi^2 u_n^g + \Pi^g L'(\partial) \Pi^g u_n^g &= F_{n,1}, \\ \Pi^k L(\partial) \Pi^k u_{n,k} &= f_{k,n}, \end{aligned}$$

where  $F_n^1 := -\Pi^g L'(\partial) (I - \Pi^g) u_n^g - i\Pi^g A_d \Upsilon^g L'(\partial) \partial_\chi u_{n-1}^g - i\Pi^g A_d \Upsilon^g A_d \partial_\chi^2 (I - \Pi^g) u_n^g$  and  $f_{k,n} := -\Pi^k L(\partial) (I - \Pi^k) u_{n,k}$  are known.

So using Lemmas 3.1 and 3.2, these equations can be put on a similar form to (56) and (57) that is:

$$-\partial_\chi^2 \Pi^g u_n^g + ic (\partial_t + v'_g \cdot \nabla_{x'}) \Pi^g u_n^g = ic F_{n,1}, \quad (74)$$

$$(\partial_t + v \cdot \nabla_x) \Pi^k u_{n,k} = f_{k,n}. \quad (75)$$

Notice that from the equation of the evanescent mode of  $\mathbb{I}_{\frac{n-2}{2}}$ , we have:

$$L(\partial_{X_d}) U_{ev,n} = \delta_{n-2,0} f_{ev} - L'(\partial) U_{ev,n-2} := F_{ev,n}. \quad (76)$$

So, using Lemma 3.3, we know the form of the evanescent profile:

$$U_{ev,n} = P_{ev} U_{ev,n} + Q_{ev} F_{ev,n}.$$

We now look at the boundary condition  $\mathbb{B}_{\frac{n}{2}}$  because we need to know  $\Pi^k u_{n,k}|_{x_d=0}$  for  $k \in \mathcal{I}$  and  $P_{ev} U_{ev,n}|_{x_d=0}$  to determine the incoming hyperbolic and the elliptic amplitude. We suppose that we know  $\Pi^k u_{n,k}$  for  $k \in \mathcal{O}$  (it is not an issue because we will determined then before the other amplitudes). We also recall that the unpolarized part of the incoming hyperbolic and  $Q_{ev} F_{ev,n}$  are known. So the boundary condition gives:

$$\begin{aligned} B \left( \sum_{k \in \mathcal{I}} \Pi^k u_{n,k}|_{x_d=0} + u_{n|_{\chi=0}}^g + P_{ev} U_{ev,n}|_{x_d=0} \right) &= -B \sum_{k \in \mathcal{O}} u_{n,k}|_{x_d=0} - B \sum_{k \in \mathcal{I}} (I - \Pi^k) u_{n,k}|_{x_d=0} \\ &\quad - B Q_{ev} F_{ev,n}|_{x_d=0}. \end{aligned}$$

Because the unpolarized part of the glancing amplitude is known and the fact that the polarized part can be write as (for the same reasons as in Section 3.2, see equation (60)):

$$\Pi^g u_{n|\chi=0}^g = \alpha_n e \text{ where } \alpha_n \text{ is a scalar function defined on } \omega_T, \quad (77)$$

the precedent equation becomes:

$$B \left( \sum_{k \in \mathcal{I}} \Pi^k u_{n,k|_{x_d=0}} + P_{ev} U_{ev,n|_{x_d=0}} \right) = G_n,$$

where  $G_n := -B \sum_{k \in \mathcal{O}} u_{n,k|_{x_d=0}} - B \sum_{k \in \mathcal{I}} (I - \Pi^k) u_{n,k|_{x_d=0}} - B Q_{ev} F_{ev,n|_{x_d=0}} - B (I - \Pi^g) u_{n|\chi=0}^g$ .

So, reversing  $B$  and using the boundary projectors (see Definition 2.5), we deduce the boundary conditions for the  $\Pi^k u_{n,k}$  for  $k \in \mathcal{I}$  and  $P_{ev} U_{ev,n}$ :

$$\begin{aligned} \forall k \in \mathcal{I}, \Pi^k u_{n,k|_{x_d=0}} &= P^k \Phi G_n, \\ P_{ev} U_{ev,n|_{x_d=0}} &= P^e \Phi G_n. \end{aligned}$$

With this result in hand, we can construct the hyperbolic and elliptic amplitudes of order  $n$ :

- The outgoing amplitudes of order  $n$  are determined by solving the transport equation without boundary condition

$$\begin{cases} (\partial_t + v_k \cdot \nabla_x) \Pi^k u_{n,k} = f_{k,n}, \\ \Pi^k u_{n,k|_{t < 0}} = 0. \end{cases}$$

Moreover, we will need to know the outgoing amplitudes of order  $n + 1$  to determine the glancing one of order  $n$ . They are given by (73) and by solving the transport equation:

$$\begin{cases} (\partial_t + v_k \cdot \nabla_x) \Pi^k u_{n+1,k} = f_{k,n+1}, \\ \Pi^k u_{n+1,k|_{t < 0}} = 0. \end{cases}$$

- The incoming amplitudes of order  $n$  are determined by solving the transport equation with boundary condition:

$$\begin{cases} (\partial_t + v_k \cdot \nabla_x) \Pi^k u_{n+1,k} = f_{k,n+1}, \\ \Pi^k u_{n,k|_{x_d=0}} = P^k \Phi G_n, \\ \Pi^k u_{n,k|_{t < 0}} = 0. \end{cases}$$

- For the evanescent amplitude, we deduce from the boundary condition that

$$U_{ev,n}(t, x', X_d) = e^{X_d A(\zeta)} P^e \Phi G_n(t, x') + Q_{ev} F_{ev,n}(t, x', X_d).$$

Furthermore, we will need to know the form of  $U_{ev,n+1}$  to determine the glancing amplitude. It solve  $L(\partial_{X_d}) U_{ev,n+1} = F_{ev,n+1}$ , so (from Lemma 3.3), we have:

$$U_{ev,n+1} = P_{ev} U_{ev,n+1} + Q_{ev} F_{ev,n+1}.$$

Now, we can construct the polarized part of the glancing amplitude. As for the term of order one, let  $v_n^g = e^{-\gamma t} \Pi^g u_n^g$  ( $\gamma > 0$ ). Using this and performing a Fourier transform, (74) gives:

$$-\partial_\chi^2 \widehat{v_n^g} + X \widehat{v_n^g} = i c e^{-\gamma t} \widehat{F_{n,1}}.$$

So, following the same steps as for the glancing amplitude of order one (see Paragraph 3.2.3), we obtain:

$$v_n^g(t, x', \chi) = F_{n,2}(t, x', \chi) + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t, x') \cdot (\tau, \eta)} e^{X \sqrt{X}} \widehat{v_n^g}|_{\chi=0}(\tau, \eta) d\tau d\eta \quad (78)$$

where  $F_{n,2} := \frac{ic}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(t, x') \cdot (\tau, \eta)} \int_0^X e^{(X-\chi') \sqrt{X}} \int_{\chi'}^\infty e^{-(X-\chi'') \sqrt{X}} e^{-\gamma t} \widehat{F_{n,1}}(\tau, \eta, \chi'') d\chi'' d\chi' d\tau d\eta$ .

So, to determine  $v_n^g$  (and so  $\Pi^g u_n^g$ ), it is sufficient to determine  $\widehat{v_n^g}|_{\chi=0}$ . Recall that from (77) we have:

$$v_n^g|_{\chi=0} = e^{-\gamma t} \alpha_n e. \quad (79)$$

Thus, it remains to determine  $\alpha_n$ . For this, we look at the boundary condition  $\mathbb{B}_{\frac{n+1}{2}}$  which can be rewritten as:

$$\begin{aligned} & B \sum_{k \in \mathcal{I}} \Pi^k u_{n+1, k|_{x_d=0}} + B u_{n+1, k|_{x_d=0}}^g + B P_{ev} U_{ev, n+1|_{x_d=0}} \\ &= -B \sum_{k \in \mathcal{O}} u_{n+1, k|_{x_d=0}} - B \sum_{k \in \mathcal{I}} (I - \Pi^k) u_{n+1, k|_{x_d=0}} - B Q_{ev} F_{ev, n+1|_{x_d=0}} \end{aligned}$$

where the right hand side term is known.

Decomposing  $u_{n+1}^g$  in terms of polarized and unpolarized part and testing against the vector  $b$  defined in (13), we obtain:

$$b \cdot B (I - \Pi^g) u_{n+1, k|_{x_d=0}}^g = G_{n,1}$$

where  $G_{n,1} := -b \cdot B \sum_{k \in \mathcal{O}} u_{n+1, k|_{x_d=0}} - b \cdot B \sum_{k \in \mathcal{I}} (I - \Pi^k) u_{n+1, k|_{x_d=0}} - b \cdot B Q_{ev} F_{ev, n+1|_{x_d=0}}$ .

Now, using (72) and multiplying by  $e^{-\gamma t}$ , it remains:

$$i b \cdot B \Upsilon^g A_d (\partial_\chi v_n^g)|_{\chi=0} = e^{-\gamma t} G_{n,1} - i e^{-\gamma t} b \cdot B \Upsilon^g A_d L'(\partial) u_{n-1|_{x_d=0}}^g.$$

Next, using (78), we deduce that:

$$i \lambda T(\alpha_n) = G_{n,2}$$

where  $G_{n,2} := e^{-\gamma t} G_{n,1} - i e^{-\gamma t} b \cdot B \Upsilon^g A_d L'(\partial) u_{n-1|_{x_d=0}}^g - i b \cdot B \Upsilon^g A_d (\partial_\chi F_{n,2})|_{\chi=0}$ . Hence, we determine  $\alpha_n \in H^\infty(\omega_T)$  vanishing for negative times as

$$\alpha_n(t, x') = \frac{1}{i\lambda} (T^{-1} G_{n,2})(t, x').$$

So, we have determined the polarized part of the glancing amplitude of order  $n$ .

To conclude this paragraph, we give the following proposition:

**Proposition 3.5.** *Under Assumptions 2.1, 2.2, 2.3, 2.4 and 2.5, for all  $n \in \mathbb{N}$ , for all  $k \in \mathcal{I} \cup \mathcal{O}$ , there exists  $u_{n,k} \in H^\infty(\Omega_T)$ ,  $u_n^g \in \mathbf{P}_g$  and  $U_{ev,n} \in \mathbf{P}_{ev}$  satisfying the cascade of equations (21), (22) and (23) at the order  $n$ .*

## 4 Discussion about the energy estimates

In this section, we consider the non-oscillatory initial boundary value problem

$$\begin{cases} L(\partial)u = f & \text{in } \Omega_T, \\ Bu|_{x_d=0} = g & \text{on } \omega_T, \\ u|_{t \leq 0} = 0 & \text{on } \mathbb{R}_+^d. \end{cases} \quad (80)$$

The aim of the following is to use the geometric optics expansion to investigate what can be the losses of derivatives for the solution to (80). Indeed because the uniform Kreiss-Lopatinskii condition fails then such losses will happen. However for a failure of the uniform Kreiss-Lopatinskii condition on a glancing frequency then the weak well-posedness theory is not achieved yet and we do not have precise energy estimates yet. The existing estimates that can apply to this framework are the ones of [11] and [10]. In [11] the result is that if the uniform Kreiss-Lopatinskii condition fails at some order  $\delta > 0$  (or that it satisfies the Kreiss-Sakamoto condition of power  $\delta$ ) meaning that we have the estimate  $|Bv| \gtrsim \gamma^\delta |v|$  then we have the following energy estimate, there exists  $C > 0$  such that for large values of  $\gamma$  there holds

$$\gamma \|u\|_{L_\gamma^2(\Omega_T)}^2 + \|u|_{x_d=0}\|_{L_\gamma^2(\omega_T)}^2 \leq C \left( \frac{1}{\gamma^{1+2\delta}} \|L(\partial)u\|_{H_\gamma^\delta(\Omega_T)}^2 + \frac{1}{\gamma^{2\delta}} \|Bu|_{x_d=0}\|_{H_\gamma^\delta(\omega_T)}^2 \right),$$

where for  $X \subset \mathbb{R}_+^d$  the space  $L_\gamma^2(X)$  is defined via the norm  $\|\cdot\|_{L_\gamma^2(X)} := \|e^{-\gamma t} \cdot\|_{L^2(X)}$ ; the Sobolev spaces  $H_\gamma^m(X)$  being defined accordingly.

In [10] it is shown that if the boundary condition is in the particular so-called conservative form then one can recover an estimate without loss of derivatives in the interior up to the price of a larger loss along the boundary meaning that the estimate reads

$$\gamma \|u\|_{L_\gamma^2(\Omega_T)}^2 + \|u|_{x_d=0}\|_{H_\gamma^{-1/2}(\omega_T)}^2 \leq \frac{C}{\gamma} \left( \|L(\partial)u\|_{L_\gamma^2(\Omega_T)}^2 + \|Bu|_{x_d=0}\|_{H_\gamma^{1/2}(\omega_T)}^2 \right).$$

For a sake of completeness we recall that conservative boundary conditions are defined in the following way

**Definition 4.1** (Conservative boundary condition). *We say that the boundary condition  $B \in \mathcal{M}_{p \times N}(\mathbb{R})$  is in conservative form if there exists a matrix  $C \in \mathcal{M}_{p \times N}(\mathbb{R})$  such that we can write*

$$A_d = \text{Re}(C^T B) := \frac{1}{2} (C^T B + B^T C).$$

However let us point that the link between the failure of the Kreiss-Lopatinskii condition in the glancing area and the fulfilment of the Kreiss-Sakamoto condition and or the fact that the boundary condition is conservative is not well-established. It seems that if the Kreiss-Lopatinskii condition fails on a glancing mode of order  $k$  then the Kreiss-Sakamoto condition of order  $1/k$  is satisfied but this conjecture is left for future studies.

Before to turn to a precise statement of the existing losses of derivatives first let us point that if one has an energy estimate for the problem (80) then it can be shown that the truncated geometric optics expansion is a good approximation of the exact solution of  $u^\varepsilon$  (we refer to the proof of Theorem 2.3 below for a precise proof of this fact (see equation (84))). This answers in particular the question of the uniqueness of the geometric optics expansion.



In order to investigate all the possible cases in the losses we will need the following extra assumption already met in [3] in order to ensure the existence of suitable some outgoing mode which can turn on the loss of derivative in the interior. Note that the existence of such modes is automatic in the so-called *WR* framework studied in [9] but that they are not in the general setting.

**Assumption 4.1.** *We assume that there exists  $\underline{k} \in \mathcal{O}$  such that  $b \cdot B \ker \mathcal{L}(d\varphi_{\underline{k}}) \neq 0$ .*

Let us also mention that the results of Theorem 2.3 agree with the conjecture about weakly well-posed problems and the failure of the uniform Kreiss-Lopatinskii condition for glancing modes of [[6]-Chapter 7.1].

We now turn to the proof of Theorem 2.3.

Proof: We first show that if Assumption 4.1 holds then the boundary value problem (5) has a loss of at least one half derivative in the interior. Then we show 3. of Theorem 2.3 that is the fact that in the general framework a loss of at least one half derivative in the interior or on the boundary occurs. In particular 2. of Theorem 2.3 (the case without loss of derivative in the interior) is obtained as a particular case of 3. and we will omit the proof.

We assume that Assumption 4.1 holds and we argue by contradiction assuming first that  $s_1 < \frac{1}{2}$  in order to show that  $s_1 \geq \frac{1}{2}$ .

Let  $u^\varepsilon$  be the solution to the problem

$$\begin{cases} L(\partial)u^\varepsilon = f^\varepsilon & \text{in } \Omega_T, \\ Bu^\varepsilon|_{x_d=0} = 0 & \text{on } \omega_T, \\ u^\varepsilon|_{t \leq 0} = 0 & \text{on } \mathbb{R}_+^d, \end{cases} \quad (81)$$

where  $f^\varepsilon := \varepsilon^{\frac{1}{2}} e^{\frac{i\varphi_{\underline{k}}}{\varepsilon}} f_{\underline{k}}$  with  $\underline{k} \in \mathcal{O}$  the index given by Assumption 4.1.

For  $N_0 \geq 0$ , let

$$u_{N_0, app}^\varepsilon := \sum_{n=0}^{N_0} \varepsilon^{\frac{n}{2}} \sum_{k \in \mathcal{I} \cup \mathcal{O}} e^{\frac{i}{\varepsilon} \varphi_k} u_{n,k} + \sum_{n=0}^{N_0} \varepsilon^{\frac{n}{2}} e^{\frac{i}{\varepsilon} \varphi^g} u_n^g + \sum_{n=0}^{N_0} \varepsilon^{\frac{n}{2}} e^{\frac{i}{\varepsilon} \psi} U_{ev,n}, \quad (82)$$

where the profiles  $(u_{n,k})_{n \in \mathbb{N}, k \in \mathcal{I} \cup \mathcal{O}} \in H^\infty(\Omega_T)$ ,  $(u_n^g)_{n \in \mathbb{N}} \in \mathbf{P}_g$  and  $(U_{ev,n})_{n \in \mathbb{N}} \in \mathbf{P}_g$  are given by Theorem 2.2.

We have, using the triangle inequality:

$$\|u_{N_0, app}^\varepsilon\|_{L^2(\Omega_T)} \leq \|u_{N_0+2, app}^\varepsilon - u_{N_0, app}^\varepsilon\|_{L^2(\Omega_T)} + \|u_{N_0+2, app}^\varepsilon - u^\varepsilon\|_{L^2(\Omega_T)} + \|u^\varepsilon\|_{L^2(\Omega_T)}, \quad (83)$$

and in the following we will estimate each term separately. The term  $u_{N_0+2, app}^\varepsilon - u^\varepsilon$  satisfies the following problem:

$$\begin{cases} L(\partial) (u_{N_0+2, app}^\varepsilon - u^\varepsilon) = f_1^\varepsilon, & \text{in } \Omega_T, \\ B (u_{N_0+2, app}^\varepsilon - u^\varepsilon)|_{x_d=0} = 0, & \text{on } \omega_T \\ (u_{N_0+2, app}^\varepsilon - u^\varepsilon)|_{t < 0} = 0 & \text{on } \mathbb{R}_+^d, \end{cases}$$

where

$$\begin{aligned} f_1^\varepsilon := & \varepsilon^{\frac{N_0+1}{2}} \left( \sum_{k \in \mathcal{I} \cup \mathcal{O}} e^{\frac{i}{\varepsilon} \varphi_k} (L(\partial)u_{N_0+1,k} + \sqrt{\varepsilon}L(\partial)u_{N_0+2,k}) \right. \\ & + e^{\frac{i}{\varepsilon} \varphi^g} (A_d \partial_\chi u_{N_0+2}^g + L'(\partial)u_{N_0+1}^g + \sqrt{\varepsilon}L'(\partial)u_{N_0+2}^g) \\ & \left. + e^{\frac{i}{\varepsilon} \psi} (L'(\partial)U_{ev, N_0+1} + \sqrt{\varepsilon}L'(\partial)U_{ev, N_0+2}) \right). \end{aligned}$$

We see that  $\|f_1^\varepsilon\|_{L^2(\Omega_T)}$  is  $O(\varepsilon^{\frac{N_0+1}{2}})$  and  $\|f_1^\varepsilon\|_{H^1(\omega_T)}$  is  $O(\varepsilon^{\frac{N_0-1}{2}})$ . So, using the interpolation inequalities, we deduce that  $\|f_1^\varepsilon\|_{L^2(H^{s_1}(\omega_T))} \leq C\varepsilon^{\frac{N_0+1}{2}-s_1}$  where  $C$  is independent of  $\varepsilon$ .

Consequently,

$$\|u_{N_0+2,app}^\varepsilon - u^\varepsilon\|_{L^2(\Omega_T)} \leq C\|f_1^\varepsilon\|_{L^2(\mathbb{R}_d^+; H_{t,x'}^{s_1}(\omega_T))} \leq C\varepsilon^{\frac{N_0+1}{2}-s_1}. \quad (84)$$

Moreover,  $\|f^\varepsilon\|_{L^2(\Omega_T)}$  is  $O(\varepsilon^{\frac{1}{2}})$  and  $\|f^\varepsilon\|_{H^1(\omega_T)}$  is  $O(\varepsilon^{-\frac{1}{2}})$ . So, using the interpolation inequalities, we deduce that  $\|f^\varepsilon\|_{H^{s_1}(\omega_T)} \leq C\varepsilon^{\frac{1}{2}-s_1}$ . Thus, from the energy estimate (16) applied to problem (81) we obtain

$$\|u^\varepsilon\|_{L^2(\Omega_T)} \leq C\varepsilon^{\frac{1}{2}-s_1}.$$

Finally  $\|u_{N_0+2,app}^\varepsilon - u_{N_0,app}^\varepsilon\|_{L^2(\Omega_T)}$  is clearly  $O(\varepsilon^{\frac{N_0+1}{2}})$ ; and so, taking  $N_0 = 0$  it follows that

$$\|u_{0,app}^\varepsilon\|_{L^2(\Omega_T)} \leq C\left(\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}-s_1} + \varepsilon^{\frac{1}{2}-s_1}\right).$$

Because  $u_{0,app}^\varepsilon = e^{\frac{i\varphi g}{\varepsilon}}u_0^g$ , we have on the one hand  $\|u_{0,app}^\varepsilon\|_{L^2(\Omega_T)} = \|u_0^g\|_{L^2(\Omega_T)}$ . On the other hand because  $s_1 < \frac{1}{2}$  it turns out that  $\|u_0^g\|_{L^2(\Omega_T)}$  tends to 0 as  $\varepsilon$  goes to 0, hence  $u_0^g = 0$ .

However  $u_0^g$  is given by (50) in which the source term  $G$  defined in (49) reads  $e^{-\gamma t}b \cdot B\Pi^k u_{1,k}|_{x_d=0}$  where  $\Pi^k u_{1,k}|_{x_d=0}$  only depends on  $f_k$  as the solution of the associated transport equation. Let us point that from Assumption 4.1 the term  $b \cdot B\Pi^k u_{1,k}|_{x_d=0}$  is non-zero. We are free to choose the interior source term  $f_k$  in order that  $\Pi^k u_{1,k}|_{x_d=0}$  is not zero at so do  $u_0^g$ . We have a contradiction and we get  $s_1 \geq \frac{1}{2}$  as desired.

Now we turn to the proof that  $s_1 \geq \frac{1}{2}$  or  $s_2 \geq \frac{1}{2}$ . Once again we argue by contradiction and assume that  $s_1 < \frac{1}{2}$  and  $s_2 < \frac{1}{2}$ .

Let  $u^\varepsilon$  be the solution to the problem

$$\begin{cases} L(\partial)u^\varepsilon = 0 & \text{in } \Omega_T, \\ Bu_{|x_d=0}^\varepsilon = g^\varepsilon, & \text{on } \omega_T, \\ u_{|t \leq 0}^\varepsilon = 0 & \text{on } \mathbb{R}_d^+, \end{cases} \quad (85)$$

where  $g^\varepsilon := \varepsilon^{\frac{1}{2}}e^{\frac{i\psi}{\varepsilon}}g$  for some regular  $g$ , specified below, vanishing for negatives times. We consider again the approximated solution  $u_{N_0,app}^\varepsilon$  given by (82) and once again we will use the triangle inequality (83) to estimate  $u_{N_0,app}^\varepsilon$ .

The term  $u_{N_0+2,app}^\varepsilon - u^\varepsilon$  satisfies the same problem as before from which we infer that

$$\|u_{N_0+2,app}^\varepsilon - u^\varepsilon\|_{L^2(\Omega_T)} \leq C\varepsilon^{\frac{N_0+1}{2}-s_1}.$$

Now, because  $\|g^\varepsilon\|_{L^2(\omega_T)}$  is  $O(\varepsilon^{\frac{1}{2}})$  and  $\|g^\varepsilon\|_{H^1(\omega_T)}$  is  $O(\varepsilon^{-\frac{1}{2}})$  so we deduce that  $\|g^\varepsilon\|_{H^{s_2}(\omega_T)} \leq C\varepsilon^{\frac{1}{2}-s_2}$ . Thus, from the energy estimate (16) applied to the problem (85) we obtain

$$\|u^\varepsilon\|_{L^2(\Omega_T)} \leq C\varepsilon^{\frac{1}{2}-s_2}.$$

Finally  $\|u_{N_0+2,app}^\varepsilon - u_{N_0,app}^\varepsilon\|_{L^2(\Omega_T)}$  is still  $O\left(\varepsilon^{\frac{N_0+1}{2}}\right)$ . Consequently<sup>2</sup> for  $N_0 = 0$ :

$$\|u_{0,app}^\varepsilon\|_{L^2(\Omega_T)} \leq C\left(\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}-s_1} + \varepsilon^{\frac{1}{2}-s_2}\right). \quad (86)$$

<sup>2</sup>Taking  $s_1 = 0$  in estimate (86) gives the proof of point 2. of Theorem 2.3

Since  $u_{0,app}^\varepsilon = e^{\frac{i\varphi^g}{\varepsilon}} u_0^g$ , we have  $\|u_{0,app}^\varepsilon\|_{L^2(\Omega_T)} = \|u_0^g\|_{L^2(\Omega_T)}$ .

Because of our working assumption on  $s_1$  and  $s_2$ , equation (86) implies that  $\|u_{0,app}^\varepsilon\|_{L^2(\Omega_T)}$  goes to zero with  $\varepsilon$  so that one should have  $u_0^g = 0$ . But recall that  $u_0^g$  is given by (50) in which the source  $G$  reads  $b \cdot g$ . In order to make sure that it gives a non zero leading order term we can for example set  $g := \chi b$  where  $\chi$  is some function in  $\mathcal{D}(\omega_T)$ . Once again we obtain the desired contradiction.  $\square$

## 5 Some examples

### 5.1 The $2d$ -wave equation

We are considering the  $2d$  classical wave equation (with without loss of generality velocity  $c = 1$ ) with Neumann boundary condition:

$$\begin{cases} \partial_t^2 u - \Delta u = f & \text{for } (t, x_1, x_2) \in ]-\infty, T] \times \mathbb{R}_+ \times \mathbb{R}, \\ (\partial_1 u)|_{x_1=0} = g & \text{for } (t, x_2) \in ]-\infty, T] \times \mathbb{R}, \\ u|_{t \leq 0} = (\partial_t u)|_{t \leq 0} = 0 & \text{for } (x_1, x_2) \in \mathbb{R} \times \mathbb{R}, \end{cases} \quad (87)$$

If we define  $U := [\partial_t u - \partial_1 u - \partial_2 u, \partial_t u + \partial_1 u - \partial_2 u]^T$  then equation (87) becomes

$$\begin{cases} \partial_t U + A_1 \partial_1 U + A_2 \partial_2 U = F & \text{for } (t, x_1, x_2) \in ]-\infty, T] \times \mathbb{R}_+ \times \mathbb{R}, \\ BU|_{x_1=0} = \frac{g}{2} & \text{for } (t, x_2) \in ]-\infty, T] \times \mathbb{R}, \\ U|_{t \leq 0} = 0 & \text{for } (x_1, x_2) \in \mathbb{R} \times \mathbb{R}, \end{cases} \quad (88)$$

where we defined  $F := [f, f]^T$  and

$$A_1 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad A_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and } B := \begin{bmatrix} -1 & 1 \end{bmatrix}.$$

We can check easily from the coefficients  $A_1$  and  $A_2$  that the boundary value problem (88) satisfy Assumptions 2.1 and 2.2. Moreover Assumption 2.3 is also satisfied because the problem is of size two.

Then in such a setting the boundary condition  $B$  breaks down the uniform Kreiss-Lopatinskii condition in the glancing area. Indeed the resolvent matrix associated to (88) is

$$\mathcal{A}(\zeta) = \begin{bmatrix} -\sigma & -i\eta \\ i\eta & \sigma \end{bmatrix},$$

whose eigenvalues  $\lambda_\pm$  ( $\lambda_-$  being the one with negative real part) are the roots of the equation

$$\lambda^2 = \sigma^2 + \eta^2,$$

and we can thus parametrize the stable subspace  $\mathbb{E}^s(\sigma, \eta) := \text{span} \left\{ [-i\eta, \lambda_- + \sigma]^T \right\}$  from which we can verify that  $\ker B \cap \mathbb{E}^s(\sigma, \eta) = \{0\}$  for all  $\gamma > 0$ . For  $\gamma = 0$  we thus obtain that the eigenvalues are solution of  $\lambda^2 = \tau^2 - \eta^2$  so that the elliptic region is given by  $\mathcal{E} := \{(i\tau, \eta) \in i\mathbb{R} \times \mathbb{R} \mid |\tau| > |\eta|\}$  and that the hyperbolic region is  $\mathcal{H} := \{(i\tau, \eta) \in i\mathbb{R} \times \mathbb{R} \mid |\tau| < |\eta|\}$ . For  $|\tau| = |\eta|$  we thus have zero as a root of multiplicity two from which we deduce that  $\mathcal{G} := \partial\mathcal{H} = \{(i\tau, \eta) \in i\mathbb{R} \times \mathbb{R} \mid |\tau| = |\eta|\}$ .

Now fix  $\eta < 0$  and choose  $\tau = -\eta$  for such parameters we have

$$\mathcal{A}(-\eta, \eta) = i\eta \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},$$

from which we can read that  $\mathbb{E}^s(-\eta, \eta) = \text{span} \left\{ [1, 1]^T \right\} = \ker B$  so that the uniform Kreiss-Lopatinskii condition fails in the glancing area. Moreover because  $\dim \ker B = \dim \mathbb{E}^s(-\eta, \eta) = 1$  we are in the framework studied in this article and Assumption 2.5 holds. Indeed because  $p = 1$  we can choose  $b = 1$  and simple computations give that

$$B\Upsilon^g A_1 e = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2\eta} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\frac{1}{2\eta} \neq 0.$$

Consequently one can use Theorem 2.2 to perform a geometric optics expansion of the solution to (88). About the existence of such a solution we refer to [11] or [10]. Then if one applies Theorem 2.3 because the system is of size two the expansion only contains one glancing amplitude (and no hyperbolic outgoing modes) then Assumption 4.1 fails. So we are in cases 2. and 3. of Theorem 2.3. However in such a configuration we have energy estimates from the work of [11] and [10].

On the one hand it is shown in [11] that the so-called Kreiss-Sakamoto condition with power  $\frac{1}{2}$  is satisfied meaning that

$$\forall \zeta \in \Xi, \forall v \in \mathbb{E}^s(\zeta), |Bv| \gtrsim \gamma^{1/2} |v|.$$

So that from [11] we have the energy estimate

$$\gamma \|u\|_{L_\gamma^2(\Omega)}^2 + \|u|_{x_d=0}\|_{L_\gamma^2(\partial\Omega)}^2 \leq C \left( \frac{1}{\gamma^2} \|f\|_{L_{x_d}^2(H_\gamma^{1/2}(\partial\Omega))}^2 + \frac{1}{\gamma} \|g\|_{H_\gamma^{1/2}(\partial\Omega)}^2 \right),$$

which is sharp because of Theorem 2.2 giving another justification of the fact that the power  $1/2$  of the Kreiss-Sakamoto condition can not be lowered in [11].

On the other hand we can show that the boundary condition  $B$  is conservative in the sense of Definition 4.1 (take  $C = B$ ) so that from [10] we also have the boundary estimate without loss of derivative in the interior

$$\gamma \|u\|_{L_\gamma^2(\Omega_T)}^2 + \gamma \|u|_{x_d=0}\|_{H_\gamma^{-1/2}(\omega_T)}^2 \leq \frac{C}{\gamma} \left( \|f\|_{L_\gamma^2(\Omega_T)}^2 + \|g\|_{H_\gamma^{1/2}(\omega_T)}^2 \right).$$

We can apply 2. of Theorem 2.3 which shows that the index  $1/2$  in the norm of the right hand side is optimal.

## 5.2 Linearisation of Euler equation

We now consider the linearisation of isentropic  $2d$ -Euler equations around some subsonic outgoing fluid. This system of equations reads

$$\begin{cases} \partial_t u + A_1 \partial_1 u + A_2 \partial_2 u = f, & \text{for } (t, x_1, x_2) \in ]-\infty, T] \times \mathbb{R} \times \mathbb{R}_+, \\ Bu|_{x_2=0} = g, & \text{for } (t, x_1) \in ]-\infty, T] \times \mathbb{R}, \\ u|_{t \leq 0} = 0, & \text{for } (x_1, x_2) \in \mathbb{R} \times \mathbb{R}_+, \end{cases} \quad (89)$$

where the coefficients  $A_1$ ,  $A_2$  and  $B$  are given by

$$A_1 := \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 := \begin{bmatrix} M & 0 & -1 \\ 0 & M & 0 \\ -1 & 0 & M \end{bmatrix} \quad \text{and} \quad B := [1 \quad b_2 \quad b_3]$$

in which  $M \in ]-1, 0[$  stands for the Mach number and where  $(b_2, b_3) \in \mathbb{R}^2$  are boundary parameters that can be chosen arbitrarily. It is rather easy to verify that the system (89) satisfies Assumptions 2.1 and 2.2. Let us mention by the way that the boundary condition  $B$  can not be conservative in the sense of Definition 4.1.

In [3] we can find a discussion about what are the influence of the parameters on the strong (or weak) well-posedness of (89).

In particular the sets of parameters  $(b_2, b_3) \in \mathbb{R}^2$  for which the Kreiss-Lopatinskii condition degenerates in the glancing area is given by

$$\Upsilon := \left\{ (b_2, b_3) \in \mathbb{R}^2 \setminus b_2 = \pm \frac{1 + Mb_3}{\sqrt{1 - M^2}} \right\} := \Omega_g^+ \cup \Omega_g^-,$$

the union of these two lines constitute the parameters  $(b_2, b_3) \in \mathbb{R}^2$  that were not cover by the analysis of [3]. Now Theorem 2.3 ends the picture.

More precisely in order to apply this theorem we should be more precise about the fulfilment of Assumption 4.1. In order to do so we should have a look to the stable and unstable subspaces for glancing frequencies. Let  $\omega$  be an eigenvalue of  $\mathcal{A}(\zeta)$  we thus have  $\det(\sigma I + i\eta A_1 + \omega A_2) = 0$  so that  $\omega$  satisfies the dispersion relation

$$(M\omega + \sigma) \cdot ((M\omega + \sigma)^2 + \eta^2 - \omega^2) = 0. \quad (90)$$

For  $\zeta \in \Xi_0$  we thus always have a hyperbolic eigenvalue  $\omega_h := -\frac{i\tau}{M}$  which differs from the roots of the second term of the left hand side of (90) for glancing frequencies  $(i\tau, \eta) \in \mathbb{G} = \{(i\tau, \eta) \in i\mathbb{R} \times \mathbb{R} \setminus |\tau| = \sqrt{1 - M^2}|\eta|\}$ . In particular Assumption 2.3 holds.

Because  $p = 1$ , the hyperbolic eigenvalue  $\omega_h$  contributes to the unstable subspace and some simple computations show that it is associated to the eigenspace  $\mathbb{E}^u(\zeta) := \text{span} \left\{ \left[ 0, \frac{\tau}{M}, \eta \right]^T \right\} = \text{span}\{v_h\}$ . In order to study Assumption 4.1 we have to compute  $Bv_h$  for  $\tau = \pm\sqrt{1 - M^2}|\eta|$  and  $(b_2, b_3) \in \Omega_g^+ \cup \Omega_g^-$  to fix the ideas we set  $\tau = \pm\sqrt{1 - M^2}|\eta|$  and  $(b_2, b_3) \in \Omega_g^+$  the other case being treated similarly. We thus have

$$Bv_h = \begin{cases} \eta(1 + b_3M + b_3) & \text{if } \eta > 0, \\ \eta(-1 - b_3M + b_3) & \text{if } \eta < 0, \end{cases}$$

from which we deduce that Assumption 4.1 is satisfied for all parameters  $b_3$  except for  $b_3 \in \left\{ \frac{-1}{M-1}, \frac{1}{1-M} \right\}$ . So that when Assumption 4.1 holds we are in 1. of Theorem 2.3 and so the equation (89) loses at least one half of derivative in the interior. This agree with the fact that the boundary conditions of (89) can not be of conservative type so that the result of [10] does not apply.

On Figure 5.2 is depicted the influence of the boundary parameters on the well-posedness of (89). In this figure  $WR$  stands for the parameter giving a problem in the  $WR$  class studied in [5],  $SU$  stands for strongly unstable that is problems that do not satisfy the weak Kreiss-Lopatinskii condition and  $SS$  stands for problem satisfying the uniform Kreiss-Lopatinskii condition.

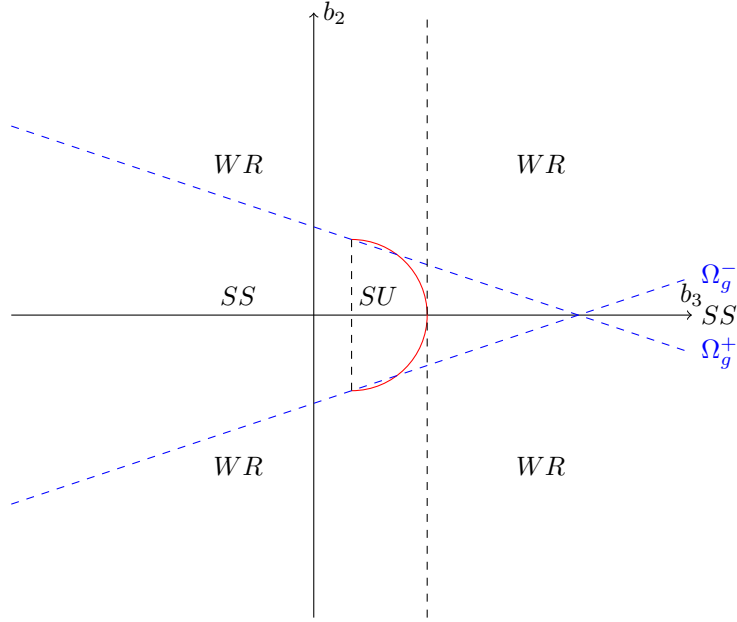


Figure 1: The different boundary parameters for (89).

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