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SATURATION, SEMINORMALIZATION AND HOMEOMORPHISMS OF ALGEBRAIC VARIETIES

FRANÇOIS BERNARD, GOULWEN FICHOU, JEAN-PHILIPPE MONNIER AND RONAN QUAREZ

ABSTRACT. We address the question under which conditions a bijective morphism between algebraic varieties over an algebraically closed field of characteristic zero is an isomorphism. Our two answers involve a study of seminormalization and saturation for morphisms between algebraic varieties, together with an interpretation in terms of continuous rational functions on the closed points of an algebraic variety. The continuity refers here to the usual Euclidean continuity in the complex case, and comes from the theory of real closed fields otherwise.

Let $\pi : Y \rightarrow X$ be a morphism between algebraic varieties over a field k such that the induced map $\pi_k : Y(k) \rightarrow X(k)$ at the level of closed points is bijective. The main question we address is under which topological conditions on π and geometric conditions on X the morphism π is an isomorphism.

We also compare bijections and isomorphisms with homeomorphisms with respect to Zariski topology. Recall from the Nullstellensatz that π is an homeomorphism if and only if π_k is an homeomorphism when k is algebraically closed, assumption we make in the paper.

Of course an isomorphism is an homeomorphism and an homeomorphism induces a bijection at the level of closed points. Conversely, starting from a bijection at the level of closed points, it is an homeomorphism in the case of irreducible curves. This is no longer true in general, however a bijection induces an isomorphism when the target variety is normal by Zariski Main Theorem. Assuming now π to be an homeomorphism, there are similar results involving the notion of seminormality in place of normality. Andreotti and Bombieri [2] proved that π is an isomorphism if X is seminormal and π is finite. Vitulli [38] managed to remove the finiteness assumption on π , by requiring that X does not have one-dimensional components. This dimensional condition is necessary, as illustrated by the normalization of a nodal curve with one of the preimage of the singular point removed : we obtain an homeomorphism onto a seminormal curve which is not an isomorphism. The dimensional condition of Vitulli assures in fact that the homeomorphism is a finite morphism.

Our results are of a similar flavour, but we manage to avoid both the finiteness assumption and the dimensional restriction, by modifying either the topological condition on π or the geometric condition on X . To state our results, assume in the first instance that k is the field of complex numbers, so that one can use additionally the strong topology on the complex points of the varieties.

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Theorem A. *Let $\pi : Y \rightarrow X$ be a morphism between complex algebraic varieties. Then π is an isomorphism if one of the following conditions is satisfied :*

- (i) *X is seminormal in Y and $\pi_{\mathbb{C}}$ is an homeomorphism with respect to the strong topology,*
- (ii) *X is saturated in Y and $\pi_{\mathbb{C}}$ is an homeomorphism with respect to Zariski topology.*

Our interpretation is that, for complex varieties at least, the relevant topology to associate to seminormality is the strong topology, whereas Zariski topology is rather related to the notion of saturation. Before discussing a generalization to any algebraically closed field of characteristic zero, let us briefly recall the notions of seminormality and saturation. The seminormalization of X is the biggest algebraic variety between X and its normalization in bijection with X . Introduced first by Andreotti and Norguet [3] for complex analytic varieties, then by Andreotti and Bombieri [2] for schemes, it appears in the study of Picard groups or as singularities in the minimal model program [25]. Seminormality corresponds algebraically to subintegral extensions, meaning an integral ring extension bijective at the spectrum level and equiresidual. Seminormalization and subintegral extensions are studied in section 1. The close notion of radicial extension, as introduced by Grothendieck [17], only requires that the (non necessarily integral) extension is injective at the spectrum level and equiresidual. We study in section 2 the saturation for varieties as the geometric counterpart of radiciality. The saturation appears first in the context of Lipschitz geometry with works of Pham and Teissier [32] in complex analytic geometry and Lipman [28] for ring extensions. For integral extensions, saturation and seminormalization coincide, providing a different approach to seminormality as proposed by Manaresi [29]. However, it is not established that the saturation produces a variety, contrarily to the seminormalization.

The study of seminormalization over non algebraically closed fields presents some difficulty, and the last three authors managed to define a sort of seminormalization for algebraic varieties over the field of real numbers [14]. This notion has to do with the central points of a real algebraic variety, that is the Euclidean closure of the set of regular points. This approach would not have been possible without the recent study of continuous rational functions in real algebraic geometry, as initiated by Kucharz [27] and Kollár and Nowak [26], and further developed in [13] as regulous functions. These regulous functions happens to be related to seminormality for complex algebraic varieties too as studied by the first author [7]. It is this approach of seminormality via continuous functions with respect to the strong topology, performed in the affine case in [7] and naturally extended to general varieties at the end of section 1, that leads us to the proof of Theorem A in section 3. Note moreover that, if the second part of Theorem A does not refer to the strong topology, our proof is a consequence of the first result where the use of the strong topology is crucial. We prove along the way that an homeomorphism with respect to the strong topology is an homeomorphism, the converse being true except for curves.

Our approach being intrinsically related to the strong topology, it looks at first sight rather improbable to extend it beyond complex algebraic varieties. However, our second main result is such a generalization for any algebraically closed fields of characteristic zero.

Theorem B. *Let $\pi : Y \rightarrow X$ be a morphism between algebraic varieties over k . Then π is an isomorphism if one of the following conditions is satisfied :*

- (i) *X is seminormal in Y and π_k is an homeomorphism with respect to the R -topology,*
- (ii) *X is saturated in Y and π_k is an homeomorphism with respect to Zariski topology.*

The point is to make sense of the R -topology which appears in the first condition in the statement of Theorem B, and plays a crucial role in the proof of the second. The Euclidean topology on \mathbb{R}^n has a basis of open sets given by semialgebraic subsets of \mathbb{R}^n , i.e. given by real polynomial equalities and inequalities. The theory of semialgebraic sets provides an algebraic way to discuss about topological question in real algebraic geometry, as developed in [8]. The great advantage of this approach is that it generalizes from \mathbb{R} to any real closed field. A real closed field is an ordered field that does not admit any ordered algebraic extension. Equivalently, adding a square root of -1 to a real closed field gives an algebraically closed field of characteristic zero. Real closed fields have been initially studied by Artin and Schreier [4] in the way of Artin's proof of Hilbert XVIIth Problem [1]. The most basic examples away from \mathbb{R} are the field of algebraic real numbers, which is the real closure of \mathbb{Q} , and the field of Puiseux series with real coefficients, which is the real closure of $\mathbb{R}((T))$ ordered by T positive and infinitely small. There are many of them, as illustrated by the fact that any algebraically closed field k of characteristic zero contains (infinitely many) real closed subfields $R \subset k$ with $k = R[\sqrt{-1}]$. Fixing such a choice of R leads to an identification $k \simeq R^2$ and equips k with an order topology, called the R -topology on k . Note that in general R is not connected, and a closed and bounded interval is not compact.

It is within this framework that we prove Theorem B in section 4. In particular, we provide a full study of the relation between seminormality and the R -topology completely parallel to the complex case in [7]. We introduce additionally the continuous rational functions over k in section 5. A remarkable fact is that the continuity of a rational function defined on an algebraic variety X over k does not depend on the choice of the real closed field R , since these functions coincide with the regular functions on the seminormalization of X , cf. Theorem 5.4. As a consequence, fixing a real closed field $R \subset k$ brings all the flexibility of semialgebraic geometry over R to algebraic geometry over k , without loosing in generality.

In the paper, k denotes a field of characteristic zero (sometimes algebraically closed), and an algebraic variety over k is a reduced and separated scheme of finite type over k .

1. SUBINTEGRAL EXTENSIONS AND SEMINORMALIZATION

After some reminders on integral extensions and normalization, we recall the notions of subintegral extensions and Traverso's construction [37] of the seminormalization for ring extensions and morphisms between affine algebraic varieties. We finally pay a special attention to the case of complex algebraic varieties, where the strong topology and regulous functions give another point of view in the spirit of [7]. We extend in particular the results from the affine setting in [7] to general complex algebraic varieties.

Notation and terminology. Let A be a ring. The Zariski spectrum $\text{Spec } A$ of A is the set of prime ideals of A . It is a topological space for the topology whose closed sets are generated by the sets $\mathcal{V}(f) = \{\mathfrak{p} \in \text{Spec } A \mid f \in \mathfrak{p}\}$ for $f \in A$. We denote by $\text{Max } A$ the subspace of maximal ideals of A . For $\mathfrak{p} \in \text{Spec } A$, we denote by $k(\mathfrak{p})$ the residue field at \mathfrak{p} .

Let (X, \mathcal{O}_X) be a variety over k . For $x \in X$ we denote by $k(x)$ the residue field at x ; for an affine neighborhood U of x then x corresponds to a prime ideal \mathfrak{p}_x of $\mathcal{O}_X(U)$ and we have $k(x) = k(\mathfrak{p}_x)$. In case X is affine then we denote by $k[X]$ the coordinate ring of X i.e. $k[X] = \mathcal{O}_X(X)$. Let K be a field containing k . We denote by $X(K)$ the set $\text{Mor}(\text{Spec } K, X)$ of K -rational points. If $K = k$ then $X(k)$ is also the set of k -closed points of X , i.e the points

of X with residue field equal to k . We have thus an inclusion

$$X(k) \hookrightarrow X$$

that makes $X(k)$ a topological space for the Zariski topology. We denote by $\mathcal{O}_{X(k)}$ the sheaf of regular functions on $X(k)$, for $x \in X(k)$ we have $\mathcal{O}_{X,x} = \mathcal{O}_{X(k),x}$. In the case k algebraically closed, for an open subset U of X , we may identify $\text{Max } \mathcal{O}_X(U)$ with $U(k)$ by the Nullstellensatz, and similarly we identify the regular functions on U with $U(k)$, namely $\mathcal{O}_X(U) = \mathcal{O}_{X(k)}(U(k))$. If T is a subset of X or $X(k)$ or $\text{Spec } A$ then we will denote by \overline{T}^Z the closure of T for the Zariski topology.

A ring extension $i : A \rightarrow B$ induces a map $\text{Spec}(i) : \text{Spec } B \rightarrow \text{Spec } A$, given by $\mathfrak{p} \mapsto (\mathfrak{p} \cap A) = i^{-1}(\mathfrak{p})$. If $\pi : Y \rightarrow X$ is a morphism between algebraic varieties over k , with $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_Y$ the associated morphism of sheaves of rings on X , then for any open subset $U \subset X$ the ring morphism $\mathcal{O}_X(U) \rightarrow \mathcal{O}_Y(\pi^{-1}(U))$ is an extension if π is dominant. For a field extension $k \rightarrow K$, we denote by $\pi_K : Y(K) \rightarrow X(K)$ the induced map. Remark that π_k is also the restriction of π to the k -closed points.

In the sequel, $\mathcal{K}(A)$ (resp. \mathcal{K}) will denote the total ring of fractions of A (resp. the sheaf of total ring of fractions on X).

1.1. Reminder on integral extensions and normalization. A ring extension $A \rightarrow B$ is said of finite type (resp. finite) if it makes B a finitely generated A -algebra (resp. A -module). The extension $A \rightarrow B$ is birational if it induces an isomorphism between $\mathcal{K}(A)$ and $\mathcal{K}(B)$.

An element $b \in B$ is integral over A if b is the root of a monic polynomial with coefficients in A , which is equivalent for $A[b]$ to be a finite A -module by [5, Prop. 5.1]. As a consequence

$$A'_B = \{b \in B \mid b \text{ is integral over } A\}$$

is a ring called the integral closure of A in B . The extension $A \rightarrow B$ is said to be integral if $A'_B = B$. In case $B = \mathcal{K}(A)$ then the ring $A'_{\mathcal{K}(A)}$ is denoted by A' and is simply called the integral closure of A . The ring A is called integrally closed (resp. in B) if $A = A'$ (resp. $A = A'_B$).

We recall that a dominant morphism $Y \rightarrow X$ between algebraic varieties over k is said of finite type (resp. finite, birational, integral) if for any open subset $U \subset X$ the ring extension $\mathcal{O}_X(U) \rightarrow \mathcal{O}_Y(\pi^{-1}(U))$ is of finite type (resp. finite, birational, integral). In this paper, a morphism between algebraic varieties is always of finite type.

Let X be an algebraic variety over k . The normalization of X , denoted by X' , is the algebraic variety over k with a finite birational morphism $\pi' : X' \rightarrow X$, called the normalization morphism such that for any open subset $U \subset X$ we have $\mathcal{O}_{X'}(\pi'^{-1}(U)) = \mathcal{O}_X(U)'$. We say that X is normal if π' is an isomorphism. A point $x \in X$ is said normal if $\mathcal{O}_{X,x}$ is integrally closed.

We will use frequently that an integral extension of rings induces surjectivity at the spectrum level (See [30, Thm. 9.3] or [5, Thm. 5.10, Cor. 5.8]).

Proposition 1.1. *Let $A \rightarrow B$ be an integral extension of rings. The maps $\text{Spec } B \rightarrow \text{Spec } A$ and $\text{Max } B \rightarrow \text{Max } A$ are surjective and closed.*

As a consequence, if π is a finite morphism between algebraic varieties over k then Proposition 1.1 implies that π and π_k are surjective.

1.2. Subintegral extensions and seminormalization. We recall the concept of subintegral extensions introduced by Traverso [37].

Definition 1.2. Let $A \rightarrow B$ be an extension of rings.

- (1) For $\mathfrak{p} \in \text{Spec } B$, we say that $\text{Spec } B \rightarrow \text{Spec } A$ is equiresidual at \mathfrak{p} if the extension $k(\mathfrak{p} \cap A) \rightarrow k(\mathfrak{p})$ is an isomorphism. Let $W \subset \text{Spec } B$, we say that $\text{Spec } B \rightarrow \text{Spec } A$ is equiresidual (resp. by restriction to W) if for any $\mathfrak{p} \in \text{Spec } B$ (resp. $\mathfrak{p} \in W$), $\text{Spec } B \rightarrow \text{Spec } A$ is equiresidual at \mathfrak{p} .
- (2) The extension $A \rightarrow B$ is said equiresidual if $\text{Spec } B \rightarrow \text{Spec } A$ is.
- (3) The extension $A \rightarrow B$ and the map $\text{Spec } B \rightarrow \text{Spec } A$ are said subintegral if the extension is integral and $\text{Spec } B \rightarrow \text{Spec } A$ is bijective and equiresidual.

Note that a field extension is equiresidual if and only if it is an isomorphism. We extend these definitions to the geometric setting, adding moreover a notion of hereditarily birational morphism that have been introduced in the real setting in [14].

Definition 1.3. Let $\pi : Y \rightarrow X$ be a dominant morphism between algebraic varieties over k .

- (1) We say that π is equiresidual if for any $y \in Y$ then the field extension $k(\pi(y)) \rightarrow k(y)$ is an isomorphism.
- (2) We say that π is subintegral if π is integral, bijective and equiresidual.
- (3) We say that $\pi : Y \rightarrow X$ is hereditarily birational if for any open subset $U \subset X$ and for any irreducible algebraic subvariety $V = \mathcal{V}(\mathfrak{p}) \simeq \text{Spec}(\mathcal{O}_Y(\pi^{-1}(U))/\mathfrak{p})$ in $\pi^{-1}(U)$, the morphism

$$\pi|_V : V \rightarrow W = \mathcal{V}(\mathfrak{p} \cap \mathcal{O}_X(U)) \simeq \text{Spec}(\mathcal{O}_X(U)/(\mathfrak{p} \cap \mathcal{O}_X(U)))$$

is birational.

Geometrically speaking, a dominant morphism $\pi : Y \rightarrow X$ is equiresidual if and only if it is hereditarily birational. Indeed, for any open subset $U \subset X$ and for any irreducible algebraic subvariety $V = \mathcal{V}(\mathfrak{p}) \simeq \text{Spec}(\mathcal{O}_Y(\pi^{-1}(U))/\mathfrak{p})$ in $\pi^{-1}(U)$, the restricted morphism

$$\pi|_V : V \rightarrow W = \mathcal{V}(\mathfrak{p} \cap \mathcal{O}_X(U)) \simeq \text{Spec}(\mathcal{O}_X(U)/(\mathfrak{p} \cap \mathcal{O}_X(U)))$$

is birational if and only if the extension $k(\mathfrak{p} \cap \mathcal{O}_X(U)) = \mathcal{K}(W) \rightarrow k(\mathfrak{p}) = \mathcal{K}(V)$ is an isomorphism.

An hereditarily birational morphism is not necessarily bijective. However, adding an integrality assumption and using Proposition 1.1, we get the following characterization.

Lemma 1.4. *Let $\pi : Y \rightarrow X$ be an integral morphism between algebraic varieties over k . The following properties are equivalent:*

- 1) π is hereditarily birational and injective.
- 2) π is subintegral.

The notion of subintegral extension leads to the notion of seminormalization, in a similar way than integral extensions lead to normalization.

In order to define the notion of seminormality, we need to consider sequences of rings extensions. A ring C is said intermediate between the rings A and B if there exists a sequence of extensions $A \rightarrow C \rightarrow B$. In that case, we say that $A \rightarrow C$ and $C \rightarrow B$ are intermediate extensions of $A \rightarrow B$ and we say in addition that $A \rightarrow C$ is a subextension of $A \rightarrow B$.

Seminormal extensions are maximal subintegral extensions.

Definition 1.5. Let $A \rightarrow C \rightarrow B$ be a sequence of two extensions of rings with $A \rightarrow C$ subintegral. We say that C is seminormal between A and B if for every intermediate ring D between C and B , with C different from D , then $A \rightarrow D$ is not subintegral.

We say that A is seminormal in B if A is seminormal between A and B . We say that A is seminormal if A is seminormal between A and A' .

Given an extension of rings $A \rightarrow B$, Traverso (see [37] or [39]) proved there exists a unique intermediate ring which is seminormal between A and B . To this purpose, he introduced the ring

$$A_B^+ = \{b \in A'_B \mid \forall \mathfrak{p} \in \text{Spec } A, b_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}((A'_B)_{\mathfrak{p}})\},$$

where Rad stands for the Jacobson radical, namely the intersection of all the maximal ideals. The idea to build A_B^+ is, for all $\mathfrak{p} \in \text{Spec } A$, to glue together all the prime ideals of A'_B lying over \mathfrak{p} (see [31]).

Theorem 1.6. [37] *Let $A \rightarrow B$ be an extension of rings. Then A_B^+ is the unique ring which is seminormal between A and B .*

Moreover, for any intermediate ring C between A and B , the extension $A \rightarrow C$ is subintegral if and only if $C \subset A_B^+$.

The ring A_B^+ is called the seminormalization of the ring extension $A \rightarrow B$ or the seminormalization of A in B . The ring A_A^+ is called the seminormalization of A and is simply denoted by A^+ . Note that when A and B are domain, then A and A_B^+ have in particular the same fraction field.

1.3. Seminormalization of a morphism between algebraic varieties. Andreatti and Bombieri [2] have introduced and built the seminormalization of a scheme in another one. In this section, we provide a different and elementary construction of the seminormalization of an affine algebraic variety in another one. The seminormalization answers the following question. Let $Y \rightarrow X$ be a dominant morphism between algebraic varieties over k . Does there exist a biggest algebraic variety Z such that $Y \rightarrow X$ factorizes through Z and $Z \rightarrow X$ is subintegral ?

We recall first the notion of normalization of a variety in another one. Let $\pi : Y \rightarrow X$ be a dominant morphism between algebraic varieties over k . The integral closure $(\mathcal{O}_X)_{\pi_*(\mathcal{O}_Y)}$ of \mathcal{O}_X in $\pi_*(\mathcal{O}_Y)$ is a coherent sheaf [36, Lem. 52.15] and by [18, II Prop. 1.3.1] it is the structural sheaf of a variety over k .

Definition 1.7. Let $Y \rightarrow X$ be a dominant morphism of finite type between algebraic varieties over k . The variety with structural sheaf equal to the integral closure of \mathcal{O}_X in $\pi_*(\mathcal{O}_Y)$ is called the normalization of X in Y and is denoted by X'_Y .

Be aware that the normalization of a variety in another one is not necessarily a normal variety, nor it admits a birational morphism onto to the original variety.

For a dominant morphism $Y \rightarrow X$ between algebraic varieties over k , we say that an algebraic variety Z over k is intermediate between X and Y if $Y \rightarrow X$ factorizes through Z . For affine varieties, it is equivalent to say that $k[Z]$ is an intermediate ring between $k[X]$ and $k[Y]$. The normalization of a variety in another one satisfies the following property :

Proposition 1.8. *Let $Y \rightarrow X$ be a dominant morphism between algebraic varieties over k . Let Z be an intermediate variety between X and Y . Then $Z \rightarrow X$ is finite if and only if it factorizes $X'_Y \rightarrow X$.*

We describe now an elementary construction of the seminormalization of an affine algebraic variety in another one. Let $Y \rightarrow X$ be a dominant morphism between affine algebraic varieties over k . We want to check that the ring $A = k[X]_{k[Y]}^+$ is the coordinate ring of an algebraic variety. We know that the morphism $X'_Y \rightarrow X$ is finite by Proposition 1.8, so we can apply next Lemma 1.9 to the extensions

$$k[X] \subset A \subset k[X'_Y] = k[X]_{k[Y]} \subset k[Y]$$

to conclude.

Lemma 1.9. *Let $\pi : Y \rightarrow X$ be a finite morphism between affine algebraic varieties over k . Let A be a ring such that $k[X] \subset A \subset k[Y]$. Then A is the coordinate ring of a unique affine algebraic variety over k and π factorizes through this variety.*

Proof. Since $k[Y]$ is a finite module over the Noetherian ring $k[X]$ then it is a Noetherian $k[X]$ -module. Thus the ring A is a finite $k[X]$ -module as a submodule of a Noetherian $k[X]$ -module. It follows that A is a finitely generated algebra over k and the proof is done. \square

For the general construction of the seminormalization, one needs to check that the seminormalization of the local charts of an affine covering glue together to give a global variety. This is done by Andreotti and Bombieri [2] using Grothendieck criterion [18, II Prop. 1.3.1] concerning the quasi-coherence of sheaves. It leads to the following definition.

Definition 1.10. Let $\pi : Y \rightarrow X$ be a dominant morphism between algebraic varieties over k . The seminormalization of X in Y is the algebraic variety X_Y^+ over k with structural sheaf equal to the seminormalization of \mathcal{O}_X in $\pi_*(\mathcal{O}_Y)$

We call X^+ the seminormalization of X in its normalization $Y = X'$. We say that X is seminormal in Y (resp. seminormal) if $X = X_Y^+$ (resp. $X = X^+$).

Remark 1.11. Note that the seminormalization of X in Y is birational to X , even if $Y \rightarrow X$ is not birational. It was not the case for the normalization of X in Y . We have in general

$$Y \rightarrow X'_Y \rightarrow X_Y^+ \rightarrow X.$$

The seminormalization of a variety in another one satisfies the following universal property :

Proposition 1.12. *Let $Y \rightarrow Z \rightarrow X$ be a sequence of dominant morphisms between algebraic varieties over k . Then $Z \rightarrow X$ is subintegral if and only if $X_Y^+ \rightarrow X$ factorizes through Z .*

Proof. It is a reformulation of the second part of Theorem 1.6. \square

1.4. Relation with the strong topology and regulous functions for complex algebraic varieties. Subintegral extensions and regulous functions are strongly related in real algebraic geometry as developed in [15]. Working with the field of complex numbers, we know from the work of Bernard [7] that the same holds true in the affine and geometric case. The aim of this section is to generalize the work of Bernard to non-necessarily affine varieties over \mathbb{C} .

Continuous rational functions and regulous functions have been originally studied in real algebraic geometry [27, 26, 13], where the continuity is regarded with respect to the Euclidean topology, which can be studied algebraically via semialgebraic open sets [8]. Let X be an algebraic variety over \mathbb{C} . In the sequel, we consider the Euclidean (or strong) topology of the complex points $X(\mathbb{C})$ seen as a topological variety (see [35]). For example, if X is affine then we have $X \subset \mathbb{A}^n$ for some $n \in \mathbb{N}$ and the strong topology is induced by the natural inclusion $X(\mathbb{C}) \subset \mathbb{R}^{2n}$.

We define the sheaf of continuous rational functions on an algebraic variety X over \mathbb{C} as follows. Let $U \subset X$ be an open subset of X . A continuous rational functions on $U(\mathbb{C})$ is a \mathbb{C} -valued function on $U(\mathbb{C})$, continuous with respect to the strong topology, that is rational on X (i.e it coincides with a regular function on a dense Zariski open subset of $U(\mathbb{C})$). The continuous rational functions on $U(\mathbb{C})$ are the sections of a presheaf of \mathbb{C} -algebras on $X(\mathbb{C})$, denoted by $\mathcal{K}_{X(\mathbb{C})}^0$ in the sequel. Since \mathcal{K} is a sheaf, and the presheaf of locally continuous functions on $X(\mathbb{C})$ for the strong topology is also a sheaf for the Zariski topology, then $\mathcal{K}_{X(\mathbb{C})}^0$ is a sheaf called the sheaf of continuous rational functions. It makes $(X(\mathbb{C}), \mathcal{K}_{X(\mathbb{C})}^0)$ a ringed space. In case X is affine then we simply denote by $\mathcal{K}^0(X(\mathbb{C}))$ the global sections of $\mathcal{K}_{X(\mathbb{C})}^0$ on $X(\mathbb{C})$. A dominant morphism $\pi : Y \rightarrow X$ between varieties over \mathbb{C} induces an extension $\mathcal{K}_{X(\mathbb{C})}^0 \rightarrow (\pi_{\mathbb{C}})_* \mathcal{K}_{Y(\mathbb{C})}^0$, hence a morphism $(Y(\mathbb{C}), \mathcal{K}_{Y(\mathbb{C})}^0) \rightarrow (X(\mathbb{C}), \mathcal{K}_{X(\mathbb{C})}^0)$ of ringed spaces.

Remark 1.13. A regulous function f is a continuous rational function that satisfies the additional property that f remains rational by restriction to any subvariety. The first author proved that it is always the case [7, Prop. 4.14], contrarily to the real case [26].

In the sequel, we also say regulous for continuous rational. An important fact is that regulous functions on a normal variety are regular. More precisely, next proposition says that regulous functions are integral on the regular ones and thus are already regular if the variety is normal.

Proposition 1.14. *Let X be an algebraic variety over \mathbb{C} . We have:*

- 1) $\mathcal{K}_{X(\mathbb{C})}^0 \subset (\pi'_{\mathbb{C}})_* \mathcal{O}_{X'(\mathbb{C})}$ where $\pi' : X' \rightarrow X$ is the normalization map.
- 2) If x is a normal closed point of $X(\mathbb{C})$ then $\mathcal{K}_{X(\mathbb{C}),x}^0 = \mathcal{O}_{X,x}$.

Proof. The properties are local, so the proof follows from the affine case proved in [7, Prop. 4.7, Prop. 4.10]. \square

Definition 1.15. Let $\pi : Y \rightarrow X$ be a dominant morphism between algebraic varieties over \mathbb{C} . We say that the map $\pi_{\mathbb{C}}$ is biregulous if it is an homeomorphism for the strong topology and the inverse map is rational (i.e coincides with a regular map on a dense Zariski open subset of $X(\mathbb{C})$).

In such a situation, the morphism $\mathcal{K}_{X(\mathbb{C})}^0 \rightarrow (\pi_{\mathbb{C}})_* \mathcal{K}_{Y(\mathbb{C})}^0$ is an isomorphism and thus the ringed spaces $(Y(\mathbb{C}), \mathcal{K}_{Y(\mathbb{C})}^0)$ and $(X(\mathbb{C}), \mathcal{K}_{X(\mathbb{C})}^0)$ are isomorphic. The following theorem explains how subintegral extensions, continuous rational functions and biregulous morphisms are related.

Theorem 1.16. *Let $\pi : Y \rightarrow X$ be a finite morphism between algebraic varieties over \mathbb{C} . The following properties are equivalent:*

- 1) π is subintegral.
- 2) $\pi_{\mathbb{C}}$ is bijective.
- 3) $\pi_{\mathbb{C}}$ is biregulous.
- 4) The ringed spaces $(Y(\mathbb{C}), \mathcal{K}_{Y(\mathbb{C})}^0)$ and $(X(\mathbb{C}), \mathcal{K}_{X(\mathbb{C})}^0)$ are isomorphic.
- 5) $\pi_{\mathbb{C}}$ is an homeomorphism for the strong topology.
- 6) $\pi_{\mathbb{C}}$ is an homeomorphism for the Zariski topology.
- 7) π is an homeomorphism.

Proof. The properties are local, so the proof follows from the affine case in [7, Thm. 3.1, Prop. 4.12]. \square

Note that a crucial point in the proof is played by properness properties : a finite morphism is proper with respect to the Zariski topology, but also with respect to the strong topology. We will come back to this point in section 4. Remark also that a morphism satisfying the conditions of Theorem 1.16 is (bijective thus) automatically birational.

In the context of complex algebraic geometry, we already get a first answer to the question regarded in this text : under which conditions a bijection is an isomorphism.

Proposition 1.17. *Let $\pi : Y \rightarrow X$ be a finite morphism between algebraic varieties over \mathbb{C} such that X is seminormal in Y . Then π is an isomorphism if and only if $\pi_{\mathbb{C}}$ is a bijection if and only if $\pi_{\mathbb{C}}$ is an homeomorphism for the strong topology if and if π is an homeomorphism.*

Proof. Assume $\pi_{\mathbb{C}}$ is bijective. Then π is subintegral by Theorem 1.16, and so $X_Y^+ \rightarrow X$ factorises through Y by Proposition 1.12. However by assumption $X = X_Y^+$, so that π is an isomorphism.

The rest of the proof follows directly from the equivalences of Theorem 1.16. \square

In the remaining of the paper, we improve this result in two different directions. First we investigate on how to drop the finiteness hypothesis, and second we aim to replace the field of complex numbers with any algebraically closed field of characteristic zero. We may already mention that the result of the previous proposition is no longer true without the finiteness hypothesis (cf. Example 2.18).

In the remaining of this section, we show how seminormalization and regulous functions are in relation for varieties over \mathbb{C} . We begin with a description of regular functions on the relative seminormalization in terms of regular functions on the relative normalization, description that is valid over any algebraically closed field.

Proposition 1.18. *Let $Y \rightarrow X$ be a dominant morphism between algebraic varieties over an algebraically closed field k . Let U be an open subset of X . Then*

$$\mathcal{O}_{X_Y^+}((\pi^+)^{-1}(U)) = \{f \in \mathcal{O}_{X'_Y}((\pi')^{-1}(U)) \mid f \text{ is constant on the fibers of } \pi'_k\}$$

where $\pi' : X'_Y \rightarrow X$ (resp. $\pi^+ : X_Y^+ \rightarrow Y$) is the relative normalization (resp. seminormalization) morphism.

Proof. By [7, Cor. 3.7] (which is written in the case U is affine and $k = \mathbb{C}$, but all section 3 there is valid for any open subset U and over any algebraically closed field of characteristic zero), we have

$$\mathcal{O}_{X_Y^+}((\pi^+)^{-1}(U)) = \{f \in \mathcal{O}_{X'_Y}((\pi')^{-1}(U)) \mid \forall x \in U(k), f_{\mathfrak{m}_x} \in \mathcal{O}_{X,x} + \text{Rad}(\mathcal{O}_{X'_Y,x})\}$$

The radical being an intersection of maximal ideals, we see that the functions in $\mathcal{O}_{X_Y^+}((\pi^+)^{-1}(U))$ correspond to the elements of $\mathcal{O}_{X'_Y}((\pi')^{-1}(U))$ constant on the fibers of π'_k . \square

We give a characterization of the structural sheaf of the seminormalization of an algebraic variety over \mathbb{C} in another one with regulous functions generalizing the main result in [7]. In order to state the result, we use the fibred product of two sheaf extensions.

Theorem 1.19. *Let $\pi : Y \rightarrow X$ be a dominant morphism between algebraic varieties over \mathbb{C} . Then*

$$(\pi_{\mathbb{C}}^+)_* \mathcal{O}_{X_Y^+(\mathbb{C})} = \mathcal{K}_{X(\mathbb{C})}^0 \times_{(\pi_{\mathbb{C}})_* \mathcal{K}_{Y(\mathbb{C})}^0} (\pi_{\mathbb{C}})_* \mathcal{O}_{Y(\mathbb{C})}$$

where $\pi^+ : X_Y^+ \rightarrow X$ is the relative seminormalization morphism.

Proof. We may assume X and Y are affine and thus we want to prove that

$$\mathbb{C}[X_Y^+] = \mathcal{K}^0(X(\mathbb{C})) \times_{\mathcal{K}^0(Y(\mathbb{C}))} \mathbb{C}[Y],$$

where the right hand side stands for the fibred product of the rings. Let $\pi' : X'_Y \rightarrow X$ be the relative normalization map.

We consider the following diagram

$$\begin{array}{ccccc} \mathbb{C}[X] & \rightarrow & \mathbb{C}[X'_Y] & \rightarrow & \mathbb{C}[Y] \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{K}^0(X(\mathbb{C})) & \rightarrow & \mathcal{K}^0(X'_Y(\mathbb{C})) & \rightarrow & \mathcal{K}^0(Y(\mathbb{C})) \end{array}$$

where the horizontal maps from the top (resp. the bottom) are given by composition with respectively π' and $Y \rightarrow X'_Y$ (resp. $\pi'_\mathbb{C}$ and $Y(\mathbb{C}) \rightarrow X'_Y(\mathbb{C})$).

Let $f \in \mathbb{C}[X_Y^+]$. We have already $f \in \mathbb{C}[Y]$ by definition. Moreover $f \in \mathcal{K}^0(X'_Y(\mathbb{C}))$, so $f \in \mathcal{K}^0(X(\mathbb{C}))$ because $\mathcal{K}^0(X'_Y(\mathbb{C}))$ and $\mathcal{K}^0(X(\mathbb{C}))$ are isomorphic by Theorem 1.16 (more precisely, as a function on $X'_Y(\mathbb{C})$, the function f is equal to the composition of a regular function on $X(\mathbb{C})$ with $X'_Y(\mathbb{C}) \rightarrow X(\mathbb{C})$). In particular $\mathbb{C}[X_Y^+] \subset \mathcal{K}^0(X(\mathbb{C})) \times_{\mathcal{K}^0(Y(\mathbb{C}))} \mathbb{C}[Y]$.

Let $f \in \mathcal{K}^0(X(\mathbb{C})) \times_{\mathcal{K}^0(Y(\mathbb{C}))} \mathbb{C}[Y]$. The continuous rational function f is integral over $\mathbb{C}[X]$ by Proposition 1.14, therefore $f \in \mathbb{C}[X'_Y]$ since additionally $f \in \mathbb{C}[Y]$. As a function on $X'_Y(\mathbb{C})$, the function f is constant on the fibers of $X'_Y(\mathbb{C}) \rightarrow X(\mathbb{C})$ since f induces a continuous function on $X(\mathbb{C})$. By Proposition 1.18, we obtain then $f \in \mathbb{C}[X_Y^+]$. It gives the reverse inclusion $\mathcal{K}^0(X(\mathbb{C})) \times_{\mathcal{K}^0(Y(\mathbb{C}))} \mathbb{C}[Y] \subset \mathbb{C}[X_Y^+]$. \square

A continuous rational function on a normal complex variety is a regular function by Proposition 1.14. The fact that the normalization of X in Y is not necessarily normal imposes to take the fibred product with $\mathcal{O}_{Y(\mathbb{C})}$ in Theorem 1.19. We state as a corollary the particular case $Y = X'$, the statement becoming much simpler by Proposition 1.14.

Corollary 1.20. *Let X be an algebraic variety over \mathbb{C} . Then $(\pi'_\mathbb{C})_* : \mathcal{O}_{X+(\mathbb{C})} \rightarrow \mathcal{K}^0_{X(\mathbb{C})}$ is an isomorphism, where $\pi^+ : X^+ \rightarrow X$ is the seminormalization morphism, and thus*

$$(X^+(\mathbb{C}), \mathcal{O}_{X+(\mathbb{C})}) = (X(\mathbb{C}), \mathcal{K}^0_{X(\mathbb{C})})$$

as ringed spaces.

This result will be generalized over any algebraically closed field in Theorem 5.4.

2. SATURATION

In the classical study of the seminormalization, some basic properties such as the local nature happen to be not so straightforward to prove. A nice algebraic approach has been proposed by Manaresi [29], in the spirit of the relative Lipschitz saturation [28], via the saturation of a ring A in another ring B . The saturation coincides with the seminormalization when the ring extension is finite. We aim to study the properties of this saturation for more general extensions, and establish its universal properties.

In this section, we work with a (non-necessarily algebraically closed) field k .

2.1. Universal property of the saturation. The saturation of a ring extension is defined as follows in [28].

Definition 2.1. Let $A \rightarrow B$ be an extension of rings. The saturation of A in B , denoted by \widehat{A}_B , is defined by

$$\widehat{A}_B = \{b \in B \mid b \otimes_A 1 - 1 \otimes_A b \in \text{NilRad}(B \otimes_A B)\}$$

where the nil radical NilRad denotes the ideal of nilpotent elements.

We say that A is saturated in B if $\widehat{A}_B = A$. The saturation of A is its saturation in A' and it is simply denoted by \widehat{A} . We say that A is saturated if $\widehat{A} = A$.

Recall that the nilradical is the intersection of all prime ideals. In order to study the saturation, we need to understand better the relation between prime ideals in A and B and prime ideals in $B \otimes_A B$. For a ring extension $A \rightarrow B$, we introduce the notation φ_1, φ_2 for the ring morphisms $\varphi_i : B \rightarrow B \otimes_A B$ defined by

$$(a) \quad \varphi_1(b) = b \otimes_A 1 \text{ and } \varphi_2(b) = 1 \otimes_A b.$$

The data of a prime ideal ω in $B \otimes_A B$, or more precisely the data of a surjective morphism $g : B \otimes_A B \rightarrow k(\omega)$ with kernel ω , is equivalent to the data of a 4-tuple of prime ideals

$$(\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}, \mathfrak{p}) \in \text{Spec } B \times \text{Spec } B \times \text{Spec } A \times \text{Spec}(k(p_1) \otimes_{k(\mathfrak{q})} k(p_2))$$

such that

$$\mathfrak{p}_1 = \ker(g \circ \varphi_1), \quad \mathfrak{p}_2 = \ker(g \circ \varphi_2), \quad \mathfrak{q} = \mathfrak{p}_1 \cap A = \mathfrak{p}_2 \cap A, \quad k(\omega) = k(\mathfrak{p})$$

and such that the composition

$$B \otimes_A B \rightarrow k(\mathfrak{p}_1) \otimes_{k(\mathfrak{q})} k(\mathfrak{p}_2) \rightarrow k(\mathfrak{p})$$

coincides with g .

We recall that the saturation is a ring, compatible with inclusion.

Lemma 2.2. *Let $A \rightarrow B$ be an extension of rings.*

- (1) \widehat{A}_B is a subring of B containing A .
- (2) If $A \subset C \subset B$, then $\widehat{A}_B \subset \widehat{C}_B$.

Proof. (1) The set \widehat{A}_B is an A -module as the kernel of the A -module morphism

$$B \xrightarrow{\varphi_1 - \varphi_2} \frac{B \otimes_A B}{\text{NilRad}(B \otimes_A B)}.$$

The stability under product comes from the identity

$$b_1 b_2 \otimes_A 1 - 1 \otimes_A b_1 b_2 = (b_1 \otimes_A 1)(b_2 \otimes_A 1 - 1 \otimes_A b_2) + (1 \otimes_A b_2)(b_1 \otimes_A 1 - 1 \otimes_A b_1)$$

and the fact that $\text{NilRad}(B \otimes_A B)$ is an ideal.

- (2) The image of a nilpotent element by the ring morphism $B \otimes_A B \rightarrow B \otimes_C B$ remains nilpotent. □

In order to give a universal property of the saturation, we recall the notion of radical extension introduced by Grothendieck [17, I, def. 3.7.2]. We also introduce a notion of radical sequence of extensions similarly to [31].

Definition 2.3. (1) An extension of rings $A \rightarrow B$ is said radical if $\text{Spec } B \rightarrow \text{Spec } A$ is injective and equiresidual.

- (2) A sequence of extensions $A \rightarrow C \rightarrow B$ of rings is said radicial if the restriction of $\text{Spec } C \rightarrow \text{Spec } A$ to the image of $\text{Spec } B \rightarrow \text{Spec } C$ is injective and equiresidual.

Remark 2.4. An extension (resp. A sequence of extensions) of fields $K \rightarrow K'$ (resp. $K \rightarrow K' \rightarrow K''$) is radicial if and only if $K \rightarrow K'$ is an isomorphism.

The saturation furnishes radicial sequences of extensions.

Proposition 2.5. *Let $A \rightarrow B$ be a ring extension. For any $C \subset \widehat{A}_B$, the sequence $A \rightarrow C \rightarrow B$ is radicial.*

Before entering into the proof, we state an elementary result about field extensions. Remark that it gives a proof of (2) implies (3) of Theorem 2.8 in the special case of a sequence of field extensions.

Lemma 2.6. *Let $K \rightarrow K' \rightarrow K''$ be a non radicial sequence of field extensions i.e such that $K \rightarrow K'$ is not an isomorphism. Then, there are two K -morphisms $K'' \rightarrow L$ into a (algebraically closed) field L whose compositions with $K' \rightarrow K''$ are distinct.*

Proof. Since $K \rightarrow K'$ is not an isomorphism, there are two distinct K -morphisms $\psi_1, \psi_2 : K' \rightarrow L'$ into a field L' by [17, Prop. I.3.7.1.c]. The point is to extend them to K'' .

For $i \in \{1, 2\}$, one can embed the field extensions $K' \rightarrow K''$ and $\psi_i : K' \rightarrow L'$ into a common extension $K' \rightarrow L'_i$ by amalgamation [9, Chap 5, §4, Prop. 2]. Denote by $\psi'_i : K'' \rightarrow L'_i$ the induced extension. By amalgamation of L'_1 and L'_2 over K'' , one can assume that the ψ'_i take values in a common field L .

The morphisms $\psi'_1, \psi'_2 : K'' \rightarrow L$ fulfil the requirements since the restriction of ψ'_i to K' coincides with ψ_i . \square

Remark 2.7. It is classical that one can choose $L = L'$ in the proof of Lemma 2.6 if the extension $K' \rightarrow K''$ is moreover algebraic, and this is used in [28] to prove that the Lipschitz saturation is stable under contraction : in the setting of Proposition 2.5, if $C \rightarrow B$ is integral, then the Lipschitz saturation of A in C is equal to the intersection of C with the Lipschitz saturation of A in B . In our context the extensions are not assumed to be integral, and this contraction property does not hold, as illustrated by Example 2.18.

Proof of Proposition 2.5. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be two prime ideals of B lying over the same ideal \mathfrak{q} of A . A first step is to prove that \mathfrak{p}_1 and \mathfrak{p}_2 lye over the same ideal of C .

Let \mathfrak{p} be a prime ideal of $k(\mathfrak{p}_1) \otimes_{k(\mathfrak{q})} k(\mathfrak{p}_2)$, and $\omega = (\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}, \mathfrak{p}) \in \text{Spec}(B \otimes_A B)$ be the corresponding element, coming with a morphism $g : B \otimes_A B \rightarrow k(\omega)$ with $\ker g = \omega$. For $c \in \mathfrak{p}_1 \cap C$, we have $g \circ \varphi_1(c) = g(c \otimes_A 1) = 0$ by construction of g . The element $1 \otimes_A c - c \otimes_A 1$ is nilpotent in $B \otimes_A B$ by assumption, so that

$$0 = g(1 \otimes_A c - c \otimes_A 1) = g \circ \varphi_1(c) - g \circ \varphi_2(c).$$

As a consequence $g \circ \varphi_2(c) = 0$ and thus $c \in \mathfrak{p}_2 \cap C$. By symmetry we obtain

$$\mathfrak{p}_1 \cap C = \mathfrak{p}_2 \cap C.$$

The second step is to prove the equiresiduality, namely that the extension $\phi : k(\mathfrak{q}) \rightarrow k(\mathfrak{p}_1 \cap C)$ is an isomorphism, where $\mathfrak{q} = \mathfrak{p}_1 \cap A$. Assume by contradiction that ϕ is not surjective, and consider the composition

$$k(\mathfrak{q}) \xrightarrow{\phi} k(\mathfrak{p}_1 \cap C) \rightarrow k(\mathfrak{p}_1).$$

By Lemma 2.6, there are two distinct $k(\mathfrak{q})$ -morphisms $\psi_1, \psi_2 : k(\mathfrak{p}_1) \rightarrow L$ into some field L , which remain distinct by restriction to $k(\mathfrak{p}_1 \cap C)$. Thus there exists $c \in C$ such that $\psi_1 \circ \pi(c) \neq \psi_2 \circ \pi(c)$, where $\pi : B \rightarrow k(\mathfrak{p}_1)$ denote the natural morphism.

The morphisms ψ_1 and ψ_2 induce a morphism $\psi : k(\mathfrak{p}_1) \otimes_{k(\mathfrak{q})} k(\mathfrak{p}_1) \rightarrow L$ given by

$$\psi(\pi(b_1) \otimes_{k(\mathfrak{q})} \pi(b_2)) = \psi_1 \circ \pi(b_1) \psi_2 \circ \pi(b_2)$$

for $b_1, b_2 \in B$. The kernel \mathfrak{p} of ψ gives rise to a prime ideal $\omega = (\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}, \mathfrak{p})$ of $B \otimes_A B$ coming with a morphism

$$g : B \otimes_A B \rightarrow k(\omega) \rightarrow L.$$

By our choice of c , the element

$$\psi(\pi(c) \otimes 1 - 1 \otimes \pi(c)) = \psi_1 \circ \pi(c) - \psi_2 \circ \pi(c)$$

is not zero, so that $1 \otimes_A c - c \otimes_A 1$ does not belong to $\ker g = \omega$, contradicting the inclusion $C \subset \widehat{A}_B$. \square

Actually the converse of the preceding result holds true, and it gives rise to universal properties of the saturation, in terms of radical sequences of extensions. This result is, up to our knowledge, not present in the literature.

Theorem 2.8. *Let $A \xrightarrow{i} C \xrightarrow{j} B$ be a sequence of extensions of rings. The following properties are equivalent:*

(1) *For any field K , the map*

$$\begin{aligned} \text{Spec}(j) \circ (\text{Mor}(\text{Spec } K, \text{Spec } B)) &\rightarrow \text{Mor}(\text{Spec } K, \text{Spec } A) \\ (\text{Spec}(j) \circ \alpha) &\mapsto \text{Spec}(i) \circ (\text{Spec}(j) \circ \alpha) \end{aligned}$$

is injective.

(2) *For any field K , if $\psi_1 : B \rightarrow K$ and $\psi_2 : B \rightarrow K$ are two field morphisms distinct by composition with j , then they are distinct by composition with $j \circ i$.*

(3) *The sequence $A \rightarrow C \rightarrow B$ is radical.*

(4) *$j(C) \subset \widehat{A}_B$.*

(5) *The kernel of the morphism $C \otimes_A C \rightarrow C$ defined by $c_1 \otimes_A c_2 \mapsto c_1 c_2$ is included in the nilradical of $B \otimes_A B$.*

Proof. The equivalence between (1) and (2) is straightforward. Since $\ker(C \otimes_A C \rightarrow C)$ is generated by the elements of the form $c \otimes_A 1 - 1 \otimes_A c$ for $c \in C$, then (4) \Leftrightarrow (5). Note that (4) implies (3) by Proposition 2.5.

Let us prove that (3) implies (2) by contraposition. Let $\psi_1 : B \rightarrow K$ and $\psi_2 : B \rightarrow K$ be two morphisms in a field K such that $\psi_1 \circ j \neq \psi_2 \circ j$ and $\psi_1 \circ j \circ i = \psi_2 \circ j \circ i$. Let $\mathfrak{p}_1, \mathfrak{p}_2$ and \mathfrak{q} denote respectively the kernels of ψ_1, ψ_2 and $\psi_1 \circ j \circ i : A \rightarrow K$. For $i = 1, 2$ we get the following commutative diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{i} & C & \xrightarrow{j} & B & \xrightarrow{\psi_i} & K \\ \downarrow & & \downarrow & & \downarrow & \nearrow & \\ k(\mathfrak{q}) & \rightarrow & k(\mathfrak{p}_i \cap C) & \rightarrow & k(\mathfrak{p}_i) & & \end{array}$$

If $\mathfrak{p}_1 \cap C$ is not equal to $\mathfrak{p}_2 \cap C$, then $\text{Spec } C \rightarrow \text{Spec } A$ is not injective on the image of $\text{Spec } B$. If $\mathfrak{p}_1 \cap C$ is equal to $\mathfrak{p}_2 \cap C$, then ψ_1 and ψ_2 induce two different $k(\mathfrak{q})$ -morphisms $\psi'_i : k(\mathfrak{p}_i \cap C) \rightarrow K$ since $\psi_1 \circ j \neq \psi_2 \circ j$. As a consequence the extension $k(\mathfrak{q}) \rightarrow k(\mathfrak{p}_1 \cap C)$ cannot be an isomorphism.

In both cases, the extension $A \rightarrow C \rightarrow B$ is not radicial.

Finally we prove that (2) implies (4) by contraposition. By assumption there are $c \in C$ and $\omega \in \text{Spec } B \otimes_A B$ such that $1 \otimes_A c - c \otimes_A 1 \notin \omega$. The ideal ω comes with a morphism $g : B \otimes_A B \rightarrow K$ with $\ker g = \omega$. Consider the composition of g with the morphisms φ_1 and φ_2 defined in (a). By construction $g \circ \varphi_1 : B \rightarrow K$ coincides with $g \circ \varphi_2 : B \rightarrow K$ when composed with $j \circ i$, but not when composed with j because $g \circ \varphi_1(j(c)) \neq g \circ \varphi_2(j(c))$. It contradicts (2). \square

If we focus on the particular case of radicial extensions rather than sequences, we recover [17, Prop. I.3.7.1].

Proposition 2.9. *Let $i : A \rightarrow B$ be an extension of rings and $\text{Spec}(i) : \text{Spec } B \rightarrow \text{Spec } A$ be the associated map. The following properties are equivalent:*

- (1) *For any field K , the map*

$$\begin{aligned} \text{Mor}(\text{Spec } K, \text{Spec } B) &\rightarrow \text{Mor}(\text{Spec } K, \text{Spec } A) \\ \alpha &\mapsto \text{Spec}(i) \circ \alpha \end{aligned}$$

is injective.

- (2) *If $\psi_1 : B \rightarrow K$ and $\psi_2 : B \rightarrow K$ are two distinct morphisms in k then the compositions $\psi_1 \circ i$ and $\psi_2 \circ i$ are different.*
(3) *$i : A \rightarrow B$ is radicial.*
(4) *$B = \widehat{A}_B$.*
(5) *The kernel of the morphism $B \otimes_A B \rightarrow B$ defined by $b_1 \otimes_A b_2 \mapsto b_1 b_2$ is included in the nilradical of $B \otimes_A B$.*

Proof. Direct consequence of Theorem 2.8, using the fact that an extension $A \rightarrow B$ is radicial if and only if the sequence of extensions $A \rightarrow B \rightarrow B$ is so. \square

2.2. Comparison between saturation and seminormalization. In general, the seminormalization is only included in the saturation.

Lemma 2.10. *Let $A \rightarrow B$ be an extension of rings. Then*

$$A_B^+ \subset \widehat{A}_B.$$

Proof. Since $A \rightarrow A_B^+$ is subintegral then $A \rightarrow A_B^+ \rightarrow B$ is radicial. The inclusion follows by Theorem 2.8. \square

Note that there is no special relationship between the saturation and the relative normalization. However saturation and seminormalization coincide when we restrict to integral extensions.

Proposition 2.11. *Let $A \rightarrow B$ be an integral extension of rings. Then*

$$A_B^+ = \widehat{A}_B.$$

Proof. The direct inclusion comes from Lemma 2.10.

The sequence $A \rightarrow \widehat{A}_B \rightarrow B$ is radicial by Theorem 2.8. Since $\widehat{A}_B \rightarrow B$ is integral then $\text{Spec } B \rightarrow \text{Spec } \widehat{A}_B$ is surjective by Proposition 1.1. It follows that $A \rightarrow \widehat{A}_B$ is radicial and integral and thus subintegral. This forces \widehat{A}_B to be equal to the seminormalization A_B^+ of A in B by Theorem 1.6. \square

2.3. Saturation for varieties. We begin with the definition of radiciality and saturation for morphisms.

- Definition 2.12.** (1) Let $\pi : Y \rightarrow X$ be a dominant morphism between algebraic varieties over k . We say that $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_Y$ is radicial if for any open subset $U \subset X$ the extension $\mathcal{O}_X(U) \rightarrow \mathcal{O}_Y(\pi^{-1}(U))$ is radicial.
- (2) Let $Y \xrightarrow{\phi} Z \xrightarrow{\psi} X$ be a sequence of dominant morphisms between algebraic varieties over k . We say that $\mathcal{O}_X \rightarrow \psi_* \mathcal{O}_Z \rightarrow (\psi \circ \phi)_* \mathcal{O}_Y$ is radicial if for any open subset $U \subset X$ the sequence of extensions $\mathcal{O}_X(U) \rightarrow \mathcal{O}_Z(\psi^{-1}(U)) \rightarrow \mathcal{O}_Y((\psi \circ \phi)^{-1}(U))$ is radicial.
- (3) We say that X is saturated in Y if \mathcal{O}_X is saturated in $\pi_* \mathcal{O}_Y$, i.e for any open subset $U \subset X$ then $\mathcal{O}_X(U)$ is saturated in $\mathcal{O}_Y(\pi^{-1}(U))$.

In order to translate the universal property of the saturation in terms of varieties, we recall the notion of universal injectivity from [17, Chap. I, 3.4.3].

- Definition 2.13.** (1) A morphism $\pi : Y \rightarrow X$ between algebraic varieties over k is said universally injective if for any field extension $k \rightarrow K$, the map $\pi_K : Y(K) \rightarrow X(K)$ is injective.
- (2) A sequence of morphisms $Y \rightarrow Z \rightarrow X$ between algebraic varieties over k is said universally injective if for any field extension $k \rightarrow K$, the map $Z(K) \rightarrow X(K)$ is injective by restriction to the image of $Y(K) \rightarrow Z(K)$.

In this setting, the universal property given in Theorem 2.8 implies the following statement.

Proposition 2.14. *Let $Y \xrightarrow{\phi} Z \xrightarrow{\psi} X$ be a sequence of dominant morphisms between algebraic varieties over k . The following properties are equivalent:*

- (1) $Y \rightarrow Z \rightarrow X$ is universally injective.
- (2) The sequence $\mathcal{O}_X \rightarrow \psi_* \mathcal{O}_Z \rightarrow (\psi \circ \phi)_* \mathcal{O}_Y$ is radicial.
- (3) For any open subset $U \subset X$ we have $\mathcal{O}_Z(\psi^{-1}(U)) \subset \widehat{\mathcal{O}_X(U)}_{\mathcal{O}_Y((\psi \circ \phi)^{-1}(U))}$.

In the case of a morphism rather than a sequence of morphisms, we obtain the analogue of Proposition 2.9 :

Proposition 2.15. *Let $\pi : Y \rightarrow X$ be a dominant morphism between algebraic varieties over k . The following properties are equivalent:*

- (1) π is universally injective.
- (2) $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_Y$ is radicial.
- (3) For any open subset $U \subset X$ we have $\mathcal{O}_Y(\pi^{-1}(U)) = \widehat{\mathcal{O}_X(U)}_{\mathcal{O}_Y(\pi^{-1}(U))}$.

Remark 2.16. Let $\pi : Y \rightarrow X$ be a dominant morphism between affine algebraic varieties over k . Contrarily to seminormalization case it is not clear if $\widehat{k[X]}_{k[Y]}$ is a finitely generated algebra over k and thus lead to the existence of a variety.

Finally we state the relation between saturation and seminormalization for varieties induced by Lemma 2.10 and Proposition 2.11.

Proposition 2.17. *Let $\pi : Y \rightarrow X$ be a dominant morphism between varieties over k . If X is saturated in Y , then X is seminormal in Y . If π is moreover integral, then the converse holds true.*

We end this section by providing explicit examples to illustrate that the notion of saturation and seminormalization do not coincide for varieties, in any dimension. The examples are built on Example 2.18, constructed from a nodal curve, for which we offer two arguments : a simple geometric one, and a direct computational one in order to construct the generalization in any dimension in Example 2.19.

Example 2.18. (1) Let X be the nodal plane curve with coordinate ring $A = k[X] = k[x, y]/(y^2 - x^2(x + 1))$. Its normalization X' has coordinate ring $A' = k[X'] = k[x, z]/(z^2 - (x + 1)) = A[y/x]$ and the inclusion $A \rightarrow A'$ is given by $(x, y) \mapsto (x, xz)$. Let Y be defined by removing one of the two points $p = (0, 1)$ and $q = (0, -1)$ of $X'(k)$ lying above the singular point of $X(k)$, say p . The coordinate ring of Y is

$$B = k[Y] = k[x, z, s]/(z^2 - (x + 1), s(z - 1) - 1) = A'[1/s],$$

and we have a sequence of inclusions $A \rightarrow A' \rightarrow B$.

Then $A_B^+ = A$ whereas $\widehat{A}_B = B$. To see the first point, note that X is seminormal in Y since the variety X is seminormal (see [16]). For the second point, note that $A \rightarrow B$ is radicial because, for irreducible curves, the prime ideals correspond to the generic point and the closed points, and here $Y \rightarrow X$ is birational with $Y(k) \rightarrow X(k)$ bijective. As a consequence $\widehat{A}_B = B$ by Proposition 2.9.

- (2) We revisit the nodal curve example proving the equality $\widehat{A}_B = B$ using the very definition of the saturation. Keeping previous notation, set $\alpha = (z \otimes_A 1) - (1 \otimes_A z)$ and $\beta = (s \otimes_A 1) - (1 \otimes_A s)$. It suffices to prove that α and β are nilpotent elements of $B \otimes_A B$. Indeed \widehat{A}_B is a ring containing x, z and s so that $\widehat{A}_B = B$ in that case.

Note that

$$\begin{aligned} x\alpha &= x((z + 1) \otimes_A 1 - 1 \otimes_A (z + 1)) \\ &= (x(z + 1) \otimes_A 1) - (1 \otimes_A x(z + 1)) \\ &= ((y + x) \otimes_A 1) - (1 \otimes_A (y + x)) \quad \text{since } y + x = x(z + 1) \text{ in } B \\ &= 0 \end{aligned}$$

hence

$$\begin{aligned} \alpha^2 &= \alpha((z + 1) \otimes_A 1 - 1 \otimes_A (z + 1)) \\ &= \alpha(xs \otimes_A 1 - 1 \otimes_A xs) \quad \text{since } xs = z + 1 \text{ in } B \\ &= x\alpha(s \otimes_A 1 - 1 \otimes_A s) \\ &= 0. \end{aligned}$$

Actually we even have $\alpha = 0$ in $B \otimes_A B$, since a straightforward computation shows that $\alpha = \frac{1}{4}\alpha^3$. Finally, using the equality $\alpha = (z - 1) \otimes_A 1 - 1 \otimes_A (z - 1)$ and the relation $s(z - 1) = 1$, we observe that

$$0 = (s \otimes_A 1)\alpha(1 \otimes_A s) = -\beta.$$

Example 2.19. Consider the curves X and Y as in Example 2.18. For $n \geq 1$, the variety $X \times \mathbb{A}^n$ is seminormal in $Y \times \mathbb{A}^n$, whereas the saturation of $X \times \mathbb{A}^n$ in $Y \times \mathbb{A}^n$ is $Y \times \mathbb{A}^n$.

To see this, note that X and \mathbb{A}^n are seminormal, so $X \times \mathbb{A}^n$ is also seminormal [16, Cor. 5.9] and thus $X \times \mathbb{A}^n$ is seminormal in $Y \times \mathbb{A}^n$.

For the saturation, if the radiciality of

$$k[X \times \mathbb{A}^n] = A[t_1, \dots, t_n] \rightarrow B[t_1, \dots, t_n] = k[Y \times \mathbb{A}^n]$$

is not so straightforward since we no longer work with curves as in Example 2.18.(1), the computations done in Example 2.18.(2) still prove that $A[\widehat{t_1, \dots, t_n}]_{B[t_1, \dots, t_n]}$ contains x, z and s , and so is equal to $B[t_1, \dots, t_n]$.

3. ISOMORPHISM VERSUS HOMEOMORPHISM FOR COMPLEX VARIETIES

Given a morphism $\pi : Y \rightarrow X$ between algebraic varieties, we are looking for topological conditions on π together with geometric conditions on X which assure that π is an isomorphism.

After some generalities on the relation between homeomorphism with respect to Zariski topology, isomorphism and normality, we focus on the particular case of complex varieties for which we provide a complete solution to this problem using the strong topology. A generalization over any algebraically closed field of characteristic zero is developed in Section 4.

From now on k is algebraically closed.

3.1. Bijection, birationality, homeomorphism. We aim to compare the notions of bijection, birational morphism, homeomorphism with respect to Zariski topology at the spectrum level and homeomorphism with respect to Zariski topology at the level of closed points.

To begin with, recall that it follows from the Nullstellensatz that the property for a morphism to be an homeomorphism with respect to the Zariski topology is already decided at the level of closed points.

Proposition 3.1. *Let $\pi : Y \rightarrow X$ be a morphism between algebraic varieties over k . Then π is an homeomorphism if and only if π_k is an homeomorphism with respect to the Zariski topology.*

It is clear that an homeomorphism induces a bijection at the level of k -rational points, the converse being false in general as illustrated by Example 3.3 below.

Restricting our attention to curves, note that the converse holds true in the case of a morphism between irreducible algebraic curves. The irreducibility of the source space is crucial here, consider for instance the disjoint union of a point with a line minus a point, sent to a line. However even for morphisms between irreducible curves, a birational homeomorphism need not be an isomorphism as illustrated by Example 2.18, where the morphism $Y(k) \rightarrow X(k)$ is a bijection between irreducible curves, so $Y \rightarrow X$ is an homeomorphism which is not an isomorphism since X is singular.

An important contribution to these questions is the fact that the bijectivity at the level of closed points induces the birationality for irreducible varieties, by Zariski Main Theorem.

Proposition 3.2. *Let X and Y be irreducible varieties over k . Then a morphism from Y to X inducing a bijection at the level of k -rational points is birational. If in addition X is normal, it is an isomorphism.*

The proof is classical, but we include it for the clarity of the exposition.

Proof. First note that $Y \rightarrow X$ is quasi-finite by [36, Lem. 20.10] and the Nullstellensatz. By Grothendieck's form of Zariski Main Theorem, a quasi-finite morphism $\pi : Y \rightarrow X$ between irreducible algebraic varieties over k factorizes into an open immersion $Y \rightarrow Z$ and a finite morphism $Z \rightarrow X$. So we identify Y with an open subset of Z and further assume that Y is Zariski dense in Z .

Assume $\pi_k : Y(k) \rightarrow X(k)$ bijective, so that X, Y and Z have the same dimension. Recall that the degree of the extension $\mathcal{K}(X) \rightarrow \mathcal{K}(Z) = \mathcal{K}(Y)$ is the cardinal of a generic fiber of $Z(k) \rightarrow X(k)$ by [34, Thm. 7]. Such a generic fiber is in general in $Y(k)$, otherwise the

dimension of $\dim(Z \setminus Y)$ would be $\geq \dim X$, in contradiction with the density of Y . Thus the finite morphism $Z \rightarrow X$ has necessarily degree one, so that $Z \rightarrow X$ is birational and thus also π .

Assuming in addition X normal implies that Z is isomorphic to X . The open immersion is surjective at the level of k -rational points and from the Nullstellensatz it follows that it is surjective, thus an isomorphism. \square

The following example shows that a morphism which gives a bijection at the level of k -rational points need not be an homeomorphism.

Example 3.3. (1) Consider the varieties of Example 2.19, for which we have an open immersion ψ and a finite morphism ϕ as follows :

$$\pi : Y \times \mathbb{A}^n \xrightarrow{\psi} X' \times \mathbb{A}^n \xrightarrow{\phi} X \times \mathbb{A}^n.$$

Even if π_k is still bijective, the morphism π is no longer an homeomorphism when $n > 0$.

To see it, it suffices to consider the case $n = 1$. Denote by $O \in (X \times \mathbb{A}^1)(k)$ the origin, and by $P = (0, 1, 0)$ and $Q = (0, -1, 0)$ the two points in the fiber $\phi^{-1}(O)$, where the coordinates are (x, z, t_1) in the notation of Example 2.19. Let C be the curve in $X' \times \mathbb{A}^1$ given by intersection with the plane $x + z + 1 = 0$ in $\mathbb{A}^2 \times \mathbb{A}^1$. Note that $C = (\{Q\} \times \mathbb{A}^1)$ and $P \notin C$, so that $C \setminus \{Q\}$ is a closed subset of $Y \times \mathbb{A}^1$. If π was an homeomorphism, then $\pi_k(C(k) \setminus \{Q\})$ should be Zariski closed in $(X \times \mathbb{A}^1)(k)$ by Proposition 3.1. However O is in the closure of $\pi_k(C(k) \setminus \{Q\})$.

- (2) This example is also interesting to consider relatively to Grothendieck's notion of universal homeomorphism [18, Defn. 3.8.1]. Recall that a morphism $Y \rightarrow X$ is a universal homeomorphism if $Y \times_X Z \rightarrow Z$ is an homeomorphism for any morphism $Z \rightarrow X$.

The morphism $Y \rightarrow X$ of Example 2.18 is an homeomorphism but not a universal homeomorphism. Indeed, let $Z = X \times \mathbb{A}^1$ and consider the base change $Z \rightarrow X$ given by the first projection. We have already checked that $Y \times_X Z = Y \times \mathbb{A}^1 \rightarrow Z = X \times \mathbb{A}^1$ is not closed.

3.2. Complex varieties and the strong topology. We focus now on the particular case of complex algebraic varieties, in order to measure the influence of the strong topology. For a finite morphism $\pi : Y \rightarrow X$ between algebraic varieties over \mathbb{C} , we know from Theorem 1.16 that the properties :

- $\pi_{\mathbb{C}}$ is bijective,
- $\pi_{\mathbb{C}}$ is an homeomorphism for the strong topology,
- π is an homeomorphism

are equivalent. This is no longer true without the finiteness hypothesis. Actually, an homeomorphism with respect to Zariski topology need not be an homeomorphism with respect to the strong topology, even for irreducible affine curves.

Example 3.4. The morphism $Y(\mathbb{C}) \rightarrow X(\mathbb{C})$ from Example 2.18 for $k = \mathbb{C}$ is not an homeomorphism with respect to the strong topology. Indeed, consider a small open ball B of $Y(\mathbb{C})$ containing the point that is sent to the singular point of $X(\mathbb{C})$ by $Y(\mathbb{C}) \rightarrow X(\mathbb{C})$. Then the image of $Y(\mathbb{C}) \setminus B$ is not closed.

Note that $Y(\mathbb{C}) \times \mathbb{C}^n \rightarrow X(\mathbb{C}) \times \mathbb{C}^n$ is not an homeomorphism with respect to the strong topology either, following the same proof as in Example 3.3.

The following two results measure the rigidity of the strong topology.

Proposition 3.5. *Let $\pi : Y \rightarrow X$ be a morphism between algebraic varieties over \mathbb{C} . If $\pi_{\mathbb{C}}$ is an homeomorphism with respect to the strong topology, then π is finite.*

Proof. It is sufficient to assume that Y and X are irreducible. By Grothendieck's form of Zariski main Theorem, π factorizes into an open immersion $g : Y \rightarrow Z$ and a finite morphism $h : Z \rightarrow X$. We consider $Y(\mathbb{C})$ embedded as an open Euclidean subset of $Z(\mathbb{C})$. We also assume Y to be Zariski dense in Z , and thus $Y(\mathbb{C})$ is dense in $Z(\mathbb{C})$ for the strong topology.

Since $\pi_{\mathbb{C}}$ is bijective, the finite morphism h is birational by Proposition 3.2. Moreover $h_{\mathbb{C}}$ is surjective by Proposition 1.1. Let us prove that $h_{\mathbb{C}}$ is also injective. If not, there exist $y \in Y(\mathbb{C})$ and $z \in Z(\mathbb{C}) \setminus Y(\mathbb{C})$ with $h_{\mathbb{C}}(y) = h_{\mathbb{C}}(z)$. Denote this point by $x \in X(\mathbb{C})$. Let V_y be a closed Euclidean neighborhood of y in $Y(\mathbb{C})$, and V_z a closed Euclidean neighborhood of z in $Z(\mathbb{C})$ disjoint from V_y . Then $V_x = h_{\mathbb{C}}(V_y)$ is a closed Euclidean neighborhood of x in $X(\mathbb{C})$ by assumption on $\pi_{\mathbb{C}}$.

By the Curve Selection Lemma [8, Thm. 2.5.5], there is a continuous curve $\gamma : [0, 1) \rightarrow Z(\mathbb{C})$ with $\gamma(0) = z$ and $\gamma(0, 1) \subset V_z \cap Y(\mathbb{C})$. Then $h_{\mathbb{C}} \circ \gamma : [0, 1) \rightarrow X(\mathbb{C})$ is a continuous curve with $h_{\mathbb{C}} \circ \gamma(0) = x$, so it meets $V_x \setminus \{x\} = h_{\mathbb{C}}(V_y \setminus \{y\})$. As a consequence V_y and V_z cannot be disjoint because $\pi_{\mathbb{C}}$ is bijective.

Therefore $h_{\mathbb{C}}$ is bijective and thus $g_{\mathbb{C}}$ is also bijective. The Nullstellensatz forces g to be a bijective open immersion, thus an isomorphism. As a consequence π is finite like h . \square

Corollary 3.6. *Let $\pi : Y \rightarrow X$ be a morphism between algebraic varieties over \mathbb{C} . If $\pi_{\mathbb{C}}$ is an homeomorphism with respect to the strong topology, then π is an homeomorphism.*

Proof. The finiteness follows from Proposition 3.5. Being a bijection on the complex points, it is an homeomorphism by Theorem 1.16. \square

Corollary 3.6 admits a converse, for varieties of dimension at least two.

Proposition 3.7. *Let $\pi : Y \rightarrow X$ be a morphism between irreducible algebraic varieties over \mathbb{C} of dimension at least two. If π is an homeomorphism, then $\pi_{\mathbb{C}}$ is an homeomorphism with respect to the strong topology.*

Proof. By [38, Theorem 2.2], the morphism π is finite (it is also birational by Proposition 3.2), so we conclude using Theorem 1.16. \square

Proposition 3.7 is however not true for curves as illustrated by Example 2.18.

As illustrated by Proposition 3.2, the normality of the target space plays a role to upgrade a bijection into an isomorphism. In the context of complex varieties, next result shows that seminormality is the correct notion to associate to an homeomorphism with respect to the strong topology in order to obtain an isomorphism.

Theorem 3.8. *Let $\pi : Y \rightarrow X$ be a morphism between algebraic varieties over \mathbb{C} such that $\pi_{\mathbb{C}}$ is an homeomorphism with respect to the strong topology. Then π is an isomorphism if and only if X is seminormal in Y .*

Proof. To prove the non trivial implication, note that π is finite by Proposition 3.5. Since X is seminormal in Y then the result follows from Proposition 1.17. \square

Note that we cannot replace the topological assumption on $\pi_{\mathbb{C}}$ by $\pi_{\mathbb{C}}$ is bijective or even π is an homeomorphism, as illustrated by Example 2.18. We are now able to give an alternative version of [38, Thm. 2.4] in the case of complex varieties, where we replace Zariski topology by the strong one. In particular our statement is valid without any restriction on dimension. It is also a generalization of Proposition 1.17.

Corollary 3.9. *Let $\pi : Y \rightarrow X$ be a morphism between algebraic varieties over \mathbb{C} . If $\pi_{\mathbb{C}}$ is an homeomorphism with respect to the strong topology and X is seminormal, then π is an isomorphism.*

Proof. If X is seminormal then X is seminormal in Y , so the result follows from Theorem 3.8. \square

We end the section by a statement with a slightly different flavour. Forgetting about the strong topology, we wonder what is the correct assumption to add to an homeomorphism in order to obtain an isomorphism. It is also an extended version of Proposition 1.17.

Theorem 3.10. *Let $\pi : Y \rightarrow X$ be a morphism between algebraic varieties over \mathbb{C} such that π is an homeomorphism. Then π is an isomorphism if and only if X is saturated in Y .*

Proof. Assume X is saturated in Y , the converse implication being trivial. We may assume X and Y irreducible, since the irreducible components of X and Y are homeomorphic one-by-one, and each saturated in the other by definition of a radicial extension or sequence.

Assume first that the dimension of X and Y is at least two. Then $\pi_{\mathbb{C}}$ is an Euclidean homeomorphism by Proposition 3.7, so π is finite by Proposition 3.5. As a consequence X is seminormal Y by Proposition 2.11, and so π is an isomorphism by Theorem 3.8.

Assume X and Y are curves. Then X and Y are birational by Proposition 3.2. The extension $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_Y$ is then radicial since $\pi_{\mathbb{C}}$ is bijective. We conclude that π is an isomorphism using Proposition 2.15. \square

Note that, even though the statement of Theorem 3.10 does not mention the strong topology, it plays a crucial role in our proof.

4. GENERALIZING THE STRONG TOPOLOGY OF \mathbb{C}

The goal of this section is to generalize Theorems 3.8 and 3.10 from complex algebraic geometry to algebraic geometry over any algebraically closed field of characteristic zero. Since there is a priori no natural strong topology in this situation, we use the theory of real closed fields to define such a topology as in [24, 21] (see also [6] for a recent cohomological use of this approach).

Let k be an algebraically closed field of characteristic zero. From Artin Schreier theory [4], we know the existence of (many) real closed subfields of k with algebraic closure k . Let $R \subset k$ denote one of these real closed fields. Then $k = R[\sqrt{-1}]$ and R comes with a unique ordering. The ordering on R gives rise to an order topology on the affine spaces R^n , in a similar way than the Euclidean topology on \mathbb{R}^n , even if the topological space R is not connected (except in the case $R = \mathbb{R}$) or the closed interval $[0, 1]$ is in general not compact.

We use this choice of R to define a topology on the closed points of an algebraic variety over k . First, for an algebraic variety X over R , choose an affine covering of X by Zariski open subsets U_i , and endow each affine sets $U_i(R)$ with the order topology. These open sets glue together to define the order topology on $X(R)$, and this topology does not depend on the choice of the covering. This topological space can be endowed additionally with the structure of a semialgebraic space by considering the sheaf of continuous semialgebraic functions [11, 12], or even of a real algebraic variety with the sheaf of regular functions on the R -points [8, 22].

Consider now the case of a quasi-projective algebraic variety X over k . By Weil restriction [40, 19], we associate to X an algebraic variety X_R over R whose R -points are in bijection

with the k -points of X . We endow $X(k)$ with the topology induced by the order topology on $X_R(R)$, and we call it the R -topology on $X(k)$.

If X is no longer quasi-projective, then the Weil restriction does not necessarily exist. Anyway choose an affine open covering $(U_i)_{i \in I}$ of X , endow the R -points of the Weil restrictions $(U_i)_R$ with the order topology, and note that these open sets glue together to define a topology on $X(k)$. This topology does not depend on the choice of the covering by [33, Lemma 5.6.1], and we call it the R -topology on $X(k)$. Again one can consider $X(k)$ as a semialgebraic space in the sense of [12] or as a real algebraic variety in the sense of [8].

The R -topology on $X(k)$ has many good properties, for instance $X(k)$ is semialgebraically connected and of pure dimension twice the dimension of X if X is irreducible [24]. For $k = \mathbb{C}$ and $R = \mathbb{R}$, the \mathbb{R} -topology is nothing more than the strong topology. The choice of a different real closed field R in k will lead to different topologies on $X(k)$ (for instance the semialgebraic fundamental group does depend on the choice of R [24]). Already with $k = \mathbb{C}$, one can choose a real closed field different from \mathbb{R} , even for instance a non-Archimedean $R \subset \mathbb{C}$. We will see however that in our setting, the choice of the real closed field is transparent.

4.1. Basics on the R -topology of k -varieties. In this section we fix a real closed field R with algebraic closure k .

Let X be a quasi-projective algebraic variety over k . Recall that by Weil restriction [19, 33] :

- (1) The variety X_R is nonsingular if X is nonsingular. More precisely, a k -point in X is singular if and only if its corresponding R -point in X_R is singular.
- (2) A Zariski open subset $U \subset X$ induces a Zariski open subset $U_R \subset X_R$.
- (3) A proper morphism $Y \rightarrow X$ between quasi-projective algebraic varieties over k induces a proper morphism $Y_R \rightarrow X_R$.
- (4) A finite morphism $Y \rightarrow X$ between quasi-projective algebraic varieties over k induces a finite morphism $Y_R \rightarrow X_R$.

Let X be an affine algebraic variety over k . A regular function on X gives rise to a continuous mapping $X_R(R) \rightarrow R^2$. Indeed the regular function is polynomial, and by Weil restriction a polynomial function to k induced a polynomial mapping to R^2 by taking the real and imaginary parts. Finally a polynomial function is continuous with respect to the R -topology.

The topological properties of R -varieties coming from k -varieties are much more moderate than for general R -varieties. For instance, if complex irreducible varieties are locally of equal dimension, irreducible algebraic subsets of \mathbb{R}^n may have isolated points. From [8], a real algebraic variety is called central if its subset of nonsingular points is dense with respect to the R -topology.

For a subset $A \subset X(k)$, we denote by \overline{A}^R the closure of A with respect to the R -topology. We denote by $\text{Reg}(X(k))$ the set of nonsingular points of $X(k)$.

Proposition 4.1. *Let X be an irreducible algebraic variety over k . Then $X(k)$ is central :*

$$\overline{\text{Reg}(X(k))}^R = X(k).$$

Proof. The question being local, it suffices to assume X is affine, and in particular the Weil restriction of X exists.

If X is nonsingular, then so is X_R by Weil restriction, and so $X_R(R)$ is central.

Otherwise, consider a resolution $\sigma : \tilde{X} \rightarrow X$ of the singularities of X which exists by [20] since k has characteristic zero. Then σ_k is surjective since k is algebraically closed, and one

can assume that σ_k induces a bijection

$$\tilde{U} = \sigma_k^{-1}(\text{Reg}(X(k))) \rightarrow \text{Reg}(X(k)) = U.$$

Let $x \in X(k)$, and choose a preimage $\tilde{x} \in \sigma_k^{-1}(x)$. The centrality of $\tilde{X}(k)$ implies the existence of a continuous semialgebraic curve $\tilde{\gamma} : [0, 1] \rightarrow \tilde{X}(k)$ with $\tilde{\gamma}(0) = \tilde{x}$ and $\tilde{\gamma}(t) \in \tilde{U}$ for $t \in (0, 1] \subset R$ by the Curve Selection Lemma [8, Theorem 2.5.5]. Its composition $\gamma = \sigma_k \circ \tilde{\gamma}$ is a continuous semialgebraic curve from $[0, 1]$ to $X(k)$ with $\gamma(0) = x$ and $\gamma(t) \in U$ for $t \in (0, 1] \subset R$. As a consequence x belongs to the closure with respect to the R -topology of $\text{Reg}(X(k))$ in $X(k)$, and so $X(k)$ is central. \square

Remark 4.2. In particular, if the irreducible algebraic variety X over k has dimension d , then the local semialgebraic dimension of $X(k)$ at any point $x \in X(k)$ is equal to $2d$.

The following result is not valid in general for algebraic varieties over R , and even for $R = \mathbb{R}$.

Lemma 4.3. *Let X be an irreducible algebraic variety over k . A non-empty Zariski open subset of $X(k)$ is dense with respect to the R -topology.*

Proof. The question being local, it suffices to assume X is affine, and in particular the Weil restriction of X exists.

A Zariski open subset remains Zariski open by Weil restriction. Combined with Proposition 4.1, it suffices to check that a non-empty Zariski open set U in a central irreducible algebraic variety over R is dense with respect to the R -topology. This last property is classical ; for instance, the complement is an algebraic subset of strictly smaller dimension, and a semialgebraic triangulation of $X(k)$ adapted to the complement shows that locally, a point in the complement is in the boundary of a semialgebraic simplex in $U(k)$. \square

Over a general real closed field, the notion of compact sets is advantageously replaced by closed and bounded semialgebraic sets. For instance, the image of a closed and bounded semialgebraic set by a continuous semialgebraic map is again closed and bounded (and semialgebraic) [8, Theorem 2.5.8]. A semialgebraic map is said to be proper with respect to the R -topology if the preimage of a closed and bounded semialgebraic set is closed and bounded.

Lemma 4.4. *Let $\sigma : \tilde{X} \rightarrow X$ be a proper morphism between irreducible varieties over k . Then σ_k is proper with respect to the R -topology. If σ is moreover birational, then σ_k is surjective.*

Proof. The notion of properness is local on the target, so that there is an affine covering of X such that for any open affine subset U in the covering, the restriction $\sigma' = \sigma|_{\sigma^{-1}(U)}$ of σ to $\sigma^{-1}(U)$ is proper. The properness of σ' is kept by Weil restriction, so that σ'_k is proper with respect to the R -topology by [11, Theorem 9.6]. Finally σ_k is proper with respect to the R -topology since that notion of properness is local on the target too by [12, Proposition 5.7].

If σ is birational, there are Zariski open sets $\tilde{U} \subset \tilde{X}$ and $U \subset X$ such that $\sigma|_{\tilde{U}}$ is a bijection onto U . Then

$$U(k) = \sigma_k(\tilde{U}(k)) \subset \sigma_k(\overline{\tilde{U}(k)}^R) = \sigma_k(\tilde{X}(k)),$$

the right hand side equality coming from Lemma 4.3. Finally $\sigma_k(\tilde{X}(k))$ is closed for the R -topology by properness of σ_k , so that taking the closure with respect to the R -topology gives the result by Lemma 4.3. \square

Lemma 4.5. *Let $\sigma : \tilde{X} \rightarrow X$ be a finite morphism between algebraic varieties over k . Then σ_k is closed with respect to the R -topology.*

Proof. By [11, Theorem 4.2], a finite morphism between algebraic varieties over R is closed with respect to the R -topology. The result follows since finiteness is local and Weil restriction preserves finite morphisms.

Alternatively when X is irreducible, a finite morphism is proper, and apply Lemma 4.4. \square

4.2. Subintegrality and homeomorphisms. We are now in position to generalize the characterization of subintegrality via homeomorphisms, as in Theorem 1.16, over any algebraically closed field of characteristic zero.

Theorem 4.6. *Let $\pi : Y \rightarrow X$ be a finite morphism between algebraic varieties over k . The following properties are equivalent:*

- 1) π is subintegral.
- 2) π_k is bijective.
- 3) π_k is an homeomorphism for the R -topology.
- 4) π_k is an homeomorphism for the Zariski topology.
- 5) π is an homeomorphism.

Proof. The equivalence between 4) and 5) is given by Proposition 3.1. Using the Nullstellensatz, by Proposition 1.1 and proceeding similarly to Bernard's proof of [7, Thm. 3.1] then we get the equivalence between 1), 2) and 5).

It is clear that 3) implies 2). The proof that one of 1), 2), 4), 5) implies 3) is a direct consequence of Lemma 4.5. Indeed assuming 2) for instance, the map π_k admits an inverse, and this inverse is continuous with respect to the R -topology since π_k is a closed map by Lemma 4.5. \square

Remark 4.7. The other equivalent statements of Theorem 1.16 make use of the notions of regulous functions on a complex algebraic variety. We provide in the last section of the paper a full treatment of this notion over k , and so we are able to provide with Theorem 5.3 a complete generalization of Theorem 1.16.

Using the Curve Selection Lemma [8, Thm. 2.5.5] over a real closed field, and repeating word by word the proof of Proposition 3.5, we get:

Proposition 4.8. *Let $\pi : Y \rightarrow X$ be a morphism between algebraic varieties over k . Let R be a real closed subfield of k such that $R[\sqrt{-1}] = k$. If π_k is an homeomorphism for the R -topology, then π is finite.*

We have now developed all the material necessary to repeat the proof of Theorem 3.8 for algebraic varieties over k .

Theorem 4.9. *Let $\pi : Y \rightarrow X$ be a morphism between algebraic varieties over k . Let R be a real closed subfield of k such that $R[\sqrt{-1}] = k$, and assume π_k is an homeomorphism for the R -topology. Then π is an isomorphism if and only if X is seminormal in Y .*

Similarly, we obtain a generalization of Theorem 4.10.

Theorem 4.10. *Let $\pi : Y \rightarrow X$ be a morphism between algebraic varieties over k such that π is an homeomorphism. Then π is an isomorphism if and only if X is saturated in Y .*

Note that there is no reference to any real closed field in the statement of Theorem 4.10, but such a real closed subfield of k plays a major role in our proof.

5. CONTINUOUS RATIONAL FUNCTIONS ON THE k -RATIONAL POINTS

The aim of this section is to define a notion of continuous rational function on an algebraic variety defined over k . More precisely, for R a real closed field such that $R[\sqrt{-1}] = k$, we prove that the continuity of a rational function with respect to the R -topology, is independent of the choice of the real closed field R . This will enable to complete Theorem 4.6 in a full generalization of Theorem 1.16.

5.1. Continuous rational functions. We fix a real closed field R such that $R[\sqrt{-1}] = k$. A rational function that is continuous with respect to the R -topology can be characterized by the fact that it becomes regular after applying a relevant proper birational map.

Proposition 5.1. *Let X be an algebraic variety over k . Let $f : X(k) \rightarrow k$ be an everywhere defined function, and assume that f coincides with a regular function on a Zariski open subset of $X(k)$.*

Then, f is continuous with respect to the R -topology if and only if there is a proper birational map $\sigma : \tilde{X} \rightarrow X$ such that $f \circ \sigma_k : \tilde{X}(k) \rightarrow k$ is regular.

Proof. Arguing similarly to [7, Lem. 4.4], we may assume X is irreducible. Assume f to be continuous, and denote by g the rational function on X that coincides with f on a Zariski open subset of $X(k)$. One can resolve the indeterminacy of the rational map g by a sequence of blowings-up along nonsingular centers, giving rise to a proper birational morphism $\sigma : \tilde{X} \rightarrow X$ such that $g \circ \sigma_k : \tilde{X}(k) \rightarrow \mathbb{P}^1(k)$ is regular. The functions $f \circ \sigma$ and $g \circ \sigma$ are equal on a Zariski dense subset of $\tilde{X}(k)$, so they are equal on a subset dense with respect to the R -topology by Lemma 4.3. Therefore they coincide on $\tilde{X}(k)$ by continuity. As a consequence the regular function $g \circ \sigma_k$ takes its values in k rather than in $\mathbb{P}^1(k)$.

Conversely, let $C \subset k$ be a closed subset with respect to the R -topology. The set $(f \circ \sigma_k)^{-1}(C)$ is closed by continuity of $f \circ \sigma_k$, and its image under σ_k is equal to $f^{-1}(C)$ by surjectivity of σ_k via Lemma 4.4. As a consequence $f^{-1}(C)$ is closed by properness of σ_k with respect to the R -topology, thanks to Lemma 4.4 again. \square

Note that the characterization of continuity given above, via a resolution of indeterminacy, does not refer to the choice of R . In particular, the continuity with respect to the R -topology of a rational function is independent of the choice of the real closed field R .

Let X be an algebraic variety over k . Let $U \subset X$ be an open subset of X . A continuous rational function on $U(k)$ is a rational function on X that admits a continuous extension on $U(k)$, when $X(k)$ is endowed with the R -topology for some real closed field $R \subset k$ whose algebraic closure is k . The continuous rational function on the closed points of an algebraic variety X over k forms a sheaf $\mathcal{K}_{X(k)}^0$ on $X(k)$ equipped with the Zariski topology, similarly to the complex case. The notion of regulus function can be defined exactly as over \mathbb{C} , as those continuous rational functions that remains rational by restriction to any subvariety. If X is affine, we denote by $\mathcal{K}^0(X(k))$ the ring of continuous rational functions on $X(k)$.

The set of indeterminacy points of a continuous rational function is related to the normal locus of the ambient variety.

Proposition 5.2. *Let X be an algebraic variety over k . We have:*

- 1) $\mathcal{K}_{X(k)}^0 \subset (\pi'_k)_* \mathcal{O}_{X'(k)}$ where $\pi' : X' \rightarrow X$ is the normalization map.
- 2) If x is a normal closed point of $X(k)$ then $\mathcal{K}_{X(k),x}^0 = \mathcal{O}_{X,x}$.

Proof. The properties are local so we may assume X is affine. The original proof in [7, Proposition 4.7] uses only one argument related to the complex setting. It is the density with respect to the strong topology of a Zariski dense open set, that can be replaced by Lemma 4.3. Note that Hartogs Lemma used in the proof is valid over k : if X is normal then the restriction map $\mathcal{O}(X(k)) \rightarrow \mathcal{O}(\text{Reg}(X(k)))$ is surjective since

$$\dim(X(k) \setminus \text{Reg}(X(k))) \leq \dim(X(k)) - 2$$

by [23, p. 124]. □

We are now in position to state the generalization of Theorem 4.6. The rest of the paper will be devoted to its proof.

Theorem 5.3. *Let $\pi : Y \rightarrow X$ be a finite morphism between algebraic varieties over k . The following properties are equivalent:*

- 1) π is subintegral.
- 2) π_k is bijective.
- 3) π_k is biregulous.
- 4) The ringed spaces $(Y(k), \mathcal{K}_{Y(k)}^0)$ and $(X(k), \mathcal{K}_{X(k)}^0)$ are isomorphic.
- 5) π_k is an homeomorphism for the R -topology.
- 6) π_k is an homeomorphism for the Zariski topology.
- 7) π is an homeomorphism.

The proof will follow the same lines as the original proof over \mathbb{C} as in [7]. We reproduce the main steps, with the adequate modifications due to the modification of the topology, since it leads to interesting auxiliary results.

5.2. Seminormalization and continuous rational functions. A first result is that the ring of continuous rational functions is isomorphic to the ring of regular functions on the seminormalization.

Theorem 5.4. *Let X be an algebraic variety over k . The ring space $(X^+(k), \mathcal{O}_{X^+(k)})$ is isomorphic to $(X(k), \mathcal{K}_{X(k)}^0)$.*

Proof. Arguing similarly to [7, Lem. 4.4], we may assume X irreducible. Moreover the property is local, so we can assume X to be affine, and what we have to prove is

$$\mathcal{K}^0(X(k)) = k[X]^+ = k[X^+].$$

Let $\pi' : X' \rightarrow X$ be the normalization map. By Proposition 1.18, we get

$$k[X^+] = \{f \in k[X'] \mid f \text{ is constant on the fibers of } \pi'_k\}.$$

Let $f \in \mathcal{K}^0(X(k))$. Clearly the continuous rational function $f \circ \pi'_k$ is constant on the fibers of π'_k . Using Proposition 5.2, we have that $f \circ \pi'_k$ is regular on $X'(k)$ and thus $f \circ \pi'_k \in k[X^+]$. We have proved

$$\mathcal{K}^0(X(k)) \subset k[X^+].$$

Conversely let $f \in k[X^+]$. Let π^+ denote the map $X^+ \rightarrow X$. Since π^+ is subintegral then it follows from Theorem 4.6 that π_k^+ is an homeomorphism with respect to the R -topology. Thus the function $h = f \circ (\pi_k^+)^{-1}$ is continuous on $X(k)$. Since π^+ is birational, there exist a non-empty Zariski open subset $U \subset X(k)$ and $g \in \mathcal{O}(U)$ such that $g \circ (\pi_k^+)^{-1}|_{(\pi_k^+)^{-1}(U)} = f$ on

$(\pi_k^+)^{-1}(U)$. Since $h \circ \pi_k^+ = f$ on $X^+(k)$ and π_k^+ is bijective then h is rational on $X(k)$, and so $h \in \mathcal{K}^0(X(k))$. It follows that the injective morphism

$$\mathcal{K}^0(X(k)) \subset k[X^+], \quad h \mapsto h \circ \pi_k^+$$

is surjective and the proof is done. \square

Remark 5.5. Since X^+ doesn't depend of the choice of the real closed field, Theorem 5.4 gives another way to check that the continuity property of a rational function does not depend on the chosen real closed field.

A morphism $\pi : Y \rightarrow X$ between algebraic varieties over k induces a morphism $\mathcal{K}_{X(k)}^0 \rightarrow (\pi_k)_* \mathcal{K}_{Y(k)}^0$ since a regular morphism induces a continuous mapping. We can characterize the subintegrality of a finite morphism π using π_k^* , generalizing the complex version in Theorem 4.6.

Proposition 5.6. *Let $\pi : Y \rightarrow X$ be a finite morphism between algebraic varieties over k . The following properties are equivalent:*

- (1) π is subintegral.
- (2) The ringed spaces $(Y(k), \mathcal{K}_{Y(k)}^0)$ and $(X(k), \mathcal{K}_{X(k)}^0)$ are isomorphic.

Proof. Assuming π to be subintegral, we know from Theorem 4.6 that it is equivalent to the property that π_k is bijective, and so π is birational. Then X and Y have the same seminormalization, and so π_k^* is an isomorphism by Theorem 5.4.

Conversely, if π_k was not bijective, we may separate two different points y, y' in the fibre $\pi_k^{-1}(x)$ of some $x \in X(k)$ by a regular function f on $Y(k)$. But such a function is continuous with respect to the R -topology, so belongs to the image of π_k^* and in particular should be constant on the fibres of π_k . \square

As a consequence, we obtain that a continuous rational function is regulous (in the sense of Remark 1.13).

Corollary 5.7. *Let X be an algebraic variety over k and let $f \in \mathcal{K}^0(X(k))$. For any Zariski closed subset V of X , the restriction $f|_{V(k)}$ belongs to $\mathcal{K}^0(V(k))$.*

Proof. The proof of [7, Proposition 4.14] works verbatim using Theorem 5.4. \square

Note that when working over a non-algebraically closed field, the class of regulous functions is different from the class of continuous rational functions, cf. [26].

Proof of Theorem 5.3. We already have the equivalence of 1), 2), 5), 6), 7) by Theorem 4.6, and these are equivalent with 4) by Proposition 5.6.

A biregulous map is bijective so 3) implies 2), and conversely by 4) we know that the inverse of π_k is continuous rational, so regulous by Corollary 5.7. \square

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