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# Boundary Control of the Kuramoto-Sivashinsky Equation Under Intermittent Data Availability

M. Maghenem, C. Prieur, and E. Witrant

**Abstract**—In this paper, two boundary controllers are proposed to stabilize the origin of the nonlinear Kuramoto-Sivashinsky equation under intermittent measurements. More precisely, the spatial domain is divided into two sub-domains. The state of the system on the first sub-domain is measured along a given interval of time, and the state on the remaining sub-domain is measured along another interval of time. Under the proposed sensing scenario, we control the considered equation by designing the value of the state at three isolated spatial points, the two extremities of the spatial domain plus one inside point. Furthermore, we impose a null value for the spatial gradient of the state at these three locations. Under such a control loop, we propose two types of controllers and we analyze the stability of the resulting closed-loop system in each case. The paper is concluded with some discussions and future works.

## I. INTRODUCTION

Partial differential equations (PDE)s have numerous applications in many engineering fields including fluid flows in conservation laws [1], flexible structures [2], electromagnetic waves, and quantum mechanics [3]. The control design for PDEs is a key step to guarantee that the related process achieves a desired behavior in closed loop, i.e., a state of interest converges (in an appropriate norm) to an invariant set [4], [5], or tracks the state of a driving process [6], [7]. Before designing the control input, it is important to know the control actions allowed by the physical process. Indeed, some processes allows to act on the dynamics at every spatial point and for all time [8]. However, in some other processes, we act intermittently in time or in space [9], [10]. Furthermore, in some scenarios, we can reset the state intermittently in time and at every spatial point [11], but in other scenarios, we reset the state only at some spatial points [6], the latter case corresponds to the well-studied boundary-control paradigm [12]. On the other hand, it is important to know the outputs available for input design. In some cases, we measure the state at every spatial point all the time [13]. However, in most realistic scenarios, we measure only intermittently in space and time [9], [14]. The feedback law, in consequence, must adapt to each of these control and sensing scenarios.

Some intermittent control strategies for PDEs are available in the literature. In [11], the Gray-Scott and the Kuramoto-

Sivashinsky equations are controlled via a periodic reset of the state using impulsive systems theory. Furthermore, in [9], the Kuramoto-Sivashinsky equation is controlled via a periodic update of the input affecting the right-hand side using sample-data control techniques. A common feature among the aforementioned works is that the PDEs are controlled at every spatial point. When controlling PDEs at isolated spatial points or intervals, the existing control literature considers only the particular case of boundary control. However, the physics community, since the late nineties, has shown an intensive interest in general spatially- and temporally-intermittent control of PDEs. For example, in [6], the Gray-Scott equation is controlled by resetting the state periodically in time and at periodically separated spatial points. In [10], [15], the Kuramoto-Sivashinsky equation is controlled by acting on the right-hand side at periodically separated spatial intervals. In [16], the Ginzburg-Landau equation is studied following the strategy in [10]. The results in the aforementioned physics literature are guaranteed via simulations, and experiments in some cases. To the best of our knowledge, a rigorous study of the aforementioned problems is not available in the literature.

The Kuramoto-Sivashinsky equation is one of the well-studied PDEs in control literature. In particular, different types of boundary controllers are proposed to stabilize the origin in a given norm. For example, in [17], the linear Kuramoto-Shavinsky equation is transformed into an equivalent finite-dimensional linear system using the Sturm-Liouville decomposition. As a consequence, a linear feedback law is assigned to one extremity of the spatial domain. Furthermore, the nonlinear equation is studied in [18], [19], [20], [21] under various boundary conditions. It is important to note that although the aforementioned results assume point-wise measurements, the equation is assumed to be either linear or intrinsically stable. To the best of our knowledge, boundary control of the nonlinear Kuramoto-Sivashinsky equation without restricting the destabilizing coefficient is studied only in [22], where the state, on the whole spacial domain, is assumed to be available all the time.

In this paper, we study boundary control of the nonlinear Kuramoto-Sivashinsky equation, without restricting the destabilizing coefficient, under intermittent measurements. Strictly speaking, we measure the state on a given spatial sub-domain along a given interval of time and, then, we measure the state on the remaining spatial sub-domain along another time interval. As a result, we do not measure the

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state, on the whole spatial domain, at the same time. Under the proposed sensing scenario, we control the considered equation by designing the state at three isolated points, the two extremities of the spatial domain plus one inside point. Furthermore, we impose a null value for the spatial gradient of the state at these three locations. Two types of design approaches are proposed. In the first one, we design feedback laws at the two extremities and set the input to zero at the inside point. In the second case, we design a feedback law at one extremity and at the inside point, and set the input to zero at the remaining extremity. The stability properties of the resulting closed-loop system are analyzed in each case using Lyapunov methods.

The remainder of the paper is organized as follows. The problem formulation is in Section II. The proposed Lyapunov-based approach is described in Section III. The main results are in Sections IV and V, respectively. Finally, the paper is concluded by some discussions and future works.

**Notation.** For  $x \in \mathbb{R}^n$ ,  $(a, b) \in \mathbb{R} \times \mathbb{R}$  with  $a < b$ , and a function  $z : [a, b] \rightarrow \mathbb{R}$ , we let  $|x| := \sqrt{x^\top x}$ ,  $\|z\| := \sqrt{\int_a^b |z(x)|^2 dx}$ ,  $z_x := \frac{\partial z}{\partial x}$ ,  $z_{xx} := \frac{\partial^2 z}{\partial x^2}$ ,  $z_{xxx} := \frac{\partial^3 z}{\partial x^3}$ , and  $z_{xxxx} := \frac{\partial^4 z}{\partial x^4}$ . Furthermore, we say that  $z \in \mathcal{L}^2(a, b)$  if  $\|z\|$  is finite,  $z \in H^1(a, b)$  if  $\|z\| + \|z_x\|$  is finite and  $z \in H^2(a, b)$  if  $\|z\| + \|z_x\| + \|z_{xx}\|$  is finite. Moreover, we say that  $z \in H_o^1(a, b)$  if  $z \in H^1(a, b)$  and  $z(a) = z(b) = 0$ , and we say that  $z \in H_o^2(a, b)$  if  $z \in H^2(a, b)$  and  $z(a) = z(b) = z_x(a) = z_x(b) = 0$ . For a matrix  $M \in \mathbb{R}^{n \times n}$ ,  $\det M$  denotes the determinant of the matrix  $M$ . To avoid heavy notations, for a function  $u : [a, b] \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ , the time dependence is implicit when we write  $u(x)$ . In other words, we use  $u(x)$ , to express  $u(x, t)$ . Moreover, we use  $\text{dom } u$  to denote the domain of definition of the map  $t \mapsto u(x, t)$  for all  $x \in [a, b]$ . Finally,  $\kappa : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is a class  $\mathcal{K}$  function if it is continuous, increasing, and  $\kappa(0) = 0$ .

## II. PROBLEM FORMULATION

Consider the nonlinear Kuramoto-Sivashinsky equation given by

$$\Sigma_1 : u_t = -uu_x - \lambda_1 u_{xx} - u_{xxxx} \quad x \in [0, L],$$

where  $L > 0$  and  $\lambda_1 \geq 0$  are constant coefficients assumed to be known. The boundary conditions of  $\Sigma_1$  will be eventually specified.

### A. Intermittent Sensing

For some  $Y \in (0, L)$ , we measure the state  $u$  on the spatial domain  $[0, Y]$  for some interval of time and, then, we measure  $u$  on the domain  $[Y, L]$  along another time interval. In particular, we do not measure  $u$  along the whole line  $[0, L]$  simultaneously. The proposed sensing scenario has the following motivations.

In network control systems [23]. Assume that we dispose of two sensors, the first one measures  $u([0, Y])$  and the second one measures  $u([Y, L])$ . The two sensors share the same channel when sending their data to the controller.

Hence,  $u([0, Y])$  is available to the controller only for some interval of time and  $u([Y, L])$  only for another interval of time. The same reasoning can be extended if a network of sensors share the same communication channel with the controller [24]. Also, when using mobile or scanning sensors; see [25] and [26], respectively. Assume that we dispose of one mobile sensor measuring the state  $u([0, Y])$  for some interval of time, then, it changes location to measure  $u([Y, L])$  for another interval of time. Note that, here, we are neglecting the dynamics of the mobile sensor compared to the dynamics of  $\Sigma_1$ . This reasoning can be extended to a network of mobile sensors as in [27], [28], [29].

Strictly speaking, we assume the existence of a sequence of times  $\{t_i\}_{i=1}^\infty$ , with  $t_1 = 0$  and  $t_{i+1} > t_i$ , such that

- $u([0, Y], t)$  is available for all  $t \in \bigcup_{k=1}^\infty [t_{2k-1}, t_{2k})$ .
- $u([Y, L], t)$  is available for all  $t \in \bigcup_{k=1}^\infty [t_{2k}, t_{2k+1})$ .

Next, we consider the following assumption:

*Assumption 1:* There exist  $\bar{T}_1, \bar{T}_2 > 0$  such that, for each  $k \in \{1, 2, \dots\}$ ,  $t_{2k} - t_{2k-1} = \bar{T}_1$  and  $t_{2k+1} - t_{2k} = \bar{T}_2$ . •

*Remark 1:* The proposed sensing scenario can be generalized if we decompose the interval  $[0, L]$  into many sub-intervals instead of only two. In this case, we define an increasing sequence  $\{Y_i\}_{i=1}^N$ , with  $Y_1 = 0$  and  $Y_N = L$  such that, for each  $i \in \{1, 2, \dots, N\}$ ,  $u([Y_i, Y_{i+1}], t)$  is available to the controller for all  $t \in \bigcup_{k=1}^\infty [t_{(N-1)(k-1)+i}, t_{(N-1)(k-1)+i+1})$ . •

*Remark 2:* Note that most of existing results on boundary stabilization of  $\Sigma_1$  assume point-wise measurements; namely, the state  $u$  is measured at the isolated points where the control is effective; see for example [18], [19], [20], [21], [17]. However, in the aforementioned results either  $\lambda_1$  is assumed to be sufficiently small or the linear version of  $\Sigma_1$  is considered. To the best of our knowledge, boundary control of the nonlinear Kuramoto-Sivashinski equation without restricting the size of  $\lambda_1$  is studied only in [22], where the availability of  $u$  on the whole line  $[0, L]$  and for all time is assumed. •

### B. Boundary Control

We propose to control  $\Sigma_1$  at three different locations; namely, at  $x = 0$ ,  $x = Y$ , and  $x = L$ . As a result, we view  $\Sigma_1$  as a system of two PDEs interconnected by a boundary constraint. That is, we consider the system

$$\Sigma_2 : \begin{cases} w_t = -ww_x - \lambda_1 w_{xx} - w_{xxxx} & x \in [0, Y] \\ v_t = -vv_x - \lambda_1 v_{xx} - v_{xxxx} & x \in [Y, L] \end{cases}$$

under the boundary constraints

$$w(Y) = v(Y) \text{ and } w_x(Y) = v_x(Y). \quad (1)$$

Condition (1) guarantees that  $w$ , defined on  $[0, Y]$ , is a continuously differentiable extension of  $v$ , defined on  $[Y, L]$ , and vice-versa.

Furthermore, each equation in  $\Sigma_2$  allows for four boundary conditions, counting the two conditions in (1), we conclude

that we are allowed to control  $\Sigma_2$  by imposing six other boundary conditions, which are given by

$$\begin{aligned} v_x(L) = 0, \quad w_x(Y) = 0, \quad w_x(0) = 0, \\ w(0) = u_1, \quad w(Y) = u_2, \quad v(L) = u_3, \end{aligned} \quad (2)$$

where  $(u_1, u_2, u_3)$  are control inputs to be designed.

Next, we specify the set of solutions to  $\Sigma_2$ .

*Definition 1:* A pair  $((w, v), \{u_i\}_{i=1}^3)$ , with  $w : [0, Y] \times \text{dom } w \rightarrow \mathbb{R}$ ,  $v : [Y, L] \times \text{dom } v \rightarrow \mathbb{R}$ , and  $u_i : \text{dom } u_i \rightarrow \mathbb{R}$ , is a solution pair to  $\Sigma_2$  if  $\text{dom } w = \text{dom } v = \text{dom } u_1 = \text{dom } u_2 = \text{dom } u_3$ ,  $(w(\cdot, t), v(\cdot, t)) \in H^4(0, Y) \times H^4(Y, L)$ ,  $(w(x, \cdot), v(x, \cdot))$  is locally absolutely continuous, (1)-(2) hold, and

$$\begin{aligned} \frac{\partial w}{\partial t}(x, t) = -w(x, t)w_x(x, t) - \lambda_1 w_{xx}(x, t) - w_{xxxx}(x, t) \\ \text{for almost all } (x, t) \in [0, Y] \times \text{dom } w, \end{aligned}$$

$$\begin{aligned} \frac{\partial v}{\partial t}(x, t) = -v(x, t)v_x(x, t) - \lambda_1 v_{xx}(x, t) - v_{xxxx}(x, t) \\ \text{for almost all } (x, t) \in [Y, L] \times \text{dom } v. \end{aligned}$$

As a consequence, we study  $\Sigma_1$  under

$$\begin{aligned} (u_x(L), u_x(Y), u_x(0)) = (0, 0, 0), \\ (u(0), u(Y), u(L)) = (u_1, u_2, u_3) \end{aligned} \quad (3)$$

by studying  $\Sigma_2$  under (1)-(2). Hence, we specify the set of solutions to  $\Sigma_1$  as follows:

*Definition 2:* A pair  $(u, \{u_i\}_{i=1}^3)$ , with  $u : [0, L] \times \text{dom } u \rightarrow \mathbb{R}$  and  $u_i : \text{dom } u_i \rightarrow \mathbb{R}$ , is a solution pair to  $\Sigma_1$  if there exists  $(w, v)$  such that  $((w, v), \{u_i\}_{i=1}^3)$  is a solution pair to  $\Sigma_2$ ,  $\text{dom } u = \text{dom } w = \text{dom } v$ ,  $u(x, t) = w(x, t)$  for all  $(x, t) \in [0, Y] \times \text{dom } u$ , and  $u(x, t) = v(x, t)$  for all  $(x, t) \in [Y, L] \times \text{dom } u$ .

*Remark 3:* In [18],  $\Sigma_1$  is considered under the boundary conditions  $u_x(L) = u_x(0) = 0$ ,  $u_{xxx}(0) = u_1$ , and  $u_{xxx}(L) = u_2$ . Furthermore, in [22], the boundary conditions  $u(L) = u(0) = 0$ ,  $u_{xx}(0) = 0$ ,  $u_{xx}(L) = u_2$  are considered. Compared to the latter two references, in our work, we use an additional control action at  $x = Y$  to handle the intermittent availability of the measurements.

In the sequel, we design  $(u_1, u_2, u_3)$  for  $\Sigma_1$ , by following two scenarios. First, we set  $u_2 = 0$  and we design  $u_1$  and  $u_3$ . Second, we set  $u_3 = 0$  and we design  $u_1$  and  $u_2$ .

### III. GENERAL APPROACH

Before designing the control inputs, we consider the following two Lyapunov function candidates:

$$V_1(w) := \frac{1}{2} \int_0^Y w(x)^2 dx, \quad V_2(v) := \frac{1}{2} \int_Y^L v(x)^2 dx. \quad (4)$$

*Lemma 1:* Along the solutions to  $\Sigma_2$  and under (1) and (2), we have

$$\begin{aligned} \dot{V}_1 = - \int_0^Y w_{xx}(x)^2 dx + \lambda_1 \int_0^Y w_x(x)^2 dx \\ - \frac{u_2^3 - u_1^3}{3} - u_2 w_{xxx}(Y) + u_1 w_{xxx}(0), \\ \dot{V}_2 = - \int_Y^L v_{xx}(x)^2 dx + \lambda_1 \int_Y^L v_x(x)^2 dx \\ - \frac{u_3^3 - u_2^3}{3} - u_3 v_{xxx}(L) + u_2 v_{xxx}(Y). \end{aligned} \quad (5)$$

□

Next, we introduce a key result that allows us to upper bound the terms  $[-\int_0^Y w_{xx}(x)^2 dx + \lambda_1 \int_0^Y w_x(x)^2 dx]$  and  $[-\int_Y^L v_{xx}(x)^2 dx + \lambda_1 \int_Y^L v_x(x)^2 dx]$  in (5) using  $V_1$ ,  $V_2$ , and the inputs  $(u_1, u_2, u_3)$ . To this end, we introduce the following eigenvalue problem.

*Problem 1:* Given  $\lambda \geq 0$ , find the smallest  $\delta \in \mathbb{R}$ , denoted  $\delta_o$ , such that

$$z_{xxxx} + \lambda z_{xx} = \delta z \quad x \in [a, b] \quad (6)$$

admits a nontrivial solution  $z : [a, b] \rightarrow \mathbb{R}$  in  $H^2(a, b)$  satisfying

$$z(a) = z(b) = z_x(a) = z_x(b) = 0. \quad (7)$$

The following result can be found in [19, Lemma 2.1, Theorem 2.1, and Remark 1].

*Lemma 2:* Given  $\lambda \geq 0$ , we let  $\delta_o \in \mathbb{R}$  be the corresponding solution to Problem 1. Then, if  $\lambda < 4\pi^2$  then  $\delta_o > 0$ . Furthermore, if  $\lambda > 4\pi^2$  then  $\delta_o < 0$ . Finally, if  $\lambda = 4\pi^2$  then  $\delta_o = 0$ .

The following lemma can be found in [19, Lemma 3.1].

*Lemma 3:* Given  $\lambda \geq 0$  and  $z \in H_o^2(a, b)$ . Let  $\delta_o \in \mathbb{R}$  be the solution to Problem 1. Then,

$$- \int_a^b z_{xx}(x)^2 dx + \lambda \int_a^b z_x(x)^2 dx \leq -\delta_o \int_a^b z(x)^2 dx. \quad (8)$$

□

Now, we propose a generalization of Lemma 3.

*Lemma 4:* Given  $\lambda_1 > 0$  and  $z \in H^2(a, b)$  with  $z_x(a) = z_x(b) = 0$ . Let  $\delta_o$  be the solution to Problem 1 with  $\lambda := 3\lambda_1$ . Then, for each  $\delta \leq \delta_o$ , we have

$$\begin{aligned} - \int_a^b z_{xx}(x)^2 dx + \lambda_1 \int_a^b z_x(x)^2 dx \leq \delta_1 \int_a^b z(x)^2 dx \\ + C_{z1}(z(a), z(b)) + \delta_2 C_{z2}(z(a), z(b)) + \lambda_1 C_{z3}(z(a), z(b)), \end{aligned} \quad (9)$$

where  $\delta_1 := (|\delta| - 2\delta)/3$ ,  $\delta_2 := (4|\delta| - 2\delta)/3$ ,

$$\begin{aligned} C_{z1}(\cdot) &:= 2 \int_a^b \kappa_{xx}(x)^2 dx, \\ C_{z2}(\cdot) &:= \int_a^b \kappa(x)^2 dx, \quad C_{z3}(\cdot) := 2 \int_a^b \kappa_x(x)^2 dx, \end{aligned} \quad (10)$$

$$\begin{aligned} \kappa(x) := & z(a) - 2[z(b) - z(a)] \left( \frac{x-a}{b} \right)^3 \\ & + 3[z(b) - z(a)] \left( \frac{x-a}{b} \right)^2. \end{aligned} \quad (11)$$

□

*Remark 4:* We can explicitly compute  $\delta$ , a lower bound of  $\delta_o$ , and numerically approach  $\delta_o$ . This is omitted due to space limitation. •

*Remark 5:* According to (10) and (11), we conclude that, for each  $i \in \{1, 2, 3\}$ ,  $C_{zi}(z(a), z(b)) := a_{zi}z(a)^2 + b_{zi}z(b)^2 + c_{zi}z(a)z(b)$ , where  $(a_{zi}, b_{zi}, c_{zi})$  are constants obtained by integrating the polynomials  $\kappa^2$ ,  $\kappa_x^2$ , and  $\kappa_{xx}^2$  on the interval  $[a, b]$ . It is also important to note that the parameters  $(a_{zi}, b_{zi}, c_{zi})$  depend only on the domain of  $z$ , which is the interval  $[a, b]$ . •

At this point, using Lemma 4, we translate the analysis of  $\Sigma_2$ , which is an infinite-dimensional system, into the analysis of a finite-dimensional system of differential inequalities.

*Lemma 5:* Along the solutions to  $\Sigma_2$  and under (1) and (2), we have

$$\begin{aligned} \dot{V}_1 &\leq 2\delta_1 V_1 + C_{w1}(u_1, u_2) + \delta_2 C_{w2}(u_1, u_2) \\ &\quad + \lambda_1 C_{w3}(u_1, u_2) - \frac{u_3^3 - u_1^3}{3} - u_2 w_{xxx}(Y) + u_1 w_{xxx}(0), \\ \dot{V}_2 &\leq 2\delta_1 V_2 + C_{v1}(u_2, u_3) + \delta_2 C_{v2}(u_2, u_3) \\ &\quad + \lambda_1 C_{v3}(u_2, u_3) - \frac{u_3^3 - u_2^3}{3} - u_3 v_{xxx}(L) + u_2 v_{xxx}(Y), \end{aligned} \quad (12)$$

where  $(\delta_1, \delta_2)$  are given in Lemma 4, and  $\{C_{wi}\}_{i=1}^3$  and  $\{C_{vi}\}_{i=1}^3$  are obtained as in Lemma 4 while substituting  $(a, b, z)$  therein by  $(0, Y, w)$  and  $(Y, L, v)$ , respectively. □

*Remark 6:* The control action  $u_2 = w(Y) = v(Y)$  will be helpful to handle the boundary terms at  $x = Y$  that appear in (12). Without this additional control action, the problem is hard to solve when  $\lambda_1 \geq 4\pi^2$ . •

#### IV. MAIN RESULT 1:

##### ACTIVE CONTROL AT $x = 0$ AND $x = L$

In this section, we let  $u_2 = 0$ ; hence, (12) becomes

$$\begin{aligned} \dot{V}_1 &\leq 2\delta_1 V_1 + C_{w1}(u_1, 0) + \delta_2 C_{w2}(u_1, 0) \\ &\quad + \lambda_1 C_{w3}(u_1, 0) + u_1 w_{xxx}(0) + u_1^3/3, \\ \dot{V}_2 &\leq 2\delta_1 V_2 + C_{v1}(0, u_3) + \delta_2 C_{v2}(0, u_3) \\ &\quad + \lambda_1 C_{v3}(0, u_3) - u_3 v_{xxx}(L) - u_3^3/3. \end{aligned}$$

Next, using Remark 5, we obtain

$$\begin{aligned} \dot{V}_1 &\leq 2\delta_1 V_1 + a_{w1}u_1^2 + \delta_2 a_{w2}u_1^2 \\ &\quad + \lambda_1 a_{w3}u_1^2 + u_1 w_{xxx}(0) + u_1^3/3, \\ \dot{V}_2 &\leq 2\delta_1 V_2 + b_{v1}u_3^2 + \delta_2 b_{v2}u_3^2 \\ &\quad + \lambda_1 b_{v3}u_3^2 - u_3 v_{xxx}(L) - u_3^3/3. \end{aligned}$$

#### A. Control Design

- When  $t \in I_1$ , we measure  $w([0, Y], t)$  and we choose  $(u_1, u_3)$  so that  $u_3 = 0$  and

$$\begin{aligned} \frac{u_1^3}{3} + (a_{w1} + \delta_2 a_{w2} + \lambda_1 a_{w3})u_1^2 + u_1 w_{xxx}(0) \\ \leq -(\alpha_1 + 2\delta_1)V_1, \end{aligned} \quad (13)$$

for some  $\alpha_1 > 0$ . Hence, we have

$$\dot{V}_1 \leq -\alpha_1 V_1 \quad \text{and} \quad \dot{V}_2 \leq 2\delta_1 V_2. \quad (14)$$

- When  $t \in I_2$ , we measure  $u([Y, L], t)$  and we choose  $(u_2, u_3)$  so that  $u_1 = 0$  and

$$\begin{aligned} -\frac{u_3^3}{3} + (b_{v1} + \delta_2 b_{v2} + \lambda_1 b_{v3})u_3^2 - u_3 v_{xxx}(L) \\ \leq -(\alpha_2 + 2\delta_1)V_2, \end{aligned} \quad (15)$$

for some  $\alpha_2 > 0$ . Hence, we obtain

$$\dot{V}_1 \leq 2\delta_1 V_1 \quad \text{and} \quad \dot{V}_2 \leq -\alpha_2 V_2. \quad (16)$$

In the following lemma, we show how to design  $u_1$  and  $u_3$  to satisfy (13) and (15), respectively.

*Remark 7:* Note that (13) involves  $w_{xxx}(0)$ , which is not guaranteed to remain bounded. Hence, it is important to design  $u_1 := \kappa_1(V_1, w_{xxx}(0))$  with  $w_{xxx}(0) \mapsto \kappa_1(V_1, w_{xxx}(0))$  globally bounded. Similarly, (15) involves  $v_{xxx}(L)$ . Hence, it is important to design  $u_3 := \kappa_3(V_2, v_{xxx}(L))$  with  $v_{xxx}(L) \mapsto \kappa_3(V_2, v_{xxx}(L))$  globally bounded. •

*Lemma 6:* To satisfy (13), we take

$$u_1 := \kappa_1(\cdot) := \begin{cases} -\text{sign}(w_{xxx}(0))V_1 & \text{if } |w_{xxx}(0)| \geq l_1(V_1) \\ k_1(V_1) & \text{otherwise,} \end{cases}$$

where  $k_1$  is such that

$$\begin{aligned} k_1^3 + 3(a_{w1} + \delta_2 a_{w2} + \lambda_1 a_{w3} + 1)k_1^2 + 3l_1(V_1)^2 \\ \leq -3(\alpha_1 + 2\delta_1)V_1, \end{aligned} \quad (17)$$

and  $l_1(V_1) := V_1^2/3 + (a_{w1} + \delta_2 a_{w2})V_1 + (\alpha_1 + 2\delta_1)$ .

Similarly, to satisfy (15), we take

$$u_3 := \kappa_3(\cdot) := \begin{cases} -\text{sign}(v_{xxx}(L))V_2 & \text{if } |v_{xxx}(L)| \geq l_3(V_2) \\ k_3(V_2) & \text{otherwise,} \end{cases}$$

where  $k_3$  is such that

$$\begin{aligned} k_3^3 + 3(b_{v1} + \delta_2 b_{v2} + \lambda_1 b_{v3} + 1)k_3^2 + 3l_3(V_2)^2 \\ \leq -3(\alpha_2 + 2\delta_1)V_2, \end{aligned} \quad (18)$$

and  $l_3(V_2) := V_2^2/3 + (b_{v1} + \delta_2 b_{v2})V_2 + (\alpha_2 + 2\delta_1)$ . □

*Remark 8:* Note that (17) and (18) always admit a solution  $k_1$  and  $k_3$  function of  $V_1$  and  $V_2$ , respectively. For example, one can take  $k_1$  as a second-order polynomial of  $V_1$  with strictly negative coefficients that are sufficiently large. •

## B. $\mathcal{L}^2$ Exponential Stability

Let  $\Sigma_1^{cl}$  be the system obtained from  $\Sigma_1$  when (3) holds,  $u_2 = 0$ ,  $(u_1, u_3) = (\kappa_1, 0)$  on  $I_1$ , and  $(u_1, u_3) = (0, \kappa_3)$  on  $I_2$ . In this section, we show how to find positive constants  $\alpha_1$  and  $\alpha_2$  such that the trivial solution to  $\Sigma_1^{cl}$  is  $\mathcal{L}^2$  globally exponentially stable.

*Definition 3:* The trivial solution to  $\Sigma_1^{cl}$  is  $\mathcal{L}^2$ -GES if there exist  $\gamma, \kappa > 0$  such that, for each solution  $u$  to  $\Sigma_1^{cl}$ , we have  $\|u(t_o + t)\| \leq \kappa e^{-\gamma t} \|u(t_o)\|$  for all  $(t_o, t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ . • According to the proposed approach, we establish the  $\mathcal{L}^2$ -GES for  $\Sigma_1^{cl}$  by showing, for an appropriate choice of  $(\alpha_1, \alpha_2)$ , GES of the origin for the switched system

$$\Sigma_3 : \begin{cases} \dot{V}_1 = -\alpha_1 V_1 \\ \dot{V}_2 = 2\delta_1 V_2 & t \in I_1, \\ \dot{V}_1 = 2\delta_1 V_1 \\ \dot{V}_2 = -\alpha_2 V_2 & t \in I_2, \end{cases} \quad (V_1, V_2) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}.$$

*Theorem 1:* Consider system  $\Sigma_1$  under the sensing scenario in Section II-A and the boundary conditions in (3). Assume that, for some  $\bar{T}_1, \bar{T}_2 > 0$ , Assumption 1 holds. Furthermore, we let  $u_2 = 0$ ,  $(u_1, u_3) = (\kappa_1, 0)$  on  $I_1$ , and  $(u_1, u_3) = (0, \kappa_3)$  on  $I_2$ , where  $\kappa_1, \kappa_3$  come from Lemma 6. Then, for  $(\alpha_1, \alpha_2)$  satisfying

$$\alpha_1 > \frac{2\delta_2 \bar{T}_2}{\bar{T}_1} \quad \text{and} \quad \alpha_2 > \frac{2\delta_2 \bar{T}_1}{\bar{T}_2}, \quad (19)$$

the trivial solution to the closed-loop system is  $\mathcal{L}^2$ -GES.  $\square$

## V. MAIN RESULT 2 :

### ACTIVE CONTROL AT $x = 0$ AND $x = Y$

In this section, we let  $u_3 = 0$ . As a result, (12) reduces to

$$\begin{aligned} \dot{V}_1 &\leq 2\delta_1 V_1 - \frac{u_2^3}{3} - u_2 w_{xxx}(Y) + \frac{u_1^3}{3} + u_1 w_{xxx}(0) \\ &\quad + C_{w1}(u_1, u_2) + C_{w2}(u_1, u_2)\delta_2 + C_{w3}(u_1, u_2)\lambda_1, \\ \dot{V}_2 &\leq 2\delta_1 V_2 + \frac{u_2^3}{3} + u_2 v_{xxx}(Y) \\ &\quad + C_{v1}(u_2, 0) + C_{v2}(u_2, 0)\delta_2 + C_{v3}(u_2, 0)\lambda_1. \end{aligned} \quad (20)$$

### A. Control Design

When  $t \in I_1$ , we set  $u_2 = 0$  and choose  $u_1$  such that (13) holds. To obtain

$$\dot{V}_1 \leq -\alpha_1 V_1 \quad \text{and} \quad \dot{V}_2 \leq 2\delta_1 V_2. \quad (21)$$

When  $t \in I_2$ , we note that the first inequality in (20) involves the term  $[u_1 w_{xxx}(0)]$ . Note that  $w_{xxx}(0)$  is unknown on the interval  $I_2$ . Hence, before designing  $(u_1, u_2)$ , we introduce the following lemma.

*Lemma 7:* Consider  $\Sigma_2$  under (1) and (2). Then

$$w_{xxx}(0) = w_{xxx}(Y) + \frac{u_2^2 - u_1^2}{2} + \dot{\gamma},$$

where  $\gamma := \left( \int_0^Y w(x) dx \right)$ .  $\square$

Using Lemma 7, (20) becomes

$$\begin{aligned} \dot{V}_1 &\leq -\frac{u_2^3}{3} + (u_1 - u_2)w_{xxx}(Y) - \frac{u_1^3}{6} \\ &\quad + C_{w1}(u_1, u_2) + \frac{u_1 u_2^2}{2} + 2\delta_1 V_1 + \delta_2 C_{w2}(u_1, u_2) \\ &\quad + \lambda_1 C_{w3}(u_1, u_2) + u_1 \dot{\gamma}, \\ \dot{V}_2 &\leq \frac{u_2^3}{3} + C_{v2}(u_2, 0)\delta_2 + C_{v3}(u_2, 0)\lambda_1 \\ &\quad + C_{v1}(u_2, 0) + u_2 v_{xxx}(Y) + 2\delta_1 V_2. \end{aligned} \quad (22)$$

Now, we introduce constants  $B > 0$  and  $C > 0$  such that

$$\begin{aligned} B u_2^2 &:= C_{v3}(u_2, 0)\lambda_1 + C_{v2}(u_2, 0)\delta_2 + C_{v2}(u_2, 0), \\ C u_2^2 &:= C_{w1}(u_2, u_2) + \delta_2 C_{w2}(u_2, u_2) + \lambda_1 C_{w3}(u_2, u_2). \end{aligned}$$

Hence, on the interval  $I_2$ , we propose to choose  $(u_1, u_2)$  such that  $u_1 = u_2$  and

$$\frac{u_2^3}{3} + B u_2^2 + u_2 v_{xxx}(Y) \leq -\alpha_2 V_2^3. \quad (23)$$

As a consequence, we obtain, for almost all  $t \in I_2$ ,

$$\begin{aligned} \dot{V}_1 &\leq 2\delta_1 V_1 + u_2 \dot{\gamma} + C u_2^2 \\ \dot{V}_2 &\leq -\alpha_2 V_2^3 + 2\delta_1 V_2. \end{aligned} \quad (24)$$

Now, we propose to find  $\kappa_2 \left( \alpha_2^{\frac{1}{3}} V_2, v_{xxx}(Y) \right)$  such that, when  $u_2 = \kappa_2 \left( \alpha_2^{\frac{1}{3}} V_2, v_{xxx}(Y) \right)$ , both (23) and the following property hold.

*Property 1:* There exists  $P > 0$  such that

$$\left| \kappa_2 \left( \alpha_2^{\frac{1}{3}} V_2, v_{xxx}(Y) \right) \right| \leq P \alpha_2^{\frac{1}{3}} V_2 \quad \forall v_{xxx}(Y) \in \mathbb{R}, \quad (25)$$

and, for almost all  $t \geq t_o \geq 0$ , we have

$$\frac{d}{dt} \kappa_2 \left( \alpha_2^{\frac{1}{3}} V_2(t_o), v_{xxx}(Y, t) \right) = 0. \quad (26)$$

*Lemma 8:* Property 1 and (23) hold for

$$\kappa_2(\cdot) := \begin{cases} -\text{sign}(v_{xxx}(Y)) \alpha_2^{\frac{1}{3}} V_2 & \text{if } |v_{xxx}(Y)| \geq 2\alpha_2^{\frac{2}{3}} V_2^2 \\ -\beta \alpha_2^{\frac{1}{3}} V_2 & \text{otherwise,} \end{cases}$$

where  $\beta > 0$  is chosen so that  $-\beta^3 + 6\beta + 3 \leq 0$ .  $\square$

### B. $\mathcal{L}^2$ -Stability Analysis

Let  $\Sigma_1^{cl}$  be the system obtained from  $\Sigma_1$  when (3) holds,  $u_3 = u_2 = 0$ ,  $u_1 = \kappa_1$  on  $I_1$ , and  $u_3 = 0$  and  $u_1 = u_2 = \kappa_2 \left( \alpha_2^{\frac{1}{3}} V_2(t_{2k}), v_{xxx}(Y) \right)$  on each interval  $[t_{2k}, t_{2k+1}] \subset I_2$ . Recall that, by definition of  $\kappa_1$  and  $\kappa_2$ ,  $\Sigma_1^{cl}$  includes  $(\alpha_1, \alpha_2)$  as free design parameters. Next, inspired by [30], we introduce some useful  $\mathcal{L}^2$ -semi-global and  $\mathcal{L}^2$ -practical-stability notions.

The trivial solution to  $\Sigma_1^{cl}$  is  $\mathcal{L}^2$  practically semi-globally attractive ( $\mathcal{L}^2$ -PSGA) if, for each  $\beta > \epsilon > 0$ , there exists  $\alpha_2^* > 0$  such that, for each  $\alpha_2 \geq \alpha_2^*$ , there exists  $\alpha_1^*$  such that, for each  $\alpha_1 \geq \alpha_1^*$ , every solution  $u$  to  $\Sigma_1^{cl}$  with  $\|u(t_o)\| \leq \beta$ , there exists  $T > 0$  such that  $\|u(t_o + T)\| \leq \epsilon$ .

$\Sigma_1^{cl}$  is  $\mathcal{L}^2$  semi-globally bounded ( $\mathcal{L}^2$ -SGB) if, for each  $\beta > 0$ , there exists  $\gamma > 0$  and  $\alpha_2^* > 0$  such that, for each  $\alpha_2 \geq \alpha_2^*$ , there exists  $\alpha_1^*$  such that, for each  $\alpha_1 \geq \alpha_1^*$ , we have, for every solution  $u$  to  $\Sigma_1^{cl}$  with  $\|u(t_o)\| \leq \beta$ ,  $\|u(t_o + t)\| \leq \gamma$  for all  $t \geq 0$ . Furthermore,  $\Sigma_1^{cl}$  is  $\mathcal{L}^2$  semi-globally ultimately bounded ( $\mathcal{L}^2$ -SGUB) if there exists  $\gamma > 0$  such that, for each  $\beta > 0$ , there exists  $\alpha_2^* > 0$  such that, for each  $\alpha_2 \geq \alpha_2^*$ , there exists  $\alpha_1^*$  such that, for each  $\alpha_1 \geq \alpha_1^*$ , for every solution  $u$  to  $\Sigma_1^{cl}$  with  $\|u(t_o)\| \leq \beta$ , there exists  $T > 0$  such that  $\|u(t_o + t)\| \leq \gamma$  for all  $t \geq T$ .

The trivial solution to  $\Sigma_1^{cl}$  is  $\mathcal{L}^2$  practically stable ( $\mathcal{L}^2$ -PS) if there exists  $\kappa \in \mathcal{K}$  such that, for each  $\epsilon > 0$ , there exists  $\alpha_2^* > 0$  such that, for each  $\alpha_2 \geq \alpha_2^*$ , there exists  $\alpha_1^*$  such that, for each  $\alpha_1 \geq \alpha_1^*$ , we have  $\|u(t)\| \leq \kappa(\|u(t_o)\|) + \epsilon$  for  $t \geq t_o$ .

*Remark 9:* Note that to guarantee  $\mathcal{L}_2$ -practical semi-global asymptotic stability, we need to guarantee  $\mathcal{L}_2$ -PSGA,  $\mathcal{L}_2$ -SGB, and  $\mathcal{L}_2$ -PS. However, in our case, we will be able to show only  $\mathcal{L}_2$ -PSGA,  $\mathcal{L}_2$ -SGB, and  $\mathcal{L}_2$ -SGUB. •

*Theorem 2:* Consider system  $\Sigma_1$  under the sensing scenario in Section II-A and the boundary conditions in (3). Assume that, for some  $\bar{T}_1, \bar{T}_2 > 0$ , Assumption 1 holds. Furthermore, we let  $u_3 = u_2 = 0$ ,  $u_1 = \kappa_1$  on  $I_1$ , and  $u_1 = u_2 = \kappa_2 \left( \alpha_2^{\frac{1}{2}} V_2(t_{2k}), v_{xxx}(Y) \right)$  on each interval  $[t_{2k}, t_{2k+1}] \subset I_2$  and  $u_3 = 0$  on  $I_2$ , where  $\kappa_1, \kappa_2$  come from Lemmas 6 and 8, respectively. Then, the trivial solution to  $\Sigma_1^{cl}$  is  $\mathcal{L}^2$ -PSGA and  $\Sigma_1^{cl}$  is  $\mathcal{L}^2$ -SGB and  $\mathcal{L}^2$ -SGUB. □

## VI. CONCLUSIONS

This paper proposed two boundary controllers to stabilize the origin of the nonlinear Kuramoto-Sivashinsky equation, under intermittent measurements. Using the first controller, we are able to provide stronger stability properties compared to the second one. In future work, we would like to improve the stability properties of the second controller and consider the case where the coefficient  $\lambda_1$  is unknown.

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