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A New Probabilistic Representation of the Alternating Zeta Function and a New Selberg-like Integral Evaluation

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Abstract

In this paper, we present two new representations of the alternating Zeta function. We show that for any $s \in \mathbb{C}$ this function can be computed as a limit of a series of determinant. We then express these determinants as the expectation of a functional of a random vector with Dixon-Anderson density. The generalization of this representation to more general alternating series allows us to evaluate a Selberg-type integral with a generalized Vandermonde determinant.

1 Introduction

The Dirichlet eta function is defined by the following Dirichlet series, which converges for any complex number having real part greater than 0

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots \quad (1)$$

This Dirichlet series is the alternating sum corresponding to the Dirichlet series expansion of the Riemann zeta function $\zeta(s)$ and for this reason the Dirichlet eta function is also known as the alternating zeta function. The following relation holds:

$$\eta(s) = (1 - 2^{1-s}) \zeta(s).$$

The starting point of our work is a result from [2] (p. 456). In their paper, they show that the sum (1) can be approximated using an array of coefficients $(a_{n,N}, 1 \leq n \leq N)$. Let

$$a_{n,N} \stackrel{\text{def}}{=} \frac{1}{2} \prod_{j=1, j \neq n}^N \frac{j^2}{j^2 - n^2} = (-1)^{n-1} \frac{\binom{2N}{N-n}}{\binom{2N}{N}}. \quad (2)$$

then

Theorem 1 (Biane and al.). *For ε_i , $1 \leq i \leq N$ independent standard exponential variables, and $\Re(s) > -2N$*

$$\mathbb{E} \left[\left(\sum_{n=1}^N \frac{\varepsilon_n}{n^2} \right)^{\frac{s}{2}} \right] = s \Gamma \left(\frac{s}{2} \right) \sum_{n=1}^N \frac{a_{n,N}}{n^s} \quad (3)$$

where the $a_{n,N}$ are defined by (2), and

$$\eta_N(s) \stackrel{\text{def}}{=} \sum_{n=1}^N \frac{a_{n,N}}{n^s} \longrightarrow \eta(s) \text{ as } N \rightarrow \infty \quad (4)$$

uniformly on every compact subset of \mathbb{C} .

Remark 1. *The signs of the coefficients $a_{n,N}$ have been modified with respect to those given in the previously cited article, in order to obtain convergence in equation (4) towards $\eta(s)$ rather than $-\eta(s)$.*

The second part of the theorem shows that, for any $s \in \mathbb{C}$ the alternating Zeta function can be obtained by weighting the first N terms of the original series which is defined only for $\Re(s) > 0$. In this paper, we will show that the weighted finite series $\eta_N(s)$ defined in (4) can be written as a determinant (section 2.1, proposition 2). We will then show that this determinant can be written as the expectation of a functional of a Dixon-Anderson random vector (section 2.2, theorem 3). This result is new (up to our knowledge) and seems to show that there is a relation between the Zeta function and the theory of random matrices.

In section 3 we give a generalization of the representations given in section 2 for general series and, by computing the expectation of these representations, we obtain the evaluation of two Selberg integrals involving a generalized Vandermonde determinant (theorem 6).

2 New Representations of the Alternating Zeta Function

2.1 Determinant Representation

We start with the following result

Proposition 2. *With the notations and conditions given in theorem (1) we have*

$$\eta_N(s) = \frac{1}{2} \begin{vmatrix} 1 & \frac{1}{3!} & \cdots & \frac{1}{(2N-1)!} \\ 2^{1-s} & \frac{2^3}{3!} & \cdots & \frac{2^{2N-1}}{(2N-1)!} \\ \vdots & \vdots & \ddots & \vdots \\ N^{1-s} & \frac{N^3}{3!} & \cdots & \frac{N^{2N-1}}{(2N-1)!} \end{vmatrix}.$$

Proof. Let $a_n = n^2$, $Q_N(x) = \prod_{n=1}^N (1 - x/a_n)$, and P_N a polynomial of degree $N - 1$ over $\mathbb{C}[X]$, with the convention $P_N(x) = \sum_{n=1}^N c_{n,N} x^{n-1}$. An adaptation of the arguments given in annex A shows that

$$\frac{P_N(x)}{Q_N(x)} = \sum_{n=1}^N P_N(a_n) \prod_{\substack{j=1 \\ j \neq n}}^N \left(\frac{1}{\left(1 - \frac{a_n}{a_j}\right)} \frac{1}{1 - \frac{x}{a_j}} \right).$$

Setting $x = 0$, we have $Q_N(0) = 1$ and we get

$$P_N(0) = 2 \sum_{n=1}^N a_{n,N} P_N(n^2).$$

We choose P_N as the polynomial of degree $N - 1$ such that $P_N(a_n) = n^{-s}$ for $n = 1, \dots, N$. The coefficients $(c_{n,N})_{n=1}^N$ of P_N are solutions of the Vandermonde system

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2^2 & \dots & 2^{2(N-1)} \\ \vdots & \vdots & & \vdots \\ 1 & N^2 & \dots & N^{2(N-1)} \end{pmatrix} \begin{pmatrix} c_{1,N} \\ c_{2,N} \\ \vdots \\ c_{N,N} \end{pmatrix} = \begin{pmatrix} 1 \\ 2^{-s} \\ \vdots \\ N^{-s} \end{pmatrix}. \quad (5)$$

The Vandermonde matrix is invertible (see annex B) showing that the polynomial P_N is uniquely determined. We are only interested in $c_{1,N} = P_N(0)$. Let us denote V_N the Vandermonde matrix in (5) and $V_N^{(s)}$ the matrix obtained by replacing the first column of V_N by the right-end term.

Using Cramer's rule, we get

$$c_{1,N} = \frac{\begin{vmatrix} 1 & 1 & \dots & 1 \\ 2^{-s} & 2^2 & \dots & 2^{2(N-1)} \\ \vdots & \vdots & & \vdots \\ N^{-s} & N^2 & \dots & N^{2(N-1)} \end{vmatrix}}{\det(V_N)} = \frac{\det(V_N^{(s)})}{\det(V_N)}. \quad (6)$$

The Vandermonde Determinant of V_N is

$$\det(V_N) = \prod_{1 \leq i < j \leq N} (j^2 - i^2) = \prod_{1 \leq i < j \leq N} (j - i)(j + i) = \prod_{j=2}^N (j - 1)! \frac{(2j - 1)!}{j!} = \frac{1}{N!} \prod_{n=1}^{N-1} (2n + 1)! \quad (7)$$

and thus, we have

$$c_{1,N} = \begin{vmatrix} 1 & \frac{1}{3!} & \dots & \frac{1}{(2N-1)!} \\ \frac{2}{2^s} & \frac{2^3}{3!} & \dots & \frac{2^{2N-1}}{(2N-1)!} \\ \vdots & \vdots & & \vdots \\ \frac{N}{N^s} & \frac{N^3}{3!} & \dots & \frac{N^{2N-1}}{(2N-1)!} \end{vmatrix}.$$

ending the proof. \square

Remark 2. Observe that using the expression (6), it is immediate that $\eta_N(s) = \frac{1}{2}$ if $s = 0$ and $\eta_N(s) = 0$ if $s = -2, -4, \dots, -2(N - 1)$.

Remark 3. An alternative (and more direct) proof could have been used using the representation of the generalized Vandermonde determinant given in lemma 3.1.

2.2 Probabilistic Representation

We observe that $\det(V_N^{(s)})$ (equation 6) is a generalized Vandermonde determinant. Using the argument used in ([8]), we get the following lemma

Lemma 2.1. For $N > 1$, let

$$\Gamma_{N-1}\left(\frac{s}{2}\right) \stackrel{\text{def}}{=} \frac{2}{s} \prod_{n=1}^{N-1} \frac{(1 + \frac{1}{n})^{s/2}}{1 + \frac{s}{2n}}, \quad s \neq 0, -2, -4, \dots$$

then for all $s \in \mathbb{C}$, $s \neq -2, -4, \dots, -2(N-1)$ we have

$$\det(V_N^{(s)}) = \frac{2(N-1)!}{s\Gamma_{N-1}(\frac{s}{2})} \int_1^{2^2} dx_1 \int_{2^2}^{3^2} dx_2 \dots \int_{(N-1)^2}^{N^2} dx_{N-1} \prod_{1 \leq i < j \leq N-1} (x_j - x_i) \prod_{n=1}^{N-1} \left(\frac{x_n}{n(n+1)} \right)^{s/2}. \quad (8)$$

Proof. Observe first that $(s\Gamma_{N-1}(\frac{s}{2}))^{-1}$ is defined for all $N > 1$ and all $s \in \mathbb{C}$. Next, we have

$$\begin{aligned} \det(V_N^{(s)}) &= \begin{vmatrix} 1 & 1 & \dots & 1 \\ 2^{-s} & 2^2 & \dots & 2^{2(N-1)} \\ 3^{-s} & 3^2 & \dots & 3^{2(N-1)} \\ \vdots & \vdots & \dots & \vdots \\ N^{-s} & N^2 & \dots & N^{2(N-1)} \end{vmatrix} = \prod_{n=2}^N \frac{1}{n^s} \times \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & 2^{2(1+s/2)} & \dots & 2^{2(N-1+s/2)} \\ 1 & 3^{2(1+s/2)} & \dots & 3^{2(N-1+s/2)} \\ \vdots & \vdots & \dots & \vdots \\ 1 & N^{2(1+s/2)} & \dots & N^{2(N-1+s/2)} \end{vmatrix} \\ &= \prod_{n=2}^N \frac{1}{n^s} \times \begin{vmatrix} 2^{2(1+s/2)} - 1 & \dots & 2^{2(N-1+s/2)} - 1 \\ 3^{2(1+s/2)} - 2^{2(1+s/2)} & \dots & 3^{2(N-1+s/2)} - 2^{2(N-1+s/2)} \\ \vdots & \vdots & \vdots \\ N^{2(1+s/2)} - (N-1)^{2(1+s/2)} & \dots & N^{2(N-1+s/2)} - (N-1)^{2(N-1+s/2)} \end{vmatrix} \\ &= \prod_{n=1}^{N-1} \frac{n+s/2}{(n+1)^s} \times \begin{vmatrix} \int_1^{2^2} x_1^{s/2} dx_1 & \dots & \int_1^{2^2} x_1^{N-2+s/2} dx_1 \\ \int_{2^2}^{3^2} x_2^{s/2} dx_2 & \dots & \int_{2^2}^{3^2} x_2^{N-2+s/2} dx_2 \\ \vdots & \vdots & \vdots \\ \int_{(N-1)^2}^{N^2} x_{N-1}^{s/2} dx_{N-1} & \dots & \int_{(N-1)^2}^{N^2} x_{N-1}^{N-2+s/2} dx_{N-1} \end{vmatrix} \\ &= \prod_{n=1}^{N-1} \frac{n+s/2}{(n+1)^s} \int_1^{2^2} dx_1 \int_{2^2}^{3^2} dx_2 \dots \int_{(N-1)^2}^{N^2} dx_{N-1} \begin{vmatrix} x_1^{s/2} & \dots & x_1^{N-2+s/2} \\ \vdots & \vdots & \vdots \\ x_{N-1}^{s/2} & \dots & x_{N-1}^{N-2+s/2} \end{vmatrix} \\ &= \prod_{n=1}^{N-1} \frac{n+s/2}{(n+1)^s} \int_1^{2^2} dx_1 \int_{2^2}^{3^2} dx_2 \dots \int_{(N-1)^2}^{N^2} dx_{N-1} \begin{vmatrix} 1 & x_1 & \dots & x_1^{N-2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{N-1} & \dots & x_{N-1}^{N-2} \end{vmatrix} \prod_{n=1}^{N-1} x_n^{s/2} \\ &= \frac{2(N-1)!}{s\Gamma_{N-1}(\frac{s}{2})} \int_1^{2^2} dx_1 \int_{2^2}^{3^2} dx_2 \dots \int_{(N-1)^2}^{N^2} dx_{N-1} \prod_{1 \leq i < j \leq N-1} (x_j - x_i) \prod_{n=1}^{N-1} \left(\frac{x_n}{n(n+1)} \right)^{s/2}. \end{aligned}$$

□

Let $\mathcal{D}_{N-1}(\mathbf{x}; \boldsymbol{\alpha}, \mathbf{a})$, with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_N)$, $\alpha_n > 0$ and $\mathbf{a} = (a_1, a_2, \dots, a_N)$ with $a_1 < a_2 < \dots < a_N$, denote the Dixon-Anderson probability density function (pdf) over the domain $\mathcal{X}_{N-1} = \{a_1 < x_1 < a_2 < \dots < x_{N-1} < a_N\}$

(see [3] page 138)

$$\mathcal{D}_{N-1}(\mathbf{x}; \boldsymbol{\alpha}, \mathbf{a}) = \frac{\Gamma\left(\sum_{n=1}^N \alpha_n\right)}{\prod_{n=1}^N \Gamma(\alpha_n)} \frac{\prod_{1 \leq i < j \leq N-1} (x_j - x_i)}{\prod_{1 \leq i < j \leq N} (a_j - a_i)^{\alpha_j + \alpha_i - 1}} \prod_{n=1}^{N-1} \prod_{i=1}^N |x_n - a_i|^{\alpha_i - 1}. \quad (9)$$

Taking $\boldsymbol{\alpha} = \mathbf{1}_N$ (i.e. $\alpha_1 = \alpha_2 = \dots = \alpha_N = 1$) and $a_n = n^2$, $n = 1, \dots, N$, we get that

$$\mathcal{D}_{N-1}(\mathbf{x}; \boldsymbol{\alpha}, \mathbf{a}) = (N-1)! \frac{\prod_{1 \leq i < j \leq N-1} (x_j - x_i)}{\prod_{1 \leq i < j \leq N} (j^2 - i^2)} \quad (10)$$

is a pdf over $\mathcal{X}_{N-1} = \{1 < x_1 < 2^2 < \dots < x_{N-1} < N^2\}$. From the previous lemma, we obtain the following theorem

Theorem 3. Let $\mathbf{X} = (X_1, \dots, X_{N-1})$ be a random vector with Dixon-Anderson distribution given by (10), then

$$\frac{1}{s} \frac{1}{\Gamma_{N-1}\left(\frac{s}{2}\right)} \mathbb{E} \left[\prod_{n=1}^{N-1} \left(\frac{X_n}{n(n+1)} \right)^{s/2} \right] \rightarrow \eta(s) \text{ as } N \rightarrow \infty \quad (11)$$

uniformly on every compact subset of \mathbb{C} .

Proof. Using the expression of $\det(V_N)$ given in equation (7) we find the expectation given in (11) for N fixed.

The Gamma function can be defined as an infinite product for all complex numbers z except the non-positive integers

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^z}{1 + \frac{z}{n}}$$

and for any $s \in \mathbb{C}$ (the case $s = 0$ is handled by continuity)

$$\frac{2}{s \Gamma_{N-1}\left(\frac{s}{2}\right)} \xrightarrow{N \rightarrow \infty} \frac{1}{\Gamma\left(1 + \frac{s}{2}\right)}$$

uniformly on every compact of \mathbb{C} . For any $s \in \mathbb{C}$, we have

$$\eta_N(s) = \frac{1}{2} \frac{\det(V_N^{(s)})}{\det(V_N)} = \frac{1}{s} \frac{1}{\Gamma_{N-1}\left(\frac{s}{2}\right)} \mathbb{E} \left[\prod_{n=1}^{N-1} \left(\frac{X_n}{n(n+1)} \right)^{s/2} \right]$$

and thus the conclusion of theorem 1 occurs as well. \square

2.3 A Result Related to Theorem 3

One interesting fact about the Dixon-Anderson distribution given in (10) is that it is invariant under some linear transformations. More precisely if \mathbf{X} is a Dixon-Anderson random vector with pdf $\mathcal{D}_N(\mathbf{x}; \boldsymbol{\alpha}, \mathbf{a})$ and $(u, v) \in \mathbb{R}^* \times \mathbb{R}$ then $\mathbf{Y} = u\mathbf{X} + v\mathbf{1}_N$ is a Dixon-Anderson random vector with pdf $\mathcal{D}_N(\mathbf{x}; \boldsymbol{\alpha}, u\mathbf{a} + v\mathbf{1}_N)$. Using this property we can renormalize the random vector \mathbf{X} over $[0, 1]$ by using the change of variable $\mathbf{Y} = (\mathbf{X} - \mathbf{1}_{N-1})/(N^2 - 1)$ giving us the identity

$$\eta_N(s) = \frac{1}{s \Gamma_{N-1}\left(\frac{s}{2}\right)} \left(\frac{(N^2 - 1)^{(N-1)}}{N!(N-1)!} \right)^{\frac{s}{2}} \mathbb{E} \left[\prod_{n=1}^{N-1} \left(Y_n + \frac{1}{N^2 - 1} \right)^{\frac{s}{2}} \right].$$

We have the following theorem

Theorem 4. Let $\psi_N(x; s)$ denote the application

$$\psi_N(x; s) = \frac{2}{\Gamma(1 + \frac{s}{2})} \left(\frac{(N^2 - 1)^{(N-1)}}{N!(N-1)!} \right)^{\frac{s}{2}} \mathbb{E} \left[\prod_{n=1}^{N-1} \left| Y_n - \frac{x^2 - 1}{N^2 - 1} \right|^{\frac{s}{2}} \right].$$

Then for all $n \in \mathbb{N}^*$

$$\psi_N(n; s) \xrightarrow{N \rightarrow \infty} \frac{1}{n^s}$$

and for $n = 0$

$$\psi_N(0; s) \xrightarrow{N \rightarrow \infty} \eta(s).$$

Proof. The case $n = 0$ is a consequences of the theorem 3. Let $b_{n,N} = \frac{n^2-1}{N^2-1}$ for $n = 1, \dots, N$ and $\mathcal{Y}_{N-1} = \{0 = b_{1,N} < y_1 < b_{2,N} < y_2 < \dots < y_{N-1} < b_{N,N} = 1\}$. Taking $x = n$ we can compute the value of $\psi_N(x; s)$. We have

$$\mathbb{E} \left[\prod_{n=1}^{N-1} |Y_n - b_{n,N}|^{\frac{s}{2}} \right] = \frac{(N-1)!}{\prod_{1 \leq i < j \leq N} (b_{j,N} - b_{i,N})} \int_{\mathcal{Y}_{N-1}} d\mathbf{y} \prod_{1 \leq i < j \leq N-1} (y_j - y_i) \prod_{k=1}^{N-1} |y_k - b_{n,N}|^{s/2}.$$

The integral of the right hand side is (Consider the pdf given in (9) with $\alpha_k = 1$ if $k \neq n$ and $\alpha_k = 1 + s/2$ otherwise)

$$\begin{aligned} \int_{\mathcal{Y}_{N-1}} d\mathbf{y} \prod_{1 \leq i < j \leq N-1} (y_j - y_i) \prod_{k=1}^{N-1} (y_k - b_{n,N})^{s/2} \\ = \frac{\Gamma(1 + \frac{s}{2})}{\Gamma(N + \frac{s}{2})} \prod_{1 \leq i < j \leq N} (b_{j,N} - b_{i,N}) \prod_{k=1}^{n-1} (b_{n,N} - b_{k,N})^{s/2} \prod_{k=n+1}^N (b_{k,N} - b_{n,N})^{s/2} \end{aligned}$$

From this we deduce that when $n \neq 1$ we have

$$\begin{aligned} \psi_N(n; s) &= \left(\frac{\prod_{k=1}^{n-1} (n^2 - k^2) \prod_{k=n+1}^N (k^2 - n^2)}{N!(N-1)!} \right)^{s/2} \frac{\Gamma(N)}{\Gamma(N + \frac{s}{2})} \\ &= \left(\frac{(N-n)!(N+n)!}{n^2 N!(N-1)!} \right)^{s/2} \frac{\Gamma(N)}{\Gamma(N + \frac{s}{2})} \\ &= \left(\frac{1}{|a_{n,N}|n^2} \right)^{s/2} N^{s/2} \frac{\Gamma(N)}{\Gamma(N + \frac{s}{2})} \end{aligned}$$

with $a_{n,N}$ defined in (2). In the case $n = 1$ we find directly

$$\psi_N(1; s) = N^{\frac{s}{2}} \frac{\Gamma(N)}{\Gamma(N + \frac{s}{2})}$$

Taking the limit and observing that $\Gamma(N + \frac{s}{2}) \sim \Gamma(N) N^{s/2}$ as $N \rightarrow +\infty$ end the proof. \square

3 Averaged Alternating Random Series

3.1 A generalization of proposition 2 and theorem 3

let $s \in \mathbb{C}$ with $s \neq -2, -4, \dots, -2(N-1)$ and let $u_1 < u_2 < \dots < u_N$ be an increasing sequence of real numbers in \mathbb{R}^* . From this sequence, we define the $N \times N$ generalized Vandermonde determinant

$$V_N^{(s/2)}(\mathbf{u}) = \begin{vmatrix} u_1^{-s/2} & u_1 & u_1^2 & \cdots & u_1^{N-1} \\ u_2^{-s/2} & u_2 & u_2^2 & \cdots & u_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_N^{-s/2} & u_N & u_N^2 & \cdots & u_N^{N-1} \end{vmatrix}$$

with \mathbf{u} denoting the ordered vector $(u_1, \dots, u_N)^1$.

Lemma 3.1. *let $u_1 < u_2 < \dots < u_N$ be an increasing sequence of real numbers in \mathbb{R}^* . The following hold*

$$\frac{V_N^{(s/2)}(\mathbf{u})}{V_N^{(0)}(\mathbf{u})} = \sum_{n=1}^N (-1)^{n-1} \frac{1}{u_n^{s/2}} \prod_{\substack{1 \leq j \leq N \\ j \neq n}} \frac{u_j}{|u_j - u_n|}. \quad (12)$$

Proof. Observe that $V_N^{(0)}(\mathbf{u})$ denotes the usual determinant of a Vandermonde matrix. Let us denote by V_N^{-n} the following determinant

$$V_N^{-n} = \begin{vmatrix} u_1 & u_1^2 & \cdots & u_1^{N-1} \\ u_2 & u_2^2 & \cdots & u_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1} & u_{n-1}^2 & \cdots & u_{n-1}^{N-1} \\ u_{n+1} & u_{n+1}^2 & \cdots & u_{n+1}^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ u_N & u_N^2 & \cdots & u_N^{N-1} \end{vmatrix}, \quad n = 1, \dots, N.$$

Then, it is obvious that

$$V_N^{-n} = \left(\prod_{\substack{1 \leq j \leq N \\ j \neq n}} u_j \right) V_{N-1}^{(0)}(u_1, \dots, u_{n-1}, u_{n+1}, \dots, u_N).$$

By looking closely at the missing products, we obtain that

$$V_N^{-n} = \left(\prod_{\substack{1 \leq j \leq N \\ j \neq n}} u_j \right) \frac{V_N^{(0)}(\mathbf{u})}{\prod_{j=1}^{n-1} (u_n - u_j) \prod_{l=n+1}^N (u_l - u_n)} = (-1)^{n-1} V_N^{(0)}(\mathbf{u}) \prod_{\substack{1 \leq j \leq N \\ j \neq n}} \frac{u_j}{u_j - u_n}.$$

We have thus

$$\frac{V_N^{(s/2)}(\mathbf{u})}{V_N^{(0)}(\mathbf{u})} = \sum_{n=1}^N (-1)^{n+1} \frac{1}{u_n^{s/2}} \frac{V_N^{-n}}{V_N^{(0)}(\mathbf{u})} = \sum_{n=1}^N \frac{1}{u_n^{s/2}} \prod_{\substack{1 \leq j \leq N \\ j \neq n}} \frac{u_j}{u_j - u_n}.$$

As the sequence $(u_n, n = 1, \dots, N)$ is strictly increasing, the sum (12) is alternating as announced. \square

¹The reader will be aware that in section 2, $V_N^{(s/2)}$ represented a **matrix**, whereas from now the notation $V_N^{(s/2)}(\mathbf{u})$ represents a **determinant**

We have the following result which generalize theorem 3

Proposition 5. *Let $0 < u_1 < u_2 < \dots < u_N$ be an arbitrary increasing sequence of positive real number, let $s \in \mathbb{C}$ with $s \neq -2, -4, \dots, -2(N-1)$ and let $\mathbf{X} = (X_1, \dots, X_{N-1})$ denote a random vector with Dixon-Anderson density $\mathcal{D}_{N-1}(\mathbf{x}; \mathbf{1}_{N-1}, \mathbf{u})$ then*

$$\frac{2}{s\Gamma_{N-1}\left(\frac{s}{2}\right)} N^{s/2} \mathbb{E} \left[\frac{\prod_{n=1}^{N-1} X_n^{s/2}}{\prod_{n=1}^N u_n^{s/2}} \right] = \frac{V_N^{(s/2)}(\mathbf{u})}{V_N^{(0)}(\mathbf{u})} = \sum_{n=1}^N (-1)^{n-1} \frac{1}{u_n^{s/2}} \prod_{\substack{1 \leq j \leq N \\ j \neq n}} \frac{u_j}{|u_j - u_n|} \quad (13)$$

The proof follows the same steps as in lemma 2.1 and is left to the reader. The reader can also note that taking $u_n = n^2$ we get the expression obtained in section 2.2.

Remark 4. *It is clear that if the left hand side of the equation (13) converges in some sense as $N \rightarrow \infty$ to a well defined function in s , then this function will be equal to 1 when $s = 0$ and equal to 0 when $s = -2k, k \in \mathbb{N}^*$.*

Finally we note that a similar lemma have been proved in [2] using exponential random variables

Lemma 3.2. *For $(\varepsilon_n, 1 \leq n \leq N)$ independent standard exponential variables, and u_1, u_2, \dots, u_N an arbitrary sequence of numbers all distincts and strictly positive, if $\Re(s/2) > -N$*

$$\mathbb{E} \left[\left(\sum_{n=1}^N \frac{\varepsilon_n}{u_n} \right)^{s/2} \right] = \Gamma\left(1 + \frac{s}{2}\right) \sum_{n=1}^N \left(\frac{1}{u_n} \right)^{s/2} \prod_{\substack{1 \leq j \leq N \\ j \neq n}} \frac{u_j}{u_j - u_n}.$$

3.2 A family of joint density probability

Let $\mathbf{u} \in \mathbb{R}^N$ and $\mathbf{x} \in \mathbb{R}^{N-1}$ be two interlacing vectors in the sense that they lie in the region \mathcal{X}'_N defined as

$$\mathcal{X}'_N = \{0 < u_1 < x_1 < u_2 < \dots < u_{N-1} < x_{N-1} < u_N\}.$$

Let g denote a positive function over \mathbb{R}_+ to be precised hereafter. We define a joint density probability over \mathcal{X}'_N by putting

$$\begin{aligned} f_{\mathbf{X}, \mathbf{U}}(\mathbf{x}, \mathbf{u}) &= \frac{(N-1)!N!}{Z_N} V_{N-1}^{(0)}(\mathbf{x}) V_N^{(0)}(\mathbf{u}) \prod_{n=1}^N g(u_n) \\ &= \mathcal{D}_{N-1}(\mathbf{x}; \mathbf{1}_{N-1}, \mathbf{u}) \frac{N!}{Z_N} \left(V_N^{(0)}(\mathbf{u}) \right)^2 \prod_{n=1}^N g(u_n). \end{aligned} \quad (14)$$

It is quite evident that if (\mathbf{X}, \mathbf{U}) are two random interlacing vectors with such distribution, then the distribution of \mathbf{X} conditional to $\mathbf{U} = \mathbf{u}$ is a Dixon-Anderson random vector of density $\mathcal{D}_{N-1}(\mathbf{x}; \mathbf{1}_{N-1}, \mathbf{u})$. The marginal distributions of \mathbf{X} and \mathbf{U} are obtained by integrating the probability density function (14) with respect to \mathbf{u} and \mathbf{x} respectively. Integrating with respect to \mathbf{x} , we find that the density of \mathbf{U} is

$$f_{\mathbf{U}}(\mathbf{u}) = \frac{N!}{Z_N} \left(V_N^{(0)}(\mathbf{u}) \right)^2 \prod_{n=1}^N g(u_n). \quad (15)$$

over the domain $\mathcal{U}_N = \{0 < u_1 < u_2 < \dots < u_N\}$. Note that, as $f_{\mathbf{U}}$ is invariant under permutation, we have

$$Z_N = \int_0^\infty du_1 \dots \int_0^\infty du_N \left(V_N^{(0)}(\mathbf{u}) \right)^2 \prod_{n=1}^N g(u_n)$$

assuming the integral exists. Thus if (\mathbf{X}, \mathbf{U}) are random vectors with joint probability density function (14) and $s \neq -2, -4, \dots, -2(N-1)$ then it follows from identity given in (13) that

$$\frac{2}{s\Gamma_{N-1}\left(\frac{s}{2}\right)} N^{s/2} \mathbb{E} \left[\frac{\prod_{n=1}^{N-1} X_n^{s/2}}{\prod_{n=1}^N U_n^{s/2}} \right] = \mathbb{E} \left[\frac{V_N^{(s/2)}(\mathbf{U})}{V_N^{(0)}(\mathbf{U})} \right] \quad (16)$$

assuming again that the expectations involved in this equality exist and are finite.

There is two obvious choices for g allowing us to compute theses expectations: the Jacobi ensemble and the Laguerre ensemble.

3.2.1 The Jacobi Ensemble

We set $g(u) = u^{a-1}(1-u)^{b-1} \mathbb{1}_{(0,1)}(u)$ with $a, b > 0$. In this case, the distribution of \mathbf{U} conditional to $\mathbf{X} = \mathbf{x}$ is a Dixon-Anderson random vector of density $D_N(\mathbf{u}; (a, \mathbf{1}_{N-1}, b), (0, \mathbf{x}, 1))$ and the marginal distribution of \mathbf{U} is a Selberg density $S_N(\mathbf{u}; a, b, 1)$ (see [4]) with $S_N(\mathbf{u}; a, b, \lambda)$ given by

$$S_N(\mathbf{u}; a, b, \lambda) = \frac{N!}{S_N(a, b, \lambda)} \left(V_N^{(0)}(\mathbf{u}) \right)^{2\lambda} \prod_{n=1}^N u_n^{a-1} (1-u_n)^{b-1} \quad (17)$$

when supported on $\mathcal{U}_N = \{0 < u_1 < u_2 < \dots < u_N < 1\}$. $S_N(a, b, \lambda)$ denotes the Selberg's integral formula (see [1], chapitre 8. We choose the definition given in this reference rather than the one given in [4]). We have thus

$$Z_N = S_N(a, b, 1) = \prod_{n=0}^{N-1} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(2+n)}{\Gamma(a+b-1+N+n)}$$

Integrating (14) with respect to \mathbf{u} gives the marginal density of \mathbf{X}

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}; a, b) &= \frac{(N-1)!N!}{S_N(a, b, 1)} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b-1+N)} \left(V_{N-1}^{(0)}(\mathbf{x}) \right)^2 \prod_{n=1}^{N-1} x_n^a (1-x_n)^b \\ &= \frac{(N-1)!}{S_{N-1}(a+1, b+1, 1)} \left(V_{N-1}^{(0)}(\mathbf{x}) \right)^2 \prod_{n=1}^{N-1} x_n^a (1-x_n)^b \end{aligned} \quad (18)$$

i.e. the marginal density of \mathbf{X} is the Selberg density $S_{N-1}(\mathbf{x}; a+1, b+1, 1)$ supported over

$$\mathcal{X}_{N-1} = \{0 < x_1 < x_2 < \dots < x_{N-1} < 1\}.$$

3.2.2 The Laguerre Ensemble

We set now $g(u) = u^{a-1}e^{-u/b} \mathbb{1}_{(0,+\infty)}(u)$ with $a, b > 0$. The joint density of (\mathbf{X}, \mathbf{U}) can be obtained as a limit of the Jacobi ensemble case by changing variables $u_n = v_n/L$, $x_n = y_n/L$, replacing $b-1$ by L/θ and by taking the limit $L \rightarrow \infty$. We have in this case

$$Z_N = W_N(a, \theta) = \lim_{L \rightarrow \infty} \frac{S_N(a, L/\theta + 1, 1)}{L^{(a+N)N}} = \theta^{(a+N)N} \prod_{n=0}^{N-1} \Gamma(a+n)\Gamma(2+n).$$

The marginal distribution of \mathbf{U} is a Laguerre density

$$L(\mathbf{u}; a, \theta) = \frac{N!}{W_N(a, \theta)} \left(V_N^{(0)}(\mathbf{u}) \right)^2 \prod_{n=1}^N u_n^a e^{-u_n/\theta}$$

supported on $\mathcal{U}_N = \{0 < u_1 < u_2 < \dots < u_N < 1\}$. Integrating (14) with respect to \mathbf{u} gives the marginal density of \mathbf{X}

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}; a, \theta) &= \frac{(N-1)!N!}{W_N(a, \theta)} \theta^{a+N} \Gamma(a) \left(V_{N-1}^{(0)}(\mathbf{x}) \right)^2 \prod_{n=1}^{N-1} x_n^a e^{-x_n/\theta} \\ &= \frac{(N-1)!}{W_{N-1}(a+1, \theta)} \left(V_{N-1}^{(0)}(\mathbf{x}) \right)^2 \prod_{n=1}^{N-1} x_n^a e^{-x_n/\theta} \end{aligned} \quad (19)$$

i.e. the marginal density of \mathbf{X} is a Laguerre density $L_{N-1}(\mathbf{x}; a+1, \theta)$.

3.3 Main Result

Theorem 6. Let $a, b, \theta > 0$ and $\Re(a - s/2) > 0$ and let \mathbf{U} be a random vector of \mathbb{R}^N . If the distribution of \mathbf{U} is the Selberg density $S_N(\mathbf{u}; a, b, 1)$ supported on $\mathcal{U}_N = \{0 < u_1 < u_2 < \dots < u_N < 1\}$ then

$$\mathbb{E} \left[\frac{V_N^{(s/2)}(\mathbf{U})}{V_N^{(0)}(\mathbf{U})} \right] = \frac{2N^{s/2}}{s\Gamma_{N-1}(s)} \frac{\Gamma(a+b-1+N)}{\Gamma(a-\frac{s}{2}+b-1+N)} \frac{\Gamma(a-\frac{s}{2})}{\Gamma(a)}.$$

If the distribution of \mathbf{U} is the Laguerre ensemble density $L(\mathbf{u}; a, \theta)$ supported on $\mathcal{U}'_N = \{0 < u_1 < u_2 < \dots < u_N\}$ then

$$\mathbb{E} \left[\frac{V_N^{(s/2)}(\mathbf{U})}{V_N^{(0)}(\mathbf{U})} \right] = \frac{2}{s\Gamma_{N-1}(s)} \left(\frac{N}{\theta} \right)^{s/2} \frac{\Gamma(a-\frac{s}{2})}{\Gamma(a)}.$$

Proof. We have

$$\begin{aligned} \mathbb{E} \left[\frac{V_N^{(s/2)}(\mathbf{U})}{V_N^{(0)}(\mathbf{U})} \right] &= \frac{2}{s\Gamma_{N-1}(\frac{s}{2})} N^{s/2} \mathbb{E} \left[\frac{\prod_{n=1}^{N-1} X_n^{s/2}}{\prod_{n=1}^N U_n^{s/2}} \right] \\ &= \frac{2N^{s/2}}{s\Gamma_{N-1}(\frac{s}{2})} \frac{N!(N-1)!}{Z_N} \int_{\mathcal{X}'_N} \frac{\prod_{n=1}^{N-1} x_n^{s/2}}{\prod_{n=1}^N u_n^{s/2}} V_{N-1}^{(0)}(\mathbf{x}) V_N^{(0)}(\mathbf{u}) \prod_{n=1}^N g(u_n) d\mathbf{u} d\mathbf{x}. \end{aligned}$$

We integrate with respect to \mathbf{u} . In the Jacobi ensemble case, we get

$$\begin{aligned} \mathbb{E} \left[\frac{V_N^{(s/2)}(\mathbf{U})}{V_N^{(0)}(\mathbf{U})} \right] &= \frac{2N^{s/2}}{s\Gamma_{N-1}(\frac{s}{2})} \frac{N!(N-1)!}{S_N(a, b, 1)} \frac{\Gamma(a-\frac{s}{2})\Gamma(b)}{\Gamma(a-\frac{s}{2}+b-1+N)} \int_{\mathcal{X}_N} \left(V_{N-1}^{(0)}(\mathbf{x}) \right)^2 \prod_{n=1}^N x_n^a (1-x_n)^b d\mathbf{x} \\ &= \frac{2N^{s/2}}{s\Gamma_{N-1}(\frac{s}{2})} \frac{N!}{S_N(a, b, 1)} \frac{\Gamma(a-\frac{s}{2})\Gamma(b)}{\Gamma(a-\frac{s}{2}+b-1+N)} S_{N-1}(a+1, b+1, 1) \end{aligned}$$

giving after some elementary simplifications the announced result. The Laguerre ensemble case can be obtain either by integration, or by replacing $b-1$ by L/θ and by taking the limit $L \rightarrow \infty$. We let the details to the reader. \square

Finally, we have the following corollary

Corollary 7. Let $a, b, \theta > 0$ and $\Re(a - s/2) > 0$. Then

$$\begin{aligned} \int_0^1 du_1 \dots \int_0^1 du_N V_N^{(s/2)}(\mathbf{u}) V_N^{(0)}(\mathbf{u}) \prod_{n=1}^N u_n^{a-1} (1-u_n)^{b-1} \\ = \frac{2N^{s/2} S_N(a, b, 1)}{s\Gamma_{N-1}(s)} \frac{\Gamma(a+b-1+N)}{\Gamma(a-\frac{s}{2}+b-1+N)} \frac{\Gamma(a-\frac{s}{2})}{\Gamma(a)} \end{aligned}$$

and

$$\int_0^\infty du_1 \dots \int_0^\infty du_N V_N^{(s/2)}(\mathbf{u}) V_N^{(0)}(\mathbf{u}) \prod_{n=1}^N u_n^{a-1} e^{-u_n/\theta} = \prod_{n=0}^{N-1} \Gamma(a+n) \Gamma(2+n) \frac{2\theta^{(a+N)N}}{s \Gamma_{N-1}(s)} \left(\frac{N}{\theta}\right)^{s/2} \frac{\Gamma(a - \frac{s}{2})}{\Gamma(a)}.$$

4 Conclusion

It is well-known, even if it is not well understood, that there is a connection between the random matrix theory and the Zeta function. For example, Keating and its co-authors ([5]) successfully use the characteristic polynomials $Z(U, \theta)$ of matrices U in the Circular Unitary Ensemble (CUE) to study the behavior of the correctly renormalized integral

$$\int_0^T |\zeta(1/2 + it)|^{2\lambda} dt$$

Our work seems to be a first step in explaining this connection. The Selberg integral plays a fundamental role in the theory of the various β -ensembles (see [3]) and the Dixon-Anderson probability distribution function is an intermediate step to the Selberg's integral evaluation.

We hope that the results presented in this article will pave the way for a deeper understanding of the links between these two fields.

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A Partial fraction decomposition

Let Q be a polynomial of degree N such that $Q(x) = \prod_{n=1}^N (x - x_n)$ with $x_i \neq x_j$ if $i \neq j$ and let $L_n(x)$, $n = 1, \dots, N$ denote the Lagrange's polynomials

$$L_n(x) = \prod_{j=1, j \neq n}^N \frac{x - x_j}{x_n - x_j}.$$

If P is a polynomial of degree strictly less than N then by the Lagrange interpolation formula, it can be written as

$$P(x) = \sum_{n=1}^N P(x_n) L_n(x).$$

From this, we deduce the partial fraction expansion of P/Q

$$\frac{P(x)}{Q(x)} = \sum_{n=1}^N P(x_n) \frac{L_n(x)}{Q(x)} = \sum_{n=1}^N \frac{c_n}{x - x_n}$$

with

$$c_n = \frac{P(x_n)}{\prod_{j=1, j \neq n}^N (x_n - x_j)} = \frac{P(x_n)}{Q'(x_n)}.$$

B Inverse of the Vandermonde's matrix

The proof of this formula can be found at https://proofwiki.org/wiki/Inverse_of_Vandermonde_Matrix and is proposed as Exercise 40 from section 1.2.3 in [6]. Consider the Vandermonde's matrix

$$V_N(x_1, \dots, x_N) = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{pmatrix} \quad (20)$$

Assume $x_i \neq x_j$ if $i \neq j$, then the Vandermonde Determinant of V_N is

$$\det(V_N(x_1, \dots, x_N)) = \prod_{1 \leq i < j \leq N} (x_j - x_i) \neq 0.$$

Since this is non-zero, the inverse matrix, denoted $W_N = [w_{ij}]$, is guaranteed to exist. Using the definition of the matrix product and the inverse matrix

$$\sum_{k=1}^N x_i^{k-1} w_{kj} = \delta_{ij}.$$

For $1 \leq n \leq N$, if $P_n(x)$ is the polynomial

$$P_n(x) := \sum_{k=1}^N w_{kn} x^{k-1}$$

then $P_n(x_1) = 0, \dots, P_n(x_{n-1}) = 0, P_n(x_n) = 1, P_n(x_{n+1}) = 0, \dots, P_n(x_N) = 0$. By the Lagrange's interpolation formula, the n th column of W_N is composed of the coefficients of the n th Lagrange basis polynomial

$$P_n(x) = \sum_{k=1}^N w_{kn} x^{k-1} = \prod_{\substack{1 \leq j \leq N \\ j \neq n}} \frac{x - x_j}{x_n - x_j}.$$

We can identify the terms w_{ij} by expanding the product. In particular, setting $x = 0$, we get that the constant of the polynomials are

$$w_{1n} = \prod_{\substack{1 \leq j \leq N \\ j \neq n}} \frac{-x_j}{x_n - x_j} = \prod_{\substack{1 \leq j \leq N \\ j \neq n}} \frac{x_j}{x_j - x_n}, \quad n = 1, \dots, N.$$

C An other determinant representation

In this part we will show the following result:

Proposition 8. *if $s \neq 0$, then*

$$\eta_N(s) = \frac{1}{2} \begin{vmatrix} 1 + \frac{1}{\lambda_{2,N}} & -1 & 0 & \dots & 0 \\ -\frac{\lambda_{2,N}}{\lambda_{3,N}} & 1 + \frac{\lambda_{2,N}}{\lambda_{3,N}} & & \ddots & 0 \\ 0 & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & & 1 + \frac{\lambda_{N-2,N}}{\lambda_{N-1,N}} & -1 \\ 0 & \dots & 0 & -\frac{\lambda_{N-1,N}}{\lambda_{N,N}} & 1 + \frac{\lambda_{N-1,N}}{\lambda_{N,N}} \end{vmatrix} \quad (21)$$

with $\lambda_{n,N}^{-1} = 2a_{n,N}(n^{-s} - 1)$.

Proof. Let W_N denote V_N^{-1} , then $|V_N^{(s)} W_N| = |V_N^{(s)}| \times |W_N| = |V_N^{(s)}|/|V_N|$. Some elementary algebra shows that $V_N^{(s)} W_N$ is equal to the matrix

$$V_N^{(s)} W_N = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ w_{11}(2^{-s} - 1) & 1 + w_{12}(2^{-s} - 1) & w_{13}(2^{-s} - 1) & \dots & w_{1N}(2^{-s} - 1) \\ w_{11}(3^{-s} - 1) & w_{12}(3^{-s} - 1) & 1 + w_{13}(3^{-s} - 1) & \dots & w_{1N}(3^{-s} - 1) \\ \vdots & \vdots & & \ddots & \vdots \\ w_{11}(N^{-s} - 1) & w_{12}(N^{-s} - 1) & w_{13}(N^{-s} - 1) & \dots & 1 + w_{1N}(N^{-s} - 1) \end{pmatrix}.$$

Appendix B reveals that $w_{1,n} = 2a_{n,N}$ for $n = 1, \dots, N$. If $s = 0$, we get the identity matrix as expected, otherwise if $s \neq 0$, $\eta_N(s)$ can be written as

$$\eta_N(s) = \frac{1}{2} \prod_{n=2}^N 2a_{n,N}(n^{-s} - 1) \begin{vmatrix} 1 + \frac{1}{2a_{2,N}(2^{-s}-1)} & 1 & \dots & 1 \\ 1 & 1 + \frac{1}{2a_{3,N}(3^{-s}-1)} & \dots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \dots & 1 + \frac{1}{2a_{N,N}(N^{-s}-1)} \end{vmatrix}. \quad (22)$$

Setting $\lambda_{n,N}^{-1} = 2a_{n,N}(n^{-s} - 1)$ and using Gaussian elimination method² we get

$$\eta_N(s) = \frac{1}{2} \prod_{n=2}^N \lambda_n^{-1} \begin{vmatrix} 1 + \lambda_{2,N} & -\lambda_{2,N} & 0 & \dots & \dots & 0 \\ -\lambda_{2,N} & \lambda_{2,N} + \lambda_{3,N} & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & & \ddots & \lambda_{N-1,N} + \lambda_{N-2,N} & -\lambda_{N-1,N} \\ 0 & \dots & 0 & -\lambda_{N-1,N} & \lambda_{N-1,N} + \lambda_{N,N} & \end{vmatrix}.$$

ending the proof. \square

Remark 5. It is tempting to find a condition on the coefficients of the determinant (22) in order to find when it is zero. Lemma D.1 is equivalent to $\eta_N(s) = 0$ if and only if $2 \sum_{n=2}^N a_{n,N}(n^{-s} - 1) = -1$. As $\sum_{n=1}^N a_{n,N} = 1/2$, this equality can be rewritten $\sum_{n=2}^N a_{n,N}n^{-s} = -a_{1,N}$ which is not really helpful !

Assume $s \neq 0$, then, setting $\Delta_{0,N} = 0$, $\Delta_{1,N} = 1$, the determinant appearing in equation (21) can be computed using the recurrence relation

$$\Delta_{n,N} = \left(1 + \frac{\lambda_{n-1,N}}{\lambda_{n,N}}\right) \Delta_{n-1,N} - \frac{\lambda_{n-1,N}}{\lambda_{n,N}} \Delta_{n-2,N}, \quad n = 2, \dots, N \quad (D)$$

with the convention $\lambda_{1,N} = 1$.

Proposition 9. Let

$$\beta_{2,N} = \frac{1}{\lambda_{2,N}} \quad \text{and} \quad \beta_{n,N} = \frac{\lambda_{n-1,N}}{\lambda_{n,N}}, \quad n = 3, \dots, N$$

and let $\tilde{\Delta}_{n,N}$ denote the solution of the recurrence equation (D) with initial conditions $\tilde{\Delta}_{0,N} = 1$ and $\tilde{\Delta}_{1,N} = 0$. Then (see [7], p. 15 for the continued fraction representation) for $n = 2, \dots, N$, we have

$$\Delta_{n,N} = 1 + \sum_{k=0}^{n-2} \beta_{n-k} \dots \beta_2 = 1 + \sum_{k=2}^n \lambda_{k,N}^{-1}, \quad n = 2, \dots, N,$$

and

$$\tilde{\Delta}_{n,N} = - \sum_{k=0}^{n-2} \beta_{n-k} \dots \beta_2 = - \sum_{k=2}^n \lambda_{k,N}^{-1}, \quad n = 2, \dots, N.$$

Thus $\Delta_{N,N} = 2\eta_N(s)$, $\tilde{\Delta}_{N,N} = 1 - 2\eta_N(s)$ and

$$\frac{1}{2\eta_N(s)} = 1 + \frac{1 - \beta_{2,N}}{\beta_{2,N} +} \frac{1 - \beta_{3,N}}{\beta_{3,N} +} \dots \frac{1 - \beta_{N-1,N}}{\beta_{N,N}}.$$

D About the rank of a class of matrix

Lemma D.1. If A is a $K \times K$ matrix with coefficients $a_{ij} = 1$ if $i \neq j$ and $a_{ii} = 1 + \lambda_i$ with $\lambda_i \neq 0$ otherwise, then $|A| = 0$ if and only if $\sum_{i=1}^K \lambda_i^{-1} = -1$.

²From last rows/columns to bottom: $(c_{N-1} - c_{N-2}, l_{N-1} - l_{N-2}, c_{N-2} - c_{N-3}, \dots, c_1 - c_2, l_1 - l_2)$

Proof. $|A| = 0$ if and only if there exists K numbers $(\alpha_1, \dots, \alpha_N)$ such that for some i , $\alpha_i \neq 0$ and such that $\sum_{j=1}^N \alpha_j A^j = 0$, where A^j denotes the j th column of A . Thus we have

$$\alpha_i \lambda_i = - \sum_{j=1}^K \alpha_j, \quad i = 1, \dots, K.$$

By assumption, all λ_i are different from zero and for some $i \in \{1, \dots, K\}$ there exists $\alpha_i \neq 0$. This implies first that $\sum_{j=1}^K \alpha_j \neq 0$ and second that in fact all α_i are different from 0. We get the identity

$$\frac{-\alpha_i}{\sum_{j=1}^K \alpha_j} = \frac{1}{\lambda_i}, \quad i = 1, \dots, K.$$

Summing we obtain the announced result. \square

Lemma D.2. *Let A is a $K \times K$ matrix with coefficients $a_{ij} = 1$ if $i \neq j$ and $a_{ii} = 1 + \lambda_i$ with $\lambda_i \neq 0$ otherwise. If $S = 1 + \sum_{i=1}^K \lambda_i^{-1} \neq 0$ then $B = A^{-1}$ exists with coefficients $b_{ij} = -\frac{1}{S\lambda_i\lambda_j}$ if $i \neq j$ and $b_{ii} = \frac{1}{\lambda_i} - \frac{1}{S\lambda_i^2}$ otherwise.*

Proof. Let $S_j = \sum_{i=1}^K b_{ij}$, if B^j denotes the j th column of B , identity $AB^j = e_j$ show that

$$\begin{cases} S_j + \lambda_i b_{ij} &= 0 & \text{if } i \neq j \\ S_j + \lambda_j b_{jj} &= 1 & \text{otherwise.} \end{cases}$$

Thus $b_{ij} = -S_j/\lambda_i$ if $i \neq j$ and $b_{jj} = (1 - S_j)/\lambda_j$. Summing and equating, we find that $S_j = 1/(S\lambda_j)$ giving the announced result. \square