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Introduction to symplectic mechanics

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Introduction to symplectic mechanics

Z Aidni Azeddine

Abstract : The aim of the present lecture note is to provide some basic notions of Hamiltonian mechanics and their relationship with symplectic geometry. The introduction is devoted to present the historical appearance of Lagrangian and Hamiltonian mechanics by using simple examples. The second section discusses the basic notions on manifolds as the phase space of a dynamical system is naturally a manifold. This manifold has a special structure known as symplectic structure. The mathematical objects we use to treat the symplectic structure are essentially the differential forms. The qualities of these forms will be discussed in sections three and four, with important concepts such as exterior and interior product, exterior differentiation, integration of forms, and so on. In section five, we shall discuss the features of a symplectic manifold as well as some useful concepts for dealing with the Hamiltonian structure, such as the Lie derivative and Poisson bracket. In section six, we present a numerical scheme for Hamilton's equations that can provide meaningful simulations in phase space. Section seven provides several examples from various disciplines that demonstrate the usefulness of the Hamiltonian formalism. This technique is fundamental and applicable in a variety of physics fields, including classical mechanics, fluid mechanics, plasma physics, optics, quantum mechanics, chemistry, and so on.

March 17, 2022

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1 Introduction

In the 18th century the only way to describe the dynamics of a system was with Newton's equation ($\vec{F} = m\vec{a}$). As a vector equation, we need a reference frame, with axes, to project the position, velocity and force vectors. In the case of a complex system the writing of Newton's second law becomes complicated. We have to think of another approach. Lagrange had the idea to write the relations on functions and quantities that characterize the system. Provided that these quantities are independent, their number corresponds to the degrees of freedom of the system. Also the functions that must be used are homogeneous to an energy. This approach allowed Lagrange to describe and solve partially the problem of motion of the moon around the earth disturbed by the sun.

1.1 Lagrangian formalism

At the time of Lagrange, the physicists only consider the kinetic energy called living force (or vis viva in Latin). By generalizing the calculations of celestial mechanics¹, Lagrange discovered that there exist a function V , of the generalized coordinates, such that its partial derivatives with respect to the generalized coordinates corresponded to the force exerted on the particle. With these new equations, Lagrange realized that the writing of the dynamic evolution of a system became much simpler. He transmitted these results to Euler, who explained to him that these equations of motion corresponded to the minimum of a quantity. This quantity would be called "Action".

In mechanics, we often study the path of a particle. For example, let us consider a particle of constant mass \mathbf{m} in rectilinear motion. We are interested in its path and thus in its position x as a function of time.

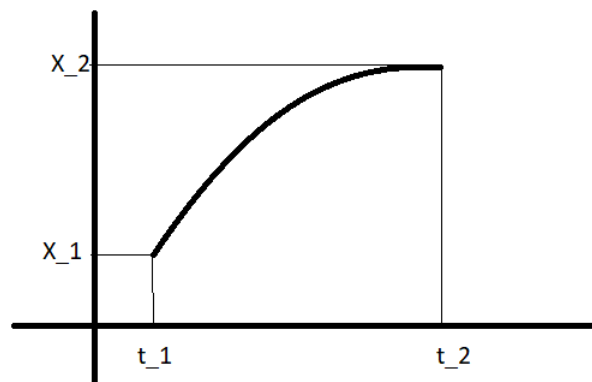


Figure 1:

Particle's velocity is given by:

$$v(t) = \frac{dx(t)}{dt} = \dot{x}(t)$$

¹Celestial mechanics describes the motion of astronomical objects such as stars and planets using physical and mathematical theories.

The evolution of the particle is given by Newton's second law. But the question is whether there is a simpler way to describe this equation.

Lagrange's idea is that there exists a function that depends only on the position. Such that its derivative with respect to the position is equal to the total force acting on the particle. We can write this equation in a more convenient way:

$$m \frac{dv(t)}{dt} = F = \frac{dV(x)}{dx}$$

The quantity V is homogeneous to an energy. Therefore Lagrange was the first to introduce "the potential energy V ".

On the other hand we can write the left term of the equation $m \frac{dv(t)}{dt}$ using kinetic energy, indeed:

$$m \frac{dv(t)}{dt} = \frac{d(mv(t))}{dt} = \frac{d}{dt} \left[\frac{d}{dv} \left(\frac{1}{2} m v^2 \right) \right] = \frac{d}{dt} \left[\frac{dT(v)}{dv} \right]$$

Where $T(v) = \frac{1}{2} m v^2 = \underline{\text{kinetic energy}}$

Then:

$$\frac{dV(x)}{dx} = F = m \frac{dv(t)}{dt} = \frac{d}{dt} \left[\frac{dT(v)}{dv} \right]$$

$$\frac{d}{dt} \left[\frac{dT(v)}{dv} \right] - \frac{dV(x)}{dx} = 0$$

We introduce the Lagrangian:

$$\mathcal{L}(x, v) = T - V$$

Since $\frac{dT(v)}{dx} = 0$ and $\frac{dV(x)}{dv} = 0$, the fundamental equation of dynamics can be written in a more compact form. The following equation, called Lagrange equation:

$$\frac{d}{dt} \left[\frac{d\mathcal{L}}{dv} \right] - \frac{d\mathcal{L}}{dx} = 0$$

We can generalize it to the 3-dimensional case:

$$\begin{cases} \frac{d}{dt} \left[\frac{d\mathcal{L}}{dv_x} \right] - \frac{d\mathcal{L}}{dx} = 0 \\ \frac{d}{dt} \left[\frac{d\mathcal{L}}{dv_y} \right] - \frac{d\mathcal{L}}{dy} = 0 \\ \frac{d}{dt} \left[\frac{d\mathcal{L}}{dv_z} \right] - \frac{d\mathcal{L}}{dz} = 0 \end{cases}$$

These equations have many advantages. Indeed, we can write the equations that govern the dynamics of the system, by writing a Lagrange equation for each degree of freedom.

Let's imagine a system composed of N particles in a 3-dimensional space. And let's suppose that there is no interaction between these particles. As each particle has three degrees of freedom in space. The system has in total $l = 3N$ degrees of freedom $(x_1, y_1, z_1, \dots, x_N, y_N, z_N)$.

We suppose now that we have constraints between these particles. which will lower the number of degrees of freedom of the system i.e. $1 \leq l \leq 3N$. As an example we can imagine 2 particles linked by a rod of negligible mass. This system has five degrees of freedom (3 translations and 2 rotations (Figure 2)).

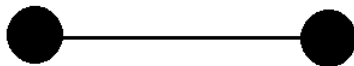


Figure 2:

Another example is the double pendulum, which has 2 degrees of freedom, corresponding to the angles of each rod with respect to the reference direction (Figure 3).

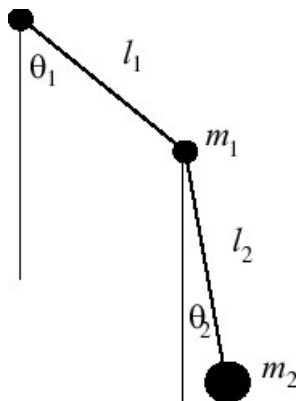


Figure 3:

If the system admits only l degrees of freedom. We can introduce independent coordinates, called generalized coordinates, noted (q_1, q_2, \dots, q_l) . The position of each particle depends only on these l generalized coordinates.

Note that a generalized coordinate does not necessarily have the dimensions of a length. For example, in the case of double pendulums, the generalized coordinates are the 2 angles (θ_1, θ_2) (Figure 3).

We can also define their generalized velocities $(\dot{q}_1, \dot{q}_2, \dots, \dot{q}_l)$. Hence, we can write the dynamics of the system, by writing a Lagrangian equation for each degree of freedom i.e. by writing a Lagrangian equation for each pair (q_k, \dot{q}_k) :

$$\begin{cases} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0 \\ k \in \{1, \dots, l\} \end{cases}$$

Let's now look at the Lagrange equations from another angle. Returning to the simple case of a constant mass particle in one dimension under a potential energy V . Figure (1) represents the real path followed by the particle. On the other hand there are many other possible paths which connect the initial and final points at the initial and final instant respectively $(x_1 = x(t_1) \rightarrow x(t_2) = x_2)$. We can pose a question: Given the couples of point $(x_1(t), x_2(t))$ and the potential energy, what is the real path followed by the particle? This path can be deduced by a variational principle, called the principle of Least Action.

Lagrange introduced the concept of action defined as:

$$S[x] = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t) dt$$

Theorem 1.1. Least Action Principle:

The path truly followed by the particle is the one that makes the action extreme i.e.

$$\delta S[x] = 0$$

1.2 Hamiltonian formalism

This formalism was also developed by Lagrange, to study celestial mechanics. Indeed, Lagrange had succeeded in writing the equations for the motion of the planets in the form of a first order equation. Known as "planetary equation", which are in principle easier to solve compared to Lagrange's equations which are of second order in t :

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right] - \frac{\partial \mathcal{L}}{\partial q_k} = 0 \quad (1)$$

We transform the previous equation (1) into a first order system in t . To do this, Lagrange introduced a new quantity called "Impulsion" defined as follows:

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k}, \quad \dot{p}_k = \frac{\partial \mathcal{L}}{\partial q_k}$$

Let develop the differential of the Lagrangian function \mathcal{L} :

$$\begin{aligned} d\mathcal{L} &= \sum_{k=1}^l \frac{\partial \mathcal{L}}{\partial q_k} dq_k + \sum_{k=1}^l \frac{\partial \mathcal{L}}{\partial \dot{q}_k} d\dot{q}_k \\ &= \sum_{k=1}^l \dot{p}_k dq_k + \sum_{k=1}^l p_k d\dot{q}_k \\ &= \sum_{k=1}^l \dot{p}_k dq_k + d \left(\sum_{k=1}^l p_k \dot{q}_k \right) - \sum_{k=1}^l \dot{q}_k dp_k \end{aligned}$$

This was an important moment in the history of science, when Lagrange decided to consider the following quantity:

$$\mathcal{H} \equiv \sum_{k=1}^l p_k \dot{q}_k - \mathcal{L} \quad (2)$$

This quantity is the Legendre transformation of the function \mathcal{L} . The function \mathcal{H} called "**The Hamiltonian**". We will see later, that the Hamiltonian represents in most cases the total energy of the system, which is a fundamental quantity in physics.

We present Hamilton's equation:

$$d\mathcal{H} = d \left(\sum_{k=1}^l p_k \dot{q}_k - \mathcal{L} \right) = - \sum_{k=1}^l \dot{p}_k dq_k + \sum_{k=1}^l \dot{q}_k dp_k$$

Then $\forall k = 1, \dots, l$

$$\begin{aligned}\dot{p}_k &= -\frac{\partial \mathcal{H}}{\partial q_k} \\ \dot{q}_k &= +\frac{\partial \mathcal{H}}{\partial p_k}\end{aligned}\tag{3}$$

Returning to the simple example: {1 particle, 1 dimension, under $V(x)$ }. In this case, we have :

$$\begin{aligned}q &= x \quad , \quad \mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x) \\ p &= \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}\end{aligned}$$

NB: This impulsion coincides with the momentum ($m\vec{v}$).

Let compute the Hamiltonian:

$$\begin{aligned}\mathcal{H} &= p\dot{x} - \mathcal{L} = m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 + V(x) = \frac{1}{2}m\dot{x}^2 + V(x) \\ &= \text{Kinetic energy} + \text{Potential energy}\end{aligned}$$

Thus the Hamiltonian represents the total energy of the particle.

Using the expression $p = m\dot{x}$, we can write the kinetic energy term as a function of the momentum:

$$\mathcal{H} = \frac{p^2}{2m} + V(x).$$

The previous expression has a capital importance in quantum physics. The description of system is done here by the variables position and impulse, not by the variables position and velocity. This remark is fundamental to pass to the formalism of quantum physics.

1.2.1 Canonical Hamilton's equations

Hamilton's equations can be summarized as follows:

$$\begin{aligned}d\mathcal{H} &= \sum_i \left[\left(\frac{\partial \mathcal{H}}{\partial q_i} \right) dq_i + \left(\frac{\partial \mathcal{H}}{\partial p_i} \right) dp_i \right] + \left(\frac{\partial \mathcal{H}}{\partial t} \right) dt \\ d\mathcal{H} &= \sum_i \left[\dot{q}_i dp_i + p_i d\dot{q}_i - \left(\frac{\partial \mathcal{L}}{\partial q_i} \right) dq_i - \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) d\dot{q}_i \right] - \left(\frac{\partial \mathcal{L}}{\partial t} \right) dt \\ \dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i} \quad ; \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \quad ; \quad \frac{\partial \mathcal{H}}{\partial t} = \frac{d\mathcal{H}}{dt} = -\frac{\partial \mathcal{L}}{\partial t}\end{aligned}$$

Note: the equality $\frac{d\mathcal{H}}{dt} = -\frac{\partial \mathcal{L}}{\partial t}$ is shown as follows:

$$\frac{d\mathcal{H}}{dt} = \frac{d(p_i \dot{q}_i - \mathcal{L})}{dt} = p_i \ddot{q}_i + \dot{p}_i \dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i - \frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

Many important results follow from the theorem on the equivalence of the equations of motion to a Hamiltonian system. For example, the energy conservation law takes the simple form :

corollary 1.1.

$$d\mathcal{H}/dt = \partial\mathcal{H}/\partial t.$$

In particular, for a system whose Hamiltonian function does not depend explicitly on time ($\partial\mathcal{H}/\partial t = 0$). The conservation law of the Hamiltonian function is given:

$$\mathcal{H}(\mathbf{p}(t), \mathbf{q}(t)) = \text{constant}$$

Demonstration.

We consider the variation of \mathcal{H} along the path $(\mathbf{p}(t), \mathbf{q}(t), t)$. Then, by Hamilton's equations:

$$\frac{d\mathcal{H}}{dt} = \frac{\partial\mathcal{H}}{\partial\mathbf{p}} \left(-\frac{\partial\mathcal{H}}{\partial\mathbf{q}} \right) + \frac{\partial\mathcal{H}}{\partial\mathbf{q}} \frac{\partial\mathcal{H}}{\partial\mathbf{p}} + \frac{\partial\mathcal{H}}{\partial t} = \frac{\partial\mathcal{H}}{\partial t}.$$

1.2.2 Liouville theorem

This theorem says that a volume of the phase space is constant along the trajectories of the system. In other words this volume remains constant in time.

To simplify, we consider the case where the Hamiltonian function does not depend explicitly on time: $\mathcal{H} = \mathcal{H}(\mathbf{p}, \mathbf{q})$.

Definition 1.1. The space of dimension $2n$ and coordinate system $p_1, \dots, p_n, q_1, \dots, q_n$ is called phase space.

Example 1.1.

If we take the simple example {1 particle, 1 dimension, under $V(x)$ },

$$q = x \quad \text{et} \quad \mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x)$$

$$p = \frac{\partial\mathcal{L}}{\partial\dot{x}} = m\dot{x}$$

We have:

$$n = 1 \quad \text{et} \quad (\mathbf{p}, \mathbf{q}) = (\dot{\mathbf{x}}, \mathbf{x})$$

Definition 1.2. (phase flow)

The phase flow is the group of transformations of the phase space defined as follows:

$$g^t : (\mathbf{p}(0), \mathbf{q}(0)) \mapsto (\mathbf{p}(t), \mathbf{q}(t))$$

Where $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are solutions of the system of Hamilton equations.

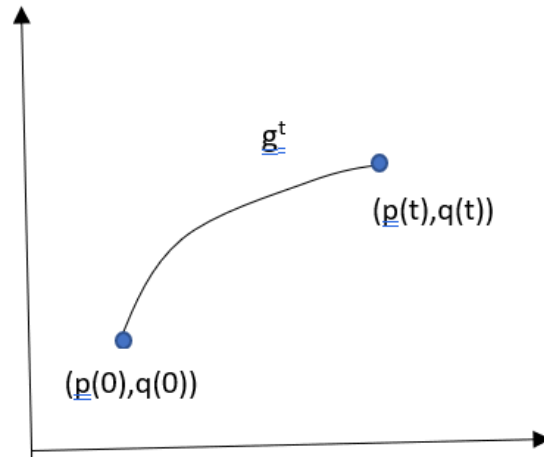


Figure 4:

Theorem 1.2. (Liouville theorem)

The phase flow preserves the volume i.e. for any region D we have:

$$\text{volume of } g^t(D) = \text{volume of } D.$$

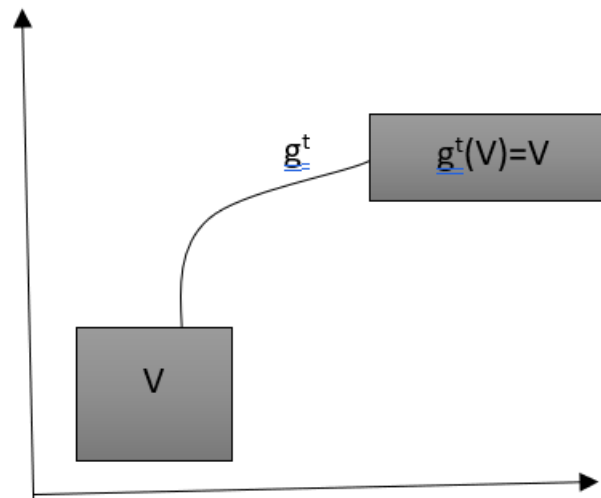


Figure 5:

Remark 1.

Liouville's theorem amounts to saying that the volume V of the phase space is time invariant:

$$\frac{dV}{dt} = 0$$

The proof of this theorem will be trivial by using Transport theorem and the Stokes formula that we will see later.

2 Manifold and sub-manifold

Hamiltonian mechanics cannot be understood without knowing some basic notions about differential manifolds. The pre-requisites we will need on manifolds are:

- Differential sub-manifold,
- Differential application,
- Tangent space,
- Tangent bundle, and cotangent bundle,
- Curves.

We will see that we can treat the notion of differentiability and integration on some specific topological spaces which are not necessarily normed vector spaces, this type of spaces called differential manifold.

2.1 Differential manifold

Definition 2.1. A topological manifold M of dimension n is a Hausdorff topological space with countable basis, such that each of its points has an open neighborhood homeomorphic to an open of the topological vector space \mathbb{R}^n .

More precisely :

$\forall x \in M$, there exists an open neighborhood U_x and a homeomorphism:

$$\varphi_x : U_x \longrightarrow \varphi_x(U_x) \subset \mathbb{R}^n$$

We say that (U_x, φ_x) is a local chart of M .

A family of charts $(U_i, \varphi_i)_i$ that covers (entirely) M forms an atlas of the manifold M .

NB: A space is called space with countable basis if its topology admits a countable basis, i.e. it is written as a union of countable opens:

$$M = \bigcup_{i \in I} U_i$$

Many of the usual spaces in analysis and many spaces in functional analysis are spaces with countable basis.

Definition 2.2. M is a differential manifold of dimension m if:

- M is a Hausdorff topological space with countable basis.
- M is provided with a family of pairs $\{(U_i, \phi_i)\}$.
- $\{U_i\}_i$ is a family of opens that covers M , i.e. $\cup_i U_i = M$.
- for all i : $\phi_i : U_i \rightarrow \phi_i(U_i) \subset \mathbb{R}^m$ is a homeomorphism.
- for all i and j such that $U_i \cap U_j \neq \emptyset$, then **the transition chart** $\psi_{ij} = \phi_i \circ \phi_j^{-1}$ from $\phi_j(U_i \cap U_j)$ to $\phi_i(U_i \cap U_j)$ is in \mathcal{C}^∞ -class.

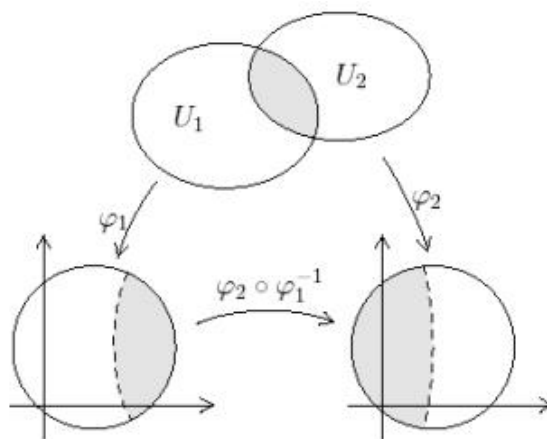


Figure 6:

Remark 2.

Two atlases are said equivalent if their union is also an atlas, the class of equivalent atlases defines what we call a **differential structure**.

Example 2.1. We give some examples of differential manifold:

- **The vector space \mathbb{R}^n :**
 \mathbb{R}^n with the only chart $(Id_{\mathbb{R}^n} : x \mapsto x, \mathbb{R}^n)$ is a differential manifold of dimension n .
- **The sphere S^n :**
 For each integer n , we denote by S^n the unit sphere of \mathbb{R}^{n+1} :

$$S^n = \{(x_1, \dots, x_n, x_{n+1}) \mid x_1^2 + \dots + x_n^2 + x_{n+1}^2 = 1\}$$

and, let us denote by $(e_i)_{1 \leq i \leq n+1}$ the canonical basis of \mathbb{R}^{n+1} and by E_n the hyperplane of \mathbb{R}^{n+1} defined by the equation $x_{n+1} = 0$ (E_n is called the equatorial hyperplane of S^n), we naturally identify the equatorial hyperplane by \mathbb{R}^n .

The point $N = (0, \dots, 0, 1)$ (resp. $S = (0, \dots, 0, -1)$), called north pole (resp. south pole) of the sphere S^n .

We consider the cover of the sphere S^n defined by the opens U_N and U_S defined by:

$$U_N = S^n - \{N\} \text{ and } U_S = S^n - \{S\}$$

We call stereographic projection of the North Pole, the mapping φ_N of U_N to an image in the equatorial hyperplane E_n , associating to an element M of $S^n - \{N\}$, the intersection $\varphi_N(M)$ of the line (NM) with the equatorial hyperplane E_n :

$$\{\varphi_N(M)\} = (NM) \cap E_n$$

In the same way, stereographic projection of the South Pole is the mapping φ_S of U_S onto an image in the equatorial hyperplane E_n , associating to an element M of $S^n - \{S\}$, the intersection $\varphi_S(M)$ of the line (SM) with the equatorial hyperplane E_n :

$$\{\varphi_S(M)\} = (SM) \cap E_n$$

Thus,

$$\varphi_N(x_1, \dots, x_n, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right)$$

for all $(x_1, \dots, x_n, x_{n+1}) \in U_N$, and

$$\varphi_S(x_1, \dots, x_n, x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}}, \dots, \frac{x_n}{1 + x_{n+1}} \right)$$

The set $\{\varphi_N; \varphi_S\}$ defines an atlas of the sphere S^n , which gives the sphere S^n a structure of differential manifold of dimension n .

• **Union of non-vertical and non-horizontal affine lines of \mathbb{R}^2 :**

The equation of a non vertical affine line is written $y = ax + b$, with $a, b \in \mathbb{R}$, this line is completely determined by the choice of the pair (a, b) . We denote by U_{nv} the set of non-vertical lines. Similarly, the equation of a non-horizontal affine line is written $x = \tilde{a}y + \tilde{b}$, with $\tilde{a}, \tilde{b} \in \mathbb{R}$, this line is completely determined by the choice of the couple (\tilde{a}, \tilde{b}) .

We denote by U_{nh} the set of non horizontal lines. We note:

$$DA(2) = U_{nv} \cup U_{nh}$$

A line which is both non-vertical and non-horizontal, i.e, an element of $DA(2)$, is written both $y = ax + b$, where $(a, b) \in \mathbb{R}^* \times \mathbb{R}$ and

$$x = \tilde{a}y + \tilde{b}, \text{ avec } (\tilde{a}, \tilde{b}) \in \mathbb{R}^* \times \mathbb{R}$$

The local coordinate systems (a, b) et (\tilde{a}, \tilde{b}) of a same line are linked by the relation:

$$\tilde{a} = \frac{1}{a} \text{ et } \tilde{b} = -\frac{b}{a}$$

Here, $DA(2)$ is a differential manifold of dimension 2.

• **Real projective space $\mathbb{R}P(n)$:**

Let's provide the open $\mathbb{R}^{n+1} - \{0\}$, by the following equivalence relation: two points u and v of $\mathbb{R}^{n+1} - \{0\}$ are equivalents, and we write $u \sim v$, if and only if, there exists $\lambda \in \mathbb{R}^*$, such that $u = \lambda v$, in other words, $u \sim v$ if and only if, they define the same vector line.

The relation " \sim " is an equivalence relation, the quotient space is called real projective space of dimension n , and is denoted by $\mathbb{R}P(n)$:

$$\mathbb{R}P(n) = \frac{\mathbb{R}^{n+1} - \{0\}}{\sim}$$

For all $i = 1, \dots, n + 1$, we denote U_i the open of \mathbb{R}^{n+1} defined by:

$$U_i = \{(x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_i \neq 0\}$$

$(U_i)_{1 \leq i \leq n+1}$ is a family of opens that covers $\mathbb{R}^{n+1} - \{0\}$. We denote by $\pi : \mathbb{R}^{n+1} - \{0\} \rightarrow \mathbb{R}P(n)$ the canonical surjection, and \bar{U}_i the image $\pi(U_i)$ of U_i by π .

The real projective space $\mathbb{R}P(n)$ with the quotient topology is covered by the family $(\bar{U}_i)_{1 \leq i \leq n+1}$.

For all $i = 1, \dots, n + 1$, we denote φ_i the mapping of U_i onto \mathbb{R}^n defined by:

$$\varphi_i(x_1, \dots, x_n, x_{n+1}) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i} \right),$$

for all $(x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$, and we denote $\bar{\varphi}_i$ the mapping of \bar{U}_i onto \mathbb{R}^n defined by:

$$\bar{\varphi}_i(\bar{u}) = \varphi_i(u)$$

for all $\bar{u} \in \bar{U}_i$, where \bar{u} is the class of u modulo the equivalence relation \sim . $\bar{\varphi}_i$ define a bijective maps, the family $(\bar{\varphi}_i)_{1 \leq i \leq n+1}$ gives to the real projective space a structure of differential manifold of dimension n .

2.2 Submanifold

Definition 2.3. Let M be a differential manifold of dimension n , a subset X of M called a submanifold of dimension $k \leq n$ if any point of X is contained in the domain of a chart $\varphi : U \hookrightarrow \mathbb{R}^n$ such that $\varphi(X \cap U) = \varphi(U) \cap (\mathbb{R}^k \times \{0\}) \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$.

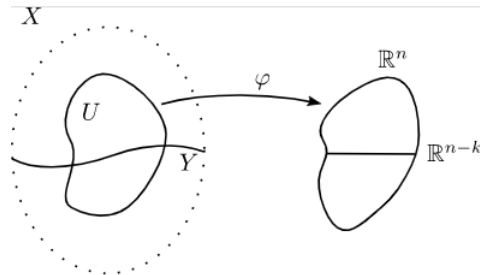


Figure 7:

Example 2.2. the circle S^1 is a submanifold of dimension 1 of the manifold \mathbb{R}^2 .

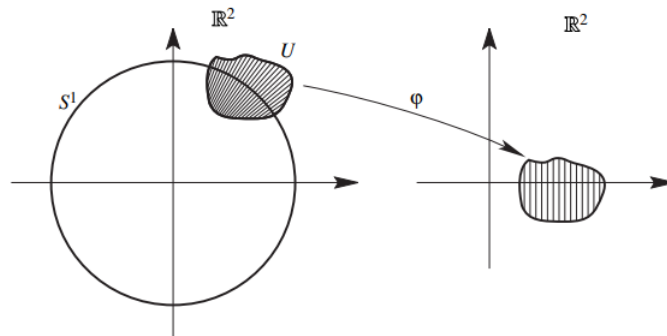


Figure 8:

2.3 Differential map

Definition 2.4. Let X and Y two differential manifolds of dimension n and p respectively.

A map $f : X \rightarrow Y$ is differential if, for any x in X , there are charts $\varphi : U \rightarrow \mathbb{R}^n$ and $\psi : V \rightarrow \mathbb{R}^p$ around of x and $f(x)$ respectively, such that $\psi \circ f \circ \varphi^{-1}$ is differential.

The map f called smooth if $\psi \circ f \circ \varphi^{-1}$ in the class C^∞ .

the map f is differential on an open set U of X , if it is differential at all point of U .

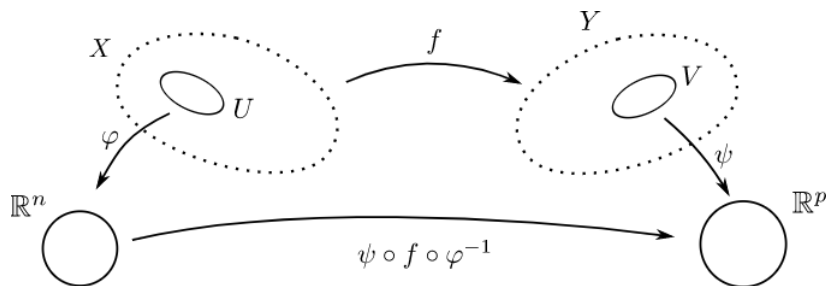


Figure 9:

Remark 3.

- In the previous definition, the differentiability does not depend on the chart, indeed, Let (U_1, φ_1) be another chart at the point x and (V_2, ψ_2) another chart at the point $f(x)$. The local expression

$$\psi_2 \circ f \circ \varphi_1^{-1} : \varphi_1(U_1 \cap V_2) \longrightarrow \psi_2(U_1 \cap V_2)$$

of f in the charts ψ_1 and ψ_2 is written as:

$$\psi_2 \circ f \circ \varphi_1^{-1} = (\psi_2 \circ \psi_1^{-1}) \circ (\psi_1 \circ f \circ \varphi_1^{-1}) \circ (\varphi_1 \circ \varphi_1^{-1})$$

The changes of charts $(\varphi \circ \varphi_1^{-1})$ and $(\psi_2 \circ \psi^{-1})$ are differential by definition, the local expression $(\varphi_2 \circ f \circ \varphi_1^{-1})$ of the map f in the charts φ_1 and φ_2 are differentials at the point $\varphi_1(x)$, therefore, $\psi_2 \circ f \circ \psi_1^{-1}$ is differential at the point $\psi_1(x)$.

- By the same argument, the rank of the linear mapping $d(\psi \circ f \circ \varphi^{-1})_{\varphi(x)}$ does not depend on local charts φ and ψ , called **the rank of f** at point x , and we note that $\mathbf{rg}_x(\mathbf{f})$, Moreover, we have:

$$\mathbf{rg}_x(\mathbf{f}) \leq \min(\mathbf{n}, \mathbf{p})$$

Definition 2.5. Let X and Y two differential manifolds of dimension n and p respectively.

A differential map $f : X \rightarrow Y$ is called:

- **An immersion** if $\forall x \in X, \mathbf{rg}_x(\mathbf{f}) = \mathbf{n}$.
- **A submersion** if $\forall x \in X, \mathbf{rg}_x(\mathbf{f}) = \mathbf{p}$.

2.4 Tangent space, tangent bundle

We can define the vectors of the tangent space at a point \mathbf{x} of a manifold M , as velocity vectors of curves in M :

$$\mathbf{v} = \lim_{t \rightarrow 0} \frac{\gamma(t) - \gamma(0)}{t} \text{ where } \gamma(0) = \mathbf{x}, \gamma(t) \in M$$

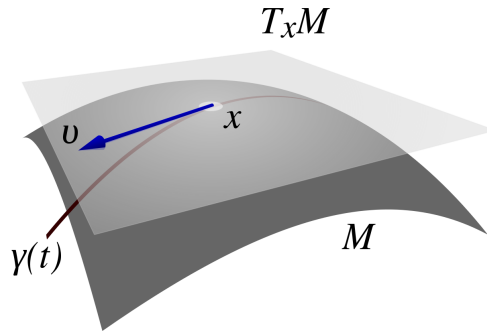


Figure 10:

The definition of tangent vectors is also intrinsic. The space formed by these vectors does not depend on the choice of chart.

Two curves $\mathbf{x} = \varphi(t)$ and $\mathbf{x} = \psi(t)$ are equivalents if $\varphi(0) = \psi(0) = \mathbf{x}$, and $\lim_{t \rightarrow 0} (\varphi(t) - \psi(t))/t = 0$ for a chart contains x , This relation is an equivalence relation, in the following we note this equivalence relation \mathcal{R}_c .

Definition 2.6. A tangent vector of a manifold M at a point \mathbf{x} is an equivalence class of the relation:

$$\{ \varphi \mathcal{R}_c \psi \text{ if and only if } \varphi(0) = \psi(0) = \mathbf{x}. \text{ and } \lim_{t \rightarrow 0} (\varphi(t) - \psi(t))/t = 0 \}$$

The tangent space TM_x of a manifold M at a point x is the set of tangent vectors at this point, i.e. TM_x is the quotient by the equivalence relation \mathcal{R}_c .

The tangent bundle TM of a manifold M is the disjoint union of TM_x for all points x of M i.e :

$$TM = \bigcup_{x \in M} TM_x$$

Remark 4.

- The tangent bundle TM has a structure of a differential manifold, its dimension is twice the dimension of M .
- For all $\mathbf{x} \in M$, TM_x is a vector space, the dual of this space called Cotangent space noted TM_x^* .
- A point of TM is a vector ξ , tangent to M at a point \mathbf{x} . The local coordinates on TM are constructed as follows: Let q_1, \dots, q_n be local coordinates on M , and ξ_1, \dots, ξ_n components of a tangent vector in this coordinate system. Then the numbers $2n$ ($q_1, \dots, q_n, \xi_1, \dots, \xi_n$) give a local coordinate system on TM . Sometimes, we write dq_i for ξ_i .

2.5 Riemannian manifold

If M is a manifold included in a Euclidean space, then we can measure the lengths of the curves, the angles between the vectors, the volumes, etc. All these quantities are expressed by the lengths of the tangent vectors, i.e. by the positive definite quadratic form given on each tangent space TM_x . (Figure11):

$$TM_x \rightarrow \mathbb{R} \quad \xi \rightarrow \langle \xi, \xi \rangle.$$

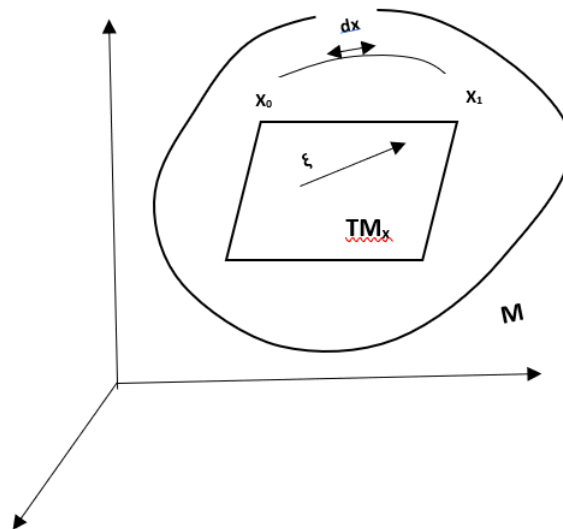


Figure 11:

For example, we can express the length on a manifold between \mathbf{x}_0 and \mathbf{x}_1 using:

$$L(\gamma) = \int_{\mathbf{x}_0}^{\mathbf{x}_1} \sqrt{\langle d\mathbf{x}, d\mathbf{x} \rangle}.$$

or if the curve is determined by a parametric function:

$$\gamma : [t_0, t_1] \rightarrow M, t \rightarrow \mathbf{x}(t) \in M.$$

Then,

$$L(\gamma) = \int_{t_0}^{t_1} \sqrt{\langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle} dt$$

Definition 2.7. A Riemannian manifold is a differential manifold with a positive definite quadratic form $\mathbf{q}(\xi) = \langle \xi, \xi \rangle$ on each tangent space $TM_{\mathbf{x}}$. The quadratic form is called the metric tensor.

Example 2.3.

Let the Euclidean space $M = \mathbb{R}^n$, with the euclidean scalar product:

$$\langle u = u^i e_i, v = v^j e_j \rangle = \delta_{ij} u^i v^j = \sum_{i=1}^n u^i v^i.$$

Let $x \in M$ with coordinates (q_1, \dots, q_n) , an element ξ of TM_x has coordinates (ξ_1, \dots, ξ_n) , so we define the quadratic form as follows:

$$\mathbf{q}(\xi) = \langle (\xi_1, \dots, \xi_n), (\xi_1, \dots, \xi_n) \rangle = \sum_{i=1}^n \xi_i^2$$

(M, \mathbf{q}) is Riemannian manifold.

Definition 2.8. (Tangent map)

Let $f : M \rightarrow N$ be differential map at a point $\mathbf{x} \in M$ the tangent map of f :

$$f_{*\mathbf{x}} : TM_{\mathbf{x}} \rightarrow TN_{f(\mathbf{x})}$$

is defined as follows (figure 12):

Let $\mathbf{v} \in TM_{\mathbf{x}}$. We consider a curve $\varphi : \mathbb{R} \rightarrow M$ where $\varphi(0) = \mathbf{x}$, and the velocity vector $\mathbf{v} = \left. \frac{d\varphi}{dt} \right|_{t=0}$.

Then $f_{*\mathbf{x}}\mathbf{v}$ is the velocity vector of the curve $f \circ \varphi : \mathbb{R} \rightarrow N$

$$f_{*\mathbf{x}}(\mathbf{v}) = \left. \frac{df(\varphi(t))}{dt} \right|_{t=0}$$

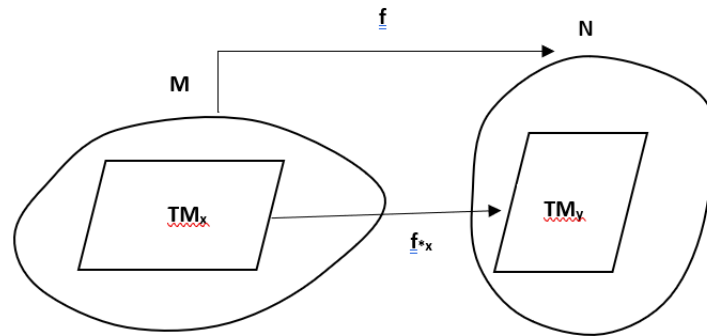


Figure 12:

Remark 5.

- The vector $f_{*x}(\mathbf{v})$ does not depend on the curve φ , but only on the vector \mathbf{v} .
- The tangent map is a linear application.
- When $M = \mathbb{R}^n$ and $N = \mathbb{R}^p$ the tangent application coincides with the usual definition of the differential of the function f , and for all $\mathbf{h} \in \mathbb{R}^n$:

$$f_{*x}(\mathbf{h}) = df(x) \cdot \mathbf{h}$$

3 Differential Form

Hamiltonian mechanics cannot be understood without differential forms, the notions we will require about differential forms are:

- Exterior product,
- Exterior differentiation,
- Interior product,
- Integration, Stokes formula.

3.1 Exterior form

Let E be a real vector space of dimension n . We will denote the vectors of this space by ξ, η, \dots

3.1.1 1-form

Definition 3.1. An exterior form of degree 1 (or 1-form) is a linear application $\omega : E \rightarrow \mathbb{R}$, i.e.

$$\omega(\lambda_1 \xi_1 + \lambda_2 \xi_2) = \lambda_1 \omega(\xi_1) + \lambda_2 \omega(\xi_2), \quad \lambda_1, \lambda_2 \in \mathbb{R}, \text{ and } \xi_1, \xi_2 \in E$$

If we define the addition of two forms by:

$$(\omega_1 + \omega_2)(\xi) = \omega_1(\xi) + \omega_2(\xi)$$

and a multiplication with a scalar by:

$$(\lambda\omega)(\xi) = \lambda\omega(\xi)$$

Then the set of all 1-forms is a real vector space denoted E^* .

Remark 6.

- The space of 1-forms on E is also a vector space of dimension n , called the dual space of E . Let $(e_i^*)_{1 \leq i \leq n}$ the dual basis of the canonical basis $(e_i)_{1 \leq i \leq n}$ of E , then each 1-form is written in the form:

$$\omega = a_1 e_1^* + \dots + a_n e_n^*, \quad a_i \in \mathbb{R}$$

- The value of ω at a point ξ is given by :

$$\omega(\xi) = a_1 e_1^*(\xi) + \dots + a_n e_n^*(\xi)$$

$$\omega(\xi) = a_1 x_1 + \dots + a_n x_n$$

where x_1, \dots, x_n are the coordinates of ξ in the basis $(e_i)_{1 \leq i \leq n}$

3.1.2 2-form

Definition 3.2. An exterior form of degree 2 (or a 2-form) is an application $\omega^2 : E \times E \rightarrow \mathbb{R}$, bilinear and antisymmetric :

$$\omega^2 (\lambda_1 \xi_1 + \lambda_2 \xi_2, \xi_3) = \lambda_1 \omega^2 (\xi_1, \xi_3) + \lambda_2 \omega^2 (\xi_2, \xi_3)$$

$$\omega^2 (\xi_1, \xi_2) = -\omega^2 (\xi_2, \xi_1)$$

$$\forall \lambda_1, \lambda_2 \in \mathbb{R}, \text{ and } \xi_1, \xi_2, \xi_3 \in \mathbb{R}^n$$

Example 3.1.

Let $S(\xi_1, \xi_2)$ the oriented area of the parallelogram formed by the vectors ξ_1 and ξ_2 of the oriented Euclidean plane \mathbb{R}^2 , i.e.,

$$S(\xi_1, \xi_2) = \begin{vmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{vmatrix}, \quad \text{tel que } \xi_1 = \xi_{11}\mathbf{e}_1 + \xi_{12}\mathbf{e}_2, \quad \xi_2 = \xi_{21}\mathbf{e}_1 + \xi_{22}\mathbf{e}_2$$

where $(\mathbf{e}_1, \mathbf{e}_2)$ is a basis of \mathbb{R}^2 . Then, $S(.,.)$ is a 2-form.

3.1.3 k-form

Definition 3.3. An exterior form of degree k , or a k -form, is an application $\omega : \underbrace{E \times \dots \times E}_{k\text{-times}} \rightarrow \mathbb{R}$ Multi-linear and antisymmetric:

$$\omega (\lambda_1 \xi'_1 + \lambda_2 \xi''_1, \xi_2, \dots, \xi_k) = \lambda_1 \omega (\xi'_1, \xi_2, \dots, \xi_k) + \lambda_2 \omega (\xi''_1, \xi_2, \dots, \xi_k)$$

For all permutation σ of $\{1, \dots, k\}$

$$\omega (\xi_{\sigma(1)}, \dots, \xi_{\sigma(k)}) = \epsilon(\sigma) \omega (\xi_1, \dots, \xi_k)$$

Where $\epsilon(\sigma)$ represents the signature of permutation σ

We denote space of k -forms by $\bigwedge^k E$

Remark 7.

$\bigwedge^k E$ is naturally provided with a real vector space structure. The direct sum $\bigwedge E = \bigoplus_{k=0}^n \bigwedge^k E$, where $\bigwedge^0 E = \mathbb{R}$ and $\bigwedge^1 E = E^*$, is called the exterior algebra of E .

Example 3.2.

Let be the determinant application:

$$\mathbf{det} (\xi_1, \dots, \xi_n) = \begin{vmatrix} \xi_{11} & \dots & \xi_{1n} \\ \vdots & & \vdots \\ \xi_{n1} & \dots & \xi_{nn} \end{vmatrix}$$

Where, $\xi_i = \xi_{i1}\mathbf{e}_1 + \dots + \xi_{in}\mathbf{e}_n$ and $\mathbf{e}_1, \dots, \mathbf{e}_n$ is a basis of E , $\mathbf{det}(., \dots, .)$ is a n -form.

3.2 Exterior product

Definition 3.4. The exterior product $\omega^k \wedge \omega^l$ of a k -form ω^k with a l -form ω^l in E is a $(k+l)$ -form in E such that the image of $k+l$ vectors $\xi_1, \dots, \xi_k, \xi_{k+1}, \dots, \xi_{k+l} \in \mathbb{R}^n$ is given by:

$$(1) \quad (\omega^k \wedge \omega^l)(\xi_1, \dots, \xi_{k+l}) = \sum \epsilon(\sigma) \omega^k(\xi_{i_1}, \dots, \xi_{i_k}) \omega^l(\xi_{j_1}, \dots, \xi_{j_l}),$$

where $i_1 < \dots < i_k, j_1 < \dots < j_l$, and $\sigma = (i_1, \dots, i_k, j_1, \dots, j_l)$ is a permutation of $(1, 2, \dots, k+l)$.

Example 3.3.

Exterior product of two 1-forms ω_1 et ω_2 is given by:

$$(\omega_1 \wedge \omega_2)(\xi_1, \xi_2) = \omega_1(\xi_1)\omega_2(\xi_2) - \omega_1(\xi_2)\omega_2(\xi_1) = \begin{vmatrix} \omega_1(\xi_1) & \omega_2(\xi_1) \\ \omega_1(\xi_2) & \omega_2(\xi_2) \end{vmatrix}$$

In the general case for the 1-forms $\omega_1, \dots, \omega_k$:

$$(\omega_1 \wedge \dots \wedge \omega_k)(\xi_1, \dots, \xi_k) = \begin{vmatrix} \omega_1(\xi_1) & \dots & \omega_k(\xi_1) \\ \vdots & & \vdots \\ \omega_1(\xi_k) & \dots & \omega_k(\xi_k) \end{vmatrix}$$

Remark 8.

For all $\xi \in E$ we have $\omega^2(\xi, \xi) = 0$, indeed, by the antisymmetry, $\omega^2(\xi, \xi) = -\omega^2(\xi, \xi)$.

We will see later that the form ω^2 has a special meaning in Hamiltonian mechanics. We can show that any non-degenerate form ² 2-form on \mathbb{R}^{2n} can be written as:

$$\omega^2 = \sum_{i=1}^n dp_i \wedge dq_i$$

in a coordinate system $(p_1, \dots, p_n, q_1, \dots, q_n)$ well chosen.

Proposition 3.1. The exterior product verifies the following properties:

i) Distributivity, if $\omega_1^k, \omega_2^k \in \wedge^k(E)$:

$$(\omega_1^k + \omega_2^k) \wedge \omega^l = \omega_1^k \wedge \omega^l + \omega_2^k \wedge \omega^l$$

ii) Anticommutativity:

$$\omega^k \wedge \omega^l = (-1)^{k \cdot l} \omega^l \wedge \omega^k$$

iii) Associativity, Let $\omega^m \in \wedge^m(E)$:

$$(\omega^k \wedge \omega^l) \wedge \omega^m = \omega^k \wedge (\omega^l \wedge \omega^m)$$

² ω^2 non-degenerate if the only vector x such that $\omega^2(x, y) = 0$ for all y in E is the null vector (in other words, only the null vector is orthogonal (according to ω) to all the vectors in space).

corollary 3.1. Every n -form on \mathbb{R}^n can be written:

$$\omega^n = a \cdot dx_1 \wedge \cdots \wedge dx_n$$

Remark 9.

Each 2-form on a space E of dimension n , can be represented in a unique way as:

$$\omega^2 = \sum_{i < j} a_{ij} e_i^* \wedge e_j^*$$

where e_1^*, \dots, e_n^* a dual basis of E

And in the general case each k -form is written

$$\omega^k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1, \dots, i_k} e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*$$

It is enough to take the basis of E $(e_i)_{1 \leq i \leq n}$. i.e, we find that $\omega(e_i, e_j) = a_{ij}$, likewise $a_{i_1, \dots, i_k} = \omega^k(e_{i_1}, \dots, e_{i_k})$.

Definition 3.5. (pullback of an exterior form).

Let E and F be two finite-dimensional vector spaces and $f : E \rightarrow F$ a linear application. Then for all k -form ω on the vector space F , there is an associated k -form on E , denoted $f^*\omega$ and called the pullback of ω under f , this k -form is defined by :

$$f^*\omega(x_1, \dots, x_p) = \omega(f(x_1), \dots, f(x_p))$$

for all $x_1, \dots, x_p \in E$.

3.3 Differential form

Let $f : M \rightarrow \mathbb{R}$ be differential map on a manifold M .

the tangent map $df|_x$ of f on \mathbf{x} is a linear application of the space TM_x in \mathbb{R} (Definition 2.8), also called the differential of f at point \mathbf{x} :

$$df_{\mathbf{x}} : TM_{\mathbf{x}} \rightarrow \mathbb{R}$$

Let $\xi \in TM_{\mathbf{x}}$, the velocity vector of the curve $\mathbf{x}(t) : \mathbb{R} \rightarrow M, \mathbf{x}(0) = \mathbf{x}$ and $\dot{\mathbf{x}}(0) = \xi$. Thus by definition:

$$df_{\mathbf{x}}(\xi) = \left. \frac{d}{dt} \right|_{t=0} f(\mathbf{x}(t))$$

Note that the differential df_x at a point $x \in M$ is a 1-form on the tangent space $TM_{\mathbf{x}}$.

The differential df of f on a manifold M is a smooth function (\mathcal{C}^∞) on the tangent bundle TM . This application is differential and linear on each tangent space $TM_{\mathbf{x}} \subset TM$

In the case where the manifold M is a vector space with coordinates x_1, \dots, x_n (i.e. M isomorphic to \mathbb{R}^n). The components ξ_1, \dots, ξ_n of a tangent vector $\xi \in T\mathbb{R}_{\mathbf{x}}^n$ are the values of the differentials dx_1, \dots, dx_n on the vector ξ . These n 1-forms on

$T\mathbb{R}_{\mathbf{x}}^n$ are linearly independent. Thus, the 1-forms dx_1, \dots, dx_n form a basis of the 1-form space of dimension n ($dx_i(h) = e_i^*(h) = h_i$ ou $h = (h_1, \dots, h_n) \in (T\mathbb{R}_{\mathbf{x}}^n)$), and each 1-form on $T\mathbb{R}_{\mathbf{x}}^n$ can be uniquely written in the form $a_1 dx_1 + \dots + a_n dx_n$, where a_i are real coefficients that depend on \mathbf{x} .

When the functions a_i depend smoothly on \mathbf{x} , then the application:

$$\mathbf{x} \longrightarrow a_1(\mathbf{x})dx_1 + \dots + a_n(\mathbf{x})dx_n$$

called differential 1-form on \mathbb{R}^n . and we generalize this definition in the following.

Let M be a differential manifold of dimension n , we denote by $\bigwedge^k(TM)$, the union of all spaces $\bigwedge^k(T_x M)$ of exterior k -forms on $T_x M$:

$$\bigwedge^k(TM) = \bigcup_{x \in M} \bigwedge^k(T_x M)$$

Definition 3.6. A k -form $\omega_{\mathbf{x}}^k$ at a point \mathbf{x} of a manifold M is an element of $\bigwedge^k(T_x M)$. Let M be a differential manifold of dimension n and an integer $k \leq n$.

We call differential form of degree k on M , (or k -form to simplify) all differential applications

$$\omega : M \longrightarrow \bigwedge^k(TM), \quad x \longmapsto \omega(x) = \omega_x^k$$

such that $\omega_x^k \in \bigwedge^k(T_x M)$ for all $x \in M$

We denote by $\bigwedge_k^m(M)$, the set of k -forms on M of class \mathcal{C}^m , and if U an open of M we denote $\bigwedge_k^m(U)$, the set of k -forms restricted to U of class \mathcal{C}^m .

Remark 10.

- A Differential function $f : M \longrightarrow \mathbb{R}$ is a differential form of degree 0 on M . All differential forms of degree $k > n = \dim M$ are zero.

- Differential forms can be multiplied by functions as well as by scalars. Therefore, the set of C^∞ differential k -forms has a natural structure as a module on the ring of real infinitely differentiable functions on M , thus equipped with operations:

$$\begin{aligned} (\omega_1 + \omega_2)(x) &= \omega_1(x) + \omega_2(x) \\ (f\omega)(x) &= f(x)\omega(x) \end{aligned}$$

$\bigwedge_k(M)$ is a $\mathfrak{F}(M, \mathbb{R})$ -module³, where $\mathfrak{F}(M, \mathbb{R})$ is the ring of differentiable applications $f : M \longrightarrow \mathbb{R}$. In particular, $\bigwedge_k(M)$ is a \mathbb{R} -vector space.

We suppose that the manifold M is a vector space (i.e. M isomorphic to \mathbb{R}^n) with a coordinate system x_1, \dots, x_n . Let fix a point \mathbf{x} , we have seen previously that the n 1-forms dx_1, \dots, dx_n form a basis of $\bigwedge^1 T M_{\mathbf{x}} = (T M_{\mathbf{x}})^*$.

³The notion of module naturally generalizes the notion of vector space, instead of being placed on a field as for these last objects, one is placed on a ring.

We consider the external product:

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad i_1 < \cdots < i_k$$

In the remark (9), we showed that these k -forms form a basis of $\bigwedge^k(T\mathbb{R}_x^n)$. Therefore, each k -form on $T\mathbb{R}_x^n$ can be uniquely written as:

$$\sum_{i_1 < \cdots < i_k} a_{i_1, \dots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Now let ω be an arbitrary k -form differential on M . At all points \mathbf{x} , the form ω can be expressed uniquely in terms of the above basis. We have the following theorem.

Theorem 3.1. Every k -form on the space \mathbb{R}^n with a given coordinate system x_1, \dots, x_n can be uniquely written as

$$\omega^k = \sum_{i_1 < \cdots < i_k} a_{i_1, \dots, i_k}(\mathbf{x}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Where $a_{i_1, \dots, i_k}(\mathbf{x})$ are differentiable functions on \mathbb{R}^n .

Example 3.4.

1) Change of variables in a form:

Let two coordinate systems on \mathbb{R}^3 : $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Let ω a 2-form on \mathbb{R}^3 . Then, by the above theorem, ω can be written in the x coordinate system as:

$$\omega = X_1 dx_2 \wedge dx_3 + X_2 dx_3 \wedge dx_1 + X_3 dx_1 \wedge dx_2$$

where X_1, X_2 , and X_3 are functions in x_1, x_2 , and x_3 , and in the y coordinate system as:

$$\omega = Y_1 dy_2 \wedge dy_3 + Y_2 dy_3 \wedge dy_1 + Y_3 dy_1 \wedge dy_2$$

where Y_1, Y_2 and Y_3 are functions in y_1, y_2 and y_3 .

2) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ in C^∞ -class, then the differential of f is:

$$\begin{aligned} Df(\text{ or } df) : \mathbb{R}^n &\rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \\ x &\mapsto Df_x(\text{ or } df(x)) \end{aligned}$$

$$\mathcal{L}(\mathbb{R}^n, \mathbb{R}) \subset L(\mathbb{R}^n, \mathbb{R}) = \bigwedge^1(\mathbb{R}^n).$$

$$\begin{aligned} \text{then: } Df(\text{ or } df) : \mathbb{R}^n &\rightarrow \bigwedge^1(\mathbb{R}^n) \\ x &\rightarrow Df_x(\text{ or } df(x)) \text{ is in } C^\infty\text{-classe} \end{aligned}$$

Hence $Df \in \bigwedge^1(\mathbb{R}^n)$. and we have:

$$Df_x = df(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) dx_i \text{ where } dx_i(h) = e_i^*(h) = h_i \text{ for } h = (h_1, \dots, h_n) \in \mathbb{R}^n.$$

3 - Let U be an open of \mathbb{R}^3 , f and $g : U \rightarrow \mathbb{R}$ in C^1 -class . We calculate $df \wedge dg$, we have:

$$df \wedge dg = \overbrace{\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}}^{\frac{D(f,g)}{D(x,y)}} dx \wedge dy + \overbrace{\begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix}}^{\frac{D(f,g)}{D(y,z)}} dy \wedge dz + \overbrace{\begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial x} \end{vmatrix}}^{\frac{D(f,g)}{D(z,x)}} dz \wedge dx.$$

If the variables are (x_1, x_2, x_3) , then:

$$df \wedge dg = \sum_{1 \leq i < j \leq 3} \frac{D(f,g)}{D(x_i, x_j)} dx_i \wedge dx_j = \sum_{1 \leq i_1 < i_2 \leq 3} \frac{D(f,g)}{D(x_{i_1}, x_{i_2})} dx_{i_1} \wedge dx_{i_2}$$

In the general case, If U is an open of \mathbb{R}^n and $f_i : U \rightarrow \mathbb{R}$ in C^1 -class where $i = 1, \dots, p \leq n$, Then:

$$df_1 \wedge \dots \wedge df_p = \sum_{1 \leq i_1 < \dots < i_p \leq n} \frac{D(f_1, \dots, f_p)}{D(x_{i_1}, \dots, x_{i_p})} dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

$$\text{If } p = n, \text{ then } df_1 \wedge \dots \wedge df_n = \frac{D(f_1, \dots, f_n)}{D(x_1, \dots, x_n)} dx_1 \wedge \dots \wedge dx_n$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} dx_1 \wedge \dots \wedge dx_n$$

If we denote $f = (f_1, \dots, f_n) : U \rightarrow \mathbb{R}^n$

$$x \mapsto f(x) = (f_1(x), \dots, f_n(x))$$

Thus:

$$(df_1 \wedge \dots \wedge df_n)_x = (\mathcal{J}ac(f))_x dx_1 \wedge \dots \wedge dx_n, \forall x \in U$$

3.4 Exterior differentiation

If $f : U \rightarrow \mathbb{R}$ in C^m -class ($m \geq 1$), where U is an open in \mathbb{R} then $df \in \Lambda_1(U)$ and df is in C^{m-1} -class, This defines an application:

$$d : \Lambda_0^{(m)}(U) \rightarrow \Lambda_1^{(m-1)}(U) \\ f \mapsto df,$$

This application called differentiation.

In this part, we treat the notion of differentiation in the case where the manifold $M = \mathbb{R}^n$. Thereafter, we generalize it for any differential manifold.

3.4.1 Exterior differentiation on \mathbb{R}^n

Theorem 3.2. Let U be an open of \mathbb{R}^n . For all $k \geq 1$, there exists one and only one unique application

$$d : \bigwedge_k^{(m)}(U) \longrightarrow \bigwedge_{k+1}^{(m-1)}(U)$$

$$\omega \longmapsto d\omega$$

verifying the following properties:

- i) $d(\omega + \eta) = d\omega + d\eta, \forall \omega, \eta \in \bigwedge_k^{(m)}(U)$.
- ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta, \forall \omega \in \bigwedge_k^{(m)}(U), \forall \eta \in \bigwedge_{k'}^{(m)}(U)$.
- iii) If $\omega = df_1 \wedge \dots \wedge df_k$ where $f_i \in C^2(U, \mathbb{R}), i = 1, \dots, k$

then:

$$d\omega = d(df_1 \wedge \dots \wedge df_k) = 0.$$

$d\omega$ is called the exterior differential of ω

Remark 11.

- If $k = \infty$ i.e ω in C^∞ -class then $d\omega$ is also in C^∞ -class.
- According to ii) if $f : U \longrightarrow \mathbb{R}$ is differentiable and $\omega \in \bigwedge_p(U)$, Then

$$d(f.\omega) = df \wedge \omega + (-1)^0 f.d\omega = df \wedge \omega + f.d\omega$$

Example 3.5.

- Let U be an open of \mathbb{R}^2 and $\omega \in \bigwedge_1(U)$, ω is written as: $\omega = Pdx + Qdy$, where:

$$P, Q : U \longrightarrow \mathbb{R}.$$

$$\omega(x, y) = P(x, y)dx + Q(x, y)dy$$

Then we have, $d\omega \in \bigwedge_2(U)$, and also:

$$d\omega = dP \wedge dx + dQ \wedge dy$$

$$= \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy \right) \wedge dx + \left(\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy \right) \wedge dy$$

$$= \left(\frac{\partial Q}{\partial x} dx - \frac{\partial P}{\partial y} dy \right) dx \wedge dy$$

- Let U be an open of \mathbb{R}^3 .
1^{er} case: $\omega \in \bigwedge_1(U)$, $\omega = Pdx + Qdy + Rdz$, then $d\omega \in \bigwedge_2(U)$
 therefore:

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

If we take the vector field⁴ V , such that, $V_{(x,y,z)} = (P(x, y, z), Q(x, y, z), R(x, y, z))$
 then:

$$\mathbf{curl}(V) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

⁴We call a vector field any application V defined on an open U of \mathbb{R}^n with values in \mathbb{R}^n . In other words, to any point of U , we associate a vector of \mathbb{R}^n .

Thus the components of $d\omega$ represent the curl of the vector field V .

2th case : $\omega \in \Lambda_2(U)$ then $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$, thus $d\omega \in \Lambda_3(U)$ and we have,

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz$$

The operator:

$$(x, y, z) \longmapsto \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)_{(x,y,z)}$$

represents the divergence of the vector field $V = (P, Q, R)$, noted:

$$\operatorname{div}(V) = \nabla \cdot V$$

Where,

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

Definition 3.7. Let U be an open of \mathbb{R}^n , a differential form ω of degree k on U ($\omega \in \Lambda_k(U)$) is called:

- 1) Closed if $d\omega \equiv 0$ in $\Lambda_{k+1}(U)$.
- 2) Exact if there exist $\eta \in \Lambda_{k-1}(U)$ such that $\omega = d\eta$, In this case η is called the primitive of ω .

Remark 12.

- Any other primitive α of ω , such that $\alpha \in \Lambda_{p-1}(U)$ implies that $\alpha - \eta$ is closed

$$\text{i.e } d(\alpha - \eta) = 0$$

- If ω exact then ω is closed, indeed:

We write:

$$\eta = \sum a_{i_1 \dots i_{p-1}} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{p-1}}$$

Then we have:

$$d\eta = \sum d(a_{i_1 \dots i_{p-1}}) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{p-1}}$$

Thus,

$$\begin{aligned} \omega = d(d\eta) &= d \left(\sum da_{i_1 \dots i_{p-1}} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{p-1}} \right) \\ &= \sum d \left(da_{i_1 \dots i_{p-1}} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{p-1}} \right) \\ &= 0 \end{aligned}$$

Example 3.6.

1) Let ω be a 1-form differential, such that, $\omega(x, y) = ydx - xdy, \omega \in \Lambda_1(\mathbb{R}^2)$ then:

$$d\omega(x, y) = dy \wedge dx - dx \wedge dy = -2dx \wedge dy \neq 0$$

Hence ω is not closed and a fortiori is not exact.

2) Soit ω tel que, $\omega(x, y) = xdx + ydy, \omega \in \Lambda_1(\mathbb{R}^2)$ then,

$$d\omega = dx \wedge dy + dy \wedge dx \equiv 0$$

thus ω is closed, moreover:

$$\omega_{(x,y)} = d\left(\frac{1}{2}(x^2 + y^2)\right) = df(x, y)$$

where $f(x, y) = \frac{1}{2}(x^2 + y^2) \cdot (f : \mathbb{R}^2 \rightarrow \mathbb{R}, f \in \mathcal{C}^\infty, \text{ donc } f \in \Lambda_0(\mathbb{R}^2))$ then ω is exact.

But a closed differential form is not always exact, it depends on the topology of the open U .

Lemma 3.1. (Poincaré lemma). Let U be an open star domain of \mathbb{R}^n . If ω is a closed p -form on U then ω is exact i.e if $\omega \in \Lambda_p(U)$ and $d\omega = 0$, there exist $\eta \in \Lambda_{p-1}(U)$ such that $\omega = d\eta$.

A subset U of \mathbb{R}^n is called star domain if there exist $x_0 \in U$ and for all $x \in U$, the segment $[x_0, x]$ is a subset of U i.e $\forall x \in U, \forall t \in [0, 1]$ we have $x_0 + t(x - x_0) \in U$. (ex. Any line or plan in \mathbb{R}^n is a star domain. Generally any non-empty convex set is a star domain. A set is convex if and only if it is a star domain with respect to any point in that set).

3.4.2 Exterior Differentiation on a manifold M

In this part we will give a definition of the exterior differentiation on manifolds, to do this we must introduce the notion of pullback of a differential form under a differential map.

Definition 3.8. Let M and N be two differential manifolds of dimension n and m respectively, and a differential map:

$$f : M \rightarrow N$$

the pullback of k -forme ω on N under f is the k -forme $f^*\omega$ on M defined by:

$$(f^*\omega)_x(\xi_1, \dots, \xi_k) = \omega_{f(x)}(df_x(\xi_1), \dots, df_x(\xi_k)).$$

for all $\xi_1, \dots, \xi_k \in TM_x$

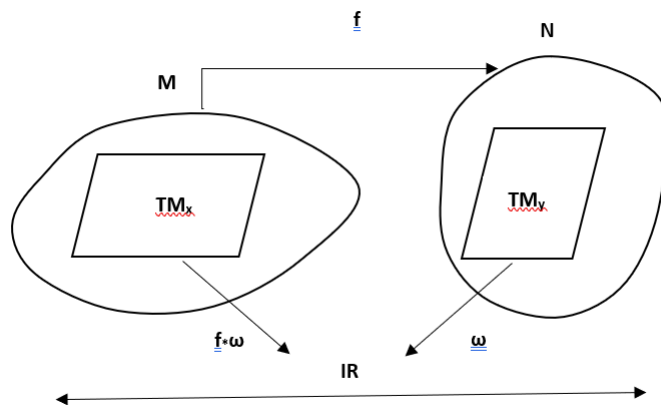


Figure 13:

Example 3.7.

If $z = f(x, y) = x^2 + y^2$ and $\omega = dz$ then $(f^*\omega)_{(x,y)} = 2xdx + 2ydy$

Remark 13.

Let :

$$\omega = \sum_{1 \leq i_1 < \dots < i_k \leq m} \alpha_{i_1 \dots i_k} dy_{i_1} \wedge \dots \wedge dy_{i_k}$$

the local expression of the k -form ω with respect to a coordinate system (y_1, \dots, y_m) defined on an open V of M_2 , where, $\alpha_{i_1 \dots i_k} : M_2 \rightarrow \mathbb{R}$ are functions defined on M_2 . Thus,

$$(f^*\omega)(x) = \sum_{1 \leq j_1 < \dots < j_k \leq n} \left(\sum_{1 \leq i_1 < \dots < i_k \leq m} \alpha_{i_1 \dots i_k}(y(x)) \frac{\partial f_{i_1}}{\partial x_{j_1}} \frac{\partial f_{i_2}}{\partial x_{j_2}} \dots \frac{\partial f_{i_k}}{\partial x_{j_k}} \right) dx_{j_1} \wedge \dots \wedge dx_{j_k}$$

We deduce the following properties :

1. $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$, for all $\omega_1, \omega_2 \in \Lambda_k(N)$.
2. $f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$, for all $\omega_1 \in \Lambda_p(N)$ et $\omega_2 \in \Lambda_q(N)$

The application $f^* : \Lambda_p(N) \rightarrow \Lambda_p(M)$ extends to an application also noted f^* :

$$f^* : \bigwedge(N) \rightarrow \bigwedge(M)$$

which is a morphism of algebra of $\bigwedge(N)$ in $\bigwedge(M)$. And if,

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

are two morphisms of differential manifolds from M_1 into M_2 and from M_2 into M_3 , then the applications:

$$\Lambda_p(M_1) \xleftarrow{f^*} \Lambda_p(M_2) \xleftarrow{g^*} \Lambda_p(M_3) \text{ et } \Lambda_p(M_1) \xleftarrow{(g \circ f)^*} \Lambda_p(M_3)$$

are linked by:

$$(g \circ f)^* = f^* \circ g^*$$

It follows immediately that if f is a diffeomorphism of a differentiable manifold M , then f^* is an isomorphism and we have :

$$(f^*)^{-1} = (f^{-1})^*$$

Definition 3.9. (Theorem)

There exists a unique application

$$d : \bigwedge_p(M) \longrightarrow \bigwedge_{p+1}(M)$$

with the following properties :

1. If $f \in \mathfrak{F}(M)$, df is the usual differential of f ,
2. $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$ for all $\omega_1, \omega_2 \in \bigwedge_p(M)$
3. $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2$ for all $\omega_1 \in \bigwedge_p(M)$ and $\omega_2 \in \bigwedge_q(M)$,
4. $d \circ d = 0$,
5. $d(f^*\omega) = f^*d\omega$.
6. If ω a p -form with local expression

$$\omega = \sum_{1 \leq i_1 < \dots < i_p \leq m} \alpha_{i_1 \dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

in a coordinate system (x_1, \dots, x_n) on a chart U of M , then:

$$d\omega = \sum_{1 \leq i_1 < \dots < i_p \leq m} d\alpha_{i_1 \dots i_p} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p}$$

$d\omega$ is called the exterior differential of ω . where $\mathfrak{F}(M, \mathbb{R})$ is the ring of differentiable maps $f : M \rightarrow \mathbb{R}$

3.5 Vector Field and Interior Product

We have previously treated some properties of vector fields defined on \mathbb{R}^n . Now we will generalize this notion on any manifold.

Definition 3.10. A vector field (also called section of the tangent bundle) on a manifold M is an application, denoted X , which associates to any point x a tangent vector $X(x) \in T_x M$.

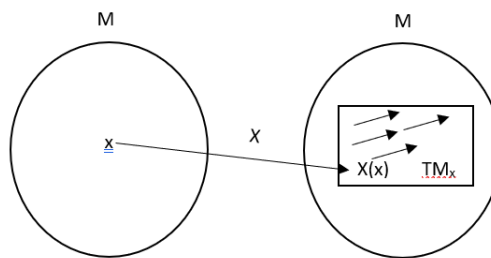


Figure 14:

Remark 14.

- Let (x_1, \dots, x_n) a local coordinate system in a chart $U \subset M$, and

$$\{e_1, \dots, e_n\} = \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$$

the basis of $T_x M$, In this system the vector field X is written in the form:

$$X(x) = \sum_{k=1}^n f_k(x) \frac{\partial}{\partial x_k}, \quad x \in U$$

where the functions

$$f_1, \dots, f_n : U \longrightarrow \mathbb{R}$$

are the local components of X with respect to (x_1, \dots, x_n) .

- A vector field X is differentiable if its components $f_k(x)$ are differentiable functions.
- This definition of differentiability does not obviously depend on the choice of the local coordinate system. Indeed, if (y_1, \dots, y_m) is another local coordinate system in U . Then,

$$X(x) = \sum_{k=1}^m h_k(x) \frac{\partial}{\partial y_k}, \quad x \in U$$

where,

$$h_1, \dots, h_m : U \longrightarrow \mathbb{R}$$

are the components X with respect to (y_1, \dots, y_m) and the result follows from the fact that:

$$h_k(x) = \sum_{l=1}^m \frac{\partial y_k}{\partial x_l} f_l(x), \quad x \in U$$

- A differentiable vector field X on M is called dynamic system.

Definition 3.11. An integral curve (or trajectory) of the vector field X is a differentiable curve

$$\varphi : I \longrightarrow M, \quad t \longmapsto \varphi(t)$$

such that,

$$\forall t \in I, \quad \frac{d\varphi(t)}{dt} = X(\varphi(t))$$

where I is an interval of \mathbb{R} .

Remark 15.

If the local expression of X is written as:

$$\sum_{k=1}^m f_k(x) \frac{\partial}{\partial x_k}$$

then the integral curves (or trajectories) of X are the solutions $\varphi(t) = \{x_k(t)\}$ of the following dynamic system:

$$\begin{cases} \frac{dx_1}{dt} &= f_1(x_1, \dots, x_m) \\ &\vdots \\ \frac{dx_m}{dt} &= f_m(x_1, \dots, x_m) \end{cases} \quad (4)$$

Definition 3.12. (Interior product)

Let ω be a k -form on a manifold M and X a vector field. The interior product $i_X\omega$ is defined by:

$$(\mathbf{i}_X\omega)_x(v_2, \dots, v_k) = \omega_x(X(x), v_2, \dots, v_k)$$

for all v_2, \dots, v_k in T_xM

Thus i_X is an application :

$$i_X : \Lambda_k(M) \rightarrow \Lambda_{k-1}(M)$$

Proposition 3.2. Let α be a k -form, β a 1-form and f a real function on a manifold M . Then:

$$\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathbf{i}_X\beta)$$

$$\mathbf{i}_f X\alpha = f \mathbf{i}_X\alpha$$

4 Integration of forms

Before treating the integration of differential forms on manifolds, it is necessary to introduce the notion of orientation for manifolds. For example, a segment of \mathbb{R} or a curve of the complex space \mathbb{C} are orientable objects, that allows us to give a precise definition, and moreover to regulate problems of sign. However, in an arbitrary manifold, the objects do not necessarily admit an orientation. We give as example the Möbius strip ⁵.

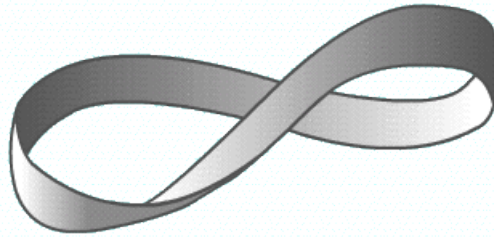


Figure 15: Möbius strip

4.1 Orientation and boundary

Definition 4.1. A manifold M of dimension n is oriented if we can choose a map system (φ_i, U_i) in which all transition maps $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ have a positive Jacobian. i.e. for all $x \in M$ we have:

$$\det \left(\frac{\partial \psi_{ij}^p}{\partial x_q}(x) \right)_{1 \leq p, q \leq n} > 0$$

where $\psi_{ij} = (\psi_{ij}^1, \dots, \psi_{ij}^n)$

Note that this definition is equivalent to defining an orientation of each tangent space $T_x M$ that depends on x :

Indeed, for each local chart $\varphi : \mathcal{O} \rightarrow U \subset \mathbb{R}^n$ and for each point $x \in M$,

$$d\varphi_x : T_x M \rightarrow \mathbb{R}^n$$

is an isomorphism that preserves the orientation.

For the following, we also need to extend the definition (4.1) of a differential manifold to the case of manifolds with boundary. For that let us note:

$$\mathbb{R}_-^n := \{t = (t_1, \dots, t_n) \in \mathbb{R}^n \mid t_1 \leq 0\} =]-\infty, 0] \times \mathbb{R}^{n-1}$$

⁵The Möbius strip is a compact surface whose boundary is homeomorphic to a circle. In other words, it has only one face unlike a classical strip which has two faces, and one cannot distinguish between the ingoing and the outgoing side so it is a non-orientable submanifold of dimension 2.

Definition 4.2. (Differential manifold with boundary)

A differential manifold with boundary M of dimension n is a topological space equipped with an atlas $\{U_i, \varphi_i\}_{i \in I}$, where the charts $\varphi_i : U_i \rightarrow \mathcal{O}_i \subset \mathbb{R}_-^n$, verify:

- Each U_i is an open of M and $M = \bigcup_{i \in I} U_i$.
- Each \mathcal{O}_i is an open of \mathbb{R}_-^n , i.e. the intersection of an open of \mathbb{R}^n with \mathbb{R}_-^n .
- Each application φ_i is a homeomorphism.
- If $U_{ij} := U_i \cap U_j$ is non-empty, then the homeomorphism $\varphi_{ij} := \varphi_j \circ \varphi_i^{-1}$ is a \mathcal{C}^∞ -diffeomorphism of $\varphi_i(U_{ij})$ into $\varphi_j(U_{ij})$.

Example 4.1.

The closed unit ball $\mathcal{B}(0, 1)$ of \mathbb{R}^n is a manifold with boundary, its boundary is the unit sphere $\mathcal{S}(0, 1)$, we write :

$$\partial\mathcal{B}(0, 1) = \mathcal{S}(0, 1)$$

Remark 16.

- The boundary of a manifold M of dimension n is the set of points sent into $\partial\mathbb{R}_-^n$ by a chart, denoted ∂M .

$$\partial M = \{x \in M \mid \exists (\varphi_i, U_i) \text{ such that } \varphi_i(x) \in \partial\mathbb{R}_-^n\}$$

- For this definition to have sense, it must be verified that this property is independent of the choice of chart. We then show that ∂M is also a differential manifold, of dimension $(n - 1)$.

More precisely, if we note $\varphi_i|_{\partial M} : \partial M \cap U_i \rightarrow \partial\mathbb{R}_-^n$ the restriction of φ_i à $\partial M \cap U_i$, $(\varphi_i|_{\partial M})_{i \in I}$ is an atlas on ∂M .

- Suppose that we can orient the manifold M . Then its boundary ∂M is also naturally provided with an orientation, which we define as follows:

Firstly we orientate $\partial\mathbb{R}_-^n$. Indeed, for all $t \in \partial\mathbb{R}_-^n$, we have (e_2, \dots, e_n) (where (e_1, \dots, e_n) is the direct canonical basis of \mathbb{R}^n) is a direct basis of $T_t\partial\mathbb{R}_-^n \simeq \mathbb{R}^{n-1}$.

Then, for each point $\mathbf{x} \in \partial M$, the tangent space $T_{\mathbf{x}}\partial M$ is provided with the orientation induced by that of $\partial\mathbb{R}_-^n$, with the charts $\varphi_i|_{\partial M} : \partial M \cap U_i \rightarrow \partial\mathbb{R}_-^n$.

4.1.1 Integration of k -form on a manifold

In this part, we will start by defining the integration of n -forms on the manifold with boundary $\partial\mathbb{R}_-^n$ and then we will generalize. Let U be an open of \mathbb{R}_-^n and ω a n -form with compact support ⁶ in U .

For any measurable set $A \subset U$, we define $\int_A \omega$ as the integral over A of the function $x \mapsto \omega_x(e_1, \dots, e_n)$. where the e_i are the vectors of the canonical basis of

⁶The support of ω on M is the adherence of the set of points x of M for which ω_x is not zero in $\bigwedge^k T_x^*M$.

\mathbb{R}^n .

The following lemma, which is related to the change of variable theorem, justifies this definition. As we will see, it allows the integration of differential forms on manifolds.

Lemma 4.1. Let $\varphi : U \rightarrow V$ be a diffeomorphism that preserves the orientation, and ω be an n -form on V , then for any measurable set $A \subset U$, we have:

$$\int_A \varphi^* \omega = \int_{\varphi(A)} \omega.$$

φ preserves the orientation if its Jacobian is positive at each point $x \in U$.

Demonstration. We denote $e = (e_1, \dots, e_n)$ the canonical basis of \mathbb{R}^n .

We write $\omega = f dx_1 \wedge \dots \wedge dx_n$, where $f : V \rightarrow \mathbb{R}$ is the function $x \mapsto \omega_x(e)$.

By definition $\int_{\varphi(A)} \omega = \int_{\varphi(A)} f(x) dx$, where dx is the Lebesgue measure on \mathbb{R}^n .

Moreover, and also by definition,

$$(\varphi^* \omega)_x(e) = \omega_{\varphi(x)}(D\varphi_x(e)) = f(\varphi(x)) dx_1 \wedge \dots \wedge dx_n (D\varphi_x(e)).$$

In the example, we have seen (3.3) that $dx_1 \wedge \dots \wedge dx_n$ is none other than the determinant in the base e .

thus $dx_1 \wedge \dots \wedge dx_n (D\varphi_x(e))$ is none other than the Jacobian determinant involved in the formula of change of variables. There is no absolute value to add because φ is supposed to preserve the orientation, which means that its Jacobian determinant is positive.

Remark 17.

This lemma implicitly uses the fact that any diffeomorphism of \mathbb{R}^n sends measurable sets into measurable sets. The same remark shows that there is a well defined notion of measurable set in a manifold. Indeed, a set $A \subset M$ is measurable if its images by all charts of a countable atlas is measurable.

Theorem 4.1. Let M be an oriented manifold of dimension n with boundary. Let A be a measurable set of M . There exists a unique linear form on the set of n -forms with compact support, denoted \int_A , which verifies the following properties:

- It coincides with the integration defined above when M is an open of \mathbb{R}^n .
- For any diffeomorphism φ that preserves the orientation between two oriented manifolds M and N , we have:

$$\int_A \varphi^* \omega = \int_{\varphi(A)} \omega.$$

Demonstration. Let $\{\varphi_i : U_i \rightarrow V_i \subset \mathbb{R}^n\}$ be an oriented atlas of M . for the next step we admit the following result :

For any cover of a manifold M by opens U_i , there exists a family of smooth functions $\rho_i : M \rightarrow \mathbb{R}_+$ verifying the following conditions:

- The support of each ρ_i is contained in U_i
- Each point of M has a neighborhood that coincides only with a finite number of

supports of ρ_i
 $-\sum \rho_i \equiv 1$

This result gives a partition of the unity $(\rho_i)_i$ subordinated to the cover U_i . We denote $\psi_i = (\varphi_i)^{-1}$ and we pose, for any n -form ω with compact support in M :

$$\int_A \omega = \sum_i \int_{\varphi_i(A)} \psi_i^* (\rho_i \omega) \quad (5)$$

We will show that this number defined in (5) depends neither on the choice of the atlas nor on the choice of the partition unity, To simplify the notations, we will suppose in all the rest that $A = M$ (so $\varphi_i(A) = V_i$).

Let now $\{\varphi'_i : U'_i \rightarrow V'_i\}$ another oriented atlas of M and ρ'_i a partition of the unity corresponding. By definition, the union of these two atlases is still an oriented atlas, so for all i and j , the change of chart $\theta_{ij} = \varphi'_j \circ (\varphi_i)^{-1}$ is a diffeomorphism that preserves the orientation of $V_{ij} = \varphi_j(U_i \cap U'_j)$ then $V'_{ij} = \varphi'_i(U_i \cap U'_j)$. In addition, we have $\psi_i = \psi'_j \circ \theta_{ij}$ on V_{ij} . We can then compare the two constructions of the integral :

$$\begin{aligned} \sum_i \int_{V_i} \psi_i^* (\rho_i \omega) &= \sum_i \int_{V_i} \psi_i^* \left(\sum_j \rho'_j \rho_i \omega \right) \\ &= \sum_{ij} \int_{V_{ij}} \psi_i^* (\rho'_j \rho_i \omega) \\ &= \sum_{ij} \int_{V_{ij}} \theta_{ij}^* ((\psi'_j)^* (\rho'_j \rho_i \omega)) \\ &= \sum_{ij} \int_{V'_{ij}} (\psi'_j)^* (\rho'_j \rho_i \omega) \\ &= \sum_{ij} \int_{V'_j} (\psi'_j)^* (\rho'_j \rho_i \omega) \\ &= \sum_j \int_{V'_j} (\psi'_j)^* \left(\sum_i \rho'_j \rho_i \omega \right) \\ &= \sum_j \int_{V'_j} (\psi'_j)^* (\rho'_j \omega) \end{aligned}$$

The formula of change of variable follows from this property by transport of structure. Indeed, if φ is a diffeomorphism of M into N then any atlas $\{\varphi_i : U_i \rightarrow V_i \subset \mathbb{R}^n\}$ and any partition of the unity $(\rho_i)_i$ on M provides an atlas $\{\varphi_i \circ \varphi^{-1} : \varphi(U_i) \rightarrow V_i \subset \mathbb{R}^n\}$ is a partition of the unity $\{\rho_i \circ \varphi^{-1}\}$ which can be used to calculate $\int_N \omega$:

$$\int_M \varphi^* \omega = \sum_i \int_{V_i} \psi_i^* (\rho_i \varphi^* \omega) = \sum_i \int_{V_i} (\varphi \circ \psi_i)^* (\rho_i \circ \varphi^{-1} \omega) = \int_N \omega.$$

Remark 18.

The previous theorem gives us in particular the following results: Let M be an oriented manifold of dimension n and $\omega \in \Lambda_n(M)$. Let (φ, U) be a chart of M

which preserves the direct orientation of \mathbb{R}_-^n such that $Supp(\omega) \subset U$ then:

$$\int_M \omega = \int_{\mathbb{R}_-^n} (\varphi^{-1})^* \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega$$

we have $\varphi(\text{supp}(\omega))$ is compact. $(\varphi^{-1})^* \omega$ is written as:

$$(\varphi^{-1})^* \omega = a(\xi_1, \dots, \xi_n) d\xi_1 \wedge \dots \wedge d\xi_n$$

(ξ_1, \dots, ξ_n) where is a coordinate system of \mathbb{R}_-^n . By definition, the integral of ω over M , is the usual multiple integral:

$$\begin{aligned} \int_M \omega &= \int_{\mathbb{R}_-^n} (\varphi^{-1})^* \omega \\ &= \int_{\mathbb{R}_-^n} a(\xi_1, \dots, \xi_n) d\xi_1 \wedge \dots \wedge d\xi_n \\ &= \int_{\mathbb{R}_-^n} a(\mathbf{x}) d\mathbf{x} \end{aligned}$$

this integral exists because $a(\xi_1, \dots, \xi_n)$ is continuous on M and has compact support.

In the general case the integral of an n -form over a measurable part $A \in M$ is given by:

$$\int_A \omega = \sum_i \int_{\varphi_i(A)} (\varphi_i^{-1})^* (\rho_i \omega)$$

where $\{\varphi_i, U_i\}_{i \in I}$ an atlas is in which all the charts respect the orientation of \mathbb{R}_-^n . $(\rho_i)_{i \in I}$ is a partition of the unity subordinate to the opens U_i .

4.2 Stokes Formula

As we have seen in the remark (16), if M is an oriented manifold of dimension n with boundary, then M is also an oriented manifold of dimension $(n - 1)$ but without boundary. On the other hand if we take ω to be a $(n - 1)$ -form we have that $d\omega$ is a n -form and we have then the following important theorem.

Theorem 4.2. Let M be an oriented manifold with boundary, For any differential form $\omega \in \Lambda_{n-1}(M)$ with compact support, we have:

$$\int_{\partial M} \omega = \int_M d\omega.$$

Demonstration. *The idea is to use a partition of the unity adapted to the problem in the definition of the integral of a differential form, and to come back to an almost obvious case.*

Let $\{U_i\}_I$ a locally finite cover of M by local chart domains $\phi_i : U_i \rightarrow \phi_i(U_i) \subset \mathbb{R}^n$, such that:

$$\phi_i(U_i \cap \partial M) = \phi_i(U_i) \cap (\{0\} \times \mathbb{R}^{n-1})$$

Let introduce a partition of the unity ρ_j subordinate to $\{U_i\}$. Since the support of ω is closed, the differential form ω is written:

$$\omega = \sum \rho_i \omega$$

where the sum is finite.

Let pose $\beta_i = \phi_i^* [\rho_i \omega]$, a differential form with compact support of $\mathbb{R}^n = \mathbb{R}_- \times \mathbb{R}^{n-1}$. The restriction $\phi_i|_{\partial M}$ is a diffeomorphism on its image preserving the outgoing orientations. Thus we have:

$$\int_{\partial M} [\rho_i \omega] = \int_{\partial \mathbb{R}_-^n} \beta_i$$

Since ϕ_i^* commutes with the differentiation operator d , we have:

$$\int_M d[\rho_i \omega] = \int_{\mathbb{R}_- \times \mathbb{R}^{n-1}} d\beta_i$$

By summation, Stokes' theorem is proved once the particular case is established $M' = \mathbb{R}_- \times \mathbb{R}^{n-1}$ a $(n-1)$ -form ω on $M' = \mathbb{R}_- \times \mathbb{R}^{n-1}$ is written:

$$\omega = \sum_{i=1}^n f_i \cdot dx_1 \wedge \cdots \wedge \widehat{dx}_i \wedge \cdots \wedge dx_n$$

where the cap means a suppression. We then find

$$\begin{aligned} d\omega &= \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j \right) \wedge dx_1 \wedge \cdots \wedge \widehat{dx}_i \wedge \cdots \wedge dx_n \\ &= \sum_{i=1}^n (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n \end{aligned}$$

Fubini's theorem gives:

$$\begin{aligned} \int_{\mathbb{R}_- \times \mathbb{R}^{n-1}} d\omega &= \sum_{i=1}^n \int_{\mathbb{R}_- \times \mathbb{R}^{n-1}} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \dots dx_n \\ &= \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^0 \frac{\partial f_1}{\partial x_1} dx_1 \right) dx_2 \dots dx_n \\ &\quad + \sum_{i=2}^n \int_{\mathbb{R}_- \times \mathbb{R}^{n-2}} (-1)^{i-1} \left(\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i} dx_i \right) dx_1 \dots \widehat{dx}_i \dots dx_n \end{aligned}$$

The hypothesis that the form ω is with compact support then allows to finish the calculation, because the terms $\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i} dx_i$ for $i \geq 2$ are all zero.

$$\int_{\mathbb{R}_-^n} d\omega = \int_{\mathbb{R}^{n-1}} f_1(0, x_2, \dots, x_n) dx_2 \dots dx_n = \int_{\partial \mathbb{R}_-^n} \omega$$

Example 4.2.

- If we take a function in C^∞ -class, $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

we denote $df = \omega_{grad f}^1$.

Now let's return to the example (3.5) and consider the following vector field:

$$A(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$$

We pose $\omega_A^1 = Pdx + Qdy + Rdz$, then $d\omega_A^1 \in \Lambda_2(U)$

thus

$$d\omega_A^1 = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy = \omega_{\text{curl} A}^2$$

Let $\omega_A^2 = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$, then $d\omega \in \Lambda_3(U)$ and we have,

$$d\omega_A^2 = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz = (\text{div } A)\omega^3$$

By the theorem of Stokes we find the following results:

- Fundamental theorem of analysis :

$$f(b) - f(a) = \int_{\Gamma} \text{grad } f d\lambda \text{ where } \partial\Gamma = \{a, b\}$$

- Divergence theorem (Green-Ostrogradski theorem):

$$\iint_S \mathbf{A} d\mathbf{n} = \iiint_D (\text{div } \mathbf{A}) \omega^3 \text{ where } \partial D = S$$

- Gauss theorem:

$$\int_{\Gamma} \mathbf{A} d\lambda = \iint_S \text{curl } \mathbf{A} \cdot d\mathbf{n} \text{ where } \partial S = \Gamma$$

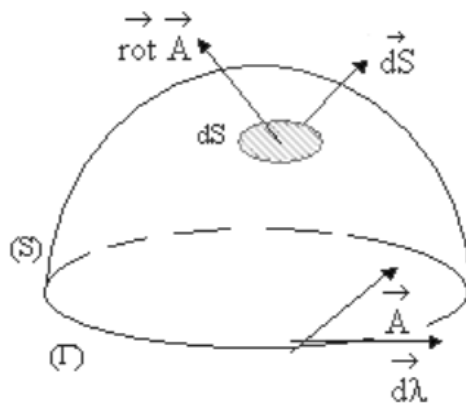


Figure 16:

Remark 19.

Stokes' formula allows us to easily show Liouville's theorem (1.2), we have :

$$\frac{dV}{dt} = \iint \mathbf{v} \cdot d\mathbf{S} \quad (\text{Transport theorem})$$

where $\mathbf{v} = \dot{\mathbf{x}} = (\dot{\mathbf{p}}, \dot{\mathbf{q}})$ is the velocity vector and \mathbf{S} a surface vector of V and using the Green-Ostrogradski theorem, we find:

$$\frac{dV}{dt} = \iint \mathbf{v} \cdot d\mathbf{S} = \iiint \text{div}(\mathbf{v}) dV$$

the divergence of the velocity vector in the Hamiltonian system is zero, indeed:

$$\text{div}(\mathbf{v}) = \frac{\partial}{\partial \mathbf{p}} \left(-\frac{\partial H}{\partial \mathbf{q}} \right) + \frac{\partial}{\partial \mathbf{q}} \left(\frac{\partial H}{\partial \mathbf{p}} \right) = 0$$

We deduce that:

$$\frac{dV}{dt} = 0$$

4.3 Integration of a k -form over a manifold of dimension n .

In this part, we will present the approach of the mathematician V. Arnold to integrate the k -form over a manifold of dimension n . But we don't necessarily have $n = k$. This approach is detailed in his book "Mathematical methods of classical mechanics" (see [1]).

Let ω be differential k -form on a manifold M of dimension n . Let D be a convex polyhedron ⁷ bounded in the euclidean space \mathbb{R}^k (Figure 17).

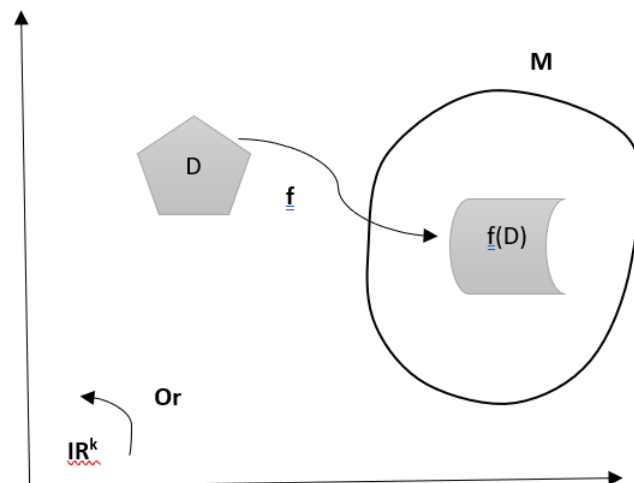


Figure 17:

⁷- A polyhedron is the union of its k -faces, and the intersection of any two k -faces of a simplex is either empty or a face of dimension $k - 1$.

- A simplex or k -simplex is the analogue of the triangle in dimension k .

- A polyhedron is said to be convex if, for each plane of the space that contains a face of the polyhedron, the polyhedron is all in one of the half-spaces bounded by the plane.

The role of "path of integration" will be played by the cell of dimension k of M , represented by a triple $\sigma = (D, f, Or)$ constituted by:

1. A polyhedron $D \subset \mathbb{R}^k$,
2. A differentiable application $f : D \rightarrow M$,
3. A choice of orientation on \mathbb{R}^k , noted by Or .

Definition 4.3. The integral of k -form ω over the cell σ is given by:

$$\int_{\sigma} \omega = \int_D f^* \omega$$

Remark 20.

- This integral is linear:

$$\int_{\sigma} \lambda_1 \omega_1 + \lambda_2 \omega_2 = \lambda_1 \int_{\sigma} \omega_1 + \lambda_2 \int_{\sigma} \omega_2$$

- The cell of dimension k that differs from σ only by the choice of orientation is called the negative of σ and denoted by $-\sigma$. we have:

$$\int_{-\sigma} \omega = - \int_{\sigma} \omega$$

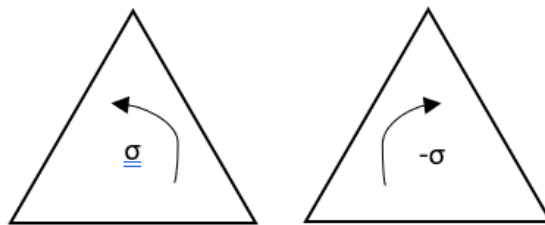


Figure 18:

4.4 Chain

The set $f(D)$ is not necessarily a submanifold of M . It could contain singularities and in particular it could be reduced to a point. However, in a way analogous to the one-dimensional case, where we divide the domains of \mathbb{R} by segments. It is useful to be able to consider contours consisting of many pieces, which can be traversed in both directions, perhaps more than once. This concept is called a chain.

Definition 4.4. A chain of dimension k (or k -chain) on a manifold M consists of a finite collection of oriented cells $\sigma_1, \dots, \sigma_r$ of dimension k in M and of integers $m_1, \dots, m_r \in \mathbb{Z}$, called the multiplicities. A chain is denoted by:

$$c_k = m_1\sigma_1 + \dots + m_r\sigma_r$$

We introduce natural identifications:

$$\begin{aligned} m_1\sigma + m_2\sigma &= (m_1 + m_2)\sigma \\ m_1\sigma_1 + m_2\sigma_2 &= m_2\sigma_2 + m_1\sigma_1 \quad 0\sigma = 0 \quad c_k + 0 = c_k \end{aligned}$$

Example 4.3.

We consider for example the full rectangle (object of dimension 2) oriented in \mathbb{R}^2 and we note it c_2 . It is constituted by two triangles σ_1 and σ_2 which have the same orientation of c_2 , thus $c_2 = \sigma_1 + \sigma_2$. The boundary of c_2 noted $\partial c_2 = \partial\sigma_1 + \partial\sigma_2$ represents the empty rectangle which is an object of dimension 1 (see (19)).

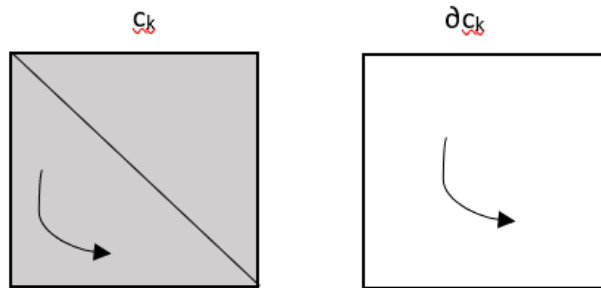


Figure 19:

NB: In general the boundary of a chain $c_k = m_1\sigma_1 + \dots + m_r\sigma_r$ is given by:

$$\partial c_k = m_1\partial\sigma_1 + \dots + m_r\partial\sigma_r$$

Definition 4.5. (Integral of a form over a chain)

Let ω^k be a k -form on M , and c_k a chain of dimension k on M , $c_k = \sum m_i\sigma_i$. The integral of a k -form ω^k on the chain c_k is the sum of the integrals on the cells, including multiplicities:

$$\int_{c_k} \omega^k = \sum m_i \int_{\sigma_i} \omega^k$$

Remark 21.

- The Stokes formula is also applicable on the chains ,indeed :

$$\int_{\partial c} \omega = \int_c d\omega$$

We give the following example: Let the 1-form ω^1 on \mathbb{R}^{2n} with coordinate system $p_1, \dots, p_n, q_1, \dots, q_n$. defined as follows:

$$\omega^1 = p_1dq_1 + \dots + p_ndq_n = \mathbf{p}d\mathbf{q}$$

Then $d\omega^1 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n = d\mathbf{p} \wedge d\mathbf{q}$, thus :

$$\iint_{c_2} d\mathbf{p} \wedge d\mathbf{q} = \int_{\partial c_2} \mathbf{p} d\mathbf{q}$$

In particular, if $(\partial c_2 = 0)$, then $\iint_{c_2} d\mathbf{p} \wedge d\mathbf{q} = 0$.

- The integral of a closed form ω^k on the boundary of any chain c_{k+1} of dimension $(k + 1)$ is zero :

$$\int_{\partial c_{k+1}} \omega^k = 0 \quad \text{if } d\omega^k = 0$$

- We say that c_{k+1} is a cycle if $\partial c_{k+1} = 0$, and we have the integral of a differential on any cycle is equal to zero :

$$\int_{c_{k+1}} d\omega^k = 0 \quad \text{if } \partial c_{k+1} = 0$$

5 Symplectic structure

Symplectic manifolds appear in the Hamiltonian reformulation using the notion of cotangent bundle of a manifold, where the configurations of a system form a manifold, whose cotangent bundle describes the phase space of the system.

The notion of symplectic manifold, and thus of symplectic geometry, was given by Jean-Marie Souriau in 1953. According to Souriau, the symplectic form would be called historically the Lagrange form.

We will define the symplectic manifolds, the Hamiltonian vector fields and the standard symplectic structure on the cotangent bundle.

5.1 Symplectic manifold

Definition 5.1. Let M^{2n} be a manifold of even dimension $2n$. A symplectic structure on M^{2n} is defined by giving a closed and non-degenerate 2-form $\omega^2 \in \Lambda_2(M)$.

$$d\omega^2 = 0 \quad \text{et} \quad \forall \xi \neq 0, \exists \eta : \omega^2(\xi, \eta) \neq 0 \quad (\xi, \eta \in TM_{\mathbf{x}})$$

We also say that (M^{2n}, ω) (or M^{2n}) is a symplectic manifold, and that ω is a symplectic form on M^{2n} .

Example 5.1.

- We consider the vector space \mathbb{R}^{2n} with coordinate (p_i, q_i) and let

$$\omega^2 = \sum dp_i \wedge dq_i$$

Obviously ω^2 is a closed form $d\omega^2 = 0$.

Let us show that it is nondegenerate, We choose the basis of \mathbb{R}^{2n} as the following :

$$\mathfrak{B} = (e_{p_1}, e_{q_1}, e_{p_2}, e_{q_2}, \dots, e_{p_n}, e_{q_n})$$

such that, $dp_i(e_{p_j}) = \delta_{ij}$, $dq_i(e_{q_j}) = \delta_{ij}$, and $dq_i(e_{p_j}) = dp_i(e_{q_j}) = 0$ for all i and j .

the matrix of ω^2 is written:

$$M(\omega, \mathfrak{B}) = (\omega(e_{p_i}, e_{q_j}))_{ij} = \begin{pmatrix} S & 0 & 0 \\ \vdots & \ddots & \dots \\ 0 & \dots & S \end{pmatrix}$$

where,

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Since the matrix of ω^2 is invertible, then it is non-degenerate.

- The complex projective space $\mathbb{C}\mathbb{P}^n$ admits a canonical symplectic structure induced from the standard symplectic structure of $\mathbb{C}^{n+1} = \{z = x + iy \mid x, y \in \mathbb{R}^{n+1}\} \simeq \mathbb{R}^{2n+2}$.

With the complex notation, the canonical structure is written :

$$\omega_0 = \sum_{i=1}^n dy_i \wedge dx_i = \frac{1}{2i} \sum_{i=1}^n dz_i \wedge d\bar{z}_i$$

Remark 22.

- Considering n times the exterior product of the symplectic 2-form ω . The manifold M is then provided with a differential $2n$ -form $\omega^{\wedge n}$. It is then possible to show, that this last differential $2n$ -form $\omega^{\wedge n}$ is a volume form on M .
- In a local chart $x = (x_1, \dots, x_{2n})$, a volume form is written:

$$\omega = \varphi \cdot dx_1 \wedge \dots \wedge dx_{2n}$$

where φ is a real differentiable function that never vanishes.

- Any symplectic manifold is canonically oriented and receives a canonical measure $\omega^{\wedge n}/n!$ called Liouville measure.
- In the remark (4), we have seen that the cotangent space at a point \mathbf{x} is defined as the dual of the tangent space at this point. Now in a similar way, we can define the cotangent bundle as the union of cotangent spaces. we write :

$$TM^* = \bigcup_{\mathbf{x} \in M} T_{\mathbf{x}}M^*$$

Moreover, TM^* is a differential manifold of dimension $2n$.

- An element of TM^* is a 1-form on the space tangent $T_{\mathbf{x}}M$ at a point \mathbf{x} of M . If $\mathbf{q} = (q_1, \dots, q_n)$ is a local coordinate system of M , then such a form is given by its n components $\mathbf{p} = (p_1, \dots, p_n)$. So the $2n$ coordinates (\mathbf{p}, \mathbf{q}) form a local coordinate system of the manifold TM^* .

We have the following theorem:

Theorem 5.1. The cotangent bundle T^*M has a natural symplectic structure. In the local coordinates described above, this symplectic structure is given by the formula:

$$\omega^2 = d\mathbf{p} \wedge d\mathbf{q} = dp_1 \wedge dq_1 + \dots + dp_n \wedge dq_n$$

Indeed, we take the 1-form $\omega^1 = \mathbf{p} d\mathbf{q}$ defined on TM^* , so $\omega^2 = d\omega^1$ is a closed non-degenerate form.

Theorem 5.2. (Darboux theorem)

Let (M, ω) be a connected symplectic manifold. The dimension of this manifold is necessarily even, let us note it $2n$.

Any point of M has an open neighborhood which is the domain of a chart whose local coordinates, denoted $(p_1, q_1, \dots, p_n, q_n)$, are such that the form ω has the expression:

$$\omega = \sum_{i=1}^n dp_i \wedge dq_i.$$

Remark 23.

- Such a chart is called canonical, or Darboux. This theorem shows that locally, in the neighborhood of each points, a symplectic manifold is isomorphic to a cotangent bundle.
- It also shows that two symplectic manifolds of the same dimension are locally isomorphic.

5.2 Hamiltonian vector field and Hamiltonian phase flow

A Riemannian structure on a manifold M establishes an isomorphism between the spaces of tangent vectors and the space of 1-forms $\Lambda_1(M)$.

$$TM \rightarrow TM^*, \xi \rightarrow \omega_\xi$$

such that $\omega_\xi(\eta) = \langle \xi, \eta \rangle$, with $\langle \cdot, \cdot \rangle$ is the metric tensor of Riemannian manifold M , a symplectic structure establishes a similar isomorphism, and we have the following definition:

Definition 5.2. In the general case the isomorphism is defined as follows, to each tangent vector ξ of a symplectic manifold (M^{2n}, ω^2) at the point \mathbf{x} , we associate a 1-form ω_ξ^1 on $TM_{\mathbf{x}}$ by the formula:

$$\omega_\xi^1(\boldsymbol{\eta}) = \omega^2(\boldsymbol{\eta}, \xi) \quad \text{pour tout } \boldsymbol{\eta} \in TM_{\mathbf{x}}.$$

Remark 24.

- We denote by I the isomorphism $I : T^*M_{\mathbf{x}} \rightarrow TM_{\mathbf{x}}$ constructed above. Since $d\mathcal{H}$ is a differential 1-form on M , and at any point there exists a tangent vector to M which is associated to it. In this way, we obtain a vector field $Id\mathcal{H}$ on M .
- From another point of view we can introduce this vector field as the following: Let (M^{2n}, ω^2) be a symplectic manifold, and $\mathcal{H} : M^{2N} \rightarrow \mathbb{R}$ a differential function, since $d\mathcal{H}$ is a 1-form, and ω is non-degenerate there exists a unique vector field on M^{2n} noted $X_{\mathcal{H}}$ such that $i_{X_{\mathcal{H}}}\omega = d\mathcal{H}$ (see the definition (3.12), and then we have:

$$Id\mathcal{H} = X_{\mathcal{H}}$$

- To express \mathcal{H} in terms of $X_{\mathcal{H}}$ and ω^2 , we integrate from $t = 0$ to $t = 1$ the identity:

$$d\mathcal{H}(t\mathbf{x}) \cdot \mathbf{x} = \omega^2(X_{\mathcal{H}}(t\mathbf{x}), \mathbf{x})$$

The fundamental theorem of analysis gives:

$$\begin{aligned} \mathcal{H}(\mathbf{x}) - \mathcal{H}(0) &= \int_0^1 \frac{d\mathcal{H}(t\mathbf{x})}{dt} dt = \int_0^1 d\mathcal{H}(t\mathbf{x}) \cdot \mathbf{x} dt \\ &= \int_0^1 \omega^2(X_{\mathcal{H}}(t\mathbf{x}), \mathbf{x}) dt \end{aligned}$$

Definition 5.3. (Hamiltonian vector field)

The vector field $I, d\mathcal{H}$ is called a Hamiltonian vector field, \mathcal{H} is called a Hamiltonian function.

We denote $A^1(M)$ the vector space of vector fields on M .

Example 5.2.

As the example (5.1) we take $M^{2n} = \mathbb{R}^{2n}$ menu of form $\omega^2 = \sum_{1 \leq i < j \leq n} dp_i \wedge dq_j$, where $(\mathbf{p}, \mathbf{q}) = (p_1, \dots, p_n, q_1, \dots, q_n)$ the usual coordinate system of \mathbb{R}^{2n} the isomorphism I is given by:

$$I : T^*M_{\mathbf{x}} \simeq (\mathbb{R}^{2n})^* \rightarrow TM_{\mathbf{x}} \simeq \mathbb{R}^{2n}$$

and

$$I^{-1} : TM_{\mathbf{x}} \simeq \mathbb{R}^{2n} \rightarrow T^*M_{\mathbf{x}} \simeq (\mathbb{R}^{2n})^* \\ \xi \rightarrow \omega^2(\xi, \cdot)$$

the matrix of I^{-1} in the usual bases is given by (see (5.1)):

$$M(I^{-1}) = \begin{pmatrix} S & 0 & 0 \\ \vdots & \ddots & \dots \\ 0 & \dots & S \end{pmatrix}$$

where

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

then matrix of I is:

$$M(I) = (M(I^{-1}))^{-1} = -M(I^{-1}) = \begin{pmatrix} -S & 0 & 0 \\ \vdots & \ddots & \dots \\ 0 & \dots & -S \end{pmatrix}$$

Now let $\mathbf{x} = (p_1, q_1, \dots, p_n, q_n)$, the associated velocity vector is given by:

$$\dot{\mathbf{x}} = (\dot{p}_1, \dot{q}_1, \dots, \dot{p}_n, \dot{q}_n)$$

On the other hand let \mathcal{H} the Hamiltonian, we have:

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial p_1} dp_1 + \frac{\partial \mathcal{H}}{\partial q_1} dq_1 + \dots + \frac{\partial \mathcal{H}}{\partial p_n} dp_n + \frac{\partial \mathcal{H}}{\partial q_n} dq_n$$

Then,

$$\dot{\mathbf{x}} = I d\mathcal{H}(\mathbf{x}) \iff {}^t(\dot{p}_1, \dot{q}_1, \dots, \dot{p}_n, \dot{q}_n) = M(I) {}^t \left(\frac{\partial \mathcal{H}}{\partial p_1}, \frac{\partial \mathcal{H}}{\partial q_1}, \dots, \frac{\partial \mathcal{H}}{\partial p_n}, \frac{\partial \mathcal{H}}{\partial q_n} \right)$$

$$\dot{\mathbf{x}} = I d\mathcal{H}(\mathbf{x}) \iff \begin{pmatrix} -S & 0 & 0 \\ \vdots & \ddots & \dots \\ 0 & \dots & -S \end{pmatrix} {}^t \left(\frac{\partial \mathcal{H}}{\partial p_1}, \frac{\partial \mathcal{H}}{\partial q_1}, \dots, \frac{\partial \mathcal{H}}{\partial p_n}, \frac{\partial \mathcal{H}}{\partial q_n} \right)$$

$$\dot{\mathbf{x}} = I d\mathcal{H}(\mathbf{x}) \iff \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \quad \text{for all } i = 1, \dots, n$$

We deduce that Hamilton's canonical equations can be reduced to a single vector equation:

$$\dot{\mathbf{x}} = I d\mathcal{H}(\mathbf{x}) = X_{\mathcal{H}}(\mathbf{x})$$

5.3 Hamiltonian phase flow

Liouville's theorem claims that the phase flow preserves the volume. The famous mathematician Poincaré found a whole series of differential forms which are preserved by the Hamiltonian phase flow, and we will see in this section that the phase flow preserves the symplectic structure.

Definition 5.4. Let (M^{2n}, ω^2) be a symplectic manifold and $\mathcal{H} : M^{2n} \rightarrow \mathbb{R}$ a function, suppose that the vector field $Id\mathcal{H} = X_{\mathcal{H}}$ corresponding to \mathcal{H} gives a group of diffeomorphisms:

$$g^t : M^{2n} \rightarrow M^{2n}$$

$$\begin{cases} \frac{d}{dt}g^t(x) = X_{\mathcal{H}}(g^t(x)) \\ \frac{d}{dt}\big|_{t=0} g^t(\mathbf{x}) = Id\mathcal{H}(\mathbf{x}) = X_{\mathcal{H}}(\mathbf{x}). \\ g^0 = id_{M^{2n}} \end{cases}$$

The group g^t is called the Hamiltonian phase flow of the Hamiltonian function \mathcal{H} .

Remark 25.

- The previous definition generalizes the definition (1.2).
- $g^t(x)$ is the curve on the manifold that passes through x and such that the tangent at this point is the vector $X_{\mathcal{H}}(g^t(x))$.
- We assume in the following that the vector field X is differentiable (of \mathcal{C}^∞ -class) and has a compact support (i.e., X is zero outside a compact of M), which will be in particular the case if the manifold M is compact.
- Given a point $x \in M^{2n}$, let $g^t(x)$ be the position of x after a displacement of time $t \in \mathbb{R}$. We thus have an application:

$$g^t : M^{2n} \longrightarrow M^{2n}, \quad t \in \mathbb{R}$$

which is a diffeomorphism, the vector field $X_{athcalH}$ is related to a group of diffeomorphisms g^t on M^{2n} i.e. a differentiable application (of \mathcal{C}^∞ -class) : $M^{2n} \times \mathbb{R} \longrightarrow M$, verifying a group law :

- $\forall t \in \mathbb{R}, g^t : M^{2n} \longrightarrow M^{2n}$ is a diffeomorphism of M on M .
- $\forall t, s \in \mathbb{R}, g^{t+s} = g^t \circ g^s$.

The condition *ii*) means that $t \longmapsto g^t$, is a homomorphism of the additive group \mathbb{R} into the group of diffeomorphisms of M^{2n} into M^{2n} . and we have:

$$g^{-t} = (g^t)^{-1}$$

NB: From the above we can give g^t the following notation:

$$g^t = \exp(tX_{\mathcal{H}})$$

Example 5.3.

- Let the symplectic manifold $(M = \mathbb{R}^2, \omega^2 = dp \wedge dq)$ (Corresponds to the phase

space for a particle in rectilinear motion 1.2).

The Hamiltonien in this case is given by:

$$\mathcal{H} = \frac{1}{2}p^2 + \frac{1}{2}q^2, \quad d\mathcal{H} = pdp + qdq$$

the vector field corresponding to \mathcal{H} is given by:

$$X_{\mathcal{H}} = q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}$$

and the group of diffeomorphisms is:

$$g^t(p, q) = (\cos(t)p, \sin(t)q)$$

Indeed,

$$\frac{d}{dt}g^t(p, q) = (-\sin(t)p, \cos(t)q) = X_{\mathcal{H}}(g^t(p, q))$$

ii) Let the symplectic manifold $(\mathbb{S}^2, \omega^2 = d\varphi \wedge dh)$, the height h represents the Hamiltonian and we have:

$$X_{\mathcal{H}} = \frac{\partial}{\partial \varphi}$$

and the group of diffeomorphisms is given by:

$$g^t(\varphi, h) = (\varphi + t, h) [2\pi]$$

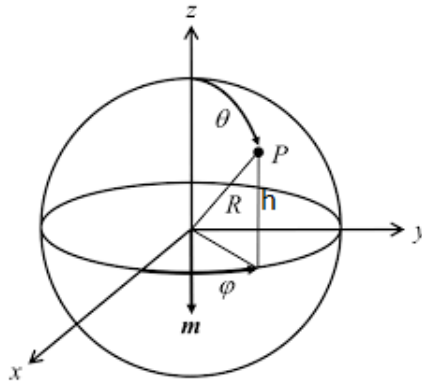


Figure 20:

Theorem 5.3. (admis)

A Hamiltonian phase flow preserves the symplectic structure i.e :

$$(g^t)^* \omega^2 = \omega^2$$

Where $(g^t)^* \omega^2$ denotes the pullback of ω^2 under the application g^t (see (3.8)).

- This result means that the quantity $(g^t)^* \omega^2$ does not depend on t i.e:

$$\frac{d}{dt}[(g^t)^* \omega^2] = 0$$

- In the case $n = 1, M^{2n} = \mathbb{R}^2$, this theorem says that the phase flow g^t preserves the area (Liouville theorem).

5.4 Integral invariants

Let $g : M \rightarrow M$ be a differentiable application.

Definition 5.5. A differential form ω of degree k is called an integral invariant of the application g if the integrals of ω on any k -chain c and on its image by g are the same i.e.

$$\int_{gc} \omega = \int_c \omega$$

Example 5.4.

If $M = \mathbb{R}^2$ and $\omega^2 = dp \wedge dq$ is the area element, then ω^2 is an integral invariant of any diffeomorphism g of Jacobian equal to 1 (change of variable theorem).

The theorem (5.3) can be reformulated as follows :

Theorem 5.4. The form ω^2 giving the symplectic structure is an integral invariant of a Hamiltonian phase flow g^t .

Indeed, according to (4.1) we have:

$$\int_{g^t c} \omega = \int_c (g^t)^* \omega = \int_c \omega$$

We now consider the exterior powers of ω^2 .

$$(\omega^2)^2 = \omega^2 \wedge \omega^2 \quad (\omega^2)^3 = \omega^2 \wedge \omega^2 \wedge \omega^2, \dots$$

corollary 5.1. Each of the forms $(\omega^2)^2, (\omega^2)^3, (\omega^2)^4, \dots$ is an integral invariant of a Hamiltonian phase flow g^t .

Remark 26.

We define a volume element on M^{2n} using $(\omega^2)^n$. Then a Hamiltonian phase flow preserves the volume. We obtain Liouville's theorem from the above corollary.

Example 5.5.

- Let us consider the symplectic coordinate space

$$M^{2n} = \mathbb{R}^{2n} = \{(\mathbf{p}, \mathbf{q})\}$$

$\omega^2 = d\mathbf{p} \wedge d\mathbf{q} = \sum dp_i \wedge dq_i$ in this case the form $(\omega^2)^k$ is proportional to the form:

$$\omega^{2k} = \sum_{i_1 < \dots < i_k} dp_{i_1} \wedge \dots \wedge dp_{i_k} \wedge dq_{i_1} \wedge \dots \wedge dq_{i_k}$$

- The integral of ω^{2k} is equal to the sum of the oriented volumes of the projections on the coordinate planes $(p_{i_1}, \dots, p_{i_k}, q_{i_1}, \dots, q_{i_k})$.

- The application $g : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ is called canonical if it has ω^2 as an integral invariant.
- A canonical application is usually called a canonical transformation.
- Each of the forms $\omega^4, \omega^6, \dots, \omega^{2n}$ is an integral invariant of any canonical transformation. Therefore, under a canonical transformation, the sum of the oriented areas of the projections onto the coordinate planes $(p_{i_1}, \dots, p_{i_k}, q_{i_1}, \dots, q_{i_k}), 1 \leq k \leq n$ is preserved. In particular, the canonical transformations preserve the volume.

The phase flow Hamiltonian given by the equations $\dot{\mathbf{p}} = -\partial H/\partial \mathbf{q}, \dot{\mathbf{q}} = \partial H/\partial \mathbf{p}$ consists of canonical transformations g^t . The integral invariants considered above are also called absolute integral invariants.

Definition 5.6. A differential k -form ω is called a relative integral invariant of the application $g : M \rightarrow M$ si $\int_{gc} \omega = \int_c \omega$ for each closed k -chain c .

Theorem 5.5. Let ω be a relative integral invariant of an application g . Then $d\omega$ is an absolute integral invariant of g .

Demonstration.

Let c be a $k + 1$ -chain, Then,

$$\int_c d\omega \stackrel{1}{=} \int_{g\partial c} \omega \stackrel{2}{=} \int_{g\partial c} \omega \stackrel{3}{=} \int_{\partial gc} \omega \stackrel{4}{=} \int_{gc} d\omega.$$

1 and 4 are by the Stokes formula, 2 by the definition of relative invariant, and 3 by the definition of the boundary.

Example 5.6.

Let a canonical application $g : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ (see 5.5) then,

$$\omega^1 = \mathbf{p} \, d\mathbf{q} = \sum_{i=1}^n p_i \, dq_i \text{ is a relative integral invariant of } g.$$

Indeed, any closed chain c on \mathbb{R}^{2n} is the boundary of a chain σ , and we find

$$\int_{gc} \omega^1 \stackrel{1}{=} \int_{g\partial\sigma} \omega^1 \stackrel{2}{=} \int_{\partial g\sigma} \omega^1 \stackrel{3}{=} \int_{g\sigma} d\omega^1 \stackrel{4}{=} \int_{\sigma} d\omega^1 \stackrel{5}{=} \int_{\partial\sigma} \omega^1 \stackrel{6}{=} \int_c \omega^1$$

1 and 6 are by definition of σ , 2 by definition of *partial*, 3 and 5 by Stokes' formula, and 4 since g is canonical and $d\omega^1 = d(pdq) = dq \wedge dq = \omega^2$.

NB: For more details on integral invariants see Arnold [1].

5.5 The Lie algebra of Hamiltonian functions

5.5.1 Lie algebra

Definition 5.7. A Lie algebra is a vector space E with a bilinear application $(x, y) \mapsto [x, y]$ from $E \times E$ into E which verifies the following properties:

1. $\forall x \in E, [x, x] = 0$
2. $\forall x, y, z \in E, [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

Remark 27.

- The product $[x, y]$ is called the Lie bracket of x and y .
- Since the bracket is an alternating bilinear application of x, y , then it is antisymmetric, i.e. $[x, y] = -[y, x]$ for all x, y in E .
- The identity (2) above is called the Jacobi identity.

Definition 5.8. A homomorphism of Lie algebras $(E, [., .]_E)$ into $(F, [., .]_F)$ is a linear application ϕ which respects the Lie bracket, i.e. such that:

$$\forall a, b \in E, \phi([a, b]_E) = [\phi(a), \phi(b)]_F$$

5.5.2 Dérivée de Lie

The Lie derivative is a natural differentiation operation on differential forms, generalizing the directional derivation of a function on an open \mathbb{R}^n or more generally on a differential manifold.

Definition 5.9. Let ω be a k -form and X be a vector field with phase flow g^t . The Lie derivative of ω along X is given by:

$$\mathcal{L}_X \omega = \lim_{t \rightarrow 0} \frac{1}{t} [((g^t)^* \omega) - \omega] = \left. \frac{d}{dt} (g^t)^* \omega \right|_{t=0} = \left. \frac{d}{dt} (\exp tX)^* \omega \right|_{t=0}$$

- If f is a real function on a manifold M (0-form) and X is a vector field on M , the Lie derivative f along X is coincident with the directional derivative:

$$\mathcal{L}_X f := \mathbf{d}f \cdot X$$

- Let (X^1, \dots, X^n) be the local components of field X we have:

$$\mathcal{L}_X f = X^i \frac{\partial f}{\partial x^i}$$

That's why we often write :

$$X = X^i \frac{\partial}{\partial x^i}$$

Then we have:

$$\mathcal{L}_X f = X[f] = X^i \frac{\partial f}{\partial x^i}$$

Theorem 5.6. (Lie derivative theorem)

$$\frac{d}{dt}\varphi_t^*\omega = \varphi_t^*\mathcal{L}_X\omega.$$

NB: In the expression $\mathcal{L}_X\omega$ the vector field X is evaluated at time t .

Definition 5.10. (Jacobi–Lie bracket)

Soient X et Y deux champs de vecteurs sur une variété M de composante locale (X^1, \dots, X^n) et (Y^1, \dots, Y^n) respectivement. Le **Crochet Jacobi–Lie** de X et Y est un champ de vecteurs sur M défini comme suite:

$$[X, Y] = \left[X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j} \right] = X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i}$$

Remark 28.

- We can define the Lie derivative of Y along X by:

$$\mathcal{L}_X Y = [X, Y]$$

- For a real function f on M we have:

$$[X, Y][f] = X[Y[f]] - Y[X[f]]$$

- The space of vector fields on M with Jacobi-Lie bracket is a Lie algebra and we have:

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$$

A very important and magical formula for the Lie derivative is given by the following theorem, under the name **Cartan formula**.

Theorem 5.7. (admitted)

Let X be a vector field and ω a k -form on a manifold M , we have:

$$\mathcal{L}_X\omega = \mathbf{d}i_X\omega + i_X\mathbf{d}\omega$$

Remark 29.

This formula and the Lie derivative theorem (5.6) proves the theorem (5.3), Indeed,

$$\begin{aligned} \frac{d}{dt}[(g^t)^*\omega] &= (g^t)^*\mathcal{L}_{X_{\mathcal{H}}}\omega = (g^t)^*(\mathbf{d}i_{X_{\mathcal{H}}}\omega + i_{X_{\mathcal{H}}}\mathbf{d}\omega) \\ &= (g^t)^*(\mathbf{d}(\mathbf{d}\mathcal{H}) + i_{X_{\mathcal{H}}}\mathbf{d}\omega) = 0 \end{aligned}$$

because $\mathbf{d} \circ \mathbf{d} = 0$ and ω is a closed form.

Proposition 5.1.

Let α and β two differential forms on a manifold M we have the following formulas:

- (a) $\mathcal{L}_f X \alpha = f \mathcal{L}_X \alpha + \mathbf{d}f \wedge \mathbf{i}_X \alpha$
- (b) $\mathcal{L}_{[X,Y]} \alpha = \mathcal{L}_X \mathcal{L}_Y \alpha - \mathcal{L}_Y \mathcal{L}_X \alpha;$
- (c) $\mathbf{i}_{[X,Y]} \alpha = \mathcal{L}_X \mathbf{i}_Y \alpha - \mathbf{i}_Y \mathcal{L}_X \alpha;$
- (d) $\mathcal{L}_X \mathbf{d} \alpha = \mathbf{d} \mathcal{L}_X \alpha$
- (e) $\mathcal{L}_X \mathbf{i}_X \alpha = \mathbf{i}_X \mathcal{L}_X \alpha$
- (f) $\mathcal{L}_X (\alpha \wedge \beta) = \mathcal{L}_X \alpha \wedge \beta + \alpha \wedge \mathcal{L}_X \beta.$

5.5.3 Poisson bracket

Definition 5.11. If f and g are two differentiable functions defined on the symplectic manifold (M, ω^2) , the function:

$$\{f, g\} = \omega^2(\text{Id}f, \text{Id}g)$$

is called the Poisson bracket of f and g .

Remark 30.

The Poisson's bracket verify the following properties:

- Linearity:

$$\{f, ag + bh\} = a\{f, g\} + b\{f, h\}$$

- Antisymmetry:

$$\{g, f\} = -\{f, g\}.$$

- Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

- Leibnez identity

$$\{f, gh\} = \{f, g\}h + \{f, h\}g$$

The Poisson bracket makes the space of differentiable functions on a symplectic manifold a Lie algebra.

Now we return to the definition (5.3), and note the vector field of any function f by X_f i.e,

$$\text{Id}f = X_f.$$

corollary 5.2. Let (M, ω^2) be a symplectic manifold, f and g two functions on M into \mathbb{R} we have:

$$\omega^2(X_f, X_g) = \omega^2(\text{Id}f, \text{Id}g) = \langle dg, \text{Id}f \rangle = \langle dg, X_f \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner scalar product.

Demonstration.

We can prove this result using the matrix notation, we use the notation of the example (5.2):

to simplify we can reason on the matrix of order 2 :

$$-S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and we have: $df = \frac{\partial f}{\partial p_1} dp_1 + \frac{\partial f}{\partial q_1} dq_1$, $dg = \frac{\partial g}{\partial p_1} dp_1 + \frac{\partial g}{\partial q_1} dq_1$

$$\text{then } -Sdf = \begin{pmatrix} -\frac{\partial f}{\partial q_1} \\ \frac{\partial f}{\partial p_1} \end{pmatrix} \quad \text{et} \quad -Sdg = \begin{pmatrix} -\frac{\partial g}{\partial q_1} \\ \frac{\partial g}{\partial p_1} \end{pmatrix}$$

Thus,

$$\begin{aligned} \omega^2(-Sdf, -Sdg) &= -\frac{\partial f}{\partial q_1} \frac{\partial g}{\partial q_1} \omega^2(e_1, e_1) - \frac{\partial f}{\partial q_1} \frac{\partial g}{\partial p_1} \omega^2(e_1, e_2) - \frac{\partial f}{\partial p_1} \frac{\partial g}{\partial q_1} \omega^2(e_2, e_1) \\ &\quad + \frac{\partial f}{\partial p_1} \frac{\partial g}{\partial p_1} \omega^2(e_2, e_2) \end{aligned}$$

$$\omega^2(-Sdf, -Sdg) = \frac{\partial f}{\partial p_1} \frac{\partial g}{\partial q_1} - \frac{\partial f}{\partial q_1} \frac{\partial g}{\partial p_1}$$

on the other hand we have:

$$\langle dg, -Sdf \rangle = \frac{\partial f}{\partial p_1} \frac{\partial g}{\partial q_1} - \frac{\partial f}{\partial q_1} \frac{\partial g}{\partial p_1}$$

and immediately we can generalize the result to the order $2n$ by taking the matrix:

$$M(I) = \begin{pmatrix} -S & 0 & 0 \\ \vdots & \ddots & \dots \\ 0 & \dots & -S \end{pmatrix}$$

Theorem 5.8. $f \mapsto X_f$ is a Lie algebra homomorphism from $C^\infty(M, \mathbb{R})$ with Poisson bracket, into $A^1(M)$ with Jacobi-Lie bracket, and we have:

$$[X_f, X_g] = X_{\{f, g\}}$$

Where $A^1(M)$ is the space of vector field.

5.6 Hamilton's equation and first integral

It is known that each vector field X defined on a differential manifold M determines a differential equation (see (4)).

$$\frac{d\varphi(t)}{dt} = X(\varphi(t))$$

Whose solutions are the differentiable parametric curves $\varphi : I \rightarrow M$, defined on an open interval $I \subset \mathbb{R}$, and the derivative at each point $t \in I$ is equal to the value of

X at point $\varphi(t)$.

On a symplectic manifold (M, ω^2) , the differential equation determined by the Hamiltonian vector field $X_{\mathcal{H}} = Id\mathcal{H}$ associated to a function \mathcal{H} is called Hamilton's equation associated to \mathcal{H} .

Let $\varphi : I \rightarrow M$ be a solution of Hamilton's equation, with Hamiltonian \mathcal{H} , i.e.

$$\frac{d\varphi(t)}{dt} = X_{\mathcal{H}}(\varphi(t))$$

The derivative of another differentiable function g on an integral curve φ of $X_{\mathcal{H}}$ is expressed by means of the Poisson bracket $\{\mathcal{H}, g\}$:

$$\begin{aligned} \frac{d(g \circ \varphi(t))}{dt} &= \left\langle dg(\varphi(t)), \frac{d\varphi(t)}{dt} \right\rangle \\ &= \langle dg(\varphi(t)), X_{\mathcal{H}}(\varphi(t)) \rangle \\ &= \omega^2(X_{\mathcal{H}}, X_g)(\varphi(t)) \\ &= \{\mathcal{H}, g\}(\varphi(t)) \end{aligned}$$

Definition 5.12. A first integral of the differential equation

$$\frac{d\varphi(t)}{dt} = X(\varphi(t))$$

is a function g that keeps a constant value on each integral curve of this equation.

$$\frac{d(g \circ \varphi(t))}{dt} = 0$$

Remark 31.

In the case of a Hamiltonian field, a differentiable function g defined on the symplectic manifold (M, ω) is a first integral of the Hamilton equation associated to the Hamiltonian \mathcal{H} if and only if:

$$\{\mathcal{H}, g\} = 0.$$

In this case, We say that the functions \mathcal{H} and g are in involution.

Theorem 5.9. (Energy theorem)

On a symplectic manifold (M, ω) , a Hamiltonian \mathcal{H} which is independent of time is the first integral of the associated Hamiltonian field $X_{\mathcal{H}}$.

This comes immediately from the antisymmetry of the Poisson bracket:

$$\{\mathcal{H}, \mathcal{H}\} = 0$$

Theorem 5.10. (Poisson theorem)

The Poisson bracket of two first integrals of the Hamiltonian vector field $X_{\mathcal{H}} = Id\mathcal{H}$ is a first integral.

This is an immediate consequence of Jacobi's identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Remark 32.

- The expression of the Hamilton equation associated with the function \mathcal{H} , with a local coordinate system $(p_1, q_1, \dots, p_n, q_n)$ in $(M^{2n}, \omega^2 = \sum_i dp_i \wedge dq_i)$, is given by:

$$\begin{cases} \frac{dp_i}{dt} = \{\mathcal{H}, p_i\} = -\frac{\partial \mathcal{H}}{\partial q_i} \\ \frac{dq_i}{dt} = \{\mathcal{H}, q_i\} = \frac{\partial \mathcal{H}}{\partial p_i} \end{cases} \quad 1 \leq i \leq n. \quad (6)$$

The expression of the Poisson's bracket of two functions f and g in this case is:

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)$$

- The local coordinates $(p_1, q_1, \dots, p_n, q_n)$ are related by the relations:

$$\{q_j, q_k\} = 0 \quad \{q_j, p_k\} = \delta_k^j \quad \{p_j, p_k\} = 0$$

- As the partial derivatives commute we can show that:

$$\frac{\partial}{\partial t} \{f, g\} = \left\{ \frac{\partial f}{\partial t}, g \right\} + \left\{ f, \frac{\partial g}{\partial t} \right\}$$

- Let f be a function on $(M^{2n}, \omega^2 = \sum_i dp_i \wedge dq_i)$. It follows from the previous relations that :

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{\mathcal{H}, f\}.$$

6 Numerical resolution

Sometimes the explicit solution of Hamiltonian system is difficult, so we must think of numerical methods to describe our dynamic system. In this section we propose the Petrov-Galerkin finite element formulation: Let be the Hamiltonian equation:

$$\begin{cases} \dot{\mathbf{q}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \\ \dot{\mathbf{p}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}} \end{cases}$$

Where $\mathbf{q}(t) = [q^1(t) \ q^2(t) \ \dots \ q^d(t)]^T$ and $\mathbf{p}(t) = [p^1(t) \ p^2(t) \ \dots \ p^d(t)]^T$.

The Petrov-Galerkin formulation between time t_n and t_{n+1} is expressed as follows:

$$\begin{cases} \int_{t_n}^{t_{n+1}} \mathbf{W}_q^T \left(\dot{\mathbf{q}} - \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \right) dt = 0 \\ \int_{t_n}^{t_{n+1}} \mathbf{W}_p^T \left(\dot{\mathbf{p}} + \frac{\partial \mathcal{H}}{\partial \mathbf{q}} \right) dt = 0 \end{cases}$$

i.e

$$\begin{cases} \int_{t_n}^{t_{n+1}} \mathbf{W}_q^T \dot{\mathbf{q}} - \mathbf{W}_q^T \frac{\partial \mathcal{H}}{\partial \mathbf{p}} dt = 0 \\ \int_{t_n}^{t_{n+1}} \mathbf{W}_p^T \dot{\mathbf{p}} + \mathbf{W}_p^T \frac{\partial \mathcal{H}}{\partial \mathbf{q}} dt = 0 \end{cases} \quad (7)$$

Where \mathbf{W}_p and \mathbf{W}_q are the weight functions with respect to \mathbf{p} and \mathbf{q} , respectively.

Let the Lagrange polynomials be defined as:

$$\left\{ L_j(t) = \frac{\prod_{i=0, i \neq j}^k (t - t_i)}{\prod_{i=0, i \neq j}^k (t_j - t_i)} \right\}_{j=0}^k$$

With $t_j = t_n + \frac{j}{k} \Delta t$ the $k + 1$ interpolation points, so we can approximate \mathbf{q} and \mathbf{p} as follows:

$$\mathbf{q}(t) = \begin{bmatrix} q^1(t) \\ q^2(t) \\ \vdots \\ q^d(t) \end{bmatrix} \approx \begin{bmatrix} \sum_{j=0}^k q_{n+\frac{j}{k}}^1 L_j(t) \\ \sum_{j=0}^k q_{n+\frac{j}{k}}^2 L_j(t) \\ \vdots \\ \sum_{j=0}^k q_{n+\frac{j}{k}}^d L_j(t) \end{bmatrix} = [\mathbf{L}_0 \ \mathbf{L}_1 \ \dots \ \mathbf{L}_k] \begin{bmatrix} \mathbf{q}_{n+\frac{0}{k}} \\ \mathbf{q}_{n+\frac{1}{k}} \\ \vdots \\ \mathbf{q}_{n+\frac{k}{k}} \end{bmatrix} \triangleq \bar{\mathbf{L}} \mathbf{q}$$

$$\mathbf{p}(t) = \begin{bmatrix} p^1(t) \\ p^2(t) \\ \vdots \\ p^d(t) \end{bmatrix} \approx \begin{bmatrix} \sum_{j=0}^k p_{n+\frac{j}{k}}^1 L_j(t) \\ \sum_{j=0}^k p_{n+\frac{j}{k}}^2 L_j(t) \\ \vdots \\ \sum_{j=0}^k p_{n+\frac{j}{k}}^d L_j(t) \end{bmatrix} = [\mathbf{L}_0 \ \mathbf{L}_1 \ \dots \ \mathbf{L}_k] \begin{bmatrix} \mathbf{p}_{n+\frac{0}{k}} \\ \mathbf{p}_{n+\frac{1}{k}} \\ \vdots \\ \mathbf{p}_{n+\frac{k}{k}} \end{bmatrix} \triangleq \bar{\mathbf{L}} \mathbf{p}$$

where $\mathbf{L}_k = L_k(t) \mathbf{I}_d$, and \mathbf{I}_d the identity matrix of order d .

$\mathbf{q}_{n+j/k} \approx \mathbf{q}(t_n + j/k \Delta t)$ and $\mathbf{p}_{n+j/k} \approx \mathbf{p}(t_n + j/k \Delta t)$ are the approximate numerical values of \mathbf{q} and \mathbf{p} at time $t_j = t_n + j/k \Delta t$.

We pose:

$$\begin{aligned} \mathbf{B}_k &= \frac{\partial \mathbf{L}_k}{\partial t} = \frac{\partial L_k(t)}{\partial t} \mathbf{I}_d \\ \bar{\mathbf{B}} &= [\mathbf{B}_0 \ \mathbf{B}_1 \ \dots \ \mathbf{B}_k] \end{aligned}$$

Hence:

$$\dot{\mathbf{q}}(t) = \overline{\mathbf{B}}\mathbf{q} \quad ; \quad \dot{\mathbf{p}}(t) = \overline{\mathbf{B}}\mathbf{p}$$

For the choice of the weight functions we take:

$$\begin{aligned} \mathbf{W}_q &= [\mathbf{B}_0 \quad \mathbf{B}_1 \quad \cdots \quad \mathbf{B}_k] + \frac{1}{\Delta t} [\beta_0^q \mathbf{L}_0 \quad \beta_1^q \mathbf{L}_1 \quad \cdots \quad \beta_k^q \mathbf{L}_k] \\ \mathbf{W}_p &= [\mathbf{B}_0 \quad \mathbf{B}_1 \quad \cdots \quad \mathbf{B}_k] + \frac{1}{\Delta t} [\beta_0^p \mathbf{L}_0 \quad \beta_1^p \mathbf{L}_1 \quad \cdots \quad \beta_k^p \mathbf{L}_k] \end{aligned}$$

with $\{\beta_k^p, \beta_k^q\}_k$ called the stabilization parameters of the algorithm, the choice of these parameters is not arbitrary and there is an algorithm to choose them well (see [17]).

In the case where the energy is conserved in time i.e $\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} = 0$, the parameters $\{\beta_k^p, \beta_k^q\}_k$ are all zero, and therefore:

$$\mathbf{W}_q = \mathbf{W}_p = [\mathbf{B}_0 \quad \mathbf{B}_1 \quad \cdots \quad \mathbf{B}_k]$$

The initial values $\mathbf{q}(t_n) = \mathbf{q}_n$ and $\mathbf{p}(t_n) = \mathbf{p}_n$ are known, so we can reduce the weight functions:

$$\hat{\mathbf{W}}_q = \hat{\mathbf{W}}_p = [\mathbf{B}_1 \quad \mathbf{B}_2 \quad \cdots \quad \mathbf{B}_k]$$

We also assume that $\frac{\partial \mathcal{H}}{\partial \mathbf{q}} = \tilde{\mathbf{K}}\mathbf{q}$, i.e $\tilde{\mathbf{K}}$ is a matrix of order d .

So the equation (7) becomes:

$$\mathbf{A}\mathbf{X} = 0$$

Where,

$$\begin{aligned} \mathbf{A} &= \int_{t_n}^{t_{n+1}} \hat{\mathbf{W}}^T (\overline{\mathbf{B}} + \mathbf{K}\overline{\mathbf{L}}) dt \\ \mathbf{X} &= \left[\mathbf{q}_{n+\frac{0}{k}} \quad \mathbf{q}_{n+\frac{1}{k}} \quad \cdots \quad \mathbf{q}_{n+1} \quad \mathbf{p}_{n+\frac{0}{k}} \quad \mathbf{p}_{n+\frac{1}{k}} \quad \cdots \quad \mathbf{p}_{n+1} \right]^T \\ \hat{\mathbf{W}}^T &= \begin{bmatrix} \hat{\mathbf{W}}_q^T & \mathbf{0}_d \\ \mathbf{0}_d & \hat{\mathbf{W}}_p^T \end{bmatrix}; \quad \mathbf{K} = \begin{bmatrix} \mathbf{0}_d & \mathbf{I}_d \\ \tilde{\mathbf{K}} & \mathbf{0}_d \end{bmatrix} \end{aligned}$$

with the change variable $s = \frac{2}{t_{n+1}-t_n} (t - t_n) - 1$, We can calculate \mathbf{A} as follows

$$\mathbf{A} = \frac{\Delta t}{2} \int_{-1}^1 \hat{\mathbf{W}}^T (\mathbf{M}\overline{\mathbf{B}} + \mathbf{K}\overline{\mathbf{L}}) ds$$

Finally we find the matrix \mathbf{X} ($d \times (2k + 2)$) and we represent the evolution of the dynamical system in the phase space as a function of time $(\mathbf{q}(t), \mathbf{p}(t))$.

Remark 33.

In the same way we can realize a numerical scheme of the Hamiltonian system using Poisson's bracket. Indeed, according to (6) the Hamiltonian system is equivalent to:

$$\dot{\mathbf{x}} = \{\mathcal{H}, \mathbf{x}\}$$

Where $\mathbf{x}(t) = (\mathbf{q}(t), \mathbf{p}(t))$, and thus the Petrov-Galerkin formulation is written:

$$\int_{t_n}^{t_{n+1}} \mathbf{W}^T \dot{\mathbf{x}}^T - \mathbf{W}^T \{\mathcal{H}, \mathbf{x}\}^T dt = 0$$

7 Examples

In this part we will treat some dynamic systems using the Hamiltonian approach, we will see that despite the complexity of a system we can determine the equation of motion or a set of equations that describes the system, the use of Hamiltonian formalism is often convenient, the difficulty is just in the determination of the Hamiltonian, is what is not the case in Newton's equation where we are looking for the vectors of forces, positions, velocities and accelerations, which contains in the majority of cases a lot of equation, variables and unknown.

7.1 Double pendulum

In this example we will show the ease of finding the equations that describe the system using the Hamiltonian approach, and compare it to the Newtonian approach.

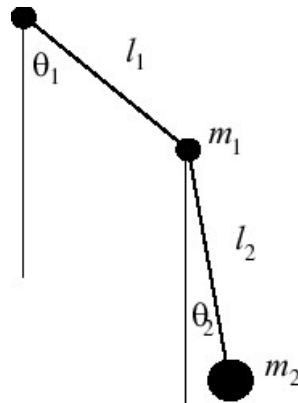


Figure 21:

7.1.1 Hamiltonian approach

the pendulum consists of two rods of zero mass and length l_1 and l_2 , and two masses m_1 and m_2 .

The kinetic energy is given by:

$$T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 \left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]$$

where θ_i is the angle to the vertical and v_i is the velocity of the pendulum i .

The potential energy is given by:

$$V = m_1gz_1 + m_2gz_2$$

(z_i is the altitude of the mass m_i),
with $V = -(m_1 + m_2)gl_1 \cos(\theta_1) - m_2gl_2 \cos(\theta_2)$

Then the Lagrangian is: $\mathcal{L} = T - V$

$$\mathcal{L} = \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ + (m_1 + m_2) g l_1 \cos(\theta_1) + m_2 g l_2 \cos(\theta_2)$$

We pose

$$\begin{cases} p_1 := \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ p_2 := \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \end{cases}$$

We see that the relation is also written;

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = M(\theta_1, \theta_2) \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

Where:

$$M(\theta_1, \theta_2) := \begin{pmatrix} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 \cos(\theta_1 - \theta_2) \\ m_2 l_1 l_2 \cos(\theta_1 - \theta_2) & m_2 l_2^2 \end{pmatrix}$$

Since :

$$\det M(\theta_1, \theta_2) = m_2 (l_1 l_2)^2 (m_1 + m_2 (1 - \cos^2(\theta_1 - \theta_2))) > 0.$$

Then :

$$\begin{cases} \dot{\theta}_1 = \frac{m_2 l_2^2 p_1 - m_2 l_1 l_2 \cos(\theta_1 - \theta_2) p_2}{m_2 (l_1 l_2)^2 (m_1 + m_2 (1 - \cos^2(\theta_1 - \theta_2)))} \\ \dot{\theta}_2 = \frac{(m_1 + m_2) l_1^2 p_2 - m_2 l_1 l_2 \cos(\theta_1 - \theta_2) p_1}{m_2 (l_1 l_2)^2 (m_1 + m_2 (1 - \cos^2(\theta_1 - \theta_2)))} \end{cases}$$

But then we know that if we construct the function:

$$\mathcal{H} := p_1 \dot{\theta}_1 + p_2 \dot{\theta}_2 - \mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \\ = (m_1 + m_2) \frac{(l_1 \dot{\theta}_1)^2}{2} + m_2 \frac{(l_2 \dot{\theta}_2)^2}{2} + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ - (m_1 + m_2) g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2)$$

The Hamilton equation is then written as :

$$\begin{cases} \dot{\theta}_1 = \frac{\partial \mathcal{H}}{\partial p_1} \\ \dot{\theta}_2 = \frac{\partial \mathcal{H}}{\partial p_2} \\ \dot{p}_1 = -\frac{\partial \mathcal{H}}{\partial \theta_1} \\ \dot{p}_2 = -\frac{\partial \mathcal{H}}{\partial \theta_2} \end{cases}$$

By replacing $\dot{\theta}_1$ and $\dot{\theta}_2$ in the expression of \mathcal{H} , we find \mathcal{H} in function of $(\theta_1, \theta_2, p_1, p_2)$:

$$\mathcal{H} = \frac{m_2 l_2^2 p_1^2 + (m_1 + m_2) l_1^2 p_2^2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2) p_1 p_2}{m_2 l_1^2 l_2^2 (m_1 + m_2 (1 - \cos^2(\theta_1 - \theta_2)))} \\ - (m_1 + m_2) g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2)$$

We finally have the system that describes the evolution of the pendulum:

$$\begin{cases} \dot{\theta}_1 = \frac{m_2 r_2^2 p_1 - m_2 l_1 l_2 \cos(\theta_1 - \theta_2) p_2}{m_2 l_1^2 l_2^2 (m_1 + m_2 (1 - \cos^2(\theta_1 - \theta_2)))} \\ \dot{\theta}_2 = \frac{(m_1 + m_2) l_1^2 p_2 - m_2 l_1 l_2 \cos(\theta_1 - \theta_2) p_1}{m_2 l_1^2 l_2^2 (m_1 + m_2 (1 - \cos^2(\theta_1 - \theta_2)))} \\ \dot{p}_1 = -m_1 g l_1 \sin(\theta_1) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g l_1 \sin(\theta_1) \\ \dot{p}_2 = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin(\theta_2) \end{cases}$$

7.1.2 Newtonian approach

The points m_1 and m_2 can be represented by their affixes z_1 and z_2 in the complex plane, such that:
$$\begin{cases} z_1 = l_1 e^{i\theta_1} \\ z_2 = l_2 e^{i\theta_2} + l_1 e^{i\theta_1} \end{cases}$$

The accelerations are given by :

$$\begin{cases} \ddot{z}_1 = \frac{d^2 z_1}{dt^2} = l_1 \left(i\ddot{\theta}_1 - \dot{\theta}_1^2 \right) e^{i\theta_1} \\ \ddot{z}_2 = \frac{d^2 z_2}{dt^2} = l_1 \left(i\ddot{\theta}_1 - \dot{\theta}_1^2 \right) e^{i\theta_1} + l_2 \left(i\ddot{\theta}_2 - \dot{\theta}_2^2 \right) e^{i\theta_2} \end{cases}$$

When a mass is suspended from a rod, a tension occurs along it. Let us call, T_1 and T_2 , the tensions exerted by the masses m_1 and m_2 and represent them as complexes t_1 and t_2 .

We then observe that t_1 , z_1 and 0 are collinear, which allows us to write: $t_1 = k_1 z_1 = k_1 l_1 e^{i\theta_1}$ ($k_1 \in \mathbb{R}$). In the same way, $t_2 = k_2 l_2 e^{i\theta_2}$ ($k_2 \in \mathbb{R}$)

The mobile 2 under the tension T_2 and the force due to gravity ($m_2 g$), giving the following relations:

$$F = m\ddot{z}_2 = t_2 + m_2 g \iff m_2 \ddot{z}_2 - m_2 g = t_2 \iff \ddot{z}_2 - g = \frac{t_2}{m_2} = \frac{k_2 z_0}{m_2} \Rightarrow \frac{\ddot{z}_2 - g}{z_0} = \frac{k_2}{m_2}$$

In the same manner:

$$\begin{aligned} F = m a_{z_1} = t_1 - t_2 + m_1 g &\iff t_1 = t_2 - m_1 g + m_1 g \ddot{z}_1 \iff t_1 = m_2 \ddot{z}_2 - m_2 g - m_1 g + m_1 \ddot{z}_1 \\ &\iff m_2 \ddot{z}_2 - m_2 g - m_1 g + m_1 \ddot{z}_1 = k_1 z_1 \iff \frac{m_2 \ddot{z}_2 - m_2 g - m_1 g + m_1 \ddot{z}_1}{z_1} = k_1 \end{aligned}$$

Concerning these two equations, the left members are complexes while the right members are real, we get:

$$\frac{\ddot{z}_2 - g}{z_0} = \frac{l_1 \left(i\ddot{\theta}_1 - \dot{\theta}_1^2 \right) e^{i\theta_1} + l_2 \left(i\ddot{\theta}_2 - \dot{\theta}_2^2 \right) e^{i\theta_2} - g}{l_2 e^{i\theta_2}}$$

$$\text{Im} \left(\frac{\ddot{z}_2 - g}{z_0} \right) = 0$$

$$l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + l_2 \ddot{\theta}_2 - l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + g \sin(\theta_2) = 0$$

Similarly for the second equation:

$$\operatorname{Im} \left(\frac{m_2 \ddot{z}_2 - m_2 g - m_1 g + m_1 \ddot{z}_1}{z_1} \right) = 0$$

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin(\theta_1) = 0$$

Finally, we obtain a system that describes the evolution of motion:

$$\begin{cases} l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + l_2 \ddot{\theta}_2 - l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + g \sin(\theta_2) = 0 \\ (m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin(\theta_1) = 0 \end{cases}$$

We notice that the Newtonian approach is more complicated, what reduces a little the calculation is just the use of the complex notation. Fortunately, this method is applicable in this case because the motion is in the plane. On the other hand, the differential system obtained is of degree 2, whereas by the Hamiltonian approach it is a system of degree 1.

7.2 Damped harmonic oscillator

The objective of this example, is to show that the Hamiltonian does not always represent the total energy of the system. Moreover, we will see that in this case the Hamiltonian depends on time .i.e

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} \neq 0$$

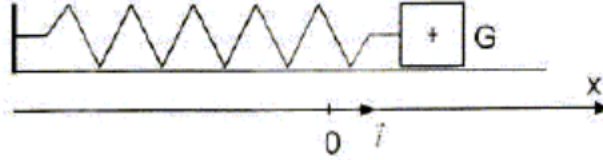


Figure 22:

The damped Lagrangian of the harmonic oscillator in rectilinear motion is given by:

$$\mathcal{L} = \mathcal{L}(q, \dot{q}, t) = e^{\gamma(t)} \left(\frac{m}{2} \dot{q}^2 - V(q) \right)$$

Where $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ is a damping function, and V represents the potential energy.

The equation of motion of the oscillator is given by:

$$m\ddot{q} + m\dot{\gamma}(t)\dot{q} + \nabla V(q) = 0$$

The mechanical energy of the system is:

$$\mathcal{E} = \frac{m}{2} \dot{q}^2 + V(q)$$

Since $\dot{\mathcal{E}} = \frac{d\mathcal{E}}{dt} = -m\dot{\gamma}\dot{q}^2 \leq 0$, the energy is dissipated.

The momentum variable is given by:

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m e^{\gamma(t)} \dot{q}$$

which now also depends on the time, therefore:

$$\dot{q} = \frac{1}{m e^{\gamma t}} p$$

The Hamiltonian is the Legend transform of the damped Lagrangian:

$$\mathcal{H} = p\dot{q} - \mathcal{L} = \frac{1}{m e^{\gamma(t)}} p^2 - e^{\gamma(t)} \left(\frac{m}{2} \left(\frac{1}{m e^{\gamma(t)}} p \right)^2 - V(q) \right) = \frac{1}{2 m e^{\gamma(t)}} p^2 + e^{\gamma(t)} V(q)$$

which we can also express in terms of q and $\dot{q} = \frac{1}{m e^{\gamma t}} p$:

$$\mathcal{H} = e^{\gamma(t)} \left(\frac{m}{2} \dot{q}^2 + V(q) \right)$$

Thus, the Hamiltonian is the sum of the kinetic energy and the potential energy (damped), while the Lagrangian is the difference, just like in the ideal case.

However, observe that the Hamiltonian is not conserved:

$$\dot{\mathcal{H}} = \dot{\gamma}(t) \mathcal{H} + e^{\gamma(t)} (m \dot{q} \ddot{q} + \nabla V(q) \dot{q}) = \dot{\gamma}(t) \mathcal{H} - m \dot{\gamma}(t) e^{\gamma(t)} \dot{q}^2 = -\dot{\gamma}(t) \mathcal{L} \neq 0$$

Hamilton's equations are follows:

$$\begin{cases} \dot{q} &= \frac{\partial \mathcal{H}}{\partial p} = \frac{1}{m e^{\gamma(t)}} p \\ \dot{p} &= -\frac{\partial \mathcal{H}}{\partial q} = -e^{\gamma(t)} \nabla V(q) \end{cases}$$

The divergence of the velocity vector $\mathbf{v} = (\dot{p}, \dot{q})$ is zero, indeed,

$$\frac{\partial}{\partial q} \left(\frac{1}{m e^{\gamma(t)}} p \right) + \frac{\partial}{\partial p} \left(-e^{\gamma(t)} \nabla V(q) \right) = 0.$$

This means that we have the volume is preserved in the phase space (see remark (19)).

Now taking the simple example where the function γ is linear (see [19]), called the damped harmonic oscillator with friction:

$$m \ddot{q} + c \dot{q} + k q = 0$$

Then: $\gamma(t) = \frac{ct}{m}$ and the potential energy is quadratic ($V(q) = \frac{k}{2} q^2$), so the Lagrangian is:

$$\mathcal{L} = e^{\frac{ct}{m}} \left(\frac{m}{2} \dot{q}^2 - \frac{k}{2} q^2 \right)$$

The Hamiltonian is given by:

$$\mathcal{H} = \frac{e^{-\frac{ct}{m}}}{2m} p^2 + \frac{k e^{ct/m}}{2} q^2$$

With the momentum $p = me^{\frac{ct}{m}} \dot{q}$.

Hamilton's equations are follows:

$$\begin{cases} \dot{q} = \frac{e^{-\frac{ct}{m}}}{m} p \\ \dot{p} = -k e^{\frac{ct}{m}} q \end{cases} \quad (8)$$

When $\omega^2 := \frac{k}{m} - \frac{c^2}{4m^2} \geq 0$, we can find the explicit solutions of the system (8):

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = M(t) \begin{pmatrix} q(0) \\ p(0) \end{pmatrix}$$

Where $M(t)$ is the matrix:

$$M(t) = \begin{pmatrix} e^{-\frac{ct}{2m}} \left(\cos(\omega t) + \frac{c}{2m\omega} \sin(\omega t) \right) & e^{-\frac{ct}{2m}} \frac{1}{m\omega} \sin(\omega t) \\ -e^{\frac{ct}{2m}} \left(m\omega + \frac{c^2}{4m\omega} \right) \sin(\omega t) & e^{\frac{ct}{2m}} \left(\cos(\omega t) - \frac{c}{2m\omega} \sin(\omega t) \right) \end{pmatrix}$$

We have seen that Liouville's theorem shows that the phase flow preserves the volume, so in this case the determinant of the matrix M must not depend on time, indeed,

$$\det(M(t)) = \cos^2(\omega t) - \frac{c^2 \sin^2(\omega t)}{4m^2\omega^2} + \sin^2(\omega t) + \frac{c^2 \sin^2(\omega t)}{4m^2\omega^2} = 1$$

.

7.3 The two-body orbital problem

We consider the two-body orbital problem with $m_1 \gg m_2$. In this case, we can treat the body of mass m_1 as essentially stationary. Moreover, with only two bodies, the coordinate system can always be chosen such that m_1 is at the origin, and $z = 0$, i.e. the motion is confined to the plane (x, y) .

the Hamiltonian is:

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m_2} - \frac{Gm_1m_2}{\sqrt{x^2 + y^2}}$$

We pose,

$$L = xp_y - yp_x$$

to be a constant of motion. If this is the case, the Poisson's bracket of \mathcal{H} with L should be zero:

$$\begin{aligned} \{\mathcal{H}, L\} &= \frac{\partial \mathcal{H}}{\partial p_x} \frac{\partial L}{\partial x} - \frac{\partial \mathcal{H}}{\partial x} \frac{\partial L}{\partial p_x} + \frac{\partial \mathcal{H}}{\partial p_y} \frac{\partial L}{\partial y} - \frac{\partial \mathcal{H}}{\partial y} \frac{\partial L}{\partial p_y} \\ &= \frac{p_x}{m_2} p_y - \frac{Gm_1m_2x}{(x^2 + y^2)^{3/2}} (-y) + \frac{p_y}{m_2} (-p_x) - \frac{Gm_1m_2y}{(x^2 + y^2)^{3/2}} x \\ &= 0. \end{aligned}$$

Thus L is the first integral of \mathcal{H} (voir (31)).

7.4 Charged particle in an electromagnetic field

To describe the motion of a charged particle in an electromagnetic field, we use the Hamiltonian mechanics. We consider the phase space coordinates \mathbf{r} and \mathbf{p} represent the position and the momentum respectively. The non-relativistic Hamiltonian, corresponding to the total energy of the system:

$$\mathcal{H}(\mathbf{r}, \mathbf{p}, t) = \frac{(\mathbf{p} - q\mathbf{A}(\mathbf{r}, t))^2}{2m} + q\Phi(\mathbf{r}, t)$$

With, m, q, A , and Φ denote the mass of the particle, its charge, the magnetic vector potential and the electric potential, respectively.

The first part is equal to the kinetic energy, but the second part is not necessarily potential energy, since in general Φ can be changed arbitrarily with a gauge.

Charged particles are treated as relativistic particles, in this case we take the relativistic Hamiltonian:

$$\begin{aligned} \mathcal{H} &= \sqrt{m^2c^4 + \|\mathbf{p} - q\mathbf{A}\|^2c^2} + q\Phi \\ &= \gamma mc^2 + q\Phi \end{aligned}$$

Where, $\gamma = \sqrt{1 + \frac{\|\mathbf{p} - q\mathbf{A}\|^2}{c^2}}$ If we take the non-relativistic case, the Hamiltonian equations are:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \frac{dr}{dt} \\ \frac{dp}{dt} \end{pmatrix} = \{\mathcal{H}, \mathbf{x}\} = \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial p} \\ -\frac{\partial \mathcal{H}}{\partial r} \end{pmatrix} = X_{\mathbf{x}(t)},$$

Where $x := (\mathbf{r}, \mathbf{p})^T$, and $X_{\mathbf{x}(t)}$ denotes the Hamiltonian vector field:

$$\mathbf{X}_{\mathbf{x}(t)} := \begin{pmatrix} \frac{\mathbf{p} - q\mathbf{A}}{m} \\ \frac{q}{m} \sum_i \frac{\partial A_i}{\partial \mathbf{r}} (p_i - qA_i) - q \frac{\partial \Phi}{\partial \mathbf{r}} \end{pmatrix}$$

For the relativistic case, the Hamiltonian vector field is given by:

$$\begin{aligned} \mathbf{X}_{\mathbf{x}(t)} &= \begin{pmatrix} \frac{\mathbf{p} - q\mathbf{A}}{\sqrt{m^2 + \frac{1}{c^2} \|\mathbf{p} - q\mathbf{A}\|^2}} \\ q \sum_i \frac{\partial A_i}{\partial \mathbf{r}} \frac{p_i - qA_i}{\sqrt{m^2 + \frac{1}{c^2} \|\mathbf{p} - q\mathbf{A}\|^2}} - q \frac{\partial \Phi}{\partial \mathbf{r}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\mathbf{p} - q\mathbf{A}}{\gamma m} \\ q \sum_i \frac{\partial A_i}{\partial \mathbf{r}} \frac{p_i - qA_i}{\gamma m} - q \frac{\partial \Phi}{\partial \mathbf{r}} \end{pmatrix} \end{aligned}$$

7.5 Fluid mechanics

In this example, we try to make a Hamiltonian formulation to describe the fluid dynamics. An important remark in this case is that the degree of freedom is infinite, To describe this type of dynamics, we introduce the notion of functional.

A functional is an application that each function gives us a real number:

$$K : \mathcal{F} \rightarrow \mathbb{R}$$

We note the image of $u(x) \in \mathcal{F}$ by $K[u]$. We want to give the variation of $K[u]$ when we make a small change around u , let $u(x) \rightarrow u(x) + \epsilon \delta u(x)$, where $u + \epsilon \delta u$ must always be in our domain. The first order variation of K induced by δu , called the first variation, noted δK , and given by:

$$\begin{aligned} \delta K[u, \delta u] &:= \lim_{\epsilon \rightarrow 0} \frac{K[u + \epsilon \delta u] - K[u]}{\epsilon} \\ &= \left. \frac{d}{d\epsilon} K[u + \epsilon \delta u] \right|_{\epsilon=0} \\ &=: \int_{x_0}^{x_1} \delta u \frac{\delta K}{\delta u} dx =: \left\langle \frac{\delta K}{\delta u}, \delta u \right\rangle. \end{aligned}$$

Example 7.1. Let the following functional:

$$T[u] = \frac{1}{2} \int_{x_0}^{x_1} \rho u^2 dx$$

The first variation of T is follows:

$$\delta T[u, \delta u] = \int_{x_0}^{x_1} \rho u \delta u dx$$

And the functional derivative is given by:

$$\frac{\delta T}{\delta u} = \rho u$$

Consider now a more general functional, one of the form

$$F[u] = \int_{x_0}^{x_1} \mathcal{F}(x, u, u_x, u_{xx}, \dots) dx$$

where \mathcal{F} is an ordinary, sufficiently differentiable, function of its arguments. Note $u_x := du/dx$, $u_{xx} := d^2u/dx^2$, etc. The first variation of F is the following:

$$\delta F[u, \delta u] = \int_{x_0}^{x_1} \left(\frac{\partial \mathcal{F}}{\partial u} \delta u + \frac{\partial \mathcal{F}}{\partial u_x} \delta u_x + \frac{\partial \mathcal{F}}{\partial u_{xx}} \delta u_{xx} + \dots \right) dx$$

which integration by parts gives:

$$\delta F[u, \delta u] = \int_{x_0}^{x_1} \delta u \left(\frac{\partial \mathcal{F}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{F}}{\partial u_x} + \frac{d^2}{dx^2} \frac{\partial \mathcal{F}}{\partial u_{xx}} - \dots \right) dx + \left(\frac{\partial \mathcal{F}}{\partial u_x} \delta u + \dots \right) \Big|_{x_0}^{x_1}$$

to make the right term, the boundary term, vanishes, we can assume the boundary conditions ; $\delta u(x_0) = \delta u(x_1) = 0$, $\delta u_x(x_0) = \delta u_x(x_1) = 0$, etc.

Sometimes the boundary term vanishes due to the nature of the function \mathcal{F} . Assuming that the boundary term vanishes. We get:

$$\delta F[u, \delta u] = \left\langle \frac{\delta F}{\delta u}, \delta u \right\rangle$$

Then,

$$\frac{\delta F}{\delta u} = \frac{\partial \mathcal{F}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{F}}{\partial u_x} + \frac{d^2}{dx^2} \frac{\partial \mathcal{F}}{\partial u_{xx}} - \dots \quad (9)$$

We can generalize what is above in the three dimensions. We assume that the function \mathcal{F} depends only on u and its first derivatives. Let $\Omega \in \mathbb{R}^3$ and $u : \Omega \rightarrow \mathbb{R}^3$. We take the following functional:

$$F[u] = \int_{\Omega} \mathcal{F}\left(u, \left(\frac{\partial u}{\partial x^i}\right)_{i=1,2,3}\right) d\mathbf{r}$$

Where, $\mathbf{r} = (x^1, x^2, x^3)$ and $d\mathbf{r} = dx^1 dx^2 dx^3$. Let compute the first variation of F :

$$\delta F[u, \delta u] = \int_{\Omega} \left(\frac{\partial \mathcal{F}}{\partial u} \delta u + \frac{\partial \mathcal{F}}{\partial \left(\frac{\partial u}{\partial x^i}\right)} \delta \left(\frac{\partial u}{\partial x^i}\right) \right) d\mathbf{r}$$

In the same way we assume the boundary conditions $\delta u = \delta \left(\frac{\partial u}{\partial x^i}\right) = 0$ on $\partial\Omega$. By integration by parts we get:

$$\delta F[u, \delta u] = \int_{\Omega} \delta u \left(\frac{\partial \mathcal{F}}{\partial u} - \frac{\partial}{\partial x^i} \frac{\partial \mathcal{F}}{\partial \left(\frac{\partial u}{\partial x^i}\right)} \right) d\mathbf{r}$$

Then,

$$\frac{\delta F}{\delta u} = \frac{\partial \mathcal{F}}{\partial u} - \frac{\partial}{\partial x^i} \frac{\partial \mathcal{F}}{\partial \left(\frac{\partial u}{\partial x^i}\right)} \quad (10)$$

(For more details see [22])

7.5.1 Eulerian variables

To describe fluid mechanics we use the Euler variables. In fact, in this description we are not interested in a point \mathbf{M} representing a particle during its motion, but we are interested in a fixed point \mathbf{M} whose coordinates are indicated by the position vector \mathbf{r} . The most important Eulerian variable is the velocity field $\mathbf{v} = \mathbf{v}(\mathbf{r}, t)$. This velocity is in fact the velocity of the particle which occupies the position \mathbf{r} at time t .

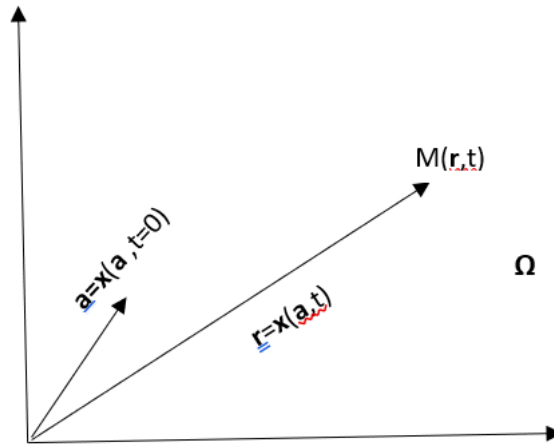


Figure 23:

Suppose that a particle is well identified by its initial position \mathbf{a} at time $t = 0$. We try to describe its velocity when it passes to the point \mathbf{M} at time t , so this position depends on \mathbf{a} and on t we note it $\mathbf{q}(\mathbf{a}, t)$. The curve $t \rightarrow \mathbf{q}(\mathbf{a}, t)$ describes the path of the fluid particle which at time $t = 0$ has the initial position $\mathbf{a} = \mathbf{q}(\mathbf{a}, 0)$. We suppose that $\mathbf{q}(\mathbf{a}, t) \neq \mathbf{q}(\mathbf{b}, t)$ for $t \geq 0$ if $\mathbf{a} \neq \mathbf{b}$, that is, the fluid particles retain their identity. It should also be noted that the $\Omega = \Omega_t$ domain depends on temp t .

We assume the assumptions:

$$\left\{ \begin{array}{l} \mathbf{q}(\mathbf{a}, 0) = \mathbf{a} \\ \mathbf{q}(\mathbf{a}, t) = \mathbf{r} \\ \text{Si } \mathbf{a} \neq \mathbf{b} \text{ then } \mathbf{q}(\mathbf{a}, t) \neq \mathbf{q}(\mathbf{b}, t) \text{ for all } t \geq 0. \\ \text{La application } \mathbf{a} \rightarrow \mathbf{q}(\mathbf{a}, t) \text{ has a smooth inverse, and } \mathbf{a} = \mathbf{q}^{-1}(\mathbf{r}, t). \end{array} \right.$$

So the velocity of this particle at the point \mathbf{M} is given by:

$$\mathbf{v}(\mathbf{r}, t) = \dot{\mathbf{q}}(\mathbf{a}, t)|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{r}, t)} = \dot{\mathbf{q}} \circ \mathbf{q}^{-1}(\mathbf{r}, t).$$

Another important variable is the density of the fluid ρ . If we note d^3a the elementary volume of the fluid in the label \mathbf{a} their corresponding in the position \mathbf{q} is noted d^3r . As the mass is preserved in the course of the motion we can write:

$$\rho(\mathbf{r}, t)d^3r = \rho(\mathbf{a}, 0)d^3r = \rho_0(\mathbf{a})d^3a$$

- $\rho_0(\mathbf{a})$: is the mass density of the particles that were at position \mathbf{a} at time $t = 0$.
- $\rho(\mathbf{r}, t)$: the density of the same particles but in the position \mathbf{r} at time t .

We can also write:

$$\rho(\mathbf{r}, t) = \int_{\Omega_t} \delta(\mathbf{r} - \mathbf{q}(\mathbf{a}, t))\rho(\mathbf{a}, 0)d^3a$$

with δ is the Dirac distribution, since $\rho(\mathbf{r}, t)d^3r = \rho_0(\mathbf{a})d^3a$ we have:

$$\rho(\mathbf{r}, t) = \int_{\Omega_0} \delta(\mathbf{r} - \mathbf{q}(\mathbf{a}, t))\rho_0(\mathbf{a})d^3a \quad (11)$$

Remark 34.

We will use the following mathematical result: Let f be a real function, and (x_1, x_2, \dots, x_N) the points of vanishing of f , such that $f'(x_i) \neq 0$ for all $i \in \{1, \dots, N\}$, Thus:

$$\delta(f(x)) = \sum_{i=1}^N \frac{\delta(x - x_i)}{|f'(x_i)|}$$

More generally:

$$\begin{aligned} f : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ \mathbf{x} = (x_1, \dots, x_n) &\longrightarrow f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_n(\mathbf{x})) \end{aligned}$$

Let $\{\mathbf{x}_0^k\}_{k=1,2,\dots,N}$ the vanishing points of the function f such that:

$\det \left(\frac{\partial f_i}{\partial x_j} \right) (\mathbf{x}_0^k) \neq 0$ for all k . Then we have:

$$\delta(f(\mathbf{x})) = \sum_{k=1}^N \frac{\delta(\mathbf{x} - \mathbf{x}_0^k)}{\left| \det \left(\frac{\partial f_i}{\partial x_j} (\mathbf{x}_0^k) \right) \right|}$$

In our case $f(\mathbf{a}) = \mathbf{r} - \mathbf{q}(\mathbf{a}, t)$, the function vanishes at the single point $\mathbf{a} = \mathbf{q}^{-1}(\mathbf{r}, t)$ and we have:

$$\mathcal{J}ac(f) = \mathbf{det} \left(\frac{\partial f}{\partial \mathbf{a}} \right) = -\mathbf{det} \left(\frac{\partial q_i}{\partial a_j} \right)$$

where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{q} = (q_1, q_2, q_3)$. Therefore, we can deduce:

$$\begin{aligned} \rho(\mathbf{r}, t) &= \int_{\Omega} \frac{\delta(\mathbf{a} - \mathbf{q}^{-1}(\mathbf{r}, t))}{|\mathcal{J}ac(f)(\mathbf{a})|} \rho_0(\mathbf{a}) d^3 a \\ &= \frac{\rho_0(\mathbf{a})}{|\mathcal{J}ac(f)(\mathbf{a})|} \Big|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{r}, t)} \\ &= \frac{\rho_0(\mathbf{a})}{\left| \mathbf{det} \left(\frac{\partial q_i}{\partial a_j} \right) \right|} \Big|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{r}, t)} \end{aligned} \quad (12)$$

The equation of conservation of mass can be written as $\rho d^3 r = \rho_0 d^3 a$. According to the above we have:

$$d^3 r = \mathcal{J} d^3 a$$

where,

$$\mathcal{J} = \left| \mathbf{det} \left(\frac{\partial q_i}{\partial a_j} \right) \right|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{r}, t)}$$

We can assume afterwards that $\mathbf{det} \left(\frac{\partial q_i}{\partial a_j} \right) \Big|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{r}, t)} \geq 0$

Now we are interested in the variable "Entropy" \mathbf{s} . It is a variable of which we can treat the conservation of energy. Let $\mathbf{s}_0(\mathbf{a})$ be the entropy per unit of mass at time $t = 0$, as the entropy is conserved we can write:

$$\mathbf{s}(\mathbf{r}, t) = \mathbf{s}_0(\mathbf{a}) \Big|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{r}, t)} = \mathbf{s}_0 \circ \mathbf{q}^{-1}(\mathbf{r}, t)$$

We introduce the entropy per unit volume (volume entropy):

$$\sigma(\mathbf{r}, t) = \rho(\mathbf{r}, t) \mathbf{s}(\mathbf{r}, t)$$

At time $t = 0$ we have $\sigma_0(\mathbf{a}) = \rho_0(\mathbf{a}) \mathbf{s}_0(\mathbf{a})$. In a similar way to the equation (11), the entropy σ is given by:

$$\sigma(\mathbf{r}, t) = \int_{\Omega} \delta(\mathbf{r} - \mathbf{q}(\mathbf{a}, t)) \sigma_0(\mathbf{a}) d^3 a$$

Finally we have:

$$\sigma(\mathbf{r}, t) = \frac{\sigma_0(\mathbf{a})}{\left| \mathbf{det} \left(\frac{\partial q_i}{\partial a_j} \right) \right|} \Big|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{r}, t)} \quad (13)$$

A last variable is the momentum $\mathbf{M} = \rho \mathbf{v}$, we have:

$$\mathbf{M}(\mathbf{r}, t) = \int_{\Omega} \dot{\mathbf{q}}(\mathbf{r}, t) \delta(\mathbf{r} - \mathbf{q}(\mathbf{a}, t)) \rho_0(\mathbf{a}) d^3 a$$

We also deduce that:

$$\mathbf{M}(\mathbf{r}, t) = \frac{\mathbf{M}_0(\mathbf{a})}{\left| \mathbf{det} \left(\frac{\partial q_i}{\partial a_j} \right) \right|} \Big|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{r}, t)} = \frac{\rho_0(\mathbf{a}) \mathbf{v}_0(\mathbf{a})}{\left| \mathbf{det} \left(\frac{\partial q_i}{\partial a_j} \right) \right|} \Big|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{r}, t)}$$

7.5.2 Hamiltonian formulation

To do the Hamiltonian formulation, we must first express the Lagrangian \mathcal{L} , and then we do the Legendre transformation (see 2).

To determine the Lagrangian one must identify the potential energy, but in the case of fluids this energy is stored in terms of pressure and temperature. On the other hand it is necessary to work on the energy per unit mass.

The potential energy U depends on the density ρ and the entropy s . The temperature and the pressure are given by the relations:

$$T = \frac{\partial U}{\partial s} \quad , \quad p = \rho^2 \frac{\partial U}{\partial \rho}$$

Example 7.2. Monatomic gaz:

$$U(\rho, s) = c\rho^{\gamma-1} \exp(\alpha s)$$

Where c, γ and α are the constants.

NB: Thermodynamic quantities can be seen as Eulerian variables.

The total potential energy on the domain Ω is given by:

$$V[q] = \int_{\Omega} \rho_0(\mathbf{a}) U(s_0, \rho_0/\mathcal{J}) d^3 a$$

The functional V only depends on $\mathcal{J} = \left| \det\left(\frac{\partial q_i}{\partial a_j}\right) \right|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{r},t)}$ and then depend on $\frac{\partial \mathbf{q}}{\partial \mathbf{a}}$.

Now we move to the construction of the kinetic energy function:

$$T[\mathbf{q}, \dot{\mathbf{q}}] = \int_{\Omega_0} \rho_0 \frac{\dot{\mathbf{q}}^2}{2} d^3 a$$

And finally, the Lagrangian is given by:

$$L[\mathbf{q}, \dot{\mathbf{q}}] = T[\dot{\mathbf{q}}, \mathbf{q}] - V[\mathbf{q}]$$

We write:

$$L[\mathbf{q}, \dot{\mathbf{q}}] = \int_{\Omega_0} \mathcal{L}\left(\mathbf{q}, \dot{\mathbf{q}}, \frac{\partial \mathbf{q}}{\partial \mathbf{a}}\right) d^3 a$$

Where \mathcal{L} denotes the Lagrangian per unit volume.

Finally, the action functional between time t_0 and t_1 is follows:

$$S[\mathbf{q}] = \int_{t_0}^{t_1} L[\mathbf{q}, \dot{\mathbf{q}}] dt = \int_{t_0}^{t_1} \int_{\Omega_0} \left[\frac{1}{2} \rho_0 \dot{\mathbf{q}}^2 - \rho_0 U(s_0, \rho_0/\mathcal{J}) \right] d^3 a dt$$

We can see that this action functional is similar to the case of a system with a finite number of degrees of freedom, but instead of taking the sum \sum we integrate on our domain Ω (As The fluid is a continuous medium).

$$\sum_i \longleftrightarrow \int_{\Omega} d^3 a$$

We fix the boundary conditions $\delta q(a, t_0) = \delta q(a, t_1) = 0$. In addition, $\delta q = 0$ or $\delta q \cdot \mathbf{n} = 0$ on $\partial\Omega$, where \mathbf{n} is a unit normal vector, and we apply the principle of least action:

$$\frac{\delta S}{\delta \mathbf{q}} = 0 \quad \text{implies:} \quad \frac{\delta L}{\delta \mathbf{q}} = \frac{d}{dt} \left(\frac{\delta L}{\delta \dot{\mathbf{q}}} \right)$$

According to the equation (10), we get :

$$\frac{\delta L}{\delta q_k} = \frac{\partial \mathcal{L}}{\partial q_k} - \frac{\partial}{\partial a_i} \left(\frac{\partial \mathcal{L}}{\partial (\partial q_k / \partial a_i)} \right)$$

Thus, the principle of last action gives:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) + \frac{\partial}{\partial a_i} \left(\frac{\partial \mathcal{L}}{\partial (\partial q_k / \partial a_i)} \right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0 \quad k = 1, 2, 3$$

If we take the matrix $\mathbf{A} = \left(\frac{\partial q_i}{\partial a_j} \right)_{i,j}$, we have:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} {}^T \text{Com}(\mathbf{A}) = \frac{1}{\mathcal{J}} {}^T \text{Com}(\mathbf{A})$$

Therefore, we deduce the following relation:

$$\frac{\partial}{\partial q_k} = \frac{1}{\mathcal{J}} \Delta_k^i \frac{\partial}{\partial a_i} \quad (14)$$

Such that : $\Delta_k^i = \frac{1}{2} \epsilon_{kjl} \epsilon_{imn} \frac{\partial q_j}{\partial a_m} \frac{\partial q_l}{\partial a_n}$ (the co-factor), with,

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i j k) \text{ is a cyclic permutation} \\ -1 & \text{if } (i j k) \text{ is an anti-cyclic permutation} \\ 0 & \text{if } i = j = k \end{cases}$$

We also present a list of formulas that can help us in the calculus:

$$\Delta_i^j = \frac{\partial \mathcal{J}}{\partial (\partial q_i / \partial a_j)} \quad (15)$$

$$\delta \mathcal{J} = \Delta_i^k \frac{\partial \delta q_i}{\partial a_k} \quad \text{Or} \quad \dot{\mathcal{J}} = \Delta_i^k \frac{\partial \dot{q}_i}{\partial a_k} \quad (16)$$

As $\mathcal{L} = (\frac{1}{2} \rho_0 \dot{\mathbf{q}}^2 - \rho_0 U(s_0, \rho_0 / \mathcal{J}))$, we obviously have $\frac{\partial \mathcal{L}}{\partial q_k} = 0$.

Using the formula (15), we get :

$$\begin{aligned} \frac{\partial}{\partial a_i} \left(\frac{\partial \mathcal{L}}{\partial (\partial q_k / \partial a_i)} \right) &= \frac{\partial}{\partial a_i} \left(\frac{\partial \mathcal{L}}{\partial \mathcal{J} / \Delta_k^i} \right) \\ &= \frac{\partial}{\partial a_i} \left(\frac{\partial (\rho_0 U(s_0, \rho_0 / \mathcal{J}))}{\partial \mathcal{J} / \Delta_k^i} \right) \\ &= -\Delta_k^i \frac{\partial}{\partial a_i} \left(\frac{\rho_0^2}{\mathcal{J}^2} \frac{\partial U(s_0, \rho_0 / \mathcal{J})}{\partial \rho} \right) \end{aligned}$$

Here we have used $\partial \Delta_k^i / \partial a^j = 0$, which we can obtain it using the expression of Δ_k^i .

Since,

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) = \rho_0 \ddot{q}_k$$

The equation of least action principle becomes:

$$\rho_0 \ddot{q}_k = -\Delta_k^i \frac{\partial}{\partial a^i} \left(\frac{\rho_0^2}{\mathcal{J}^2} \frac{\partial U}{\partial \rho} \right) \quad (17)$$

Using (14) the equation of motion can be written in the form

$$\rho_0 \ddot{q}_k \frac{\partial q_k}{\partial a^i} + \mathcal{J} \frac{\partial}{\partial a^i} \left(\frac{\rho_0^2}{\mathcal{J}^2} \frac{\partial U}{\partial \rho} \right) = 0 \quad (18)$$

Remark 35.

$$\begin{aligned} d\dot{\mathbf{q}}(\mathbf{q}(\mathbf{a}, t), t) &= \frac{\partial \dot{\mathbf{q}}}{\partial t} dt + \frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{r}} d\mathbf{q}(\mathbf{a}, t) = \frac{\partial \dot{\mathbf{q}}}{\partial t} dt + \frac{\partial \mathbf{q}(\mathbf{a}, t)}{\partial t} \frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{r}} dt \\ \frac{d\dot{\mathbf{q}}(\mathbf{q}(\mathbf{a}, t), t)}{dt} &= \left(\frac{\partial \dot{\mathbf{q}}}{\partial t} + \dot{\mathbf{q}} \cdot \nabla \dot{\mathbf{q}} \right) (\mathbf{q}(\mathbf{a}, t), t) \end{aligned}$$

Then,

$$\ddot{\mathbf{q}} = \frac{\partial v}{\partial t} + v \cdot \nabla v \quad (19)$$

According to (14) and (19), we can also show that the equation (17) is equivalent to the conservation of momentum equation, or what is called the Euler equation:

$$\rho \left(\frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p$$

The equation of conservation of mass and the advection of entropy (12,13) are given by:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) &= 0 \\ \frac{\partial \sigma}{\partial t} + \nabla \cdot (\rho \sigma) &= 0 \end{aligned}$$

7.5.3 Canonical Hamiltonian formulation

To determine the Hamiltonian, we apply the Legendre transformation (see 2). Indeed, we pose:

$$\pi(\mathbf{a}, t) = \frac{\delta L}{\delta \dot{\mathbf{q}}} = \rho_0 \dot{\mathbf{q}}$$

Then :

$$\begin{aligned} \mathcal{H} &= \int_{\Omega_0} (\pi \dot{\mathbf{q}} - \mathcal{L}) d^3 a \\ &= \int_{\Omega_0} \left(\pi \dot{\mathbf{q}} - \frac{\rho_0}{2} \dot{\mathbf{q}}^2 + \rho_0 U \right) d^3 a \\ &= \int_{\Omega_0} \left(\frac{\pi^2}{2\rho_0} + \rho_0 U \right) d^3 a \end{aligned}$$

Hamilton's equations are written as:

$$\dot{\pi}_i = -\frac{\delta\mathcal{H}}{\delta q_i} \quad , \quad \dot{q}_i = \frac{\delta\mathcal{H}}{\delta\pi_i}$$

These equations can be reformulated using the canonical Poisson bracket:

$$\{F, G\} = \int_{\Omega_0} \left[\frac{\delta F}{\delta \mathbf{q}} \frac{\delta G}{\delta \pi} - \frac{\delta G}{\delta \pi} \frac{\delta F}{\delta \mathbf{q}} \right] d^3 a$$

with F and G two functional depending on π and \mathbf{q} ($F[\pi, \mathbf{q}], G[\pi, \mathbf{q}]$). Then the Hamilton equation is equivalent to:

$$\dot{\pi}_i = \{\pi_i, \mathcal{H}\} \quad , \quad \dot{q}_i = \{q_i, \mathcal{H}\}$$

where $\frac{\delta q_i(a)}{\delta q_j(a')} = \delta_{ij} \delta(a - a')$ (similar to $\frac{\partial q_i}{\partial q_j} = \delta_{ij}$).

L'equation $\dot{\pi}_i = \{\pi_i, \mathcal{H}\}$, est equivalent au equation de mouvement 17. Indeed:

$$\{\pi_i, \mathcal{H}\} = -\Delta_k^i \frac{\partial}{\partial a^i} \left(\frac{\rho_0^2}{\mathcal{J}^2} \frac{\partial U}{\partial \rho} \right)$$

7.5.4 Non-canonical Hamiltonian formulation

The non-canonical Hamiltonian formulation⁸ is the construction of another Poisson bracket, noted $\{.,.\}^{n-c}$, in order that the Euler variables ($\mathbf{M} = \rho \mathbf{v}, \rho, \sigma = \rho \mathbf{s}$) verify equations of the type:

$$\dot{\mathbf{x}} = \{\mathbf{x}, \mathcal{H}\}^{n-c}$$

This amounts to developing the chain rule for the functional (see [22]). By doing this, we get the following non-canonical bracket:

$$\begin{aligned} \{F, G\}^{n-c} = & - \int_{\Omega} \left[M_i \left(\frac{\delta F}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta F}{\delta M_i} \right) \right. \\ & + \rho \left(\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \rho} \right) \\ & \left. + \sigma \left(\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \sigma} \right) \right] d^3 r \end{aligned}$$

Where: $\mathbf{M} = (M_1, M_2, M_3)$ and $\mathbf{r} = (x^1, x^2, x^3)$.

The equations of motion of the fluid can be written as:

$$\frac{\partial \mathbf{M}}{\partial t} = \{\mathbf{M}, \mathcal{H}\}^{n-c}, \quad \frac{\partial \rho}{\partial t} = \{\rho, \mathcal{H}\}^{n-c}, \quad \frac{\partial \sigma}{\partial t} = \{\sigma, \mathcal{H}\}^{n-c}$$

⁸The word non-canonical means that the bracket is not the classic commutator and the variables do not represent positions and momentum.

7.6 Plasma dynamic

In this example, we will treat the dynamics of a medium consisting of a mixture of a fluid (neutral particles), positive ions and negative electrons. This type of state of matter is called **Plasma**.

The equations related to the electromagnetic part of a plasma are the following:

$$\begin{aligned}
 \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{e}{m} \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial f}{\partial \mathbf{v}} &= 0 \\
 \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} &= -\text{curl } \mathbf{E}, \\
 \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} &= \text{curl } \mathbf{B} - \frac{1}{c} \mathbf{j}_f, \\
 \text{div } \mathbf{E} &= \rho_f, \\
 \text{div } \mathbf{B} &= 0.
 \end{aligned} \tag{20}$$

Called Maxwell–Vlasov system, where:

$f(\mathbf{r}, \mathbf{v}, t)$: The density of the particles of plasma in the phase space

$\mathbf{j}_f = e \int \mathbf{v} f(\mathbf{r}, \mathbf{v}, t) d^3v$: The electric current

$\rho_f = e \int f(\mathbf{r}, \mathbf{v}, t) d^3v$: density of the charge

$\mathbf{E}(\mathbf{r}, t)$: The electric field.

$\mathbf{B}(\mathbf{r}, t)$: The magnetic field.

m , e and c is the mass of the particle, its charge and the speed of light (respectively).

To complete the equations (20) that describe the dynamics of the plasma, we add the Euler equation for the fluid. But this time we must add the effect of the Lorentz force:

$$\begin{aligned}
 \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) &= 0 \\
 \rho \left(\frac{\partial v}{\partial t} + v \cdot \nabla v \right) &= -\nabla p + \rho_f (\mathbf{E} + v \wedge \mathbf{B}) \\
 \frac{\partial \sigma}{\partial t} + v \cdot \nabla \sigma &= 0
 \end{aligned} \tag{21}$$

7.6.1 Hamiltonian formulation

For the fluid part we take the variables of the previous example $(\rho, \mathbf{M}, \sigma)$. For the electromagnetic part, we take the variables \mathbf{E} and \mathbf{B} . To simplify, we take $c = e = 1$. The total energy of the system on a domain Ω is given by:

$$\mathcal{H}[\mathbf{M}, \rho, \sigma, \mathbf{E}, \mathbf{B}] = \int_{\Omega} \frac{1}{2\rho} \mathbf{M}^2 d^3r + \int_{\Omega} \rho U(s, \rho) d^3r + \frac{1}{2} \int_{\Omega} (\|\mathbf{E}(\mathbf{r}, t)\|^2 + \|\mathbf{B}(\mathbf{r}, t)\|^2) d^3r$$

We define the following Poisson brackets:

- The bracket that gives the electromagnetic description:

$$\{F, G\}^{Maxwel} = \int \left(\frac{\delta F}{\delta \mathbf{E}} \cdot \text{curl } \frac{\delta G}{\delta \mathbf{B}} - \frac{\delta G}{\delta \mathbf{E}} \cdot \text{curl } \frac{\delta F}{\delta \mathbf{B}} \right) d^3r$$

- The bracket that gives the Eulerian description of fluid (see previous example):

$$\begin{aligned} \{F, G\}^{Euler} = \{F, G\}^{n-c} = & - \int_{\Omega} \left[M_i \left(\frac{\delta F}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta F}{\delta M_i} \right) \right. \\ & + \rho \left(\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \rho} \right) \\ & \left. + \sigma \left(\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \sigma} \right) \right] d^3r \end{aligned}$$

- The poisson bracket which gives the lorentz effect:

$$\{F, G\}^{Lorentz} = \int_{\Omega} \rho_f \left(\frac{\delta F}{\delta \mathbf{M}} \cdot \frac{\delta G}{\delta \mathbf{E}} - \frac{\delta G}{\delta \mathbf{M}} \cdot \frac{\delta F}{\delta \mathbf{E}} \right) d^3r + \int_{\Omega} \rho_f \mathbf{B} \cdot \left(\frac{\delta F}{\delta \mathbf{M}} \wedge \frac{\delta G}{\delta \mathbf{M}} \right) d^3r$$

The Poisson bracket for the Plasma dynamics is the sum of the three:

$$\{F, G\}^{plasma} = \{F, G\}^{Maxwel} + \{F, G\}^{Euler} + \{F, G\}^{Lorentz}.$$

Therefore, the equation of dynamic systems (20,21), is reformulated as follows:

$$\begin{aligned} \frac{\partial M}{\partial t} &= \{M, \mathcal{H}\}^{plasma} \\ \frac{\partial \rho}{\partial t} &= \{\rho, \mathcal{H}\}^{plasma} \\ \frac{\partial \sigma}{\partial t} &= \{\sigma, \mathcal{H}\}^{plasma} \\ \frac{\partial \mathbf{E}}{\partial t} &= \{\mathbf{E}, \mathcal{H}\}^{plasma} \\ \frac{\partial \mathbf{B}}{\partial t} &= \{\mathbf{B}, \mathcal{H}\}^{plasma} \end{aligned}$$

8 Conclusion

To describe the dynamics of a system, we have three formalism: Newtonian, Lagrangian, and Hamiltonian. All three are equivalent, but depending on the situation, we chose what makes it easier for us to get the desired goals. On the other hand, formalism allows a deeper understanding of the dynamics of a system is certainly the Hamiltonian formalism, in which it is clear that the total energy of the system that governs its evolution in time, this remark has allowed solving completely a series of mechanical problems that do not give solutions by other means (for example, the problem of attraction by two stationary centers and the problem of geodesics on the triaxial ellipsoid). The Hamiltonian point of view is even more valuable for approximate methods of perturbation theory (celestial mechanics), for understanding the general character of motion in complicated mechanical systems (ergodic theory, statistical mechanics) and in connection with other fields of mathematical physics (optics, quantum mechanics, etc.).

References

- [1] V.I.Arnold,Mathematical methods of classical mechanics, Springer, Volume 60, (1989)
- [2] V. I. Arnold, S. P. Novikov, Dynamical systems 04 Symplectic geometry and its applications, Springer, (1990).
- [3] Abraham, Ralph, and Jerrold E. Marsden. Foundations of mechanics. No. 364. American Mathematical Soc., 2008.
- [4] Da Silva, Ana Cannas, and A. Cannas Da Salva. Lectures on symplectic geometry. Berlin: Springer, 2008.
- [5] V.I.Arnold, Equations différentielles ordinaires, éditions Mir (1988)
- [6] M.BERGER, B.Gostiaux, Differential Geometry: Manifolds, Curves, and Surfaces (1992)
- [7] C.Godbillon géométrie différentielle et mécanique analytique collection méthodes Hermann, paris (1969)
- [8] Frédéric HÉLEIN, cours de Calcul différentiel extérieur et géométrie, (2016)
- [9] Boschetto, Davide, and Jérôme Perez. "Physique quantique ENSTA 2010-2011."
- [10] Davide Boschetto, Mécanique Analytique , 2020, Davide Boschetto
- [11] François Labourie, géométrie différentielle (2013)
- [12] LESFARI, A. Géométrie symplectique et Mécanique hamiltonienne.
- [13] Azzouz Awane. Cours de géométrie différentielle. DEA. 2001-2005 à la Faculté des Sciences Ben M'sik. Casablanca. Maroc, 2005, pp.214. cel-00277648.
- [14] Kern, Felix, et al. "A Hamiltonian mechanics framework for charge particle optics in straight and curved systems." *Optik* (2021): 167242.
- [15] Pritula, G. M., Petrenko, E. V., Usatenko, O. (2018). Adiabatic dynamics of one-dimensional classical Hamiltonian dissipative systems. *Physics Letters A*, 382(8), 548-553.
- [16] Pavelka, Michal, Ilya Peshkov, and Vaclav Klika. "On Hamiltonian continuum mechanics." *Physica D: Nonlinear Phenomena* 408 (2020): 132510.
- [17] Xue, Tao, et al. "On a generalized energy conservation/dissipation time finite element method for Hamiltonian mechanics." *Computer Methods in Applied Mechanics and Engineering* 373 (2021): 113509.
- [18] Marsden, Jerrold E., and Tudor S. Ratiu. "Introduction to mechanics and symmetry." *Physics Today* 48.12 (1995): 65.
- [19] Volker Perlick. "Lagrangian and Hamiltonian Dynamics",Lancaster University, Volker Perlick.

- [20] Weisstein, Eric. "Double pendulum. March 11, 2007." (1996). Eric Weisstein.
- [21] P.J. Morrison, " Hamiltonian Fluid Mechanics", in Encyclopedia of Mathematical Physics, vol. 2, (Elsevier, Amsterdam, 2006) p. 593.
- [22] Morrison, Philip J. "Hamiltonian description of the ideal fluid." Reviews of modern physics 70.2 (1998): 467.
- [23] Brizard, Alain J., and Cristel Chandre. "Hamiltonian formulations for perturbed dissipationless plasma equations." Physics of Plasmas 27.12 (2020): 122111.
- [24] Spencer, Richard G., and Allan N. Kaufman. "Hamiltonian structure of two-fluid plasma dynamics." Physical Review A 25.4 (1982): 2437.
- [25] Morrison, P. J. "Hamiltonian and action principle formulations of plasma physics." Physics of plasmas 12.5 (2005): 058102.
- [26] Morrison, P. J., T. Andreussi, and F. Pegoraro. "Lagrangian and Dirac constraints for the ideal incompressible fluid and magnetohydrodynamics." Journal of Plasma Physics 86.3 (2020).