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# LIMIT LAWS FOR LARGE $k$ TH-NEAREST NEIGHBOR BALLS

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A PREPRINT

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## ABSTRACT

Let  $X_1, X_2, \dots, X_n$  be a sequence of independent random points in  $\mathbb{R}^d$  with common Lebesgue density  $f$ . Under some conditions on  $f$ , we obtain a Poisson limit theorem, as  $n \rightarrow \infty$ , for the number of large probability  $k$ th-nearest neighbor balls of  $X_1, \dots, X_n$ . Our result generalizes Theorem 2.2 of [11], which refers to the special case  $k = 1$ . Our proof is completely different since it employs the Chen-Stein method instead of the method of moments. Moreover, we obtain a rate of convergence for the Poisson approximation.

## 1 Introduction and main results

The starting point of this paper is the following result, see [11]. Let  $X, X_1, \dots, X_n, \dots$  be a sequence of independent and identically distributed (i.i.d.) random points in  $\mathbb{R}^d$ ,  $d \geq 2$ , that are defined on a common probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . We assume that the distribution of  $X$ , which is denoted by  $\mu$ , is absolutely continuous with respect to Lebesgue measure  $\lambda$ , and we denote the density of  $\mu$  by  $f$ . Writing  $\|\cdot\|$  for the Euclidean norm in  $\mathbb{R}^d$ , and putting  $\mathcal{X}_n := \{X_1, \dots, X_n\}$ , let  $R_{i,n} := \min_{j \neq i, j \leq n} \|X_i - X_j\|$  be the distance from  $X_i$  to its nearest neighbor in the set  $\mathcal{X}_n \setminus \{X_i\}$ . Moreover, let  $\mathbf{1}\{A\}$  denote the indicator function of a set  $A$ , and write  $B(x, r) = \{y \in \mathbb{R}^d : \|x - y\| \leq r\}$  for the closed ball centered at  $x$  with radius  $r$ . Finally, let

$$C_n := \sum_{i=1}^n \mathbf{1}\left\{\mu(B(X_i, R_{i,n})) > \frac{t + \log n}{n}\right\}$$

denote the number of exceedances of probability volumes of nearest neighbor balls that are larger than the threshold  $(t + \log n)/n$ . The main result of [11] is Theorem 2.2 of that paper, which states that, under a weak condition on the density  $f$ , for each fixed  $t \in \mathbb{R}$ , we have

$$C_n \xrightarrow{\mathcal{D}} \text{Po}(\exp(-t)) \tag{1}$$

as  $n \rightarrow \infty$ , where  $\xrightarrow{\mathcal{D}}$  denotes convergence in distribution, and  $\text{Po}(\xi)$  is the Poisson distribution with parameter  $\xi > 0$ .

Since the *maximum* probability content of these nearest balls, denoted by  $P_n$ , is at most  $(t + \log n)/n$  if, and only if,  $C_n = 0$ , we immediately obtain a Gumbel limit  $\lim_{n \rightarrow \infty} \mathbb{P}(nP_n - \log n \leq t) = \exp(-\exp(-t))$  for  $P_n$ .

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To state a sufficient condition on  $f$  that guarantees (1), let  $\text{supp}(\mu) := \{x \in \mathbb{R}^d : \mu(B(x, r)) > 0 \text{ for each } r > 0\}$  denote the support of  $\mu$ . Theorem 2.2 of [11] requires that there are  $\beta \in (0, 1)$ ,  $c_{max} < \infty$  and  $\delta > 0$  such that, for any  $r, s > 0$  and any  $x, z \in \text{supp}(\mu)$  with  $\|x - z\| \geq \max\{r, s\}$  and  $\mu(B(x, r)) = \mu(B(z, s)) \leq \delta$ , one has

$$\frac{\mu(B(x, r) \cap B(z, s))}{\mu(B(z, s))} \leq \beta$$

and  $\mu(B(z, 2s)) \leq c_{max}\mu(B(z, s))$ .

These conditions hold if  $\text{supp}(f)$  is a compact set  $K$  (say), and there are  $f_-, f_+ \in (0, \infty)$  such that

$$f_- \leq f(x) \leq f_+, \quad x \in K. \quad (2)$$

Thus, the density  $f$  of  $X$  is bounded and bounded away from zero.

The purpose of this paper is to generalize (1) to  $k$ th-nearest neighbors, and to derive a rate of convergence for the Poisson approximation of the number of exceedances.

Before stating our main results, we give some more notation. For fixed  $k \leq n - 1$ , we denote by  $R_{i,n,k}$  the Euclidean distance of  $X_i$  to its  $k$ th-nearest neighbor among  $\mathcal{X}_n \setminus \{X_i\}$ , and we write  $B(X_i, R_{i,n,k})$  for the  $k$ th-nearest neighbor ball centered at  $X_i$  with radius  $R_{i,n,k}$ . For fixed  $t \in \mathbb{R}$ , put

$$v_{n,k} := v_{n,k}(t) := \frac{t + \log n + (k - 1) \log \log n - \log(k - 1)!}{n}, \quad (3)$$

and let

$$C_{n,k} := \sum_{i=1}^n \mathbf{1}\{\mu(B(X_i, R_{i,n,k})) > v_{n,k}\} \quad (4)$$

denote the *number of exceedances* of probability contents of  $k$ th-nearest neighbor balls over the threshold  $v_{n,k}$  defined in (3).

The term  $\log \log n$ , which shows up in the case  $k > 1$ , is typical in extreme value theory. It occurs, e.g., in the affine transformation of the maximum of  $n$  i.i.d. standard normal random variables, which has a Gumbel limit distribution (see Example 3.3.29 of [10]), or in a recent Poisson limit theorem for the number of cells having at most  $k - 1$  particles in the coupon collector's problem (see Theorem 1 of [19]).

The threshold  $v_{n,k}$  is in some sense *universal* in dealing with the number of exceedances of probability contents of  $k$ th-nearest neighbor balls. To this end, suppose that, in much more generality than considered so far,  $X, X_1, X_2, \dots$  are i.i.d. random elements taking values in a separable metric space  $(S, \rho)$ . We retain the notation  $\mu$  for the distribution of  $X$  and  $B(x, r) := \{y \in S : \rho(x, y) \leq r\}$  for the closed ball with radius  $r$  centered at  $x \in S$ . Regarding the distribution  $\mu$ , we assume that

$$\mu(\{y \in S : \rho(x, y) = r\}) = 0, \quad x \in S, r \geq 0. \quad (5)$$

As a consequence, the distances  $\rho(X_i, X_j)$ , where  $j \in \{1, \dots, n\} \setminus \{i\}$ , are different with probability one for each  $i \in \{1, \dots, n\}$ . Thus, for fixed  $k \leq n - 1$ , there is almost surely a unique  $k$ th-nearest neighbor of  $X_i$ , and we also retain the notation  $R_{i,n,k}$  for the distance of  $X_i$  to its  $k$ th-nearest neighbor among  $\mathcal{X}_n \setminus \{X_i\}$  and  $B(X_i, R_{i,n,k})$  for the ball centered at  $X_i$  with radius  $R_{i,n,k}$ . Notice that the condition (5) excludes discrete metric spaces (see, e.g., Section 4 of [20]), but not function spaces like, e.g., the space  $C[0, 1]$  of continuous functions on  $[0, 1]$  with the supremum metric, and with Wiener measure  $\mu$ .

In what follows, for sequences  $(a_n)_{n \geq 0}$  and  $(b_n)_{n \geq 0}$  of real numbers, write  $a_n = O(b_n)$  if  $|a_n| \leq C|b_n|$ ,  $n \geq 1$ , for some positive constant  $C$ .

**Theorem 1.1.** *If  $X_1, X_2, \dots$  are i.i.d. random elements of a metric space  $(S, \rho)$ , and if (5) holds, then the sequence  $(C_{n,k})$  satisfies*

$$\mathbb{E}[C_{n,k}] = e^{-t} + O\left(\frac{\log \log n}{\log n}\right).$$

In particular, the mean number of exceedances  $C_{n,k}$  converges to  $e^{-t}$  as  $n$  goes to infinity. By Markov's inequality, this result implies the tightness of the sequence  $(C_{n,k})_{n \geq 1}$ . Thus, at least a subsequence converges in distribution. The next result states convergence of  $C_{n,k}$  to a Poisson distribution if  $(S, \rho) = (\mathbb{R}^d, \|\cdot\|)$  and (2) holds. To this end, let  $d_{TV}(Y, Z)$  be the total variation between two integer-valued random variables  $Y$  and  $Z$ , i.e.,

$$d_{TV}(Y, Z) = 2 \sup_{A \subset \mathbb{N}} |\mathbb{P}(Y \in A) - \mathbb{P}(Z \in A)|.$$

**Theorem 1.2.** *Let  $Z$  be a Poisson random variable with parameter  $e^{-t}$ . If  $X, X_1, X_2, \dots$  are i.i.d. in  $\mathbb{R}^d$  with density  $f$ , and if the distribution  $\mu$  of  $X$  has compact support  $[0, 1]^d$  and satisfies (2), then, as  $n \rightarrow \infty$ ,*

$$d_{TV}(C_{n,k}, Z) = O\left(\frac{\log \log n}{\log n}\right).$$

Theorem 1.2 is not only a generalization of Theorem 2.2 of [11] over all  $k \geq 1$ : it also provides a rate of convergence for the Poisson approximation of  $C_{n,k}$ . Our theorem is stated in the particular case that the support of  $\mu$  is  $[0, 1]^d$  but we think that it can be extended to any measure  $\mu$  whose support is a general convex body. For the sake of readability of the manuscript, we did not deal with such a generalization.

**Remark 1.3.** The study of extremes of  $k$ th-nearest neighbor balls is classical in stochastic geometry, and it has various applications, see e.g. [17]. In Section 4 in [16], bounds for the total variation distance of the process of Poisson points with large  $k$ th-nearest neighbor ball (with respect to the intensity measure) and a Poisson process were obtained. Parallel to our work, these results have been extended by Bobrowski *et al.* to the Kantorovich-Rubinstein distance and generalized to the binomial process in Section 6.2 of a paper which has just been submitted [5]. Theorem 6.5 in [5] implies our Theorem 1.2. Nevertheless, the approaches in [5] and [16] and in the present paper are conceptionally different. While the results in [5] and [16] rely on Palm couplings of a thinned Poisson/binomial process and employ distances of point processes, we derive a bound on the total variation distance of the number of large  $k$ th-nearest neighbor balls and a Poisson-distributed random variable. Our approach permits to build arguments on classical Poisson approximation theory [2] and an asymptotic independence property stated in Lemma 2.2 below, and it thus results in a considerably shorter and less technical proof.

**Remark 1.4.** From Theorem 1.2 we can deduce an analogous Poisson approximation result for Poisson input (instead of  $X_1, X_2, \dots$ ). Assume w.l.o.g. that  $\mu(\mathbb{R}^d) = 1$ , and let  $\eta_n$  be a Poisson process with intensity measure  $n\mu$ . By Proposition 3.8 in [15], there are i.i.d. random points  $X_1, X_2, \dots$  in  $\mathbb{R}^d$ , where  $X_1$  has the distribution  $\mu$ , and a Poisson random variable  $N(n)$  with expectation  $n$  that is independent of  $X_1, X_2, \dots$ , such that  $\eta_n = \sum_{i=1}^{N(n)} \delta_{X_i}$ . Here,  $\delta_x$  denotes a unit mass at  $x \in \mathbb{R}^d$ . Let

$$D_{n,k} := \sum_{i=1}^{N(n)} \mathbf{1}\{\mu(B(X_i, R_{i,N(n),k})) > v_{n,k}\} \quad (6)$$

be the number of exceedances of probability contents of  $k$ th-nearest neighbor balls over the threshold  $v_{n,k}$  for the process  $\eta_n$ . By the triangle inequality, we have

$$d_{TV}(D_{n,k}, Z) \leq d_{TV}(D_{n,k}, C_{n,k}) + d_{TV}(C_{n,k}, Z),$$

where  $d_{TV}(D_{n,k}, C_{n,k})$  is at most

$$\mathbb{E} \left| \sum_{i=1}^{N(n)} \mathbf{1}\{\mu(B(X_i, R_{i,N(n),k})) > v_{n,k}\} - \sum_{i=1}^n \mathbf{1}\{\mu(B(X_i, R_{i,n,k})) > v_{n,k}\} \right|.$$

The last term can be bounded using a concentration inequality for the Poisson distribution, see e.g. Lemma 1.4 in [18] (we omit the details). Together with Theorem 1.2, it follows that

$$d_{TV}(D_{n,k}, Z) = O\left(\frac{\log \log n}{\log n}\right)$$

as  $n \rightarrow \infty$ . This result is also implied by Theorem 4.2 in [16] and by Theorem 6.4 in [5].

Now, let  $P_{n,k} = \max_{1 \leq i \leq n} \mu(B(X_i, R_{i,n,k}))$  be the maximum probability content of the  $k$ th-nearest neighbor balls. Since  $C_{n,k} = 0$  if, and only if,  $P_{n,k} \leq v_{n,k}$ , we obtain the following corollary.

**Corollary 1.5.** *Under the conditions of Theorem 1.2, we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}(nP_{n,k} - \log n - (k-1) \log \log n + \log(k-1)! \leq t) = G(t), \quad t \in \mathbb{R},$$

where  $G(t) = \exp(-\exp(-t))$  is the distribution function of the Gumbel distribution.

**Remark 1.6.** If, in the Euclidean case, the density  $f$  is continuous, then  $\mu(B(X_i, R_{i,n,k}))$  is approximately equal to  $f(X_i) \kappa_d R_{i,n,k}^d$ , where  $\kappa_d = \pi^{d/2} / \Gamma(1 + d/2)$  is the volume of the unit ball in  $\mathbb{R}^d$ . Under additional smoothness assumptions on  $f$  and (2), [12, 13] proved that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \max_{i=1, \dots, n} f(X_i) \kappa_d \min(R_{i,n,k}^d, \|X_i - \partial K\|^d) \leq v_{n,k} \right) = G(t), \quad (7)$$

where  $K$  is the support of  $\mu$ . Here, the distance  $\|X_i - \partial K\|$  of  $X_i$  to the boundary of  $K$  is important to overcome edge effects. These effects dominate the asymptotic behavior of the maximum of the  $k$ th-nearest neighbor distances if  $k \geq d$ , see [8, 9]. In fact, [12] proved convergence of the factorial moments of

$$\tilde{C}_{n,k} := \sum_{i=1}^n \mathbf{1} \left\{ f(X_i) \kappa_d \min(R_{i,n,k}^d, \|X_i - \partial K\|^d) > v_{n,k} \right\}$$

to the corresponding factorial moments of a random variable with the Poisson distribution  $\text{Po}(e^{-t})$  and thus, by the method of moments, more than (7), namely  $\tilde{C}_{n,k} \xrightarrow{D} \text{Po}(e^{-t})$ . However, our proof of Theorem 1.2 is completely different thereof, since it is based on the Chen-Stein method and provides a rate of convergence.

## 2 Proofs

### 2.1 Proof of Theorem 1.1

*Proof.* By symmetry, we have

$$\begin{aligned} \mathbb{E}[C_{n,k}] &= n \mathbb{P}(\mu(B(X_1, R_{1,n,k})) > v_{n,k}) \\ &= n \mathbb{E}[\mathbb{P}(\mu(B(X_1, R_{1,n,k})) > v_{n,k} | X_1)]. \end{aligned}$$

For a fixed  $x \in S$ , let

$$H_x(r) := \mathbb{P}(\rho(x, X) \leq r), \quad r \geq 0,$$

be the cumulative distribution function of  $\rho(x, X)$ . Due to the condition (5), the function  $H_x$  is continuous, and by the probability integral transform (see e.g. [4], p. 8), the random variable

$$H_x(\rho(x, X)) = \mu(B(x, \rho(x, X)))$$

is uniformly distributed in the unit interval  $[0, 1]$ . Put  $U_j := H_x(\rho(x, X_{j+1}))$ ,  $j = 1, \dots, n-1$ . Then  $U_1, \dots, U_{n-1}$  are i.i.d. random variables with a uniform distribution in  $(0, 1)$ . Hence, conditionally on  $X_1 = x$ , the random variable  $\mu(B(X_1, R_{1,n,k}))$  has the same distribution as  $U_{k:n-1}$ , where  $U_{1:n-1} < \dots < U_{n-1:n-1}$  are the order statistics of  $U_1, \dots, U_{n-1}$ , and this distribution does not depend on  $x$ . Now, because of a well-known relation between the distribution of order statistics from the uniform distribution on  $(0, 1)$  and the binomial distribution (see, e.g., [1], p. 16), we have

$$\mathbb{P}(U_{k:n-1} > s) = \sum_{j=0}^{k-1} \binom{n-1}{j} s^j (1-s)^{n-1-j}$$

and thus

$$\mathbb{E}[C_{n,k}] = n \sum_{j=0}^{k-1} \binom{n-1}{j} v_{n,k}^j (1-v_{n,k})^{n-1-j}. \quad (8)$$

Here, the summand for  $j = k-1$  equals

$$n \binom{n-1}{k-1} v_{n,k}^{k-1} (1-v_{n,k})^{n-k} = \frac{n}{(k-1)!} (nv_{n,k})^{k-1} \prod_{i=1}^{k-1} \frac{n-i}{n} (1-v_{n,k})^{n-k}.$$

Using Taylor expansions, (3) yields

$$nv_{n,k} = \log n + O(\log \log n), \quad \prod_{i=1}^{k-1} \frac{n-i}{n} = 1 + O\left(\frac{1}{n}\right)$$

and

$$(1-v_{n,k})^{n-k} = \frac{(k-1)!}{n} \exp\left(-t - (k-1) \log \log n + O\left(\frac{\log^2(n)}{n}\right)\right).$$

Straightforward computations now give

$$n \binom{n-1}{k-1} v_{n,k}^{k-1} (1-v_{n,k})^{n-k} = e^{-t} + O\left(\frac{\log \log n}{\log n}\right).$$

Regarding the remaining summands on the right hand side of (8), it is readily seen that

$$\sum_{j=0}^{k-2} \binom{n-1}{j} v_{n,k}^j (1-v_{n,k})^{n-1-j} = O\left(n \binom{n-1}{k-1} v_{n,k}^{k-1} (1-v_{n,k})^{n-k} \cdot \frac{1}{nv_{n,k}}\right),$$

with the convention that the sum equals 0 if  $k = 1$ . From the above computations and from (3), it follows that this sum equals  $O(1/\log n)$ , which concludes the proof of Theorem 1.1.  $\square$

**Remark 2.1.** In the proof given above, we conditioned on the realizations  $x$  of  $X_1$ . Since the distribution of  $H_x(\rho(x, X)) = \mu(B(x, \rho(x, X)))$  does not depend on  $X$ , we obtain as a by-product that

$$\mathbb{P}(\mu(B(X_1, R_{1,n,k})) > v_{n,k}) = \sum_{j=0}^{k-1} \binom{n-1}{j} v_{n,k}^j (1-v_{n,k})^{n-1-j} \sim \frac{e^{-t}}{n},$$

if  $X_1, X_2, \dots, X_n$  are independent and  $X_2, \dots, X_n$  are i.i.d. according to  $\mu$ . Here,  $X_1$  may have an arbitrary distribution and  $a_n \sim b_n$  means that  $a_n/b_n \rightarrow 1$  as  $n \rightarrow \infty$ .

## 2.2 Proof of Theorem 1.2

The main idea to derive Theorem 1.2 is to discretize  $\text{supp}(\mu) = [0, 1]^d$  into finitely many ‘‘small sets’’ and then to employ the Chen-Stein method. To apply this method, we will have to check an *asymptotic independence property* and a *local property* which ensures that, with high probability, two exceedances cannot appear in the same neighborhood. We introduce these properties below and recall a result due to Arratia *et al.* [2] on the Chen-Stein method.

**The asymptotic independence property** Fix  $\varepsilon > 0$ . Writing  $\lfloor \cdot \rfloor$  for the floor function, we partition  $[0, 1]^d$  into a set  $\mathcal{V}_n$  of  $N_n^d$  *subcubes* (i.e., subsets that are cubes) of equal size that can only have boundary points in common, where  $N_n = \lfloor (n/\log(n)^{1+\varepsilon})^{1/d} \rfloor$ . The subcubes are indexed by the set  $[1, N_n]^d = \{\mathbf{j} := (j_1, \dots, j_d) : j_m \in \{1, \dots, N_n\} \text{ for } m \in \{1, \dots, d\}\}$ . With a slight abuse of notation, we identify a cube with its index. Let

$$\mathcal{E}_n = \bigcap_{\mathbf{j} \in \mathcal{V}_n} \{\mathcal{X}_n \cap \mathbf{j} \neq \emptyset\}$$

be the event that each of the subcubes contains at least one of the points of  $\mathcal{X}_n$ . The event  $\mathcal{E}_n$  is extensively used in stochastic geometry to derive central limit theorems or to deal with extremes [3, 6, 7], and it will play a crucial role throughout the rest of the paper. The following lemma, which captures the idea of ‘‘asymptotic independence’’, is at the heart of our development.

**Lemma 2.2.** *For each  $\alpha > 0$ , we have  $\mathbb{P}(\mathcal{E}_n^c) = o(n^{-\alpha})$  as  $n \rightarrow \infty$ .*

*Proof.* By subadditivity and independence, it follows that

$$\begin{aligned} \mathbb{P}(\mathcal{E}_n^c) &\leq \sum_{\mathbf{j} \in \mathcal{V}_n} \mathbb{P}(\mathcal{X}_n \cap \mathbf{j} = \emptyset) \\ &= \sum_{\mathbf{j} \in \mathcal{V}_n} (\mathbb{P}(X_1 \notin \mathbf{j}))^n \\ &= \sum_{\mathbf{j} \in \mathcal{V}_n} (1 - \mu(\mathbf{j}))^n \\ &\leq \sum_{\mathbf{j} \in \mathcal{V}_n} \exp(-n\mu(\mathbf{j})). \end{aligned}$$

Here, the last inequality holds since  $\log(1-x) \leq -x$  for each  $x \in [0, 1)$ . Since  $f \geq f_- > 0$  on  $K$ , we have  $\mu(\mathbf{j}) = \int_{\mathbf{j}} f d\lambda \geq f_- \lambda(\mathbf{j})$ , whence – writing  $\#M$  for the cardinality of a finite set  $M$  –

$$\begin{aligned} \mathbb{P}(\mathcal{E}_n^c) &\leq \sum_{\mathbf{j} \in \mathcal{V}_n} \exp(-nf_- \lambda(\mathbf{j})) \\ &\leq \#\mathcal{V}_n \exp(-f_- (\log n)^{1+\varepsilon}). \end{aligned}$$

Since  $\#\mathcal{V}_n \leq n/(\log n)^{1+\varepsilon}$ , it follows that  $n^\alpha \mathbb{P}(\mathcal{E}_n^c) \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

**The local property** We now define a metric  $d$  on  $\mathcal{V}_n$  by putting  $d(\mathbf{j}, \mathbf{j}') := \max_{1 \leq s \leq d} |j_s - j'_s|$  for any two different subcubes  $\mathbf{j}$  and  $\mathbf{j}'$ , and  $d(\mathbf{j}, \mathbf{j}) := 0$ ,  $\mathbf{j} \in \mathcal{V}_n$ . Let  $S(\mathbf{j}, r) = \{\mathbf{j}' \in \mathcal{V}_n : d(\mathbf{j}, \mathbf{j}') \leq r\}$  be the ball of subcubes of radius  $r$  centered at  $\mathbf{j}$ . For any  $\mathbf{j} \in \mathcal{V}_n$ , put

$$M_{\mathbf{j}} := \max_{i \leq n, X_i \in \mathbf{j}} \mu(B(X_i, R_{i,n,k})),$$

with the convention  $M_{\mathbf{j}} = 0$  if  $\mathcal{X}_n \cap \mathbf{j} = \emptyset$ . Conditionally on the event  $\mathcal{E}_n$ , and provided that  $d(\mathbf{j}, \mathbf{j}') \geq 2k + 1$ , the random variables  $M_{\mathbf{j}}$  and  $M_{\mathbf{j}'}$  are independent. Lemma 2.2 is referred to as the *asymptotic independence property*: conditionally on the event  $\mathcal{E}_n$ , which occurs with high probability, the extremes  $M_{\mathbf{j}}$  and  $M_{\mathbf{j}'}$  attained on two subcubes which are sufficiently distant from each other are independent.

The following lemma claims that, with high probability, two exceedances cannot occur in the same neighborhood.

**Lemma 2.3.** *With the notation  $a \wedge b := \min(a, b)$  for  $a, b \in \mathbb{R}$ , let*

$$R(n) = \sup_{\mathbf{j} \in \mathcal{V}_n} \sum_{i \neq i' \leq n} \mathbb{P}(X_i, X_{i'} \in S(\mathbf{j}, 2k); \mu(B(X_i, R_{i,n,k})) \wedge \mu(B(X_{i'}, R_{i',n,k})) > v_{n,k}).$$

Then  $R(n) = O(n^{-1}(\log n)^{2-d+\varepsilon})$  as  $n \rightarrow \infty$ .

Here, with a slight abuse of notation, we have identified the family of subcubes  $S(\mathbf{j}, 2k) = \{\mathbf{j}' \in \mathcal{V}_n : d(\mathbf{j}, \mathbf{j}') \leq 2k\}$  with the set  $\bigcup \{\mathbf{j}' : \mathbf{j}' \in \mathcal{V}_n \text{ and } d(\mathbf{j}, \mathbf{j}') \leq 2k\}$ .

We prepare the proof of Lemma 2.3 with the following result that gives the volume of two  $d$ -dimensional balls.

**Lemma 2.4.** *If  $x \in B(0, 2)$  then*

$$\lambda(B(0, 1) \cup B(x, 1)) = 2 \left( \kappa_d \left( 1 - \frac{\arccos(\|x\|/2)}{\pi} \right) + \frac{\|x\| \kappa_{d-1}}{2d} \left( \sqrt{1 - (\|x\|/2)^2} \right)^{d-1} \right).$$

*Proof.* We calculate the volume of  $\lambda(B(0, 1) \cup B(x, 1))$  as the sum of the volumes of the following two congruent sets. The first one, say  $B$ , is given by the set of all points in  $B(0, 1) \cup B(x, 1)$  that are closer to 0 than to  $x$  and for the second one we change the roles of 0 and  $x$ . The set  $B$  is the union of a cone  $C$  with radius  $\sqrt{1 - (\|x\|/2)^2}$ , height  $\|x\|/2$  and apex at the origin and a set  $D := B(0, 1) \setminus S$ , where  $S$  is a simplicial cone with external angle  $\arccos(\|x\|/2)$ . From elementary geometry, we obtain that the volumes of  $C$  and  $D$  are given by

$$\lambda(C) = \frac{\|x\| \kappa_{d-1}}{2d} \left( \sqrt{1 - (\|x\|/2)^2} \right)^{d-1}, \quad \lambda(D) = \kappa_d \left( 1 - \frac{\arccos(\|x\|/2)}{\pi} \right).$$

This finishes the proof of the lemma.  $\square$

*Proof of Lemma 2.3.* For  $z \in [0, 1]^d$ , let

$$r_{n,k}(z) := \inf\{r > 0 : \mu(B(z, r)) > v_{n,k}\}.$$

Writing  $\#\mathcal{Y}(A)$  for the number of points of a finite set  $\mathcal{Y}$  of random points in  $\mathbb{R}^d$  that fall into a Borel set  $A$ , we have

$$\mu(B(z, R_{n,k}(z))) > v_{n,k} \iff \#\mathcal{X}_n(B(z, r_{n,k}(z))) \leq k - 1.$$

In the following, we assume that  $r_{n,k}(X_{i'}) \leq r_{n,k}(X_i)$  (which is at the cost of a factor 2) and distinguish the two cases  $X_{i'} \in B(X_i, r_{n,k}(X_i))$  and  $X_{i'} \in S(\mathbf{j}, 2k) \setminus B(X_i, r_{n,k}(X_i))$ . This distinction of cases gives

$$\begin{aligned} & \mathbb{P}(X_i, X_{i'} \in S(\mathbf{j}, 2k); \mu(B(X_i, R_{i,n,k})) \wedge \mu(B(X_{i'}, R_{i',n,k})) > v_{n,k}) \\ & \leq 2\mathbb{P}(X_i, X_{i'} \in S(\mathbf{j}, 2k); r_{n,k}(X_{i'}) \leq r_{n,k}(X_i); \mu(B(X_i, R_{i,n,k})) \wedge \mu(B(X_{i'}, R_{i',n,k})) > v_{n,k}) \\ & \leq 2\mathbb{P}(X_i \in S(\mathbf{j}, 2k); X_{i'} \in B(X_i, r_{n,k}(X_i)); r_{n,k}(X_{i'}) \leq r_{n,k}(X_i); \\ & \quad \mu(B(X_i, R_{i,n,k})) \wedge \mu(B(X_{i'}, R_{i',n,k})) > v_{n,k}) \end{aligned} \tag{9}$$

$$\begin{aligned} & + 2\mathbb{P}(X_i \in S(\mathbf{j}, 2k), X_{i'} \in S(\mathbf{j}, 2k) \setminus B(X_i, r_{n,k}(X_i)); r_{n,k}(X_{i'}) \leq r_{n,k}(X_i); \\ & \quad \mu(B(X_i, R_{i,n,k})) \wedge \mu(B(X_{i'}, R_{i',n,k})) > v_{n,k}). \end{aligned} \tag{10}$$

We bound the summands (9) and (10) separately. Since  $X_i$  and  $X_{i'}$  are independent, (9) takes the form

$$\begin{aligned} & 2 \int_{S(\mathbf{j}, 2k)} \int_{B(x, r_{n,k}(x))} \mathbb{1}\{r_{n,k}(y) \leq r_{n,k}(x)\} \mathbb{P}(\#\mathcal{X}_n \setminus \{X_i, X_{i'}\} \cup \{x\})(B(y, r_{n,k}(y))) \leq k - 1; \\ & \quad \#\mathcal{X}_n \setminus \{X_i, X_{i'}\} \cup \{y\})(B(x, r_{n,k}(x))) \leq k - 1) \mu(dy) \mu(dx). \end{aligned} \tag{11}$$

For  $y \in B(x, r_{n,k}(x))$ , the probability in the integrand figuring above is bounded from above by

$$\begin{aligned} & \mathbb{P}(\#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(y, r_{n,k}(y))) \leq k-1; \\ & \quad \#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(x, r_{n,k}(x))) \leq k-2) \\ & \leq \mathbb{P}(\#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(y, r_{n,k}(y))) \leq k-1; \\ & \quad \#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(y))) \leq k-2). \end{aligned} \quad (12)$$

Since the random vector

$$(\#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(y, r_{n,k}(y))), \#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(y))))$$

is negatively quadrant dependent (see [14, Section 3.1]), Equation (12) has the upper bound

$$\begin{aligned} & \mathbb{P}(\#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(y, r_{n,k}(y))) \leq k-1) \\ & \quad \times \mathbb{P}(\#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(y))) \leq k-2) \\ & \leq \mathbb{P}(\#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(y, r_{n,k}(y))) \leq k-1) \\ & \quad \times \mathbb{P}(\#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(x))) \leq k-2), \end{aligned} \quad (13)$$

where the last inequality holds since  $r_{n,k}(y) \leq r_{n,k}(x)$ . Analogously to Remark 2.1, the first probability is

$$\mathbb{P}(\#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(y, r_{n,k}(y))) \leq k-1) = \sum_{j=0}^{k-1} \binom{n-2}{j} v_{n,k}^j (1-v_{n,k})^{n-2-j} \sim \frac{e^{-t}}{n}.$$

The latter probability in (13) is given by

$$\sum_{\ell=0}^{k-2} \binom{n-2}{\ell} \mu(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(x)))^\ell (1 - \mu(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(x))))^{n-2-\ell}. \quad (14)$$

In a next step, we estimate  $\mu(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(x)))$ . Since  $f(x) \geq f_- > 0$ ,  $x \in [0, 1]^d$ , and by the homogeneity of  $d$ -dimensional Lebesgue measure  $\lambda$ , we obtain

$$\begin{aligned} \mu(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(x))) & \geq f_- \lambda(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(x))) \\ & = f_- r_{n,k}(x)^d \lambda(B(0, 1) \setminus B(r_{n,k}(x)^{-1}(y-x), 1)) \\ & = f_- r_{n,k}(x)^d (\lambda(B(0, 1) \cup B(r_{n,k}(x)^{-1}(y-x), 1)) - \kappa_d). \end{aligned}$$

For  $y \in B(x, r_{n,k}(x))$ , Lemma 2.4 yields

$$\begin{aligned} & \mu(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(x))) \\ & \geq f_- r_{n,k}(x)^d \left( \kappa_d \left( 1 - \frac{2 \arccos(\|x-y\|/2r_{n,k}(x))}{\pi} \right) + \frac{\|x-y\| \kappa_{d-1}}{2dr_{n,k}(x)} \left( \sqrt{1 - (\|x-y\|/2r_{n,k}(x))^2} \right)^{d-1} \right). \end{aligned}$$

Since  $\inf_{s>0} s^{-1}(1 - 2 \arccos(s)/\pi) > 0$ , there is  $c_0 > 0$  such that

$$\mu(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(x))) \geq c_0 \|x-y\| r_{n,k}(x)^{d-1}, \quad x \in S(\mathbf{j}, 2k), y \in B(x, r_{n,k}(x)).$$

Equation (14) and the bound  $f(x) \leq f_+$ ,  $x \in [0, 1]^d$ , give

$$\begin{aligned} & \int_{B(x, r_{n,k}(x))} \mathbf{1}\{r_{n,k}(y) \leq r_{n,k}(x)\} \mathbb{P}(\#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(x))) \leq k-1) \mu(dy) \\ & \leq f_+ \sum_{\ell=0}^{k-2} \binom{n-2}{\ell} \int_{B(x, r_{n,k}(x))} (c_0 \|x-y\| r_{n,k}(x)^{d-1})^\ell (1 - c_0 \|x-y\| r_{n,k}(x)^{d-1})^{n-2-\ell} \lambda(dy). \end{aligned}$$

We now introduce spherical coordinates and obtain

$$\begin{aligned} & f_+ d\kappa_d \sum_{\ell=0}^{k-2} \binom{n-2}{\ell} \int_0^{r_{n,k}(x)} (c_0 t r_{n,k}(x)^{d-1})^\ell (1 - c_0 t r_{n,k}(x)^{d-1})^{n-2-\ell} t^{d-1} dt \\ & = f_+ d\kappa_d \sum_{\ell=0}^{k-2} \binom{n-2}{\ell} \int_0^{r_{n,k}(x)} (c_0 t r_{n,k}(x)^{d-1})^\ell \exp((n-2-\ell) \log(1 - c_0 t r_{n,k}(x)^{d-1})) t^{d-1} dt \\ & \leq f_+ d\kappa_d \sum_{\ell=0}^{k-2} \binom{n-2}{\ell} \int_0^{r_{n,k}(x)} (c_0 t r_{n,k}(x)^{d-1})^\ell \exp(-c_0(n-2-\ell)t r_{n,k}(x)^{d-1}) t^{d-1} dt. \end{aligned}$$



Here, the last line follows from the inequality  $\log s \leq s - 1$ ,  $s > 0$ . Next, we apply the change of variables

$$t := (c_0(n - 2 - \ell))^{-1} r_{n,k}(x)^{1-d} s \quad (\text{i.e., } s = c_0(n - 2 - \ell) t r_{n,k}(x)^{d-1}),$$

which shows that the last upper bound takes the form

$$f_+ d \kappa_d c_0^{-d} r_{n,k}(x)^{d(1-d)} \sum_{\ell=0}^{k-2} \binom{n-2}{\ell} (n-2-\ell)^{-d-\ell} \int_0^{c_0(n-2-\ell)r_{n,k}(x)^d} s^{\ell+d-1} e^{-s} ds. \quad (15)$$

We now use the bounds  $f_- \kappa_d r_{n,k}(x)^d \leq v_{n,k}$ ,  $\binom{n-2}{\ell} \leq n^\ell / \ell!$ , and the fact that the integral figuring in (15) converges as  $n \rightarrow \infty$ . Hence, the expression in (15) is bounded from above by  $c_1 n^{-1} (\log n)^{1-d}$ , where  $c_1$  is some positive constant. Consequently, (9) is bounded from above by

$$\begin{aligned} c_1 n^{-1} (\log n)^{1-d} \lambda(S(\mathbf{j}, 2k)) \sup_{y \in S(\mathbf{j}, 2k)} \mathbb{P}(\#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(y, r_{n,k}(y)))) \leq k-1 \\ \sim c_2 n^{-3} (\log n)^{2-d+\varepsilon} \end{aligned} \quad (16)$$

for some  $c_2 > 0$ .

By analogy with the reasoning above, (10) is given by the integral

$$\begin{aligned} 2 \int_{S(\mathbf{j}, 2k)} \int_{S(\mathbf{j}, 2k) \setminus B(x, r_{n,k}(x))} \mathbb{1}\{r_{n,k}(y) \leq r_{n,k}(x)\} \mathbb{P}(\#(\mathcal{X}_n \setminus \{X_i, X_{i'}\} \cup \{x\})(B(y, r_{n,k}(y)))) \leq k-1 \\ \times \mathbb{P}(\#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(y)))) \leq k-1 \mu(dy) \mu(dx). \end{aligned} \quad (17)$$

If  $y \notin B(x, r_{n,k}(x))$  and  $r_{n,k}(x) \geq r_{n,k}(y)$ , we have the lower bound

$$\lambda(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(y))) \geq \frac{\lambda(B(x, r_{n,k}(x)))}{2}.$$

Since  $f_+ \kappa_d r_{n,k}(x)^d \geq v_{n,k}$ , we find a constant  $c_3 > 0$  such that

$$\lambda(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(y))) \geq c_3 v_{n,k},$$

whence

$$\begin{aligned} \mathbb{P}(\#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(x, r_{n,k}(x)) \setminus B(y, r_{n,k}(y)))) \leq k-1 \\ \leq \sum_{\ell=0}^{k-1} \binom{n-2}{\ell} (c_3 v_{n,k})^\ell (1 - c_3 v_{n,k})^{n-2-\ell} \\ \sim \frac{c_3^{k-1}}{(k-1)!} (\log n)^{k-1} \exp(n \log(1 - c_3 v_{n,k})) \end{aligned}$$

as  $n \rightarrow \infty$ . Since  $\log s \leq s - 1$  for  $s > 0$ , (17) is bounded from above by

$$\begin{aligned} c_4 n^{-c_3} \lambda(S(\mathbf{j}, 2k))^2 \sup_{y \in S(\mathbf{j}, 2k)} \mathbb{P}(\#(\mathcal{X}_n \setminus \{X_i, X_{i'}\})(B(y, r_{n,k}(y)))) \leq k-1 \\ \sim c_5 (4k+1)^{2d} \frac{(\log n)^{2+2\varepsilon}}{n^{3+c_3}}, \end{aligned} \quad (18)$$

where  $c_4$  and  $c_5$  are positive constants. Summing over all  $i \neq i' \leq n$ , it follows from (16) and (18) that  $R(n) = O(n^{-1} (\log n)^{2-d+\varepsilon})$  as  $n \rightarrow \infty$ , which finishes the proof of Lemma 2.3.  $\square$

**A Poisson approximation result based on the Chen-Stein method** In this paragraph, we recall a Poisson approximation result due to Arratia *et al.* [2], which is based on the Chen-Stein method. To this end, we consider a finite or countable collection  $(Y_\alpha)_{\alpha \in I}$  of  $\{0, 1\}$ -valued random variables and we let  $p_\alpha = \mathbb{P}(Y_\alpha = 1) > 0$ ,  $p_{\alpha\beta} = \mathbb{P}(Y_\alpha = 1, Y_\beta = 1)$ . Moreover, suppose that for each  $\alpha \in I$ , there is a set  $B_\alpha \subset I$  that contains  $\alpha$ . The set  $B_\alpha$  is regarded as a neighborhood of  $\alpha$  that consists of the set of indices  $\beta$  such that  $Y_\alpha$  and  $Y_\beta$  are *not* independent. Finally, put

$$b_1 = \sum_{\alpha \in I} \sum_{\beta \in B_\alpha} p_\alpha p_\beta, \quad b_2 = \sum_{\alpha \in I} \sum_{\alpha \neq \beta \in B_\alpha} p_{\alpha\beta}, \quad b_3 = \sum_{\alpha \in I} \mathbb{E}[|\mathbb{E}[Y_\alpha - p_\alpha | \sigma(Y_\beta : \beta \notin B_\alpha)]|]. \quad (19)$$

**Theorem 2.5.** (Theorem 1 of [2]) Let  $W = \sum_{\alpha \in I} Y_\alpha$ , and assume  $\lambda := \mathbb{E}(W) \in (0, \infty)$ . Then

$$d_{TV}(W, \text{Po}(\lambda)) \leq 2(b_1 + b_2 + b_3).$$

**Proof of Theorem 1.2** Recall  $v_{n,k}$  from (3) and  $C_{n,k}$  from (4). Put

$$\widehat{C}_{n,k} := \sum_{\mathbf{j} \in \mathcal{V}_n} \mathbf{1}\{M_{\mathbf{j}} > v_{n,k}\}.$$

The following lemma claims that the number  $C_{n,k}$  of exceedances is close to the number of subcubes for which there exists at least one exceedance, i.e.  $\widehat{C}_{n,k}$ , and that  $\widehat{C}_{n,k}$  can be approximated by a Poisson random variable.

**Lemma 2.6.** *We have*

- a)  $\mathbb{P}(C_{n,k} \neq \widehat{C}_{n,k}) = O\left((\log n)^{1-d}\right),$
- b)  $d_{TV}(\widehat{C}_{n,k}, \text{Po}(\mathbb{E}[\widehat{C}_{n,k}])) = O\left((\log n)^{1-d}\right),$
- c)  $\mathbb{E}[\widehat{C}_{n,k}] = e^{-t} + O\left(\frac{\log \log n}{\log n}\right).$

*Proof.* Assertion a) is a direct consequence of Lemma 2.3 and of the inequalities

$$\begin{aligned} \mathbb{P}(C_{n,k} \neq \widehat{C}_{n,k}) &= \mathbb{P}(\exists \mathbf{j} \in \mathcal{V}_n, \exists i, \ell \text{ s.t. } X_i, X_\ell \in \mathbf{j}; \mu(B(X_i, R_{i,n,k})) \wedge \mu(B(X_\ell, R_{\ell,n,k})) > v_{n,k}) \\ &\leq \sum_{\mathbf{j} \in \mathcal{V}_n} \sum_{i \neq \ell \leq n} \mathbb{P}(X_i, X_\ell \in \mathbf{j}; \mu(B(X_i, R_{i,n,k})) \wedge \mu(B(X_\ell, R_{\ell,n,k})) > v_{n,k}) \\ &\leq \frac{n}{(\log n)^{1+\varepsilon}} \times R(n). \end{aligned}$$

To prove b), we apply Theorem 2.5 to the collection  $(Y_\alpha)_{\alpha \in I} = (M_{\mathbf{j}})_{\mathbf{j} \in \mathcal{V}_n}$ . Recall that, conditionally on the event  $\mathcal{E}_n$ , the random variables  $M_{\mathbf{j}}$  and  $M_{\mathbf{j}'}$  are independent provided that  $d(\mathbf{j}, \mathbf{j}') \geq 2k + 1$ . With a slight abuse of notation, we omit to condition on  $\mathcal{E}_n$  since this event occurs with probability tending to 1 as  $n \rightarrow \infty$  (Lemma 2.2) at a rate which is at least polynomial. The first two terms in (19) are

$$b_1 = \sum_{\mathbf{j} \in \mathcal{V}_n} \sum_{\mathbf{j}' \in S(\mathbf{j}, 2k)} p_{\mathbf{j}} p_{\mathbf{j}'}, \quad b_2 = \sum_{\mathbf{j} \in \mathcal{V}_n} \sum_{\mathbf{j}' \notin S(\mathbf{j}, 2k)} p_{\mathbf{j} \mathbf{j}'},$$

where

$$p_{\mathbf{j}} = \mathbb{P}(M_{\mathbf{j}} > v_{n,k}), \quad p_{\mathbf{j} \mathbf{j}'} = \mathbb{P}(M_{\mathbf{j}} > v_{n,k}, M_{\mathbf{j}'} > v_{n,k}).$$

The term  $b_3$  figuring in (19) equals 0 since, conditionally on  $\mathcal{E}_n$ , the random variable  $M_{\mathbf{j}}$  is independent of the  $\sigma$ -field  $\sigma(M_{\mathbf{j}'} : \mathbf{j}' \notin S(\mathbf{j}, 2k))$ . Thus, according to Theorem 2.5, we have

$$d_{TV}(\widehat{C}_{n,k}, \text{Po}(\mathbb{E}[\widehat{C}_{n,k}])) \leq 2(b_1 + b_2).$$

First, we deal with  $b_1$ . As for the first assertion, notice that for each  $\mathbf{j} \in \mathcal{V}_n$ , using symmetry, we obtain

$$\begin{aligned} p_{\mathbf{j}} &= \mathbb{P}\left(\bigcup_{i \leq n} \{X_i \in \mathbf{j}, \mu(B(X_i, R_{i,n,k})) > v_{n,k}\}\right) \\ &\leq n \cdot \mathbb{P}(X_1 \in \mathbf{j}, \mu(B(X_1, R_{1,n,k})) > v_{n,k}) \\ &= n \cdot \int_{\mathbf{j}} \mathbb{P}(\mu(B(x, R_{1,n,k})) > v_{n,k} | X_1 = x) f(x) \, dx \\ &\leq n f^+ \lambda(\mathbf{j}) \int_{\mathbf{j}} \mathbb{P}(\mu(B(x, R_{1,n,k})) > v_{n,k} | X_1 = x) \frac{1}{\lambda(\mathbf{j})} \, dx \\ &= n f^+ \lambda(\mathbf{j}) \mathbb{P}(\mu(B(\widetilde{X}_1, R_{1,n,k})) > v_{n,k}), \end{aligned}$$

where  $\widetilde{X}_1$  is independent of  $X_2, \dots, X_n$  and has a uniform distribution over  $\mathbf{j}$ . Invoking Remark 2.1, the probability figuring in the last line is asymptotically equal to  $e^{-t}/n$  as  $n \rightarrow \infty$ . Since  $\lambda(\mathbf{j}) = O((\log n)^{1+\varepsilon}/n)$ , we thus have

$$p_{\mathbf{j}} \leq C \cdot \frac{(\log n)^{1+\varepsilon}}{n},$$

where  $C$  is a constant that does not depend on  $\mathbf{j}$ . Since  $\#\mathcal{V}_n \leq \frac{n}{(\log n)^{1+\varepsilon}}$  and  $\#S(\mathbf{j}, 2k) \leq (4k+1)^d$ , summing over  $\mathbf{j}, \mathbf{j}'$  gives

$$b_1 \leq C^2 \sum_{\mathbf{j} \in \mathcal{V}_n} \sum_{\mathbf{j}' \in S(\mathbf{j}, 2k)} \left( \frac{(\log n)^{1+\varepsilon}}{n} \right)^2 = O\left( \frac{(\log n)^{1+\varepsilon}}{n} \right).$$

To deal with  $b_2$ , notice that, for each  $\mathbf{j}, \mathbf{j}' \in \mathcal{V}_n$  and  $\mathbf{j}' \in S(\mathbf{j}, 2k)$ , we have

$$\begin{aligned} p_{\mathbf{j}\mathbf{j}'} &= \mathbb{P}\left( \bigcup_{i \neq i' \leq n} \{X_i \in \mathbf{j}, X_{i'} \in S(\mathbf{j}, 2k), \mu(B(X_i, R_{i,n,k})) \wedge \mu(B(X_{i'}, R_{i',n,k})) > v_{n,k}\} \right) \\ &\leq \mathbb{P}\left( \bigcup_{i \neq i' \leq n} \{X_i, X_{i'} \in S(\mathbf{j}, 2k); \mu(B(X_i, R_{i,n,k})) \wedge \mu(B(X_{i'}, R_{i',n,k})) > v_{n,k}\} \right). \end{aligned}$$

Using subadditivity, and taking the supremum, we obtain

$$\begin{aligned} b_2 &\leq \sum_{\mathbf{j} \in \mathcal{V}_n} \sum_{\mathbf{j}' \in S(\mathbf{j}, 2k)} \sup_{\mathbf{j} \in \mathcal{V}_n} \sum_{i \neq i' \leq n} \mathbb{P}(X_i, X_{i'} \in S(\mathbf{j}, 2k), \mu(B(X_i, R_{i,n,k})) \wedge \mu(B(X_{i'}, R_{i',n,k})) > v_{n,k}) \\ &\leq \frac{n}{(\log n)^{1+\varepsilon}} \times (4k+1)^d \times R(n). \end{aligned}$$

According to Lemma 2.3, the last term equals  $O\left((\log n)^{1-d}\right)$ , which concludes the proof of b).

To prove c), observe that

$$|\mathbb{E}[\widehat{C}_{n,k}] - e^{-t}| \leq |\mathbb{E}[\widehat{C}_{n,k}] - \mathbb{E}[C_{n,k}]| + |\mathbb{E}[C_{n,k}] - e^{-t}|.$$

By Theorem 1.1, the last summand is  $O\left(\frac{\log \log n}{\log n}\right)$ . Since  $C_{n,k} \geq \widehat{C}_{n,k}$ , we further have

$$\begin{aligned} |\mathbb{E}[\widehat{C}_{n,k}] - \mathbb{E}[C_{n,k}]| &= \mathbb{E}[C_{n,k} - \widehat{C}_{n,k}] \\ &= \mathbb{E}\left( \sum_{i \leq n} \mathbf{1}\{\mu(B(X_i, R_{i,n,k})) > v_{n,k}\} - \sum_{\mathbf{j} \in \mathcal{V}_n} \mathbf{1}\{M_{\mathbf{j}} > v_{n,k}\} \right) \\ &= \sum_{\mathbf{j} \in \mathcal{V}_n} \mathbb{E}\left[ \left( \sum_{i \leq n} \mathbf{1}\{X_i \in \mathbf{j}\} \mathbf{1}\{\mu(B(X_i, R_{i,n,k})) > v_{n,k}\} - 1 \right) \mathbf{1}\{M_{\mathbf{j}} > v_{n,k}\} \right] \\ &\leq \sum_{\mathbf{j} \in \mathcal{V}_n} \sum_{i \neq i' \leq n} \mathbb{P}(X_i, X_{i'} \in \mathbf{j}, \mu(B(X_i, R_{i,n,k})) \wedge \mu(B(X_{i'}, R_{i',n,k})) > v_{n,k}) \\ &\leq \#\mathcal{V}_n \times R(n). \end{aligned}$$

According to Lemma 2.3, the last term equals  $O\left((\log n)^{1-d}\right)$ . This concludes the proof of Lemma 2.6 and thus of Theorem 1.2.  $\square$

### 3 Concluding remarks

When dealing with limit laws for large  $k$ th-nearest neighbor *distances* of a sequence of i.i.d. random points in  $\mathbb{R}^d$  with density  $f$ , which take values in a bounded region  $K$ , the modification of the  $k$ th-nearest neighbor distances made in (7) (by introducing the ‘‘boundary distances’’  $\|X_i - \partial K\|$ ) and the condition that  $f$  is bounded away from zero, which have been adopted in [12] and [13], seem to be crucial, since boundary effects play a decisive role ([8, 9]). Regarding  $k$ th-nearest neighbor balls with *large probability volume*, there is no need to introduce  $\|X_i - \partial K\|$ . It is an open problem, however, whether Theorem 1.2 continues to hold for densities that are not bounded away from zero.

A second open problem refers to Theorem 1.1, which states convergence of expectations of  $C_{n,k}$  in a setting beyond the finite-dimensional case. Since  $C_{n,k}$  is non-negative, the sequence  $(C_{n,k})_k$  is tight by Markov’s inequality. Can one find conditions on the underlying distribution that ensure convergence in distribution to some random element of the metric space?

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