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► **To cite this version:**

Tomasz Skalski, Piotr Graczyk, Bartosz Kolodziejek, Maciej Wilczyński. Pattern recovery and signal denoising by SLOPE when the design matrix is orthogonal. 2022. hal-03610548

HAL Id: hal-03610548

<https://hal.science/hal-03610548>

Preprint submitted on 16 Mar 2022

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Pattern recovery and signal denoising by SLOPE when the design matrix is orthogonal ^{*}

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February 17, 2022

Abstract

Sorted ℓ_1 Penalized Estimator (SLOPE) is a relatively new convex regularization method for fitting high-dimensional regression models. SLOPE allows to reduce the model dimension by nullifying some of the regression coefficients and by equalizing some of nonzero coefficients. This allows to identify situations where some of true regression coefficients are equal. In this article we will introduce the SLOPE pattern, i.e., the set of relations between the true regression coefficients, which can be identified by SLOPE. We will also present new results on the strong consistency of SLOPE estimators and on the strong consistency of pattern recovery by SLOPE when the design matrix is orthogonal and illustrate advantages of the SLOPE clustering in the context of high frequency signal denoising.

^{*}The authors want to thank Małgorzata Bogdan, Patrick Tardivel, Wojciech Rejchel and Tomasz Żak for references, comments and discussion. First Author was supported by a French Government Scholarship. This research was carried out while the first author spent the summer semester of 2021 at Laboratoire de Mathématiques LAREMA and at Université de Bourgogne.

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1 Introduction

1.1 Introduction and motivations

The Linear Multiple Regression concerns the model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $\mathbf{Y} \in \mathbb{R}^n$ is an output vector, $\mathbf{X} \in \mathbb{R}^{n \times p}$ is a fixed design matrix, $\boldsymbol{\beta} \in \mathbb{R}^p$ is an unknown vector of predictors and $\boldsymbol{\varepsilon} \in \mathbb{R}^n$ is a noise vector. The primary goal is to estimate $\boldsymbol{\beta}$. In the low-dimensional setting, i.e., when the number of predictors p is not larger than the number of explanatory variables n and \mathbf{X} is of full rank, the ordinary least squares estimator $\hat{\boldsymbol{\beta}}^{OLS}$ has an exact formula $\hat{\boldsymbol{\beta}}^{OLS} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$. For practical reasons there is an urge to avoid the high-dimensionality curse, therefore we want the estimate to be sparse, i.e., to be describable by a smaller number of parameters. Several solutions were proposed to deal with such problem. One of them, the Least Absolute Shrinkage and Selection Operator (LASSO [6, 24]) involves penalizing the residual sum of squares $\|\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}\|_2^2$ with an ℓ_1 norm of $\hat{\boldsymbol{\beta}}$ multiplied by a tuning parameter λ :

$$\hat{\boldsymbol{\beta}}^{LASSO} := \arg \min_{\mathbf{b} \in \mathbb{R}^p} \left[\frac{1}{2} \|\mathbf{Y} - \mathbf{X}\mathbf{b}\|_2^2 + \lambda \|\mathbf{b}\|_1 \right].$$

The LASSO estimator is not unbiased, but is a shrinkage estimator which reduces some of the coefficients of $\boldsymbol{\beta}$ to zero, resulting in a sparser estimate. In the case of \mathbf{X} being an orthogonal matrix, i.e. $\mathbf{X}'\mathbf{X} = \mathbf{I}_p$, the exact formula for $\hat{\boldsymbol{\beta}}^{LASSO}$ introduced by Tibshirani [24] is based on $\hat{\boldsymbol{\beta}}^{OLS}$:

$$\hat{\beta}_i^{LASSO} = \text{sign}(\hat{\beta}_i^{OLS}) \max\{|\hat{\beta}_i^{OLS}| - \lambda, 0\}.$$

Another approach to reduce the dimensionality is the Sorted ℓ_1 Penalized Estimator (SLOPE [3, 2, 25]), which not only generalizes the LASSO method, but also allows to clusterize the similar coefficients of $\boldsymbol{\beta}$. In SLOPE, ℓ_1 -norm is replaced by its sorted version $J_{\boldsymbol{\Lambda}}$, which depends on the tuning vector $\boldsymbol{\Lambda} = (\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p$, where $\lambda_1 \geq \dots \geq \lambda_p \geq 0$:

$$J_{\boldsymbol{\Lambda}}(\boldsymbol{\beta}) := \sum_{i=1}^p |\boldsymbol{\beta}|_{(i)} \lambda_i,$$

where $\{|\boldsymbol{\beta}|_{(i)}\}_{i=1}^p$ is a decreasing permutation of absolute values of $\boldsymbol{\beta}_i$, namely

$$\hat{\boldsymbol{\beta}}^{SLOPE} := \arg \min_{\mathbf{b} \in \mathbb{R}^p} \left[\frac{1}{2} \|\mathbf{Y} - \mathbf{X}\mathbf{b}\|_2^2 + \gamma J_{\boldsymbol{\Lambda}}(\mathbf{b}) \right].$$

The case of $\boldsymbol{\Lambda}$ being an arithmetic sequence was studied by Bondell and Reich [4] and called the Octagonal Shrinkage and Clustering Algorithm for Regression (OSCAR). The special case of SLOPE with

$$\lambda_1 = \lambda_2 = \dots = \lambda_p > 0$$

is LASSO. For $\boldsymbol{\Lambda} = (0, \dots, 0)$ we obtain the OLS estimator.

Clustering the predictors allows for additional dimension reduction by identifying variables with the same values of regression coefficients. One may recently observe the rise of interest

in methods, which cluster highly correlated predictors [5, 11, 13, 16, 17, 23]. SLOPE is ideal for this task, since it is capable to identify the low-dimensional structure, which is called the SLOPE pattern, defined by Schneider and Tardivel with the subdifferential of the SLOPE norm J_Λ [19]. For the convention of this article we let $\text{sign}(0) = 0$. As k we will denote the number of clusters of $\text{patt}(\mathbf{b}) = (m_1, \dots, m_p)'$ i.e., the number of nonzero components of $|\mathbf{b}|$.

Definition 1.0.1 (SLOPE pattern [19]). *The SLOPE pattern is a function $\text{patt} : \mathbb{R}^p \rightarrow \mathbb{Z}^p$ such that*

$$\text{patt}(\mathbf{b})_i = \text{sign}(b_i) \text{rank}(|b_i|),$$

where $\text{rank}(|b_i|) \in \{1, 2, \dots, k\}$ is a rank of $|b_i|$ in a vector of distinct nonzero values among $\{|b_1|, \dots, |b_p|\}$. We adopt the convention that $\text{rank}(0) = 0$.

As \mathcal{M}_p we denote the set of all possible SLOPE patterns of $\mathbf{b} \in \mathbb{R}^p$.

Fact 1.0.1 (Basic properties of SLOPE pattern).

- (a) for every $1 \leq l \leq \|\text{patt}(\mathbf{b})\|_\infty$ there exists j such that $|\text{patt}(\mathbf{b})_j| = l$,
- (b) $\text{sign}(\text{patt}(\mathbf{b})) = \text{sign}(\mathbf{b})$ (sign preservation),
- (c) $|b_i| = |b_j| \Rightarrow |\text{patt}(\mathbf{b})_i| = |\text{patt}(\mathbf{b})_j|$ (clustering preservation),
- (d) $|b_i| > |b_j| \Rightarrow |\text{patt}(\mathbf{b})_i| > |\text{patt}(\mathbf{b})_j|$ (hierarchy preservation).

Example 1.0.1. $\text{patt}(4, 0, -1.5, 1.5, -4) = (2, 0, -1, 1, -2)$.

Remark 1.0.1 (Subdifferential description of the SLOPE pattern [19]).

Let $\Lambda = (\lambda_1, \dots, \lambda_p)$ satisfy $\lambda_1 > \dots > \lambda_p > 0$. Then

$$\text{patt}(\mathbf{b}_1) = \text{patt}(\mathbf{b}_2) \iff \partial_{J_\Lambda}(\mathbf{b}_1) = \partial_{J_\Lambda}(\mathbf{b}_2),$$

where $\partial_f(\mathbf{b})$ is a subdifferential of the function $f : \mathbb{R}^p \rightarrow \mathbb{R}$ in \mathbf{b} , i.e.:

$$\partial_f(\mathbf{b}) = \{v \in \mathbb{R}^p : f(\mathbf{z}) \geq f(\mathbf{b}) + v'(\mathbf{z} - \mathbf{b}) \quad \forall \mathbf{z} \in \mathbb{R}^p\}.$$

The subdifferential approach may be applied to a wider class of penalizers being polyhedral gauges, cf. [22].

Definition 1.0.2 (Pattern recovery by SLOPE). *We say that the SLOPE estimator $\widehat{\beta}^{\text{SLOPE}}$ recovers the pattern of β when*

$$\text{patt}(\widehat{\beta}^{\text{SLOPE}}) = \text{patt}(\beta).$$

The clustering properties of SLOPE have been studied before, cf. [4, 10], but the researchers consider strongly correlated predictors, which are being used in financial mathematics to group the assets with respect to their partial correlation with the hedge fund return times series [12]. In our article we present the clusterization by SLOPE when the predictors are orthogonal, i.e. $\mathbf{X}'\mathbf{X} = \mathbf{I}_p$. Such class of matrices is being widely used in signal analysis, [18, 7]. For general \mathbf{X} the problem is considered in our parallel article [20].

To study the properties of SLOPE we often use the unit ball in the dual norm of J_Λ , which was studied e.g. by Zeng and Figueiredo [25]. This dual ball is also known as a signed permutahedron, see e.g. [15, 19].

$$C_\Lambda = \{\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_p) \in \mathbb{R}^p : \sum_{j \leq i} |\pi_{(j)}| \leq \sum_{j \leq i} \lambda_j : i = 1, 2, \dots, p\}. \quad (1)$$

In this article we prove novel results on the strong consistency of SLOPE both in estimation and in pattern recovery. We also introduce a new, based on minimaxity, approach to relations between $\hat{\boldsymbol{\beta}}^{SLOPE}$ and $\hat{\boldsymbol{\beta}}^{OLS}$.

1.2 Outline of the paper

In Section 2 we derive the connections between $\hat{\boldsymbol{\beta}}^{SLOPE}$ and $\hat{\boldsymbol{\beta}}^{OLS}$ in the orthogonal design. We use the minimax theorem of Sion, cf. [1]. In Section 3 we focus on the properties of $\hat{\boldsymbol{\beta}}^{SLOPE}$. We use the geometric interpretation of SLOPE to explain its ability to identify the SLOPE pattern and provide new theoretical results on the support recovery and clustering properties using a representation of SLOPE as a function of the ordinary least squares (OLS) estimator. Similar approach for LASSO was used by Ewald and Schneider, cf. [9]. In Section 4 we discuss asymptotic properties of the SLOPE estimators in the low-dimensional regression model in which p is fixed and the sample size n tends to infinity. To be more precise, for each $n \geq 1$ we consider a linear regression model

$$\mathbf{Y}^{(n)} = \mathbf{X}^{(n)}\boldsymbol{\beta} + \boldsymbol{\varepsilon}^{(n)}$$

with the vector of observations $\mathbf{Y}^{(n)} \in \mathbb{R}^n$ and the design matrix $\mathbf{X}^{(n)} \in \mathbb{R}^{n \times p}$. Here $\boldsymbol{\varepsilon}^{(n)} = (\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, \dots, \varepsilon_n^{(n)})' \in \mathbb{R}^n$ is a noise term, which has the normal distribution $N(0, \sigma^2 \mathbf{I}_n)$. We make no assumptions about the relations between $\boldsymbol{\varepsilon}^{(n)}$ and $\boldsymbol{\varepsilon}^{(m)}$ for $n \neq m$. We consider the low-dimensional setup when p is fixed and n goes to infinity and provide the conditions under which the SLOPE estimator is strongly consistent. Additionally, in case when for each n the design matrix is orthogonal, we provide the conditions on the sequence of tuning parameters such that SLOPE is strongly consistent in the pattern recovery. Compared to similar results in [20], dealing with the general design matrices, our results are obtained using substantially simpler techniques and impose weaker restrictions on the sequence of tuning parameters required for the pattern recovery. In Section 5 we show the applications of the SLOPE clustering in terms of high frequency signal denoising and illustrate them with simulations. The Appendix covers the proofs of technical results.

2 Approach by minimax theorem

2.1 Technical results

Let r_{SLOPE} denote the minimum value of the SLOPE criterion, attained by $\hat{\boldsymbol{\beta}}^{SLOPE}$, i.e.

$$r_{SLOPE} := \min_{\mathbf{b} \in \mathbb{R}^p} \left[\frac{1}{2} \|\mathbf{Y} - \mathbf{X}\mathbf{b}\|_2^2 + J_\Lambda(\mathbf{b}) \right] = \frac{1}{2} \|\mathbf{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}^{SLOPE}\|_2^2 + J_\Lambda(\hat{\boldsymbol{\beta}}^{SLOPE}).$$

Since

$$\|\widehat{\boldsymbol{\beta}}^{SLOPE}\|_2 \leq \sqrt{p} \|\widehat{\boldsymbol{\beta}}^{SLOPE}\|_\infty \quad \text{and} \quad \lambda_1 \|\widehat{\boldsymbol{\beta}}^{SLOPE}\|_\infty \leq J_\Lambda(\widehat{\boldsymbol{\beta}}^{SLOPE}) \leq r_{SLOPE},$$

it follows that

$$\lambda_1 \|\widehat{\boldsymbol{\beta}}^{SLOPE}\|_2 \leq \sqrt{p} r_{SLOPE} \leq \sqrt{p} \left[\frac{1}{2} \|\mathbf{Y} - \mathbf{X}\mathbf{0}\|_2^2 + J_\Lambda(\mathbf{0}) \right] = \frac{\sqrt{p}}{2} \|\mathbf{Y}\|_2^2.$$

We immediately get the following result.

Corollary 2.0.1. $\|\widehat{\boldsymbol{\beta}}^{SLOPE}\|_2^2 \leq M_0$, where

$$M_0 = \left(\frac{p \|\mathbf{Y}\|_2^4}{4\lambda_1^2} \right).$$

From this corollary it is seen that we can clearly limit our search to vectors $\boldsymbol{\beta}$ from the compact set $\mathcal{M} \subset \mathbb{R}^p$ defined by

$$\mathcal{M} := \{\mathbf{b} \in \mathbb{R}^p : \|\mathbf{b}\|_2^2 \leq M_0\}.$$

Therefore, we can equivalently define a SLOPE solution by

$$\widehat{\boldsymbol{\beta}}^{SLOPE} = \arg \min_{\mathbf{b} \in \mathcal{M}} \left[\frac{1}{2} \|\mathbf{Y} - \mathbf{X}\mathbf{b}\|_2^2 + J_\Lambda(\mathbf{b}) \right]. \quad (2)$$

Proposition 2.0.1. [3, Proposition 1.1] Let C_Λ be the unit ball in the dual SLOPE norm. Then, for each $\mathbf{b} \in \mathbb{R}^p$,

$$J_\Lambda(\mathbf{b}) = \max_{\boldsymbol{\pi} \in C_\Lambda} \boldsymbol{\pi}'\mathbf{b}.$$

This statement follows from the definition of C_Λ (1) and from the reflexivity of finite-dimensional spaces.

2.2 Saddle point

Let the function $r : \mathcal{M} \times C_\Lambda \rightarrow \mathbb{R}$ be defined by

$$r(\mathbf{b}, \boldsymbol{\pi}) := \frac{1}{2} \|\mathbf{Y} - \mathbf{X}\mathbf{b}\|_2^2 + \boldsymbol{\pi}'\mathbf{b}.$$

As an immediate consequence of ((2)) and Proposition 2.0.1 we obtain

$$\begin{aligned} r_{SLOPE} &= \min_{\mathbf{b} \in \mathbb{R}^p} \left[\frac{1}{2} \|\mathbf{Y} - \mathbf{X}\mathbf{b}\|_2^2 + J_\Lambda(\mathbf{b}) \right] = \min_{\mathbf{b} \in \mathcal{M}} \left[\frac{1}{2} \|\mathbf{Y} - \mathbf{X}\mathbf{b}\|_2^2 + J_\Lambda(\mathbf{b}) \right] \\ &= \min_{\mathbf{b} \in \mathcal{M}} \max_{\boldsymbol{\pi} \in C_\Lambda} \left[\frac{1}{2} \|\mathbf{Y} - \mathbf{X}\mathbf{b}\|_2^2 + \boldsymbol{\pi}'\mathbf{b} \right] = \min_{\mathbf{b} \in \mathcal{M}} \max_{\boldsymbol{\pi} \in C_\Lambda} r(\mathbf{b}, \boldsymbol{\pi}). \end{aligned}$$

It turns out that the order of the maximization over $\boldsymbol{\pi} \in C_\Lambda$ and the minimization over $\mathbf{b} \in \mathcal{M}$ can be switched without affecting the result. To see this, note that both C_Λ and \mathcal{M}

are convex and compact. Moreover, for each fixed $\boldsymbol{\pi} \in C_{\Lambda}$, $r(\mathbf{b}, \boldsymbol{\pi})$ is a convex continuous function with respect to $\mathbf{b} \in \mathcal{M}$ and, for each fixed $\mathbf{b} \in \mathcal{M}$, $r(\mathbf{b}, \boldsymbol{\pi})$ is concave with respect to $\boldsymbol{\pi} \in C_{\Lambda}$ (in fact, it is linear). Therefore, all assumptions of the Sion's minimax theorem are fulfilled (see [1, p. 218]) and thus there exists a saddle point $(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*) \in \mathcal{M} \times C_{\Lambda}$ such that

$$\begin{aligned} \max_{\boldsymbol{\pi} \in C_{\Lambda}} \min_{\mathbf{b} \in \mathcal{M}} r(\mathbf{b}, \boldsymbol{\pi}) &= \min_{\mathbf{b} \in \mathcal{M}} r(\mathbf{b}, \boldsymbol{\pi}^*) = r(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*) \\ &= \max_{\boldsymbol{\pi} \in C_{\Lambda}} r(\boldsymbol{\beta}^*, \boldsymbol{\pi}) = \min_{\mathbf{b} \in \mathcal{M}} \max_{\boldsymbol{\pi} \in C_{\Lambda}} r(\mathbf{b}, \boldsymbol{\pi}) = r_{SLOPE}. \end{aligned}$$

In the next section we shall see that the first coordinate of any saddle point $(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*)$ is the SLOPE estimator.

2.3 SLOPE solution in the orthogonal design

Since for each fixed $\boldsymbol{\pi} \in C_{\Lambda}$, the function $r(\mathbf{b}, \boldsymbol{\pi})$ is convex with respect to $\mathbf{b} \in \mathcal{M}$, any point $\mathbf{b}_{\boldsymbol{\pi}} \in \mathcal{M}$, at which the gradient $\frac{\partial r(\mathbf{b}, \boldsymbol{\pi})}{\partial \mathbf{b}}$ is zero, is a global minimum. If we rewrite $r(\mathbf{b}, \boldsymbol{\pi})$ as

$$r(\mathbf{b}, \boldsymbol{\pi}) = \frac{1}{2} \mathbf{Y}' \mathbf{Y} - \mathbf{Y}' \mathbf{X} \mathbf{b} + \frac{1}{2} \mathbf{b}' \mathbf{b} + \boldsymbol{\pi}' \mathbf{b}$$

and differentiate with respect to \mathbf{b} , we obtain

$$\frac{\partial r(\mathbf{b}, \boldsymbol{\pi})}{\partial \mathbf{b}} = -\mathbf{X}'(\mathbf{Y} - \mathbf{X} \mathbf{b}) + \boldsymbol{\pi}.$$

Equating this gradient to $\mathbf{0}$ gives the following equation for the optimum point $\mathbf{b}_{\boldsymbol{\pi}}$:

$$\mathbf{b}_{\boldsymbol{\pi}} = \mathbf{X}' \mathbf{Y} - \boldsymbol{\pi} = \hat{\boldsymbol{\beta}}^{OLS} - \boldsymbol{\pi}.$$

Substituting this into the equation for $r(\mathbf{b}_{\boldsymbol{\pi}}, \boldsymbol{\pi})$, we find that

$$\begin{aligned} r(\mathbf{b}_{\boldsymbol{\pi}}, \boldsymbol{\pi}) &= \frac{1}{2} \mathbf{Y}' \mathbf{Y} - \mathbf{b}'_{\boldsymbol{\pi}} \mathbf{X}' \mathbf{Y} + \frac{1}{2} \mathbf{b}'_{\boldsymbol{\pi}} \mathbf{b}_{\boldsymbol{\pi}} + \boldsymbol{\pi}' \mathbf{b}_{\boldsymbol{\pi}} \\ &= \frac{1}{2} \mathbf{Y}' \mathbf{Y} - \mathbf{b}'_{\boldsymbol{\pi}} \mathbf{X}' \mathbf{Y} + \mathbf{b}'_{\boldsymbol{\pi}} \mathbf{b}_{\boldsymbol{\pi}} + \mathbf{b}'_{\boldsymbol{\pi}} \boldsymbol{\pi} - \frac{1}{2} \mathbf{b}'_{\boldsymbol{\pi}} \mathbf{X}' \mathbf{X} \mathbf{b}_{\boldsymbol{\pi}} \\ &= \frac{1}{2} \mathbf{Y}' \mathbf{Y} - \frac{1}{2} \mathbf{b}'_{\boldsymbol{\pi}} \mathbf{b}_{\boldsymbol{\pi}} = \frac{1}{2} \mathbf{Y}' \mathbf{Y} - \frac{1}{2} \mathbf{b}'_{\boldsymbol{\pi}} \mathbf{b}_{\boldsymbol{\pi}} \\ &= \frac{1}{2} \mathbf{Y}' \mathbf{Y} - \frac{1}{2} (\mathbf{X}' \mathbf{Y} - \boldsymbol{\pi})' (\mathbf{X}' \mathbf{Y} - \boldsymbol{\pi}). \end{aligned}$$

Let $p_j = |\{i : |m_i| = k+1-j\}|$ be the number of elements of the j^{th} cluster of $\boldsymbol{\beta}$, $P_j = \sum_{i \leq j} p_i$ and $P_{k+1} = p$.

Lemma 2.1. *Let $\boldsymbol{\pi}^* = (\pi_1^*, \dots, \pi_p^*)' \in C_{\Lambda}$ be any solution of*

$$\boldsymbol{\pi}^* = \arg \min_{\boldsymbol{\pi} \in C_{\Lambda}} \left[(\hat{\boldsymbol{\beta}}^{OLS} - \boldsymbol{\pi})' (\hat{\boldsymbol{\beta}}^{OLS} - \boldsymbol{\pi}) \right]$$

and let $\boldsymbol{\beta}^* = (\beta_1^*, \dots, \beta_p^*)'$ be the corresponding point from \mathcal{M} given by

$$\boldsymbol{\beta}^* = \hat{\boldsymbol{\beta}}^{OLS} - \boldsymbol{\pi}^*.$$

Then, $(\boldsymbol{\pi} - \boldsymbol{\pi}^*)' \boldsymbol{\beta}^* \leq 0$, for all $\boldsymbol{\pi} \in C_{\Lambda}$ and hence

- (a) $\text{sign}(\beta_i^*) \cdot \text{sign}(\pi_i^*) \geq 0$, $i = 1, 2, \dots, p$,
- (b) $(|\pi_1^*|, \dots, |\pi_p^*|)$ and $(|\beta_1^*|, \dots, |\beta_p^*|)$ are similarly sorted, i.e.
if $|(patt(\beta))_i| = k + 1 - j$, then $|\pi^*|_i \in \{|\pi^*|_{(P_{j-1}+1)}, \dots, |\pi^*|_{(P_j)}\}$,
- (c) if there is a $k \in \{2, \dots, p\}$ such that

$$\sum_{i=1}^{k-1} |\pi_{\sigma(i)}^*| < \sum_{i=1}^{k-1} \lambda_i \quad \text{and} \quad |\pi_{\sigma(k)}^*| > 0,$$

$$\text{then } |\beta_{\sigma(k-1)}^*| = |\beta_{\sigma(k)}^*|.$$

The proof is given in the Appendix. An immediate consequence of the Lemma is the following result.

Lemma 2.2. *The point $(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*)$ defined as in Lemma 2.1 is the saddle point of the function $r(\mathbf{b}, \boldsymbol{\pi})$.*

The proof is given in the Appendix. We use the last lemma to prove the main result of this section.

Theorem 2.3. *Let the point $\boldsymbol{\beta}^*$ be defined as in Lemma 2.1. Then $\boldsymbol{\beta}^*$ is the SLOPE estimator of $\boldsymbol{\beta}$.*

Proof. Using the fact that $\max_{\boldsymbol{\pi} \in C_{\Lambda}} r(\boldsymbol{\beta}^*, \boldsymbol{\pi}) = \min_{\mathbf{b} \in \mathcal{M}} \max_{\boldsymbol{\pi} \in C_{\Lambda}} r(\mathbf{b}, \boldsymbol{\pi})$ (see previous lemma) we have

$$\begin{aligned} \frac{1}{2} \|Y - X\boldsymbol{\beta}^*\|_2^2 + J_{\Lambda}(\boldsymbol{\beta}^*) &= \max_{\boldsymbol{\pi} \in C_{\Lambda}} \left[\frac{1}{2} \|Y - X\boldsymbol{\beta}^*\|_2^2 + \lambda \boldsymbol{\pi}' \boldsymbol{\beta}^* \right] \\ &= \max_{\boldsymbol{\pi} \in C_{\Lambda}} r(\boldsymbol{\beta}^*, \boldsymbol{\pi}) = \min_{\mathbf{b} \in \mathcal{M}} \max_{\boldsymbol{\pi} \in C_{\Lambda}} r(\mathbf{b}, \boldsymbol{\pi}) = \min_{\mathbf{b} \in \mathbb{R}^p} \left[\frac{1}{2} \|Y - X\mathbf{b}\|_2^2 + J_{\Lambda}(\mathbf{b}) \right]. \quad \square \end{aligned}$$

Corollary 2.3.1. [14]

$$\hat{\boldsymbol{\beta}}^{OLS} - \hat{\boldsymbol{\beta}}^{SLOPE} = \boldsymbol{\pi}^* = \arg \min_{\boldsymbol{\pi} \in C_{\Lambda}} \left\| \hat{\boldsymbol{\beta}}^{OLS} - \boldsymbol{\pi} \right\|_2^2$$

is the proximal projection of $\hat{\boldsymbol{\beta}}^{OLS}$ onto C_{Λ} .

Remark 2.3.1. For each $\boldsymbol{\pi} \in C_{\Lambda}$, the point $\hat{\boldsymbol{\beta}}^{SLOPE} = \mathbf{b}_{\boldsymbol{\pi}} = \hat{\boldsymbol{\beta}}^{OLS} - \boldsymbol{\pi}$ belongs to

$$\mathcal{M} := \{\mathbf{b} \in \mathbb{R}^p : \|\mathbf{b}\|_2^2 \leq M\}.$$

Therefore, the constant M is chosen so that $M > \max\{M_0, M_1\}$, where

$$M_1 := \max_{\boldsymbol{\pi} \in C_{\Lambda}} \left\| \hat{\boldsymbol{\beta}}^{OLS} - \boldsymbol{\pi} \right\|_2^2 \leq M.$$

3 Properties of SLOPE in the orthogonal design

3.1 SLOPE vs. OLS

By the Theorem 2.3 and Corollary 2.3.1, when $\mathbf{X}'\mathbf{X} = \mathbf{I}_p$, the orthogonal projection of the ordinary least squares estimator $\hat{\boldsymbol{\beta}}^{OLS} = \mathbf{X}'\mathbf{Y}$ onto the unit ball C_{Λ} is equal to $\hat{\boldsymbol{\beta}}^{OLS} - \hat{\boldsymbol{\beta}}^{SLOPE}$. For $\Lambda = (4, 2)$ this property is illustrated on Figure 1. The figure represents $\hat{\boldsymbol{\beta}}^{SLOPE}$ (black arrows) depending on the localization of $\hat{\boldsymbol{\beta}}^{OLS}$ in the orthogonal design. For $\hat{\boldsymbol{\beta}}^{OLS}$ being the blue point located on the area labelled by (1, 0) the first component of $\hat{\boldsymbol{\beta}}^{SLOPE}$ is positive and the second is null. For $\hat{\boldsymbol{\beta}}^{OLS}$ being the yellow point located on the area labelled by (-1, 1) both components of $\hat{\boldsymbol{\beta}}^{SLOPE}$ have equal absolute value (clusterization), but their signs are opposite. For $\hat{\boldsymbol{\beta}}^{OLS}$ being the red point located on the area labelled by (1, 2) both components of $\hat{\boldsymbol{\beta}}^{SLOPE}$ are positive and the first component is smaller than the second one. The blue polytope is the dual SLOPE unit ball C_{Λ} and labels

$$\mathcal{M}_2 = \{(0, 0), (\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1), (\pm 2, \pm 1), (\pm 1, \pm 2)\}$$

associated to the areas of this figure correspond to all SLOPE patterns for $n \geq p = 2$. In the orthogonal design, one may also explicitly compute the SLOPE estimator. Indeed, by the Corollary 2.3.1, $\hat{\boldsymbol{\beta}}^{SLOPE}$ is the image of $\hat{\boldsymbol{\beta}}^{OLS}$ by the proximal operator of the SLOPE norm. Therefore, this operator has a closed form formula [2, 21, 8]. This explicit expression gives an analytical way to learn that SLOPE solution is sparse and built of clusters.

Lemma 3.1. [3, Equation (1.14)] *In the linear model with orthogonal design $\mathbf{X}'\mathbf{X} = \mathbf{I}_p$ we have*

$$\arg \min_{\mathbf{b} \in \mathbb{R}^p} \left[\frac{1}{2} \|\mathbf{Y} - \mathbf{X}\mathbf{b}\|_2^2 + J_{\Lambda}(\mathbf{b}) \right] = \arg \min_{\mathbf{b} \in \mathbb{R}^p} \left[\frac{1}{2} \|\hat{\boldsymbol{\beta}}^{OLS} - \mathbf{b}\|_2^2 + J_{\Lambda}(\mathbf{b}) \right]. \quad (3)$$

The next theorem gives a sufficient condition for the clustering effect of the SLOPE estimator in the orthogonal design.

Theorem 3.2. *Consider a linear model with orthogonal design $\mathbf{X}'\mathbf{X} = \mathbf{I}_p$. Let π be a permutation of $(1, 2, \dots, p)$ such that*

$$\left| \hat{\boldsymbol{\beta}}_{\pi(1)}^{OLS} \right| \geq \left| \hat{\boldsymbol{\beta}}_{\pi(2)}^{OLS} \right| \geq \dots \geq \left| \hat{\boldsymbol{\beta}}_{\pi(p)}^{OLS} \right|.$$

For $i \in \{1, 2, \dots, p-1\}$,

$$\text{if } \left| \hat{\boldsymbol{\beta}}_{\pi(i)}^{OLS} \right| - \left| \hat{\boldsymbol{\beta}}_{\pi(i+1)}^{OLS} \right| \leq \lambda_i - \lambda_{i+1}, \text{ then } \left| \hat{\boldsymbol{\beta}}_{\pi(i)}^{SLOPE} \right| = \left| \hat{\boldsymbol{\beta}}_{\pi(i+1)}^{SLOPE} \right|.$$

Proof. By Lemma 3.1, in the orthogonal design, the calculation of SLOPE reduces to the application of the proximal algorithm of SLOPE to the $\hat{\boldsymbol{\beta}}^{OLS}$. The result may be inferred from [2, Lemma 2.3]. \square

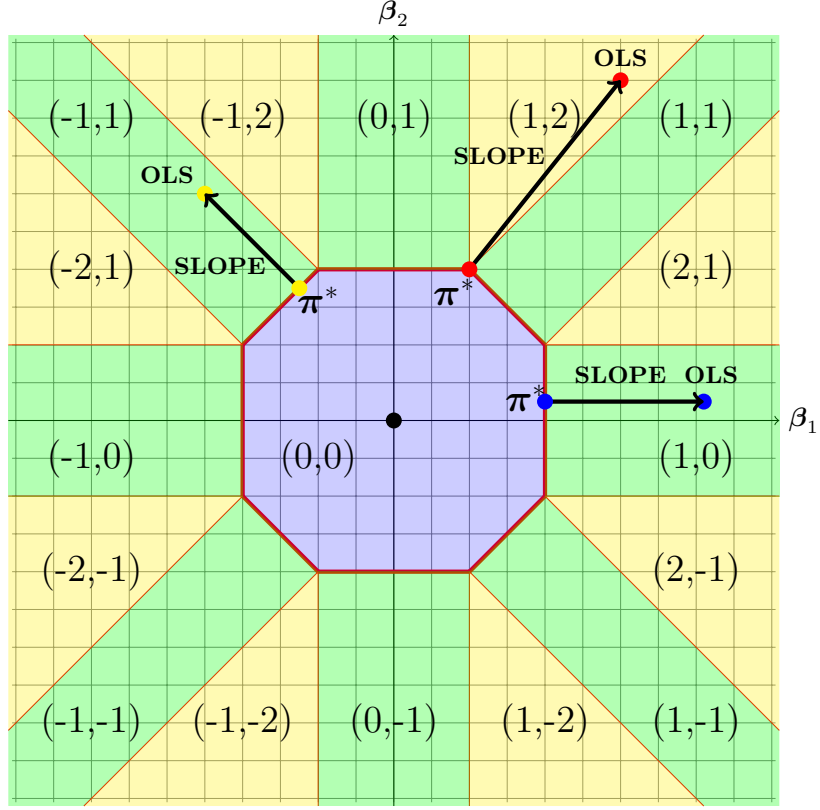


Figure 1: The dual unit ball C_{Λ} for $\Lambda = (4, 2)'$ and examples of $\hat{\beta}^{SLOPE}$ and $\hat{\beta}^{OLS}$ in the orthogonal design for $n \geq p = 2$. The labels of each colored set refer to the pattern of $\hat{\beta}^{SLOPE}$ for $\hat{\beta}^{OLS}$ lying in this set. The arrows point from $(\hat{\beta}^{OLS} - \hat{\beta}^{SLOPE})$ to $\hat{\beta}^{OLS}$.

In the following theorem we derive necessary and sufficient conditions under which SLOPE in the orthogonal design recovers the support of the vector $\beta = (\beta_1, \dots, \beta_p)'$, i.e.

$$\hat{\beta}_i^{SLOPE} = 0 \iff \beta_i = 0.$$

Theorem 3.3. *Under orthogonal design, let π be a permutation of $(1, 2, \dots, p)$ satisfying $|\hat{\beta}_{\pi(1)}^{OLS}| \geq |\hat{\beta}_{\pi(2)}^{OLS}| \geq \dots \geq |\hat{\beta}_{\pi(p)}^{OLS}|$. Without loss of generality suppose that $\text{supp}(\beta) = \{1, 2, \dots, p_0\}$ with $p_0 < p$. The necessary and sufficient condition for SLOPE to identify the set of relevant covariables is the following:*

- (a) $\min_{1 \leq i \leq p_0} |\hat{\beta}_i^{OLS}| > \max_{p_0+1 \leq i \leq p} |\hat{\beta}_i^{OLS}|$,
- (b) $\sum_{i=k}^{p_0} |\hat{\beta}_{\pi(i)}^{OLS}| > \sum_{i=k}^{p_0} \lambda_i$, for $k = 1, 2, \dots, p_0$,

$$(c) \sum_{i=p_0+1}^k \left| \widehat{\beta}_{\pi(i)}^{\text{OLS}} \right| \leq \sum_{i=p_0+1}^k \lambda_i, \quad \text{for } k = p_0 + 1, p_0 + 2, \dots, p.$$

Proof. The result may be inferred from the properties of the proximal SLOPE [3, Lemma 2.3 and Lemma 2.4] and from Lemma 3.1. \square

4 Asymptotic properties of SLOPE

In this section we discuss several asymptotic properties of SLOPE estimators in the low-dimensional regression model in which p is fixed and the sample size n tends to infinity. For each $n \geq 1$ we consider a linear model

$$\mathbf{Y}^{(n)} = \mathbf{X}^{(n)}\boldsymbol{\beta} + \boldsymbol{\varepsilon}^{(n)}, \quad (4)$$

where $\mathbf{Y}^{(n)} = (y_1, y_2, \dots, y_n)' \in \mathbb{R}^n$ is a vector of observations, $\mathbf{X}^{(n)} \in \mathbb{R}^{n \times p}$ is a deterministic design matrix with $\text{rank}(\mathbf{X}^{(n)}) = p$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_p)' \in \mathbb{R}^p$ is a vector of unknown regression coefficients and $\boldsymbol{\varepsilon}^{(n)} = (\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, \dots, \varepsilon_n^{(n)})' \in \mathbb{R}^n$ is a noise term, which has the normal distribution $N(0, \sigma^2 \mathbf{I}_n)$. We make no assumptions about the dependence between $\boldsymbol{\varepsilon}^{(n)}$ and $\boldsymbol{\varepsilon}^{(m)}$ for $n \neq m$. In particular, $\boldsymbol{\varepsilon}^{(n)}$ does not need to be a subsequence of $\boldsymbol{\varepsilon}^{(m)}$.

When defining the sequence $(\widehat{\boldsymbol{\beta}}_n^{\text{SLOPE}})$ of SLOPE estimators, we assume that the tuning vector varies with n . More precisely, for each $n \geq 1$ its coefficients $\lambda_1^{(n)} \geq \lambda_2^{(n)} \geq \dots \geq \lambda_p^{(n)} \geq 0$ are fixed and $\lambda_1^{(n)} > 0$. By $\widehat{\boldsymbol{\beta}}_n^{\text{SLOPE}}$ we denote the SLOPE estimator corresponding to the tuning vector $\boldsymbol{\Lambda}^{(n)} = (\lambda_1^{(n)}, \dots, \lambda_p^{(n)})'$:

$$\widehat{\boldsymbol{\beta}}_n^{\text{SLOPE}} = \arg \min_{\mathbf{b} \in \mathbb{R}^p} \left[\frac{1}{2} \left\| \mathbf{Y}^{(n)} - \mathbf{X}^{(n)}\mathbf{b} \right\|_2^2 + J_{\boldsymbol{\Lambda}^{(n)}}(\mathbf{b}) \right]. \quad (5)$$

4.1 Strong consistency of the SLOPE estimator

Below we discuss consistency of the sequence $(\widehat{\boldsymbol{\beta}}_n^{\text{SLOPE}})$ of SLOPE estimators, defined by (5).

Theorem 4.1. *Consider the linear regression model ((4)) and assume that*

$$\lim_n n^{-1} \left(\mathbf{X}^{(n)} \right)' \mathbf{X}^{(n)} = \mathbf{C},$$

where \mathbf{C} is a positive definite matrix. Let $\widehat{\boldsymbol{\beta}}_n^{\text{SLOPE}}$, $n \geq 1$, be the SLOPE estimator corresponding to the tuning vector $\boldsymbol{\Lambda}^{(n)} = (\lambda_1^{(n)}, \lambda_2^{(n)}, \dots, \lambda_p^{(n)})'$.

(a) *If $\lim_{n \rightarrow \infty} \frac{\lambda_1^{(n)}}{n} = 0$, then $\widehat{\boldsymbol{\beta}}_n^{\text{SLOPE}} \xrightarrow{a.s.} \boldsymbol{\beta}$.*

(b) *If $\lim_{n \rightarrow \infty} \frac{\lambda_1^{(n)}}{n} = \lambda_0 > 0$, then $\widehat{\boldsymbol{\beta}}_n^{\text{SLOPE}}$ is not strongly consistent for $\boldsymbol{\beta}$.*

Before proving the above theorems we start with stating a simple technical lemma. It follows quickly from the Borel-Cantelli Lemma and the well-known tail inequality:

If $Z \sim N(0, 1)$, then $\mathbb{P}(Z > t) \leq t^{-1} e^{-t^2/2} / \sqrt{2\pi}$, $t > 0$.

Lemma 4.2. Assume that $(Q_n)_{n \in \mathbb{N}}$ is a sequence of Gaussian random variables, defined on the same probability space, which converges in distribution to $N(0, \sigma^2)$ for some $\sigma \in (0, \infty)$. Then, for any $\delta > 0$,

$$\lim_{n \rightarrow \infty} \frac{Q_n}{(\log(n))^{1/2+\delta}} = 0 \quad a.s.$$

Our proof of the strong consistency of SLOPE is based on the strong consistency of the OLS estimator. The latter result is a folklore and we prove it in our setting.

Proposition 4.2.1. Consider the linear regression model ((4)).

If $\lim_n n^{-1}(\mathbf{X}^{(n)})' \mathbf{X}^{(n)} = \mathbf{C}$, where \mathbf{C} is positive definite, then $\widehat{\boldsymbol{\beta}}_n^{OLS} \xrightarrow{a.s.} \boldsymbol{\beta}$.

Proof. We have

$$\widehat{\boldsymbol{\beta}}_n^{OLS} - \boldsymbol{\beta} = ((\mathbf{X}^{(n)})' \mathbf{X}^{(n)})^{-1} (\mathbf{X}^{(n)})' \mathbf{Y}^{(n)} - \boldsymbol{\beta} = ((\mathbf{X}^{(n)})' \mathbf{X}^{(n)})^{-1} (\mathbf{X}^{(n)})' \boldsymbol{\varepsilon}^{(n)}.$$

Then $\sqrt{n} \left(\widehat{\boldsymbol{\beta}}_n^{OLS} - \boldsymbol{\beta} \right)$ has the normal distribution $N(0, (n^{-1}(\mathbf{X}^{(n)})' \mathbf{X}^{(n)})^{-1})$ and its components satisfy the assumptions of Lemma 4.2. Since $\log(n)^{1/2+\delta} = o(\sqrt{n})$, we get the assertion by Lemma 4.2. \square

Proof of Theorem 4.1. (a) It follows from Theorem 2.1 that there exists a vector $\boldsymbol{\pi}_n^* \in C_{(\boldsymbol{\Lambda}^{(n)})}$ such that

$$\widehat{\boldsymbol{\beta}}_n^{SLOPE} = ((\mathbf{X}^{(n)})' \mathbf{X}^{(n)})^{-1} ((\mathbf{X}^{(n)})' \mathbf{Y}^{(n)} - \boldsymbol{\pi}_n^*).$$

Since $\boldsymbol{\pi}_n^*$ takes values in $C_{\boldsymbol{\Lambda}^{(n)}}$, it follows that $\|\boldsymbol{\pi}_n^*\|_\infty \leq \lambda_1^{(n)}$. Hence,

$$\frac{\boldsymbol{\pi}_n^*}{n} \xrightarrow{a.s.} \mathbf{0}, \quad (6)$$

because $\left\| \frac{\boldsymbol{\pi}_n^*}{n} \right\|_\infty \leq \frac{\lambda_1^{(n)}}{n} \rightarrow 0$. The assumption that $\text{rank}(\mathbf{X}^{(n)}) = p$ implies that the matrix $(\mathbf{X}^{(n)})' \mathbf{X}^{(n)}$ is invertible and hence the least squares estimator of $\boldsymbol{\beta}$ is unique and has the form $\widehat{\boldsymbol{\beta}}_n^{OLS} = ((\mathbf{X}^{(n)})' \mathbf{X}^{(n)})^{-1} (\mathbf{X}^{(n)})' \mathbf{Y}^{(n)}$. Combining with ((6)) the fact that $\widehat{\boldsymbol{\beta}}_n^{OLS} \xrightarrow{a.s.} \boldsymbol{\beta}$, we conclude that

$$\begin{aligned} \widehat{\boldsymbol{\beta}}_n^{SLOPE} &= ((\mathbf{X}^{(n)})' \mathbf{X}^{(n)})^{-1} ((\mathbf{X}^{(n)})' \mathbf{Y}^{(n)} - \boldsymbol{\pi}_n^*) = \widehat{\boldsymbol{\beta}}_n^{OLS} - ((\mathbf{X}^{(n)})' \mathbf{X}^{(n)})^{-1} \boldsymbol{\pi}_n^* \\ &= \widehat{\boldsymbol{\beta}}_n^{OLS} - \left(\frac{(\mathbf{X}^{(n)})' \mathbf{X}^{(n)}}{n} \right)^{-1} \frac{\boldsymbol{\pi}_n^*}{n} \xrightarrow{a.s.} \boldsymbol{\beta} - \mathbf{C}^{-1} \mathbf{0} = \boldsymbol{\beta}. \end{aligned}$$

(b) Since $\widehat{\boldsymbol{\beta}}_n^{SLOPE}$ minimizes over $\mathbf{b} \in \mathbb{R}^p$ the function

$$l(\mathbf{b}) := \frac{1}{2} \|\mathbf{Y}^{(n)} - \mathbf{X}^{(n)} \mathbf{b}\|_2^2 + J_{\boldsymbol{\Lambda}^{(n)}}(\mathbf{b})$$

and since $\lambda_1^{(n)} \|\mathbf{b}\|_\infty \leq J_{\mathbf{A}^{(n)}}(\mathbf{b})$, it follows that

$$\begin{aligned}
0 &\leq l(0) - l(\widehat{\boldsymbol{\beta}}_n^{SLOPE}) = (\widehat{\boldsymbol{\beta}}_n^{SLOPE})'(\mathbf{X}^{(n)})' \mathbf{Y}^{(n)} \\
&\quad - \frac{1}{2}(\widehat{\boldsymbol{\beta}}_n^{SLOPE})'(\mathbf{X}^{(n)})' \mathbf{X}^{(n)} \widehat{\boldsymbol{\beta}}_n^{SLOPE} - J_{\mathbf{A}^{(n)}}(\widehat{\boldsymbol{\beta}}_n^{SLOPE}) \\
&\leq (\widehat{\boldsymbol{\beta}}_n^{SLOPE})'(\mathbf{X}^{(n)})' \mathbf{Y}^{(n)} - \frac{1}{2}(\widehat{\boldsymbol{\beta}}_n^{SLOPE})'(\mathbf{X}^{(n)})' \mathbf{X}^{(n)} \widehat{\boldsymbol{\beta}}_n^{SLOPE} \\
&\quad - \lambda_1^{(n)} \|\widehat{\boldsymbol{\beta}}_n^{SLOPE}\|_\infty = (\widehat{\boldsymbol{\beta}}_n^{SLOPE})'(\mathbf{X}^{(n)})' \mathbf{X}^{(n)} \widehat{\boldsymbol{\beta}}_n^{OLS} \\
&\quad - \frac{1}{2}(\widehat{\boldsymbol{\beta}}_n^{SLOPE})'(\mathbf{X}^{(n)})' \mathbf{X}^{(n)} \widehat{\boldsymbol{\beta}}_n^{SLOPE} - \lambda_1^{(n)} \|\widehat{\boldsymbol{\beta}}_n^{SLOPE}\|_\infty.
\end{aligned}$$

Suppose to the contrary that $\widehat{\boldsymbol{\beta}}_n^{SLOPE} \xrightarrow{a.s.} \boldsymbol{\beta}$. Then, using the facts that $\widehat{\boldsymbol{\beta}}_n^{OLS} \xrightarrow{a.s.} \boldsymbol{\beta}$ and that $\lim_n n^{-1}(\mathbf{X}^{(n)})' \mathbf{X}^{(n)} = \mathbf{C}$, we have

$$0 \leq \frac{l(0) - l(\widehat{\boldsymbol{\beta}}_n^{SLOPE})}{n} \xrightarrow{a.s.} \boldsymbol{\beta}' \mathbf{C} \boldsymbol{\beta} - \frac{1}{2} \boldsymbol{\beta}' \mathbf{C} \boldsymbol{\beta} - \lambda_0 \|\boldsymbol{\beta}\|_\infty = \frac{1}{2} \boldsymbol{\beta}' \mathbf{C} \boldsymbol{\beta} - \lambda_0 \|\boldsymbol{\beta}\|_\infty.$$

For $\lambda_0 > 0$ this provides a contradiction since the inequality $\lambda_0 \|\boldsymbol{\beta}\|_\infty \leq \frac{1}{2} \boldsymbol{\beta}' \mathbf{C} \boldsymbol{\beta}$ does not hold when the value of $\boldsymbol{\beta}$ is sufficiently close to 0. \square

4.2 Asymptotical pattern recovery in the orthogonal design

We again consider a sequence of linear models ((4)) but this time we assume that for each n the deterministic design matrix $\mathbf{X}^{(n)}$ of size $n \times p$ satisfies

$$(\mathbf{X}^{(n)})' \mathbf{X}^{(n)} = n \mathbf{I}_p. \quad (7)$$

As usual, we assume Gaussian errors, $\boldsymbol{\varepsilon}^{(n)} \sim \mathbf{N}(0, \sigma^2 \mathbf{I}_n)$.

Let $\widehat{\boldsymbol{\beta}}_n^{SLOPE} = (\widehat{\beta}_1^{SLOPE}(n), \dots, \widehat{\beta}_p^{SLOPE}(n))'$ be the SLOPE estimator defined by (5).

With the above notation we present the main result of this section.

Theorem 4.3. *Assume that*

$$\lim_{n \rightarrow \infty} \frac{\lambda_1^{(n)}}{n} = 0$$

and that there exists $\delta > 0$ such that

$$\liminf_{n \rightarrow \infty} \frac{\lambda_i^{(n)} - \lambda_{i+1}^{(n)}}{\sqrt{n} (\log(n))^{1/2+\delta}} = m > 0 \quad \text{for } i = 1, \dots, p-1. \quad (8)$$

Then we have

$$\text{patt}(\widehat{\boldsymbol{\beta}}_n^{SLOPE}) \xrightarrow{a.s.} \text{patt}(\boldsymbol{\beta}).$$

Proof. Without loss of generality we may assume that $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)'$ and $\beta_1 \geq \beta_2 \geq \dots \geq \beta_p \geq 0$. Indeed, we can always achieve such condition by permuting the columns of $\mathbf{X}^{(n)}$ and changing their signs.

Since the space of models is discrete, we have to show that for large n ,

$\text{patt}(\widehat{\boldsymbol{\beta}}_n^{SLOPE}) = \text{patt}(\boldsymbol{\beta})$ a.s. We divide the proof into the following four parts:

- (a) $\beta_i = \beta_j > 0 \implies \widehat{\beta}_i^{\text{SLOPE}}(n) = \widehat{\beta}_j^{\text{SLOPE}}(n)$ a.s. for large n ,
- (b) $\beta_i > \beta_{i+1} \implies \widehat{\beta}_i^{\text{SLOPE}}(n) > \widehat{\beta}_{i+1}^{\text{SLOPE}}(n)$ a.s. for large n ,
- (c) $\beta_i = 0 \implies \widehat{\beta}_i^{\text{SLOPE}}(n) = 0$ a.s. for large n ,
- (d) $\beta_i > 0 \implies \widehat{\beta}_i^{\text{SLOPE}}(n) > 0$ a.s. for large n .

The points (b) and (d) follow quickly by the strong consistency of $\widehat{\beta}^{\text{SLOPE}}(n)$. To prove (a) and (c) we reduce the problem to the orthogonal design case. We have

$$\begin{aligned} & \arg \min_{\mathbf{b} \in \mathbb{R}^p} \left[\frac{1}{2} \|\mathbf{Y}^{(n)} - \mathbf{X}^{(n)} \mathbf{b}\|_2^2 + J_{\mathbf{\Lambda}^{(n)}}(\mathbf{b}) \right] \\ &= \arg \min_{\mathbf{b} \in \mathbb{R}^p} \left[\frac{1}{2} \|(\widetilde{\mathbf{Y}}^{(n)})' - (\widetilde{\mathbf{X}}^{(n)})' \mathbf{b}\|_2^2 + J_{\widetilde{\mathbf{\Lambda}}^{(n)}}(\mathbf{b}) \right], \end{aligned}$$

where $\widetilde{\mathbf{Y}}^{(n)} = \mathbf{Y}_n/\sqrt{n}$, $\widetilde{\mathbf{X}}^{(n)} = \mathbf{X}^{(n)}/\sqrt{n}$ and $\widetilde{\mathbf{\Lambda}}^{(n)} = \mathbf{\Lambda}^{(n)}/n$. Clearly, (7) implies that $(\widetilde{\mathbf{X}}^{(n)})' \widetilde{\mathbf{X}}^{(n)} = \mathbf{I}_p$, which allows to use results from the orthogonal design. However, we note that the OLS estimators $\widehat{\beta}_n^{\text{OLS}} = (\widehat{\beta}_1^{\text{OLS}}(n), \dots, \widehat{\beta}_p^{\text{OLS}}(n))$ are the same in the original model and its scaled version $\widetilde{\mathbf{Y}}^{(n)} = \widetilde{\mathbf{X}}^{(n)} \boldsymbol{\beta} + \boldsymbol{\varepsilon}^{(n)}/\sqrt{n}$.

Let π_n be a permutation of $(1, 2, \dots, p)$ satisfying

$$|\widehat{\beta}_{\pi_n(1)}^{\text{OLS}}(n)| \geq |\widehat{\beta}_{\pi_n(2)}^{\text{OLS}}(n)| \geq \dots \geq |\widehat{\beta}_{\pi_n(p)}^{\text{OLS}}(n)|.$$

By the strong consistency of the OLS estimator, taking n sufficiently large, we may ensure that the clusters of $\boldsymbol{\beta}$ do not interlace in $\widehat{\beta}_n^{\text{OLS}}$ in the sense that if $\beta_i > \beta_j$, then $\widehat{\beta}_i^{\text{OLS}}(n) > \widehat{\beta}_j^{\text{OLS}}(n)$ a.s. for n sufficiently large.

Let us now consider point (i). Let S_i denote the cluster containing $\beta_i > 0$, that is, the set $S_i = \{j \in \{1, \dots, p\} : \beta_j = \beta_i\}$. In view of the ordering of $\boldsymbol{\beta}$, there exists $k_i \in \{1, \dots, p\}$ such that

$$S_i = \{\pi_n(j) : j \in \{k_i, k_i + 1, \dots, k_i + \#S_i - 1\}\}.$$

We will show that if $\pi_n(k), \pi_n(k+1) \in S_i$, then for large n

$$\widehat{\beta}_{\pi_n(k)}^{\text{SLOPE}}(n) = \widehat{\beta}_{\pi_n(k+1)}^{\text{SLOPE}}(n) \quad \text{a.s.}, \quad (9)$$

which implies that $\widehat{\beta}_j^{\text{SLOPE}}(n) = \widehat{\beta}_k^{\text{SLOPE}}(n)$ for $j, k \in S_i$ and finishes the proof of (a).

Now assume that $\pi_n(k), \pi_n(k+1) \in S_i$. Then, by Theorem 3.2, the condition (9) is satisfied if

$$\left| \widehat{\beta}_{\pi_n(k)}^{\text{OLS}}(n) \right| - \left| \widehat{\beta}_{\pi_n(k+1)}^{\text{OLS}}(n) \right| \leq \widetilde{\Lambda}_k^{(n)} - \widetilde{\Lambda}_{k+1}^{(n)} = \frac{1}{n} \left(\lambda_k^{(n)} - \lambda_{k+1}^{(n)} \right) \quad (10)$$

holds for large n and both $\widehat{\beta}_{\pi_n(k)}^{\text{OLS}}(n)$ and $\widehat{\beta}_{\pi_n(k+1)}^{\text{OLS}}(n)$ have the same sign. The latter is ensured by the strong consistency of the OLS estimator and the fact that $\beta_i > 0$.

If $\pi_n(k), \pi_n(k+1) \in S_i$, then we have the following bound

$$\left| \widehat{\beta}_{\pi_n(k)}^{\text{OLS}}(n) - \widehat{\beta}_{\pi_n(k+1)}^{\text{OLS}}(n) \right| \leq \sum_{j \in S_i} \left| \widehat{\beta}_j^{\text{OLS}}(n) - \widehat{\beta}_i^{\text{OLS}}(n) \right|. \quad (11)$$

Take any $j \in S_i$. Since both $\widehat{\beta}_j^{\text{OLS}}(n)$ and $\widehat{\beta}_i^{\text{OLS}}(n)$ have the normal distribution with the same mean, by Lemma 4.2, we have

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n} \left(\widehat{\beta}_j^{\text{OLS}}(n) - \widehat{\beta}_i^{\text{OLS}}(n) \right)}{(\log(n))^{1/2+\delta}} = 0 \quad \text{a.s.}$$

In view of (11) and (8), this implies that (10) holds true for large n . Hence, (a) follows. It remains to establish (c). Assume that $\beta_{p_0} > 0 = \beta_{p_0+1} = \dots = \beta_p$. Clearly, condition (a) from Theorem 3.3 is satisfied thanks to the strong consistency of the OLS estimator. For (b), we have for $k = 1, 2, \dots, p_0$,

$$\sum_{i=k}^{p_0} \widetilde{\Lambda}_i^{(n)} = \frac{1}{n} \sum_{i=k}^{p_0} \lambda_i^{(n)} \leq p_0 \frac{\lambda_1^{(n)}}{n},$$

which converges to 0. On the other hand, the left-hand side of (b) converges a.s. to $\sum_{i=k}^{p_0} \beta_i$, which is positive. Thus, condition (b) from Theorem 3.3 holds for large n . Condition (c) from Theorem 3.3 follows from Lemma 4.2. Indeed, we have for $\delta > 0$ and $k = p_0 + 1, \dots, p$,

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{(\log(n))^{1/2+\delta}} \sum_{i=p_0+1}^k |\widehat{\beta}_{\pi_n(i)}^{\text{OLS}}(n)| = \sum_{i=p_0+1}^k \lim_{n \rightarrow \infty} \frac{|\sqrt{n} \widehat{\beta}_{\pi_n(i)}^{\text{OLS}}(n)|}{(\log(n))^{1/2+\delta}} = 0 \quad \text{a.s.},$$

while

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{(\log(n))^{1/2+\delta}} \sum_{i=p_0+1}^k \widetilde{\Lambda}_i^{(n)} \geq \sum_{i=p_0+1}^k \lim_{n \rightarrow \infty} \frac{\lambda_i^{(n)} - \lambda_{i+1}^{(n)}}{\sqrt{n}(\log(n))^{1/2+\delta}} = m > 0$$

Thus, all assumptions of Theorem 3.3 are verified and the proof is complete. \square

5 Applications and simulations

Below we present an application of SLOPE in signal denoising. In our example $\mathbf{X} \in \mathbb{R}^{300 \times 100}$ is an orthogonal system of trigonometric functions, i.e.

$X_{i,(2*j-1)} = \sin(2\pi i j/n)$ and $X_{i,(2*j)} = \cos(2\pi i j/n)$ for $i = 1, \dots, 100$

and $j = 1, \dots, 150$. Here $\beta \in \mathbb{R}^p$ is a vector consisting of two clusters: 20 coordinates with absolute value 100 and 20 coordinates with absolute value 80. To avoid large bias caused by the shrinkage nature of LASSO and SLOPE, we debias them by combining with the Ordinary Least Squares method in the following way. Firstly we recover the support and clusters of a true vector β and secondly we apply the OLS method to the regression model $\mathbf{Y} = \mathbf{X}\mathbf{U}\beta + \varepsilon$, where the rows of $\mathbf{U} \in \mathbb{R}^{n \times 2}$ are defined as

$$\mathbf{U}_{i,\bullet} = \begin{cases} [\text{sign}(\widehat{\beta}_i^{\text{SLOPE}}), 0], & 1 \leq i \leq k, \\ [0, \text{sign}(\widehat{\beta}_i^{\text{SLOPE}})], & k+1 \leq i \leq 2k, \\ [0, 0], & \text{else.} \end{cases}$$

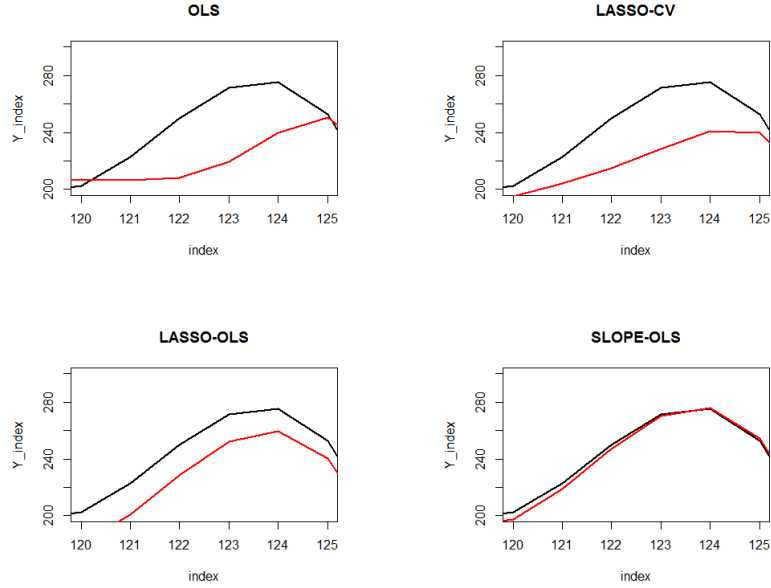


Figure 3: Comparison of signal denoising by OLS, LASSO, debiased LASSO and debiased SLOPE (respective images from left) on the coordinates $[120, 125]$ of the regression model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$. The black lines correspond to the true values of $\mathbf{X}\boldsymbol{\beta}$. The red lines correspond to the estimators $\mathbf{Y} = \mathbf{X}\hat{\boldsymbol{\beta}} + \boldsymbol{\varepsilon}$.

We compare the Mean Square Error and the signal denoising of the classical OLS estimation, the LASSO with the tuning parameter λ_{cv} minimizing the cross-validated error, the denoised version of LASSO with $\lambda = 5\lambda_{cv}$ and the denoised version of SLOPE with the tuning vector $\boldsymbol{\Lambda}$ chosen with respect to the scaled arithmetic sequence ($\lambda_i = 3.5(p - 1)$). We also compare the Mean Square Error of the SLOPE estimator with others.

	OLS	LASSO-CV	LASSO-LS	SLOPE-LS
$MSE(\boldsymbol{\beta}, \cdot)$	613.6797	417.0071	171.7957	20.74967

Table 1: Comparison of MSE between different regression methods.

A Appendix

Proof of Lemma 2.1. It follows that the function $g : C_{\boldsymbol{\Lambda}} \rightarrow [0, \infty)$ defined by

$$g(\boldsymbol{\pi}) := (\hat{\boldsymbol{\beta}}^{OLS} - \boldsymbol{\pi})'(\hat{\boldsymbol{\beta}}^{OLS} - \boldsymbol{\pi})$$

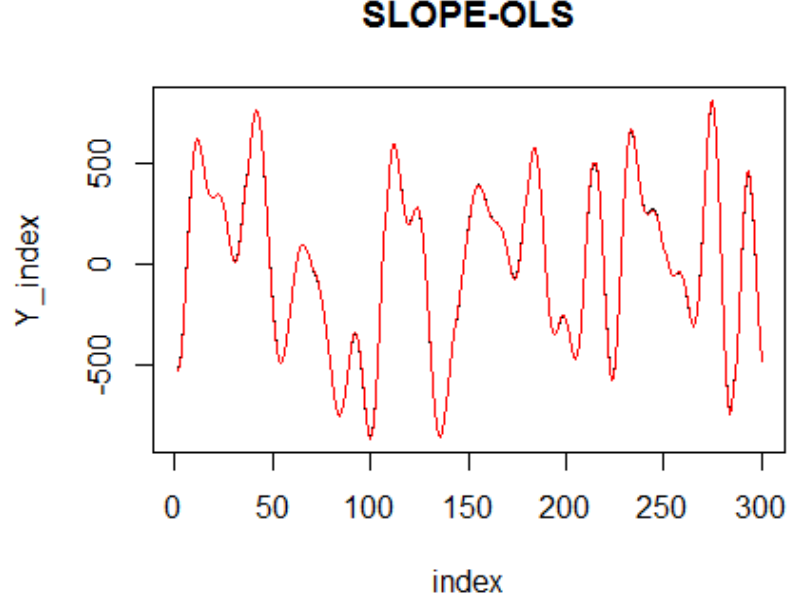


Figure 4: Signal denoising by debiased SLOPE on all coordinates of the regression model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$. The (almost overlapping) black line and the red line correspond respectively to the true values of $\mathbf{X}\boldsymbol{\beta}$ and to $\mathbf{Y} = \mathbf{X}\hat{\boldsymbol{\beta}}^{SLOPE} + \boldsymbol{\varepsilon}$.

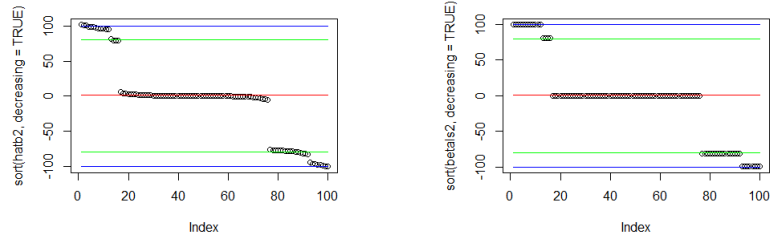


Figure 5: Pattern recovery by LASSO (left image) and by debiased SLOPE (right image) in the same setting as above. The horizontal lines correspond to the true values of $\boldsymbol{\beta}$. As one may observe, in the presented setting LASSO does not recover the true support, while debiased SLOPE perfectly recovers support, sign and clusters.

is convex in $\boldsymbol{\pi}$. Therefore, at the point $\boldsymbol{\pi}^* = (\pi_1^*, \dots, \pi_p^*)'$, where g attains its global minimum over $C_{\boldsymbol{\Lambda}}$, the gradient ∇g of g satisfies

$$[\nabla g(\boldsymbol{\pi}^*)]'(\boldsymbol{\pi} - \boldsymbol{\pi}^*) \geq 0, \quad \text{for all } \boldsymbol{\pi} \in C_{\boldsymbol{\Lambda}}.$$

This implies $(\boldsymbol{\pi} - \boldsymbol{\pi}^*)' \boldsymbol{\beta}^* \leq 0$, for all $\boldsymbol{\pi} \in C_{\Lambda}$, because

$$\nabla g(\boldsymbol{\pi}^*) = -2(\hat{\boldsymbol{\beta}}^{OLS} - \boldsymbol{\pi}^*) = -2\boldsymbol{\beta}^*.$$

In the proof of parts (a), (b) and (c) we use the fact that $\boldsymbol{\pi}^*$ maximizes $\boldsymbol{\pi}' \boldsymbol{\beta}^*$ over $\boldsymbol{\pi} \in C_{\Lambda}$. To prove part (a) suppose that $\text{sign}(\beta_i^*) \cdot \text{sign}(\pi_i^*) < 0$ for some i and define

$$\boldsymbol{\pi} = (\pi_1^*, \dots, \pi_{i-1}^*, -\pi_i^*, \pi_{i+1}^*, \dots, \pi_p^*)'.$$

Then we have $(\boldsymbol{\pi}^*)' \boldsymbol{\beta}^* < \boldsymbol{\pi}' \boldsymbol{\beta}^*$, which is impossible since $\boldsymbol{\pi} \in C_{\Lambda}$.

To prove part (b), consider a permutation τ of $(1, 2, \dots, p)$ such that

$(|\pi_{\tau(1)}^*|, \dots, |\pi_{\tau(p)}^*|)$ and $(|\beta_1^*|, \dots, |\beta_p^*|)$ are similarly sorted. Define the point

$\boldsymbol{\pi} = (s_1 \cdot \pi_{\tau(1)}^*, s_2 \cdot \pi_{\tau(2)}^*, \dots, s_p \cdot \pi_{\tau(p)}^*)$, where $s_i = \text{sign}(\beta_i^*)$, for $i = 1, 2, \dots, p$. If $(|\pi_{\tau(1)}^*|, \dots, |\pi_{\tau(p)}^*|) \neq (|\pi_1^*|, \dots, |\pi_p^*|)$, then, by the Hardy-Littlewood-Pólya rearrangement inequality,

$$\boldsymbol{\pi}' \boldsymbol{\beta}^* = \sum_{i=1}^p |\pi_{\tau(i)}^*| |\beta_i^*| > \sum_{i=1}^p |\pi_i^*| |\beta_i^*| \geq (\boldsymbol{\pi}^*)' \boldsymbol{\beta}^*,$$

which is impossible since $\boldsymbol{\pi} \in C_{\Lambda}$.

Finally, to prove part (c), suppose that $\sum_{i=1}^{k-1} |\pi_{\sigma(i)}^*| < \sum_{i=1}^{k-1} \lambda_i$, and that $|\pi_{\sigma(k)}^*| > 0$. In this case there is a sufficiently small $\delta > 0$, such that

$$\boldsymbol{\pi} = (\pi_1^*, \dots, \pi_{i-2}^*, \pi_{i-1}^* + \delta s_{i-1}, \pi_i^* - \delta s_i, \pi_{i+1}^*, \dots, \pi_p^*)' \in C_{\Lambda}.$$

If $|\beta_{\sigma(k-1)}^*| > |\beta_{\sigma(k)}^*|$ then

$$\boldsymbol{\pi}' \boldsymbol{\beta}^* = (\boldsymbol{\pi}^*)' \boldsymbol{\beta}^* + \delta(|\beta_{\sigma(k-1)}^*| - |\beta_{\sigma(k)}^*|) > (\boldsymbol{\pi}^*)' \boldsymbol{\beta}^*,$$

which is impossible. □

Proof of Lemma 2.2. At first we note that for all $\boldsymbol{\pi} \in C_{\Lambda}$

$$\begin{aligned} r(\boldsymbol{\beta}^*, \boldsymbol{\pi}) &= \frac{1}{2} \|\mathbf{Y} - \mathbf{X} \boldsymbol{\beta}^*\|_2^2 + \boldsymbol{\pi}' \boldsymbol{\beta}^* = \frac{1}{2} \|\mathbf{Y} - \mathbf{X} \boldsymbol{\beta}^*\|_2^2 + (\boldsymbol{\pi}^*)' \boldsymbol{\beta}^* \\ &\quad + (\boldsymbol{\pi} - \boldsymbol{\pi}^*)' \boldsymbol{\beta}^* = r(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*) + (\boldsymbol{\pi} - \boldsymbol{\pi}^*)' \boldsymbol{\beta}^* \leq r(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*), \end{aligned}$$

where the last inequality follows from the fact that $(\boldsymbol{\pi} - \boldsymbol{\pi}^*)' \boldsymbol{\beta}^* \leq 0$, for all $\boldsymbol{\pi} \in C_{\Lambda}$, see the proof of 2.1. Therefore, $\max_{\boldsymbol{\pi} \in C_{\Lambda}} r(\boldsymbol{\beta}^*, \boldsymbol{\pi}) = r(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*)$. Moreover, from the definition of the point $\boldsymbol{\beta}^*$ it is seen that $r(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*) = \min_{\boldsymbol{\beta} \in \mathcal{M}} r(\boldsymbol{\beta}, \boldsymbol{\pi}^*)$. These two facts imply that

$$\begin{aligned} \min_{\boldsymbol{\beta} \in \mathcal{M}} \max_{\boldsymbol{\pi} \in C_{\Lambda}} r(\boldsymbol{\beta}, \boldsymbol{\pi}) &\leq \max_{\boldsymbol{\pi} \in C_{\Lambda}} r(\boldsymbol{\beta}^*, \boldsymbol{\pi}) = r(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*) \\ &= \min_{\boldsymbol{\beta} \in \mathcal{M}} r(\boldsymbol{\beta}, \boldsymbol{\pi}^*) \leq \max_{\boldsymbol{\pi} \in C_{\Lambda}} \min_{\boldsymbol{\beta} \in \mathcal{M}} r(\boldsymbol{\beta}, \boldsymbol{\pi}). \end{aligned}$$

Since $\max_{\boldsymbol{\pi} \in C_{\Lambda}} \min_{\boldsymbol{\beta} \in \mathcal{M}} r(\boldsymbol{\beta}, \boldsymbol{\pi}) \leq \min_{\boldsymbol{\beta} \in \mathcal{M}} \max_{\boldsymbol{\pi} \in C_{\Lambda}} r(\boldsymbol{\beta}, \boldsymbol{\pi})$ (by the max-min inequality), we have the equality throughout. This completes the proof. □

Proof of Lemma 3.1. Let $\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ be a projection matrix on $\text{Col}(\mathbf{X})$ - the linear subspace spanned by the columns of \mathbf{X} . Then

$$\begin{aligned}\|\mathbf{Y} - \mathbf{X}\mathbf{b}\|_2^2 &= \|(\mathbf{I}_p - \mathbf{H})\mathbf{Y} + \mathbf{H}\mathbf{Y} - \mathbf{X}\mathbf{b}\|_2^2 = \|(\mathbf{I}_p - \mathbf{H})\mathbf{Y} + \mathbf{H}(\mathbf{Y} - \mathbf{X}\mathbf{b})\|_2^2 \\ &= \|(\mathbf{I}_p - \mathbf{H})\mathbf{Y}\|_2^2 + 2[(\mathbf{I}_p - \mathbf{H})\mathbf{Y}]' \mathbf{H}(\mathbf{Y} - \mathbf{X}\mathbf{b}) + \|\mathbf{H}(\mathbf{Y} - \mathbf{X}\mathbf{b})\|_2^2 \\ &= \|(\mathbf{I}_p - \mathbf{H})\mathbf{Y}\|_2^2 + 2\mathbf{Y}'(\mathbf{I}_p - \mathbf{H})\mathbf{H}(\mathbf{Y} - \mathbf{X}\mathbf{b}) + \|\mathbf{H}(\mathbf{Y} - \mathbf{X}\mathbf{b})\|_2^2 \\ &= \|(\mathbf{I}_p - \mathbf{H})\mathbf{Y}\|_2^2 + \|\mathbf{H}(\mathbf{Y} - \mathbf{X}\mathbf{b})\|_2^2,\end{aligned}$$

because $\mathbf{H} = \mathbf{H}'$, $\mathbf{H}(\mathbf{I}_p - \mathbf{H}) = \mathbf{0}$ and $\mathbf{H}\mathbf{X} = \mathbf{X}$. In the orthogonal design we have $\mathbf{H} = \mathbf{X}\mathbf{X}'$ and $\hat{\boldsymbol{\beta}}^{\text{OLS}} = \mathbf{X}'\mathbf{Y}$, which yields

$$\|\mathbf{H}(\mathbf{Y} - \mathbf{X}\mathbf{b})\|_2^2 = \|\mathbf{X}'\mathbf{Y} - \mathbf{X}'\mathbf{X}\mathbf{b}\|_2^2 = \|\hat{\boldsymbol{\beta}}^{\text{OLS}} - \mathbf{b}\|_2^2.$$

Therefore

$$\|\mathbf{Y} - \mathbf{X}\mathbf{b}\|_2^2 = \|(\mathbf{I}_p - \mathbf{H})\mathbf{Y}\|_2^2 + \|\hat{\boldsymbol{\beta}}^{\text{OLS}} - \mathbf{b}\|_2^2.$$

Since the difference $\|\mathbf{Y} - \mathbf{X}\mathbf{b}\|_2^2 - \|\hat{\boldsymbol{\beta}}^{\text{OLS}} - \mathbf{b}\|_2^2$ does not depend on \mathbf{b} , it follows that the equality ((3)) holds. \square

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