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# ON THE MEAN PROJECTION THEOREM FOR DETERMINANTAL POINT PROCESSES

ADRIEN KASSEL AND THIERRY LÉVY

ABSTRACT. In this short note, we extend a mean projection theorem for discrete determinantal point processes to the continuous case, thus strengthening a known result in random linear algebra due to Ermakov and Zolotukhin. We also give a new formula for the variance of the exterior power of the random projection.

## 1. INTRODUCTION

Kirchhoff's work on electrical networks [Kir47] seems to be one of the earliest works in the literature where linear algebra and graph-theoretical combinatorial methods were put together. Later on, linear algebra problems, and classical determinantal methods for solving them, gave rise to various statistical approaches, notably linked to the so-called determinantal point processes (introduced by Macchi in 1975 [Mac75], and named like this by Borodin, only around 2000 which saw a blossoming of results on those processes from various authors, see [Sos00, ST03, Lyo03, Joh06, Bor11]). These methods recently became an active field in randomized numerical linear algebra [DM21].

In his work, Kirchhoff solved a linear algebra system on an electrical network seen as a finite graph, by expressing the current induced by an external battery hooked on the network, as an average over spanning trees of a certain current associated to the tree. In modern terms, he expressed an orthogonal projection as the expectation of a certain random projection associated to a random spanning tree. Such a mean projection theorem appeared in several guises in the literature, and more or less independently, in works of Maurer [Mau76], Lyons [Lyo03], Catanzaro–Chernyak–Klein [CCK13], and probably others that we are unaware of.

In our work [KL19, Theorem 5.9], we extended the mean projection formula for determinantal point processes on finite sets, thus putting the statements of [Kir47, Mau76, Lyo03, CCK13] in a unified geometric framework, and strengthening the result by proving a mean projection theorem for the exterior powers of the projections, that is, for minors of their matrices in a fixed basis.

Let us quickly recall our statement. Let  $\mathbb{K}$  be  $\mathbb{R}$  or  $\mathbb{C}$ , let  $E$  be a finite dimensional Euclidean space on  $\mathbb{K}$  of dimension  $d$ , and let  $(e_i)_{1 \leq i \leq d}$  be an orthonormal basis of  $E$ . We let  $S = \{1, \dots, d\}$  and consider  $H$  a subspace of  $E$  of dimension  $n$ . Let  $\mathbf{X}$  be the determinantal point process on  $S$  associated to the matrix  $K = (\langle e_i, \Pi^H e_j \rangle)_{1 \leq i, j \leq d}$ , where  $\Pi^H$  is the orthogonal projection on  $H$ . For each  $X \subseteq S$ , let  $E_X = \bigoplus_{x \in X} \mathbb{K}e_x$  be the corresponding coordinate subspace of  $E$ .

**Theorem 1.1.** *Almost surely, the equality  $E = H \oplus E_X^\perp$  holds, and denoting by  $P_X$  the projection on  $H$  parallel to  $E_X^\perp$ , we have*

$$\mathbb{E}[\wedge P_X] = \wedge \Pi^H.$$

In words, in a fixed basis of  $E$ , the expectation of any minor of the matrix of  $P_X$  is equal to the same minor of  $\Pi^H$ .

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A short while ago, it came to our attention while reading the recent statistics paper [GBV19] on Monte–Carlo integration methods, that such a mean projection formula had also appeared in [EZ60] in the case of  $S = \mathbb{R}$ , in a different guise, although the relation to the above-cited works was not mentioned there. The goal of this short note is to emphasize this link, by extending Theorem 1.1 to the case of a determinantal point process associated to a finite rank orthogonal projection on any Polish space  $S$ , so that it applies for instance to any orthogonal polynomial ensemble, see [Lyo14, Section 3.8]. This extension is the content of Theorem 2.2. An extension of Theorem 1.1 to the case of a projection with infinite range (both in the case where  $S$  is discrete or continuous) would be interesting. It would probably be harder to prove in view of [BQ22] which studies the continuous analogue of  $\mathbf{P}_X$  in the case of the Bergman kernel.

## 2. THE MEAN PROJECTION THEOREM

Let  $S$  be a Polish space and  $\lambda$  a positive Radon measure on  $S$ . Let us consider the space  $E = L^2(S, \lambda)$  and the space  $\mathcal{C}(S)$  of continuous functions on  $S$ .<sup>1</sup> Let  $H \subseteq E \cap \mathcal{C}(S)$  be a linear subspace of finite dimension  $n$ .

Let  $\text{Conf}_n(S)$  be the set of collections of  $n$  distinct points in  $S$ , and let  $\mu$  be the determinantal probability measure on  $\text{Conf}_n(S)$  associated with the orthogonal projection on  $H$ . This means that if we choose an orthonormal basis  $(\varphi_j)_{1 \leq j \leq n}$  of  $H$ , then we have for any bounded continuous symmetric test function  $T : S^n \rightarrow \mathbb{C}$  the equality

$$(1) \quad \int_{\text{Conf}_n(S)} T(X) \, d\mu(X) = \frac{1}{n!} \int_{S^n} T(x_1, \dots, x_n) |\det(\varphi_j(x_i)_{1 \leq i, j \leq n})|^2 \, d\lambda^{\otimes n}(x_1, \dots, x_n),$$

in which the right-hand side does not depend on the choice of the orthonormal basis. We will denote by  $\mathbf{X}$  a random subset of  $S$  distributed according to  $\mu$ , and use the notation  $\mathbb{E}[T(\mathbf{X})]$  for either of the two sides of the equality above.

It follows from (1) that  $\mu$ -almost every  $X$  is a *uniqueness set* for  $H$ , in the sense that two elements of  $H$  that coincide on  $X$  are equal.<sup>2</sup> This fact can be used to define a random projection onto  $H$ , as follows. For every  $X \in \text{Conf}_n(S)$ , let us define  $\mathcal{C}(S; X) = \{f \in \mathcal{C}(S) : f|_X = 0\}$ .

**Lemma 2.1.** *For  $\mu$ -almost every  $X \in \text{Conf}_n(S)$ , the decomposition  $\mathcal{C}(S) = H \oplus \mathcal{C}(S; X)$  holds.*

*Proof.* Let  $f$  be an element of  $\mathcal{C}(S)$ . Let  $(\varphi_j)_{1 \leq j \leq n}$  be an orthonormal basis of  $H$ . For  $\mu$ -almost every  $X = \{x_1, \dots, x_n\}$  in  $\text{Conf}_n(S)$ , we have  $\det(\varphi_j(x_i)_{1 \leq i, j \leq n}) \neq 0$ , so that the system

$$\alpha_1 \varphi_1(x_i) + \dots + \alpha_n \varphi_n(x_i) = f(x_i), \quad \forall i \in \{1, \dots, n\}$$

admits a unique solution. Then  $\mathbf{P}_X f = \alpha_1 \varphi_1 + \dots + \alpha_n \varphi_n$  is the unique element of  $H$  which takes the same values as  $f$  on  $X$ .  $\square$

For the rest of this note, we will keep the notation  $\mathbf{P}_X$  introduced in the previous proof for the projection on  $H$  parallel to  $\mathcal{C}(S; X)$ . Let us emphasize that the decomposition given by Lemma 2.1 depends on  $H$  and  $X$ , but is independent of the Euclidean structure of  $E$ . In particular, the projection  $\mathbf{P}_X$  is independent of this Euclidean structure.

For example, if  $S = \mathbb{R}$ ,  $\lambda$  is a measure with infinite support which admits moments of all orders, and  $\varphi_1, \dots, \varphi_n$  are the first  $n$  orthogonal polynomials with respect to  $\lambda$ , then  $H$  is the space of polynomial functions of degree at most  $n - 1$  and  $\mathbf{P}_X f$  is the *interpolating polynomial* of the restriction of  $f$  to  $X$ .

<sup>1</sup>The space of continuous functions plays for us the role usually devoted to a reproducing kernel Hilbert space (RKHS), namely that of a space of functions that can be evaluated at points. However, we do not need this extra structure, because we do not need evaluation at a point to be a continuous linear form. Moreover, it seems that in many examples of interest, the RKHS is a subspace of continuous functions, so that our result applies.

<sup>2</sup>The uniqueness property is true for all determinantal processes associated with an orthogonal projection of possibly infinite range, that is with infinitely many points ( $n = \infty$ ), as proved in the discrete case by Lyons [Lyo03], and recently by Bufetov–Qiu–Shamov [BQS21] in the general case, following partial results by Ghosh [Gho15].

For all  $g_1, \dots, g_m \in E \cap \mathcal{C}(S)$ , let us define  $g_1 \wedge \dots \wedge g_m \in L^2(S^m, \frac{1}{m!} \lambda^{\otimes m}) \cap \mathcal{C}(S^m)$  by setting, for all  $y_1, \dots, y_m \in S$ ,

$$(2) \quad (g_1 \wedge \dots \wedge g_m)(y_1, \dots, y_m) = \det (g_j(y_i)_{1 \leq i, j \leq m}).$$

We will use several times the Andreieff–Heine identity, which is a continuous analogue of the Cauchy–Binet identity, and can be phrased as follows: if  $h_1, \dots, h_m$  belong to  $E \cap \mathcal{C}(S)$ , then

$$(3) \quad \langle g_1 \wedge \dots \wedge g_m, h_1 \wedge \dots \wedge h_m \rangle_{L^2(S^m, \frac{1}{m!} \lambda^{\otimes m})} = \det (\langle g_i, h_j \rangle)_{1 \leq i, j \leq m}.$$

This equality justifies, for instance, the fact that the measure  $\mu$  defined by (1) is a probability measure.

Let us write  $H^0 = H$  and  $H^1 = H^\perp$ . The isomorphism of vector spaces  $L^2(S^m, \frac{1}{m!} \lambda^{\otimes m}) \simeq L^2(S, \lambda)^{\otimes m}$  is  $\sqrt{m!}$  times an isometry, and the orthogonal decomposition  $L^2(S) = H^0 \oplus H^1$  gives rise to an orthogonal decomposition

$$(4) \quad L^2(S^m) \simeq L^2(S)^{\otimes m} = \bigoplus_{\varepsilon_1, \dots, \varepsilon_m \in \{0,1\}} H^{\varepsilon_1} \otimes \dots \otimes H^{\varepsilon_m} = \bigoplus_{k=0}^m \left[ \bigoplus_{\substack{\varepsilon_1, \dots, \varepsilon_m \in \{0,1\} \\ \varepsilon_1 + \dots + \varepsilon_m = k}} H^{\varepsilon_1} \otimes \dots \otimes H^{\varepsilon_m} \right].$$

Let us denote by  $\Pi_k$  the orthogonal projection of  $L^2(S^m)$  on the  $k$ -th summand of the last expression. In order to describe this operator more concretely, recall that we denote by  $\Pi^H$  the orthogonal projection on  $H$  in  $E$ . For all real  $t$ , let us define the linear operator  $D_t = \Pi^H + t\Pi^{H^\perp}$  on  $E$ . Then

$$D_t g_1 \wedge \dots \wedge D_t g_m = \sum_{k=0}^m t^k \Pi_k(g_1 \wedge \dots \wedge g_m).$$

In words,  $\Pi_k(g_1 \wedge \dots \wedge g_m)$  is the sum of all the functions obtained from  $g_1 \wedge \dots \wedge g_m$  by replacing  $k$  of the  $g_i$ 's by their projections on  $H^\perp$ , and the others by their projection on  $H$ .

**Theorem 2.2.** *For all  $m \geq 1$ , and all  $f_1, \dots, f_m \in E \cap \mathcal{C}(S)$ , we have*

$$(5) \quad \mathbb{E}[\mathsf{P}_X f_1 \wedge \dots \wedge \mathsf{P}_X f_m] = \Pi^H f_1 \wedge \dots \wedge \Pi^H f_m,$$

$$(6) \quad \text{Var}(\mathsf{P}_X f_1 \wedge \dots \wedge \mathsf{P}_X f_m) = \sum_{k=1}^m \binom{n-m+k}{k} \|\Pi_k(f_1 \wedge \dots \wedge f_m)\|^2.$$

The variance in the second assertion is that of a random element of  $L^2(S^m, \frac{1}{m!} \lambda^{\otimes m})$ , that is, to be explicit, and in view of the first assertion,

$$\text{Var}(\mathsf{P}_X f_1 \wedge \dots \wedge \mathsf{P}_X f_m) = \mathbb{E} \left[ \left\| \mathsf{P}_X f_1 \wedge \dots \wedge \mathsf{P}_X f_m - \Pi^H f_1 \wedge \dots \wedge \Pi^H f_m \right\|_{L^2(S^m, \frac{1}{m!} \lambda^{\otimes m})}^2 \right].$$

Further note that the quadratic identity (6) may be polarized to obtain information on covariances.

Given the remark made after Lemma 2.1, one can view Theorem 2.2 as providing a statistical estimator of part of the Euclidean structure of  $E$  given  $H$  and a realisation  $X$ .

When  $m = 1$ , this is the theorem of Ermakov–Zolotukhin [EZ60], rephrased by [GBV19]:

$$\mathbb{E}[\mathsf{P}_X f] = \Pi^H f \quad \text{and} \quad \text{Var}(\mathsf{P}_X f) = n \|\Pi^{H^\perp} f\|^2.$$

In order to prove Theorem 2.2, we will use the following generalization of Cramer's formula, which surprisingly enough, we have not encountered in our undergraduate linear algebra class.

For all integers  $n$  and  $m$ , we denote by  $\llbracket n \rrbracket$  the set  $\{1, \dots, n\}$  and by  $\mathcal{P}_m(\llbracket n \rrbracket)$  the set of its subsets with  $m$  elements. Given a  $p \times q$  matrix  $M$  and two subsets  $I \subseteq \llbracket p \rrbracket$  and  $J \subseteq \llbracket q \rrbracket$ , we define

$$M_J^I = (M_{ij})_{i \in I, j \in J} \quad \text{and} \quad M^I = M_{\llbracket q \rrbracket}^I.$$

**Proposition 2.3** (Cramer's identity for minors). *Let  $1 \leq m \leq n$  be two integers. Let  $M$  be an  $n \times n$  invertible square matrix, and  $F$  an  $n \times m$  rectangular matrix. Let  $A$  be the  $n \times m$  rectangular matrix solving  $MA = F$ . Then for all  $I \in \mathcal{P}_m(\llbracket n \rrbracket)$ , the  $m \times m$  submatrix  $A^I$  has determinant*

$$(7) \quad \det A^I = (\det M)^{-1} \det M_{[I \leftarrow F]},$$

where  $M_{[I \leftarrow F]}$  is the  $n \times n$  square matrix obtained by replacing in  $M$  the columns indexed by  $I$  by the columns of the matrix  $F$ .

If  $I = \{i_1 < \dots < i_m\}$ , then  $(M_{[I \leftarrow F]})_{ij} = M_{ij}$  for  $j \notin I$ , and  $(M_{[I \leftarrow F]})_{ij} = F_{ik}$  for  $j = i_k$ .

*Proof.* Let us write  $A = M^{-1}F$  and use the Cauchy–Binet formula:

$$\det A^I = \sum_{J \in \mathcal{P}_m(\llbracket n \rrbracket)} \det(M^{-1})_J^I \det F^J.$$

Now, by Jacobi's complementary minor formula,

$$\det(M^{-1})_J^I = (-1)^{\sum_{i \in I} i + \sum_{j \in J} j} (\det M)^{-1} \det M_{I^c}^{J^c}.$$

Combining the two previous equations and checking signs, we now recognize the Laplace expansion of  $\det M_{[I \leftarrow F]}$  with respect to all columns in  $I$ :

$$\det A^I = (\det M)^{-1} \sum_{J \in \mathcal{P}_m(\llbracket n \rrbracket)} (-1)^{\sum_{i \in I} i + \sum_{j \in J} j} \det M_{I^c}^{J^c} \det F^J = (\det M)^{-1} \det M_{[I \leftarrow F]},$$

which concludes the proof.  $\square$

*Proof of Theorem 2.2.* Let  $(\varphi_i)_{1 \leq i \leq n}$  be an orthonormal basis of  $H$ . Let  $X = (x_1, \dots, x_n) \in S^n$  be such that  $\det(\varphi_j(x_i)_{1 \leq i, j \leq n}) \neq 0$ . Let us introduce the following matrices:

- $M = (\varphi_j(x_i)) \quad 1 \leq i, j \leq n$ ,
- $F = (f_j(x_i)) \quad 1 \leq i \leq n, 1 \leq j \leq m$ ,
- $A = (\alpha_{ij}) \quad 1 \leq i \leq n, 1 \leq j \leq m$ , the solution to  $MA = F$ ,
- $G = (\langle \varphi_i, f_j \rangle) \quad 1 \leq i \leq n, 1 \leq j \leq m$ .

For each  $I = \{i_1 < \dots < i_k\} \subseteq \{1, \dots, n\}$ , let us write  $\varphi_I = \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$ .

For each  $1 \leq i \leq m$ , we have

$$\mathbb{P}_X f_i = \sum_{k=1}^n A_{ki} \varphi_k \quad \text{and} \quad \Pi^H f_i = \sum_{k=1}^n G_{ki} \varphi_k,$$

so that

$$(8) \quad \mathbb{P}_X f_1 \wedge \dots \wedge \mathbb{P}_X f_m = \sum_{I \in \mathcal{P}_m(\llbracket n \rrbracket)} \det A^I \varphi_I \quad \text{and} \quad \Pi^H f_1 \wedge \dots \wedge \Pi^H f_m = \sum_{I \in \mathcal{P}_m(\llbracket n \rrbracket)} \det G^I \varphi_I.$$

In order to prove the first assertion of the theorem, namely (5), we are thus left to show that for all  $I \in \mathcal{P}_m(\llbracket n \rrbracket)$ , we have

$$(9) \quad \mathbb{E}[\det A^I] = \det G^I,$$

where we view  $A$  as a function of the subset  $X \subseteq S$  and the expectation is with respect to  $\mu$ .

By Proposition 2.3, we can write  $\det A^I = (\det M)^{-1} \det M_{[I \leftarrow F]}$ . Using the form (1) of the density of  $\mu$  and the Andreieff–Heine identity (3), we find

$$\begin{aligned} \mathbb{E}[\det A^I] &= \frac{1}{n!} \int_{S^n} \det A^I |\det M|^2 d\lambda^{\otimes n} \\ &= \frac{1}{n!} \int_{S^n} \det M_{[I \leftarrow F]} \overline{\det M} d\lambda^{\otimes n} \\ &= \det (\langle \varphi_a, \psi_{I,b} \rangle)_{1 \leq a, b \leq n}, \end{aligned}$$

where  $(\psi_{I,1}, \dots, \psi_{I,n})$  is the list  $(\varphi_1, \dots, \varphi_n)$  in which the terms labelled by elements of  $I$  have been replaced by  $f_1, \dots, f_m$ . In symbols,  $\psi_{I,b} = \varphi_b$  if  $b \notin I$  and  $\psi_{I,b} = f_k$  if  $I = \{i_1 < \dots < i_m\}$  and  $b = i_k$ .

It is readily seen that the last determinant is equal to  $\det G^I$ , which proves (9) and thus (5).

We now turn to the computation of the variance. An important observation is that the family  $\{\varphi_I : I \in \mathcal{P}_m(\llbracket n \rrbracket)\}$  is orthonormal in  $L^2(S^m, \frac{1}{m!} \lambda^{\otimes m})$ . Thus, using (8), Pythagoras’s theorem, and (5), we find that

$$(10) \quad \text{Var}(\text{P}_X f_1 \wedge \dots \wedge \text{P}_X f_m) = \sum_{I \in \mathcal{P}_m(\llbracket n \rrbracket)} \mathbb{E}[(\det A^I)^2] - \|\Pi^H f_1 \wedge \dots \wedge \Pi^H f_m\|^2.$$

Using the same strategy as before, we compute, for each set  $I$  of cardinality  $m$ ,

$$\begin{aligned} \mathbb{E}[|\det A^I|^2] &= \frac{1}{n!} \int_{S^n} |\det A^I|^2 |\det M|^2 d\lambda^{\otimes n} \\ &= \frac{1}{n!} \int_{S^n} |\det M_{[I \leftarrow F]}|^2 d\lambda^{\otimes n} \\ &= \det (\langle \psi_{I,a}, \psi_{I,b} \rangle)_{1 \leq a, b \leq n}. \end{aligned}$$

We now use the Schur complement formula to obtain that the last determinant is equal to

$$\det (\langle f_i, (\Pi^{H^\perp} + \Pi^{H_I}) f_j \rangle)_{1 \leq i, j \leq m},$$

where for all  $J \subseteq \{1, \dots, n\}$ , we set  $H_J = \text{Vect}(\varphi_j, j \in J)$ . Using the Andreieff–Heine identity, we rewrite this determinant as

$$\mathbb{E}[|\det A^I|^2] = \langle f_1 \wedge \dots \wedge f_m, (\Pi^{H^\perp} + \Pi^{H_I}) f_1 \wedge \dots \wedge (\Pi^{H^\perp} + \Pi^{H_I}) f_m \rangle$$

and what we need now is to sum this quantity over all  $I \in \mathcal{P}_m(\llbracket n \rrbracket)$ .

For each  $i \in \{1, \dots, m\}$ , let us decompose  $f_i$  as  $f_{i,0} + f_{i,1} + \dots + f_{i,n}$ , where  $f_{i,0} = \Pi^{H^\perp} f_i$  and for all  $j \in \{1, \dots, n\}$ ,  $f_{i,j} = \langle \varphi_j, f_i \rangle \varphi_j$ . By multilinearity, we find

$$\mathbb{E}[|\det A^I|^2] = \sum_{j_1, \dots, j_m=0}^n \langle f_1 \wedge \dots \wedge f_m, \underbrace{(\Pi^{H^\perp} + \Pi^{H_I}) f_{1,j_1} \wedge \dots \wedge (\Pi^{H^\perp} + \Pi^{H_I}) f_{m,j_m}}_R \rangle.$$

Let us call  $R$  the function in the right-hand side of the scalar product. If among the integers  $j_1, \dots, j_m$  two are positive and equal, then  $R$  vanishes, and so does the corresponding term of the sum. Let us now assume that the positive indices among  $j_1, \dots, j_m$  are pairwise distinct, and let us list them as  $\{l_1, \dots, l_{m-k}\}$ , where  $k = 1_{\{j_1=0\}} + \dots + 1_{\{j_m=0\}}$ . We make three observations. Firstly, for  $R$  not to be zero, it is necessary that  $\{l_1, \dots, l_{m-k}\} \subseteq I$ . Secondly, if this condition is satisfied, then  $R = f_{1,j_1} \wedge \dots \wedge f_{m,j_m}$ , and in particular does not depend on  $I$ . Finally, the condition  $\{l_1, \dots, l_{m-k}\} \subseteq I$  is verified for  $\binom{n-m+k}{k}$  subsets  $I$  of  $\{1, \dots, n\}$  with  $m$  elements. Putting these observations together, we find

$$\sum_{I \in \mathcal{P}_m(\llbracket n \rrbracket)} \mathbb{E}[|\det A^I|^2] = \sum_{j_1, \dots, j_m} \binom{n-m+k}{k} \langle f_1 \wedge \dots \wedge f_m, f_{1,j_1} \wedge \dots \wedge f_{m,j_m} \rangle.$$

The sum runs over those  $j_1, \dots, j_m$  between 0 and  $n$  among which no two are positive and equal, but lifting this condition only adds null terms to the sum. Therefore, we let  $j_1, \dots, j_m$  run freely between 0 and  $n$ , and  $k$  is the number of them that are zero.

Let us sort the terms of the last sum according to which of the indices  $j_1, \dots, j_m$  are zero and which are not: calling  $B$  the set  $\{p \in \llbracket m \rrbracket : j_p = 0\}$  and with the notation  $H^0 = H$  and  $H^1 = H^\perp$ , this resummation yields

$$(11) \quad \sum_{I \in \mathcal{P}_m(\llbracket n \rrbracket)} \mathbb{E}[|\det A^I|^2] = \sum_{k=0}^m \binom{n-m+k}{k} \left\langle f_1 \wedge \dots \wedge f_m, \sum_{B \in \mathcal{P}_k(\llbracket m \rrbracket)} \Pi^{H^{1_B(1)}} f_1 \wedge \dots \wedge \Pi^{H^{1_B(m)}} f_m \right\rangle.$$

The sum over  $B$  yields exactly the function  $\Pi_k(f_1 \wedge \dots \wedge f_m)$ . The result follows from the orthogonality of the decomposition (4) and the observation that the term corresponding to  $k = 0$  is exactly the last term of (10).  $\square$

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