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► To cite this version:

Adrien Kassel, Thierry Lévy. On the mean projection theorem for determinantal point processes. 2022. hal-03609991v2

HAL Id: hal-03609991

<https://hal.science/hal-03609991v2>

Preprint submitted on 25 Oct 2022 (v2), last revised 21 Feb 2023 (v3)

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ON THE MEAN PROJECTION THEOREM FOR DETERMINANTAL POINT PROCESSES

ADRIEN KASSEL AND THIERRY LÉVY

ABSTRACT. In this short note, we extend a mean projection theorem for discrete determinantal point processes to the continuous case, thus strengthening a known result in random linear algebra due to Ermakov and Zolotukhin. We also give a new formula for the variance of the exterior power of the random projection.

1. INTRODUCTION

Kirchhoff's work on electrical networks [Kir47] seems to be one of the earliest works in the literature where linear algebra and graph-theoretical combinatorial methods were put together. Later on, linear algebra problems, and classical determinantal methods for solving them, gave rise to various statistical approaches, notably linked to the so-called determinantal point processes (introduced by Macchi in 1975 [Mac75], and named like this by Borodin, only around 2000 which saw a blossoming of results on those processes from various authors, see [Sos00, ST03, Lyo03, Joh06, Bor11]). These methods recently became an active field in randomized numerical linear algebra [DM21].

In his work, Kirchhoff solved a linear algebra system on an electrical network seen as a finite graph, by expressing the current induced by an external battery hooked on the network, as an average over spanning trees of a certain current associated to the tree. In modern terms, he expressed an orthogonal projection as the expectation of a certain random projection associated to a random spanning tree. Such a mean projection theorem appeared in several guises in the literature, and more or less independently, in works of Maurer [Mau76], Lyons [Lyo03], Catanzaro–Chernyak–Klein [CCK13], and probably others that we are unaware of.

In our work [KL19, Theorem 5.9], we extended the mean projection formula for determinantal point processes on finite sets, thus putting the statements of [Kir47, Mau76, Lyo03, CCK13] in a unified geometric framework, and strengthening the result by proving a mean projection theorem for the exterior powers of the projections, that is, for minors of their matrices in a fixed basis.

Let us quickly recall our statement. Let \mathbb{K} be \mathbb{R} or \mathbb{C} , let E be a finite dimensional Euclidean space on \mathbb{K} of dimension d , and let $(e_i)_{1 \leq i \leq d}$ be an orthonormal basis of E . We let $S = \{1, \dots, d\}$ and consider H a subspace of E of dimension n . Let \mathbf{X} be the determinantal point process on S associated to the matrix $K = (\langle e_i, \Pi^H e_j \rangle)_{1 \leq i, j \leq d}$, where Π^H is the orthogonal projection on H . For each $X \subseteq S$, let $E_X = \bigoplus_{x \in X} \mathbb{K} e_x$ be the corresponding coordinate subspace of E .

Theorem 1.1. *Almost surely, the equality $E = H \oplus E_X^\perp$ holds, and denoting by \mathbf{P}_X the projection on H parallel to E_X^\perp , we have*

$$\mathbb{E}[\bigwedge \mathbf{P}_X] = \bigwedge \Pi^H.$$

In words, in a fixed basis of E , the expectation of any minor of the matrix of \mathbf{P}_X is equal to the same minor of Π^H .

Date: October 25, 2022.

2010 Mathematics Subject Classification. 60G55, 15A75.

Key words and phrases. determinantal point processes, random projection, exterior algebra.

A short while ago, it came to our attention while reading the recent statistics paper [GBV19] on Monte–Carlo integration methods, that such a mean projection formula had also appeared in [EZ60] in the case of $S = \mathbb{R}$, in a different guise, although the relation to the above-cited works was not mentioned there. The goal of this short note is to emphasize this link, by extending Theorem 1.1 to the case of a determinantal point process associated to a finite rank orthogonal projection on any Polish space S , so that it applies for instance to any orthogonal polynomial ensemble, see [Lyo14, Section 3.8]. This extension is the content of Theorem 2.2. An extension of Theorem 1.1 to the case of a projection with infinite range (both in the case where S is discrete or continuous) would be interesting. It would probably be harder to prove in view of [BQ22] which studies the continuous analogue of P_X in the case of the Bergman kernel.

2. THE MEAN PROJECTION THEOREM

Let S be a Polish space and λ a positive Radon measure on S . Let us consider the space $E = L^2(S, \lambda)$ and the space $\mathcal{C}(S)$ of continuous functions on S .¹ Let $H \subseteq E \cap \mathcal{C}(S)$ be a linear subspace of finite dimension n .

Let $\text{Conf}_n(S)$ be the set of collections of n distinct points in S , and let μ be the determinantal probability measure on $\text{Conf}_n(S)$ associated with the orthogonal projection on H . This means that if we choose an orthonormal basis $(\varphi_j)_{1 \leq j \leq n}$ of H , then we have for any bounded continuous symmetric test function $T : S^n \rightarrow \mathbb{C}$ the equality

$$(1) \quad \int_{\text{Conf}_n(S)} T(X) \, d\mu(X) = \frac{1}{n!} \int_{S^n} T(x_1, \dots, x_n) |\det(\varphi_j(x_i)_{1 \leq i, j \leq n})|^2 \, d\lambda^{\otimes n}(x_1, \dots, x_n),$$

in which the right-hand side does not depend on the choice of the orthonormal basis. We will denote by X a random subset of S distributed according to μ , and use the notation $\mathbb{E}[T(X)]$ for either of the two sides of the equality above.

It follows from (1) that μ -almost every X is a *uniqueness set* for H , in the sense that two elements of H that coincide on X are equal.² This fact can be used to define a random projection onto H , as follows. For every $X \in \text{Conf}_n(S)$, let us define $\mathcal{C}(S; X) = \{f \in \mathcal{C}(S) : f|_X = 0\}$.

Lemma 2.1. *For μ -almost every $X \in \text{Conf}_n(S)$, the decomposition $\mathcal{C}(S) = H \oplus \mathcal{C}(S; X)$ holds.*

Proof. Let f be an element of $\mathcal{C}(S)$. Let $(\varphi_j)_{1 \leq j \leq n}$ be an orthonormal basis of H . For μ -almost every $X = \{x_1, \dots, x_n\}$ in $\text{Conf}_n(S)$, we have $\det(\varphi_j(x_i)_{1 \leq i, j \leq n}) \neq 0$, so that the system

$$\alpha_1 \varphi_1(x_i) + \dots + \alpha_n \varphi_n(x_i) = f(x_i), \quad \forall i \in \{1, \dots, n\}$$

admits a unique solution. Then $P_X f = \alpha_1 \varphi_1 + \dots + \alpha_n \varphi_n$ is the unique element of H which takes the same values as f on X . \square

For the rest of this note, we will keep the notation P_X introduced in the previous proof for the projection on H parallel to $\mathcal{C}(S; X)$. Let us emphasize that the decomposition given by Lemma 2.1 depends on H and X , but is independent of the Euclidean structure of E . In particular, the projection P_X is independent of this Euclidean structure.

For example, if $S = \mathbb{R}$, λ is a measure with infinite support which admits moments of all orders, and $\varphi_1, \dots, \varphi_n$ are the first n orthogonal polynomials with respect to λ , then H is the space of polynomial functions of degree at most $n - 1$ and $P_X f$ is the *interpolating polynomial* of the restriction of f to X .

¹The space of continuous functions plays for us the role usually devoted to a reproducing kernel Hilbert space (RKHS), namely that of a space of functions that can be evaluated at points. However, we do not need this extra structure, because we do not need evaluation at a point to be a continuous linear form. Moreover, it seems that in many examples of interest, the RKHS is a subspace of continuous functions, so that our result applies.

²The uniqueness property is true for all determinantal processes associated with an orthogonal projection of possibly infinite range, that is with infinitely many points ($n = \infty$), as proved in the discrete case by Lyons [Lyo03], and recently by Bufetov–Qiu–Shamov [BQS21] in the general case, following partial results by Ghosh [Gho15].

For all $g_1, \dots, g_m \in E \cap \mathcal{C}(S)$, let us define $g_1 \wedge \dots \wedge g_m \in L^2(S^m, \frac{1}{m!} \lambda^{\otimes m}) \cap \mathcal{C}(S^m)$ by setting, for all $y_1, \dots, y_m \in S$,

$$(2) \quad (g_1 \wedge \dots \wedge g_m)(y_1, \dots, y_m) = \det(g_j(y_i)_{1 \leq i, j \leq m}).$$

We will use several times the Andreieff–Heine identity, which is a continuous analogue of the Cauchy–Binet identity, and can be phrased as follows: if h_1, \dots, h_m belong to $E \cap \mathcal{C}(S)$, then

$$(3) \quad \langle g_1 \wedge \dots \wedge g_m, h_1 \wedge \dots \wedge h_m \rangle_{L^2(S^m, \frac{1}{m!} \lambda^{\otimes m})} = \det(\langle g_i, h_j \rangle)_{1 \leq i, j \leq m}.$$

This equality justifies, for instance, the fact that the measure μ defined by (1) is a probability measure.

Let us write $H^0 = H$ and $H^1 = H^\perp$. The isomorphism of vector spaces $L^2(S^m, \frac{1}{m!} \lambda^{\otimes m}) \simeq L^2(S, \lambda)^{\otimes m}$ is $\sqrt{m!}$ times an isometry, and the orthogonal decomposition $L^2(S) = H^0 \oplus H^1$ gives rise to an orthogonal decomposition

$$(4) \quad L^2(S^m) \simeq L^2(S)^{\otimes m} = \bigoplus_{\varepsilon_1, \dots, \varepsilon_m \in \{0,1\}} H^{\varepsilon_1} \otimes \dots \otimes H^{\varepsilon_m} = \bigoplus_{k=0}^m \left[\bigoplus_{\substack{\varepsilon_1, \dots, \varepsilon_m \in \{0,1\} \\ \varepsilon_1 + \dots + \varepsilon_m = k}} H^{\varepsilon_1} \otimes \dots \otimes H^{\varepsilon_m} \right].$$

Let us denote by Π_k the orthogonal projection of $L^2(S^m)$ on the k -th summand of the last expression. In order to describe this operator more concretely, recall that we denote by Π^H the orthogonal projection on H in E . For all real t , let us define the linear operator $D_t = \Pi^H + t\Pi^{H^\perp}$ on E . Then

$$D_t g_1 \wedge \dots \wedge D_t g_m = \sum_{k=0}^m t^k \Pi_k(g_1 \wedge \dots \wedge g_m).$$

In words, $\Pi_k(g_1 \wedge \dots \wedge g_m)$ is the sum of all the functions obtained from $g_1 \wedge \dots \wedge g_m$ by replacing k of the g_i 's by their projections on H^\perp , and the others by their projection on H .

Theorem 2.2. *For all $m \geq 1$, and all $f_1, \dots, f_m \in E \cap \mathcal{C}(S)$, we have*

$$(5) \quad \mathbb{E}[\mathbf{P}_X f_1 \wedge \dots \wedge \mathbf{P}_X f_m] = \Pi^H f_1 \wedge \dots \wedge \Pi^H f_m,$$

$$(6) \quad \text{Var}(\mathbf{P}_X f_1 \wedge \dots \wedge \mathbf{P}_X f_m) = \sum_{k=1}^m \binom{n-m+k}{k} \|\Pi_k(f_1 \wedge \dots \wedge f_m)\|^2.$$

The variance in the second assertion is that of a random element of $L^2(S^m, \frac{1}{m!} \lambda^{\otimes m})$, that is, to be explicit, and in view of the first assertion,

$$\text{Var}(\mathbf{P}_X f_1 \wedge \dots \wedge \mathbf{P}_X f_m) = \mathbb{E} \left[\left\| \mathbf{P}_X f_1 \wedge \dots \wedge \mathbf{P}_X f_m - \Pi^H f_1 \wedge \dots \wedge \Pi^H f_m \right\|_{L^2(S^m, \frac{1}{m!} \lambda^{\otimes m})}^2 \right].$$

Further note that the quadratic identity (6) may be polarized to obtain information on covariances.

Given the remark made after Lemma 2.1, one can view Theorem 2.2 as providing a statistical estimator of part of the Euclidean structure of E given H and a realisation \mathbf{X} .

When $m = 1$, this is the theorem of Ermakov–Zolotukhin [EZ60], rephrased by [GBV19]:

$$\mathbb{E}[\mathbf{P}_X f] = \Pi^H f \quad \text{and} \quad \text{Var}(\mathbf{P}_X f) = n \|\Pi^{H^\perp} f\|^2.$$

In order to prove Theorem 2.2, we will use the following generalization of Cramer's formula, which surprisingly enough, we have not encountered in our undergraduate linear algebra class.

For all integers n and m , we denote by $\llbracket n \rrbracket$ the set $\{1, \dots, n\}$ and by $\mathcal{P}_m(\llbracket n \rrbracket)$ the set of its subsets with m elements. Given a $p \times q$ matrix M and two subsets $I \subseteq \llbracket p \rrbracket$ and $J \subseteq \llbracket q \rrbracket$, we define

$$M_J^I = (M_{ij})_{i \in I, j \in J} \quad \text{and} \quad M^I = M_{\llbracket q \rrbracket}^I.$$

Proposition 2.3 (Cramer's identity for minors). *Let $1 \leq m \leq n$ be two integers. Let M be an $n \times n$ invertible square matrix, and F an $n \times m$ rectangular matrix. Let A be the $n \times m$ rectangular matrix solving $MA = F$. Then for all $I \in \mathcal{P}_m(\llbracket n \rrbracket)$, the $m \times m$ submatrix A^I has determinant*

$$(7) \quad \det A^I = (\det M)^{-1} \det M_{[I \leftarrow F]},$$

where $M_{[I \leftarrow F]}$ is the $n \times n$ square matrix obtained by replacing in M the columns indexed by I by the columns of the matrix F .

If $I = \{i_1 < \dots < i_m\}$, then $(M_{[I \leftarrow F]})_{ij} = M_{ij}$ for $j \notin I$, and $(M_{[I \leftarrow F]})_{ij} = F_{ik}$ for $j = i_k$.

Proof. Let us write $A = M^{-1}F$ and use the Cauchy–Binet formula:

$$\det A^I = \sum_{J \in \mathcal{P}_m(\llbracket n \rrbracket)} \det(M^{-1})_J^I \det F^J.$$

Now, by Jacobi's complementary minor formula,

$$\det(M^{-1})_J^I = (-1)^{\sum_{i \in I} i + \sum_{j \in J} j} (\det M)^{-1} \det M_{I^c}^{J^c}.$$

Combining the two previous equations and checking signs, we now recognize the Laplace expansion of $\det M_{[I \leftarrow F]}$ with respect to all columns in I :

$$\det A^I = (\det M)^{-1} \sum_{J \in \mathcal{P}_m(\llbracket n \rrbracket)} (-1)^{\sum_{i \in I} i + \sum_{j \in J} j} \det M_{I^c}^{J^c} \det F^J = (\det M)^{-1} \det M_{[I \leftarrow F]},$$

which concludes the proof. \square

Proof of Theorem 2.2. Let $(\varphi_i)_{1 \leq i \leq n}$ be an orthonormal basis of H . Let $X = (x_1, \dots, x_n) \in S^n$ be such that $\det(\varphi_j(x_i)_{1 \leq i, j \leq n}) \neq 0$. Let us introduce the following matrices:

- $M = (\varphi_j(x_i)) \quad 1 \leq i, j \leq n$,
- $F = (f_j(x_i)) \quad 1 \leq i \leq n, 1 \leq j \leq m$,
- $A = (\alpha_{ij}) \quad 1 \leq i \leq n, 1 \leq j \leq m$, the solution to $MA = F$,
- $G = (\langle \varphi_i, f_j \rangle) \quad 1 \leq i \leq n, 1 \leq j \leq m$.

For each $I = \{i_1 < \dots < i_k\} \subseteq \{1, \dots, n\}$, let us write $\varphi_I = \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$.

For each $1 \leq i \leq m$, we have

$$P_X f_i = \sum_{k=1}^n A_{ki} \varphi_k \quad \text{and} \quad \Pi^H f_i = \sum_{k=1}^n G_{ki} \varphi_k,$$

so that

$$(8) \quad P_X f_1 \wedge \dots \wedge P_X f_m = \sum_{I \in \mathcal{P}_m(\llbracket n \rrbracket)} \det A^I \varphi_I \quad \text{and} \quad \Pi^H f_1 \wedge \dots \wedge \Pi^H f_m = \sum_{I \in \mathcal{P}_m(\llbracket n \rrbracket)} \det G^I \varphi_I.$$

In order to prove the first assertion of the theorem, namely (5), we are thus left to show that for all $I \in \mathcal{P}_m(\llbracket n \rrbracket)$, we have

$$(9) \quad \mathbb{E}[\det A^I] = \det G^I,$$

where we view A as a function of the subset $X \subseteq S$ and the expectation is with respect to μ .

By Proposition 2.3, we can write $\det A^I = (\det M)^{-1} \det M_{[I \leftarrow F]}$. Using the form (1) of the density of μ and the Andreieff–Heine identity (3), we find

$$\begin{aligned} \mathbb{E}[\det A^I] &= \frac{1}{n!} \int_{S^n} \det A^I |\det M|^2 d\lambda^{\otimes n} \\ &= \frac{1}{n!} \int_{S^n} \det M_{[I \leftarrow F]} \overline{\det M} d\lambda^{\otimes n} \\ &= \det(\langle \varphi_a, \psi_{I,b} \rangle)_{1 \leq a, b \leq n}, \end{aligned}$$

where $(\psi_{I,1}, \dots, \psi_{I,n})$ is the list $(\varphi_1, \dots, \varphi_n)$ in which the terms labelled by elements of I have been replaced by f_1, \dots, f_m . In symbols, $\psi_{I,b} = \varphi_b$ if $b \notin I$ and $\psi_{I,b} = f_k$ if $I = \{i_1 < \dots < i_m\}$ and $b = i_k$.

It is readily seen that the last determinant is equal to $\det G^I$, which proves (9) and thus (5).

We now turn to the computation of the variance. An important observation is that the family $\{\varphi_I : I \in \mathcal{P}_m(\llbracket n \rrbracket)\}$ is orthonormal in $L^2(S^m, \frac{1}{m!} \lambda^{\otimes m})$. Thus, using (8), Pythagoras's theorem, and (5), we find that

$$(10) \quad \text{Var}(\mathbf{P}_X f_1 \wedge \dots \wedge \mathbf{P}_X f_m) = \sum_{I \in \mathcal{P}_m(\llbracket n \rrbracket)} \mathbb{E}[(\det A^I)^2] - \|\Pi^H f_1 \wedge \dots \wedge \Pi^H f_m\|^2.$$

Using the same strategy as before, we compute, for each set I of cardinality m ,

$$\begin{aligned} \mathbb{E}[|\det A^I|^2] &= \frac{1}{n!} \int_{S^n} |\det A^I|^2 |\det M|^2 d\lambda^{\otimes n} \\ &= \frac{1}{n!} \int_{S^n} |\det M_{[I \leftarrow F]}|^2 d\lambda^{\otimes n} \\ &= \det(\langle \psi_{I,a}, \psi_{I,b} \rangle)_{1 \leq a, b \leq n}. \end{aligned}$$

We now use the Schur complement formula to obtain that the last determinant is equal to

$$\det(\langle f_i, (\Pi^{H^\perp} + \Pi^{H_I}) f_j \rangle)_{1 \leq i, j \leq m},$$

where for all $J \subseteq \{1, \dots, n\}$, we set $H_J = \text{Vect}(\varphi_j, j \in J)$. Using the Andreieff–Heine identity, we rewrite this determinant as

$$\mathbb{E}[|\det A^I|^2] = \langle f_1 \wedge \dots \wedge f_m, (\Pi^{H^\perp} + \Pi^{H_I}) f_1 \wedge \dots \wedge (\Pi^{H^\perp} + \Pi^{H_I}) f_m \rangle$$

and what we need now is to sum this quantity over all $I \in \mathcal{P}_m(\llbracket n \rrbracket)$.

For each $i \in \{1, \dots, m\}$, let us decompose f_i as $f_{i,0} + f_{i,1} + \dots + f_{i,n}$, where $f_{i,0} = \Pi^{H^\perp} f_i$ and for all $j \in \{1, \dots, n\}$, $f_{i,j} = \langle \varphi_j, f_i \rangle \varphi_j$. By multilinearity, we find

$$\mathbb{E}[|\det A^I|^2] = \sum_{j_1, \dots, j_m=0}^n \langle f_1 \wedge \dots \wedge f_m, \underbrace{(\Pi^{H^\perp} + \Pi^{H_I}) f_{1,j_1} \wedge \dots \wedge (\Pi^{H^\perp} + \Pi^{H_I}) f_{m,j_m}}_R \rangle.$$

Let us call R the function in the right-hand side of the scalar product. If among the integers j_1, \dots, j_m two are positive and equal, then R vanishes, and so does the corresponding term of the sum. Let us now assume that the positive indices among j_1, \dots, j_m are pairwise distinct, and let us list them as $\{l_1, \dots, l_{m-k}\}$, where $k = 1_{\{j_1=0\}} + \dots + 1_{\{j_m=0\}}$. We make three observations. Firstly, for R not to be zero, it is necessary that $\{l_1, \dots, l_{m-k}\} \subseteq I$. Secondly, if this condition is satisfied, then $R = f_{1,j_1} \wedge \dots \wedge f_{m,j_m}$, and in particular does not depend on I . Finally, the condition $\{l_1, \dots, l_{m-k}\} \subseteq I$ is verified for $\binom{n-m+k}{k}$ subsets I of $\{1, \dots, n\}$ with m elements. Putting these observations together, we find

$$\sum_{I \in \mathcal{P}_m(\llbracket n \rrbracket)} \mathbb{E}[|\det A^I|^2] = \sum_{j_1, \dots, j_m} \binom{n-m+k}{k} \langle f_1 \wedge \dots \wedge f_m, f_{1,j_1} \wedge \dots \wedge f_{m,j_m} \rangle.$$

The sum runs over those j_1, \dots, j_m between 0 and n among which no two are positive and equal, but lifting this condition only adds null terms to the sum. Therefore, we let j_1, \dots, j_m run freely between 0 and n , and k is the number of them that are zero.

Let us sort the terms of the last sum according to which of the indices j_1, \dots, j_m are zero and which are not: calling B the set $\{p \in \llbracket m \rrbracket : j_p = 0\}$ and with the notation $H^0 = H$ and $H^1 = H^\perp$, this resummation yields

$$(11) \quad \sum_{I \in \mathcal{P}_m(\llbracket n \rrbracket)} \mathbb{E}[|\det A^I|^2] = \sum_{k=0}^m \binom{n-m+k}{k} \left\langle f_1 \wedge \dots \wedge f_m, \sum_{B \in \mathcal{P}_k(\llbracket m \rrbracket)} \Pi^{H^{1_{B^{(1)}}}} f_1 \wedge \dots \wedge \Pi^{H^{1_{B^{(m)}}}} f_m \right\rangle.$$

The sum over B yields exactly the function $\Pi_k(f_1 \wedge \dots \wedge f_m)$. The result follows from the orthogonality of the decomposition (4) and the observation that the term corresponding to $k = 0$ is exactly the last term of (10). \square

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