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# Zonotopic Unknown Input State Estimator for Discrete-Time Linear Systems

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## Abstract

In this work state estimation for unknown input discrete-time linear systems is tackled in a bounded error context. First, under the boundedness assumption on both process and measurement noise as well as on system's initial state, the estimation problem is reformulated as a reachability problem. Then, by preprocessing the set-valued expression of the general solution of the state equation and applying zonotopic set computation rules, a non-conservative and computationally tractable algorithm is proposed to characterize efficiently the solution set of this problem. Numerical examples are presented to show the performance of the proposed zonotopic unknown input state estimator.

*Keywords:* Discrete-time linear systems, Unknown input observers, Zonotopic set computation, Reachability analysis, Set-membership estimation

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## 1. Introduction

In many real world applications, online knowledge of the internal state of dynamical systems is essential to perform efficiently control and monitoring tasks. Thus, state estimation has gained great interest in control system theory. Numerous approaches have been proposed to solve this problem for different class of systems, where assumptions on the nature of the system uncertainty are needed. In general, state estimation is formulated as an online inversion-like problem, where the internal state of a system has to be deduced from the model, the input and the measurements. In the framework

of discrete-time linear systems, most of the available techniques are based on: *Kalman Filter* (Kalman, 1960; Sorenson, 1985) for the stochastic case, and *Luenberger Observer* (Luenberger, 1964, 1966, 1971) for the deterministic case. However, this problem is more complicated in the case where some parts of the system are poorly-known or uncertain, for example when there are no statistical distributions capable to accurately describe the behavior of the uncertain parts. As examples of uncertainties one can emphasize actuator and sensor noises and deterministic modeling errors. To face these complex situations, set-valued estimation offers a promising framework. In this context, the uncertainties are modeled by bounded sets, and as a result the set of consistent values of the state vector is provided at each time instant. It is worth pointing out that, in the literature, there exist two families of methods designed to solve this set-valued estimation problem. The first one is a natural extension of the classical Luenberger observer to the interval case (J. L. Gouzé and Hadj-Sadok, 2000; Meslem and Ramdani, 2011; Mazenc et al., 2014; Cacace et al., 2015; Efimov and Raïssi, 2016; Meslem et al., 2018), while the second one relies on an extension of the prediction-correction principle to the set-valued case (Chisci et al., 1996; Kurzhanski and Vályi, 1996; Jaulin et al., 2001; Combastel, 2003; Alamo et al., 2005; Ben Chabane et al., 2014; Meslem and Ramdani, 2019).

On the other hand, systems may be subject to completely unknown inputs, for instance actuator faults or exogenous state disturbances. To manage the impact of the unknown inputs on the estimation error, unknown input observer design approaches have been proposed in the literature (Wang et al., 1978; Wang and Davison, 1978; Chen and Patton, 1999; Valcher, 1999; Hou and Müller, 1992; Maquin et al., 1994; Meditch and Hostetter, 1973; Patton et al., 1989). These approaches have also been extended to the set-valued case (Xu et al., 2016; Robinson et al., 2017; Meyer et al., 2018). The idea behind this extension is to build state estimators robust against bounded uncertainties and insensitive to unknown inputs. In (Robinson et al., 2017), this problem is tackled in the framework of positive systems. A reduced-order unknown input interval observer is proposed, where the dynamics of its estimation error is described by a stable positive system. To reach this purpose, specific state-coordinate transformations are applied. In (Meyer et al., 2018), a full-order unknown input interval observer is introduced. In their work the authors established LMIs conditions to ensure both positivity and stability of the estimation error of their proposed unknown input interval observer. In (Xu et al., 2016), this problem is tackled differently, where zonotopic set

computations (Shephard, 1974; Kühn, 1998; Combastel, 2003) are applied to characterize enclosures of the actual state vector.

In this present work, based on an explicit reachability method, we propose a zonotopic extension of the full-order unknown input observers introduced in (Chen and Patton (1999), Theorem 3.1, page 74). More precisely, based on zonotopic set computation, the developed reachability method allows one to obtain a guaranteed enclosure of all the possible estimation errors of the unknown input observer. It is worth pointing out that the proposed reachability algorithm is converging, computationally tractable and free from conservatism linked to the *wrapping effect*.

The remaining parts of this paper are organized as follows. In Section 2, a brief introduction about zonotopic set computation is given and a new reachability method for discrete-time linear systems is proposed. This latter represents the first contribution of the paper. Section 3 formulates the considered set-valued state estimation problem. Then, a zonotopic extension of an unknown input observer is introduced in Section 4, which is the main novelty of this work. Section 5 is dedicated to show, throughout a numerical example, the performance of the proposed set-valued state estimator, while Section 6 is devoted to compare our method with other approaches selected from the literature. A conclusion and perspectives are given in Section 7.

## 2. Preliminaries and a new reachability analysis method

To model bounded uncertainties and to manage their propagation in the context of dynamical system, zonotope shapes have been proposed (Kühn, 1998; Combastel, 2003; Alamo et al., 2005). By definition, a zonotope is a symmetric convex polytope. Given a vector  $\mathbf{p} \in \mathbb{R}^n$  and a matrix  $\mathbf{M} \in \mathbb{R}^{n \times m}$ , the zonotope  $\mathcal{Z}(\mathbf{p}, \mathbf{M})$  of order  $m$  in  $\mathbb{R}^n$  (also called  $m$ -zonotope) is the bounded set:

$$\mathcal{Z}(\mathbf{p}, \mathbf{M}) = \mathbf{p} \oplus \mathbf{M}\mathcal{B}^m = \{\mathbf{p} + \mathbf{M}\mathbf{z} \mid \mathbf{z} \in \mathcal{B}^m\} \quad (1)$$

where  $\mathcal{B}^m$  is a box or interval vector of dimension  $m$  where all its entries are unitary intervals  $\mathcal{B} = [-1, 1]$  and by  $\oplus$  one denotes the Minkowski sum of sets. The complexity of zonotopes is increasing with respect to the number of its generators  $m$  (the columns of  $\mathbf{M}$ ) and the dimension of the space  $n$ . Thus, based on (1), one can characterize a complex geometrical form by using one vector and one matrix. Notice, an interval vector or box of dimension  $m$  is

a degenerated zonotope where the generator matrix  $\mathbf{M}$  is a diagonal matrix or when  $m = 1$ .

Let  $\mathcal{Z}_1(\mathbf{p}_1, \mathbf{M}_1) = \mathbf{p}_1 \oplus \mathbf{M}_1 \mathcal{B}^{m_1}$  and  $\mathcal{Z}_2(\mathbf{p}_2, \mathbf{M}_2) = \mathbf{p}_2 \oplus \mathbf{M}_2 \mathcal{B}^{m_2}$  be two zonotopes in  $\mathbb{R}^n$ . The Minkowski sum of  $\mathcal{Z}_1$  and  $\mathcal{Z}_2$  is a zonotope given by

$$\mathcal{Z}_1(\mathbf{p}_1, \mathbf{M}_1) \oplus \mathcal{Z}_2(\mathbf{p}_2, \mathbf{M}_2) = \mathcal{Z}_3(\mathbf{p}_3, \mathbf{M}_3) \quad (2)$$

where  $\mathbf{p}_3 = \mathbf{p}_1 + \mathbf{p}_2$ ,  $\mathbf{M}_3 = [\mathbf{M}_1, \mathbf{M}_2]$  and  $m_3 = m_1 + m_2$ .

Now, let  $\mathcal{Z}(\mathbf{p}, \mathbf{M}) = \mathbf{p} \oplus \mathbf{M} \mathcal{B}^m$  be a zonotope in  $\mathbb{R}^n$  and  $\mathbf{A} \in \mathbb{R}^{n \times n}$  be a real matrix, the linear transformation of  $\mathcal{Z}$  by the matrix  $\mathbf{A}$  is a zonotope in  $\mathbb{R}^n$   $\mathcal{Z}_a(\mathbf{p}_a, \mathbf{M}_a)$  defined by

$$\mathcal{Z}_a(\mathbf{p}_a, \mathbf{M}_a) = \mathbf{A} \mathcal{Z}(\mathbf{p}, \mathbf{M}) \quad (3)$$

where  $\mathbf{p}_a = \mathbf{A} \mathbf{p}$  and  $\mathbf{M}_a = \mathbf{A} \mathbf{M}$ .

Let  $\mathcal{Z}(\mathbf{p}, \mathbf{M}) = \mathbf{p} \oplus \mathbf{M} \mathcal{B}^m$  be a zonotope in  $\mathbb{R}^n$ . The smallest box (interval vector) containing this zonotope is computed by:

$$\square \mathcal{Z}(\mathbf{p}, \mathbf{M}) = \mathbf{p} \oplus \mathbf{D}_M \mathcal{B}^n. \quad (4)$$

where  $\mathbf{D}_M$  is a diagonal matrix such that

$$(\mathbf{D}_M)_{ii} = \sum_{j=1}^m |(\mathbf{M})_{ij}|, \quad i = 1, \dots, n \quad (5)$$

More precisely, the endpoints of the box  $\square \mathcal{Z}(\mathbf{p}, \mathbf{M})$  can be computed as follows:

$$\begin{aligned} \overline{\mathcal{Z}} &= \mathbf{p} + \mathbf{D}_M \mathbf{1}_n \\ \underline{\mathcal{Z}} &= \mathbf{p} - \mathbf{D}_M \mathbf{1}_n \end{aligned} \quad (6)$$

where  $\mathbf{1}_n$  is a vector-column of dimension  $n$  and all its entries are equal to 1.

*Reduction criterion.* (Alamo et al., 2005)

Let  $\mathcal{Z}(\mathbf{p}, \mathbf{M}) = \mathbf{p} \oplus \mathbf{M} \mathcal{B}^m$  be a zonotope in  $\mathbb{R}^n$ . Consider an integer  $s$  such that  $n < s < m$  and denote by  $\hat{\mathbf{M}} = [\hat{M}_1, \hat{M}_2, \dots, \hat{M}_m]$  the matrix resulting from the permutation of the columns of  $\mathbf{M}$  in decreasing order of their euclidean norms. Then, the following inclusion holds:

$$\mathcal{Z}(\mathbf{p}, \mathbf{M}) \subseteq \mathbf{p} \oplus \mathbf{M}_r \mathcal{B}^s \quad (7)$$

where  $\mathbf{M}_r = [\hat{\mathbf{M}}_{s-n}, \mathbf{Q}]$ , with  $\hat{\mathbf{M}}_{s-n}$  is an horizontal concatenation of the first  $s - n$  columns of the matrix  $\hat{\mathbf{M}}$  and  $\mathbf{Q} \in \mathbb{R}^{n \times n}$  is a diagonal matrix computed as follows

$$(\mathbf{Q})_{ii} = \sum_{j=s-n+1}^m |(\hat{\mathbf{M}})_{ij}|, i = 1, \dots, n. \quad (8)$$

Note that, inclusion (7)-(8) allows one to reduce the complexity of zonotopes, especially in iterative numerical schemes. It manages the growth of the number of segments used to describe zonotopes, which is an important issue in the context of uncertain dynamical systems.

**Remark 1.** *In the literature there exist other zonotope order reduction methods, for instance interval hull method, parallelotope hull method and cascade reduction method, see (Le et al., 2013) and references therein.*

In what follows, for sake of simplicity, we denote by,

- *Size-Reduction*(.,.) a function that implements the reduction operator. That is,  $\mathbf{M}_r = \text{Size-Reduction}(\mathbf{M}, s)$ .
- *dim*( $\mathbf{M}$ ) a function that returns the number of columns of the matrix  $\mathbf{M}$ .
- *size*( $\mathcal{Z}$ ) a function that computes the volume of the smallest box containing the zonotope  $\mathcal{Z}$ .

### 2.1. Zonotopic reachability analysis method

Characterizing the reachable set of dynamical systems is a fundamental issue that arises in many engineering areas, especially when the safety requirement is crucial. The reachability problem consists in computing outer approximation of all the possible state trajectories generated from a given set of initial states and governed by uncertain dynamics. In the case of discrete-time linear systems described by,

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{E}\mathbf{w}_k \quad (9)$$

zonotopic set computation represents an efficient approach to solve this problem. Note that, the state disturbance  $\mathbf{w}_k$  and the initial state  $\mathbf{x}_0$  of (9) are

assumed to be unknown but belonging into bounded sets  $\mathcal{W}$  and  $\mathcal{X}_0$ , respectively. In other words, (9) can be considered as a difference inclusion. Therefore, there exist zonotopes that include these sets. That is,

$$\mathbf{x}_0 \in \mathcal{X}_0 \subseteq \mathcal{Z}_0(\mathbf{p}_0, \mathbf{M}_0) \subset \mathbb{R}^n \quad (10)$$

and there exists a centered zonotope  $\mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w)$  in  $\mathbb{R}^n$  with  $\mathbf{M}_w \in \mathbb{R}^{n \times m}$  such that,

$$\forall \mathbf{w}_k \in \mathcal{W}, \mathbf{E}\mathbf{w}_k \in \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w) \quad (11)$$

Thus, at every time instant  $k + 1$  the exact reachable set  $\mathcal{X}_{k+1}$  of (9) is included in the following zonotope

$$\mathcal{Z}_{k+1}(\mathbf{p}_{k+1}, \mathbf{M}_{k+1}) = \mathbf{p}_{k+1} \oplus \mathbf{M}_{k+1} \mathcal{B}^{m+r} \quad (12)$$

where

$$\begin{aligned} \mathbf{p}_{k+1} &= \mathbf{A}\mathbf{p}_k \\ \mathbf{M}_{k+1} &= [\mathbf{A}\mathbf{M}_k, \mathbf{M}_w] \end{aligned} \quad (13)$$

Note that, the direct application of the *sum of two zonotopes* and *Affine transformation of a zonotope* properties allows one to ensure,

$$\forall k \geq 0, \mathcal{X}_k \subseteq \mathcal{Z}_k(\mathbf{p}_k, \mathbf{M}_k)$$

where  $\mathcal{X}_k$  stands for the exact reachable set of (9) at the time instant  $k$  and  $\mathcal{Z}_k(\mathbf{p}_k, \mathbf{M}_k)$  is its outer approximation.

**Remark 2.** In (13) the size of the matrix  $\mathbf{M}_k$  is growing up with respect to  $k$ , which increases the complexity of this method. Thus, to manage this issue, the criterion-based reduction techniques is used.

**Remark 3.** For the sake of simplicity, we have assumed that the center of zonotope  $\mathcal{Z}_w$  is a zero vector. However, this method can be also applied for the case where the center of this zonotope is not equal to zero.

## 2.2. Explicit reachability method

This subsection presents the first contribution of this work. Unlike the classical reachability methods that apply set-valued computation to the dynamics (9), this work proposes to extend set-valued computation to the general solution of (9). That is, to the following analytic expression,

$$\mathbf{x}_k = \mathbf{A}^k \mathbf{x}_0 + \sum_{i=0}^{k-1} \mathbf{A}^{k-1-i} \mathbf{E}\mathbf{w}_i \quad (14)$$

This allows to avoid, by construction, the conservatism generated by the *wrapping effect* that appears in iterative numerical schemes, as those described in (9). Note that, by the *wrapping effect* one means the conservatism introduced at each iteration by over-approximating the exact solution set by a simpler form which is usually larger (Kühn, 1998).

**Proposition 1.** *Given the zonotopes  $\mathcal{Z}_0(\mathbf{p}_0, \mathbf{M}_0)$  and  $\mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w)$  defined in (10) and (11). Then, Algorithm 1 provides the vectors  $\mathbf{p}_k$  and the matrices  $\mathbf{M}_k$  such that:*

- for all  $k \geq 0$

$$\mathcal{X}_k \subseteq \mathcal{Z}_k(\mathbf{p}_k, \mathbf{M}_k) \quad (15)$$

where  $\mathcal{X}_k$  is the exact reachable set of (9), and

- for all  $\mathbf{x}_k \in \mathcal{X}_k$

$$\underline{\mathbf{x}}_k \leq \mathbf{x}_k \leq \bar{\mathbf{x}}_k \quad (16)$$

where  $\underline{\mathbf{x}}_k$  and  $\bar{\mathbf{x}}_k$  stand for the endpoints of the smallest box containing  $\mathcal{Z}_k(\mathbf{p}_k, \mathbf{M}_k)$ .

**Proof 1.** *On the one hand, based on the sum of two zonotopes and affine transformation of a zonotope properties, one can use the steps 1 to 3 of Algorithm 1 to define the following Minkowski sums,*

$$\begin{aligned} \mathcal{Z}_{k+1}(\mathbf{p}_{k+1}, \mathbf{M}_{k+1}) &= \mathbf{A}^{k+1} \mathcal{Z}_0(\mathbf{p}_0, \mathbf{M}_0) \oplus \mathcal{R}_k(\mathbf{0}, \mathbf{S}_k) \\ \mathcal{R}_{k+1}(\mathbf{0}, \mathbf{S}_{k+1}) &= \mathbf{A}^{k+1} \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w) \oplus \mathcal{R}_k(\mathbf{0}, \mathbf{S}_k) \end{aligned} \quad (17)$$

with  $\mathcal{R}_0(\mathbf{0}, \mathbf{S}_0) = \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w)$ . *On the other hand, by applying zonotopic set computation to (14), one obtains:*

$$\mathcal{Z}_{k+1}(\mathbf{p}_{k+1}, \mathbf{M}_{k+1}) = \mathbf{A}^{k+1} \mathcal{Z}_0(\mathbf{p}_0, \mathbf{M}_0) \oplus \sum_{i=0}^{k-1} \mathbf{A}^{k-1-i} \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w) \quad (18)$$

*Thus, one can state that all the possible solutions to (9) at  $k+1$  are included inside the zonotope computed by (18). Now, to complete this proof we have to show that (18) is equivalent to (17). To achieve that, first, by direct computation one can develop the second equation in (17) to obtain the following*



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**Algorithm 1:** Explicit-Reachability-Method

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- **Require:**  $\mathbf{p}_0, \mathbf{M}_0, \mathbf{M}_w, \mathbf{A}, s$
  - **Set:**  $\mathbf{S}_0 := \mathbf{M}_w$
  - **While**  $k \geq 0$  **do**
    1.  $\mathbf{p}_{k+1} := \mathbf{A}^{k+1} \mathbf{p}_0$
    2.  $\mathbf{M}_{k+1} := [\mathbf{A}^{k+1} \mathbf{M}_0, \mathbf{S}_k]$
    3.  $\mathbf{S}_{k+1} := [\mathbf{A}^{k+1} \mathbf{M}_w, \mathbf{S}_k]$
    4.  $\bar{\mathbf{x}}_{k+1} := \mathbf{p}_{k+1} + \mathbf{D}_{\mathbf{M}_{k+1}} \mathbf{1}_s$
    5.  $\underline{\mathbf{x}}_{k+1} := \mathbf{p}_{k+1} - \mathbf{D}_{\mathbf{M}_{k+1}} \mathbf{1}_s$
    6. **if**  $\dim(\mathbf{M}_{k+1}) > s$ 
      - $\mathbf{M}_{k+1} := \text{Size-Reduction}(\mathbf{M}_{k+1}, s)$
    7. **end**
    8. **if**  $\dim(\mathbf{S}_{k+1}) > s$ 
      - $\mathbf{S}_{k+1} := \text{Size-Reduction}(\mathbf{S}_{k+1}, s)$
    9. **end**
  - **end**
-

expression:

$$\begin{aligned}
\mathcal{R}_k(\mathbf{0}, \mathbf{S}_k) &= \mathbf{A}^k \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w) \oplus \mathcal{R}_{k-1}(\mathbf{0}, \mathbf{S}_{k-1}) \\
&= \mathbf{A}^k \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w) \oplus \mathbf{A}^{k-1} \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w) \oplus \mathcal{R}_{k-2}(\mathbf{0}, \mathbf{S}_{k-2}) \\
&\quad \vdots \\
&= \mathbf{A}^k \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w) \oplus \mathbf{A}^{k-1} \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w) \oplus \cdots \oplus \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w)
\end{aligned} \tag{19}$$

Then, by substituting  $\mathcal{R}_k(\mathbf{0}, \mathbf{S}_k)$  by its expression (19) in the first equation of (17) we obtain:

$$\begin{aligned}
\mathcal{Z}_{k+1}(\mathbf{p}_{k+1}, \mathbf{M}_{k+1}) &= \mathbf{A}^{k+1} \mathcal{Z}_0(\mathbf{p}_0, \mathbf{M}_0) \\
&\quad \oplus \mathbf{A}^k \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w) \oplus \mathbf{A}^{k-1} \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w) \oplus \cdots \oplus \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w) \\
&= \mathbf{A}^{k+1} \mathcal{Z}_0(\mathbf{p}_0, \mathbf{M}_0) \oplus \sum_{i=0}^{k-1} \mathbf{A}^{k-1-i} \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w)
\end{aligned} \tag{20}$$

That is exactly the zonotopic sum (18). Thus, one can affirm that the steps 1 to 3 of Algorithm 1 compute a center and a matrix shape of a zonotope that encloses all the possible solutions of (14).

Now, thanks to the interval hull properties (4), we have

$$\mathcal{X}_k \subseteq \mathcal{Z}_k(\mathbf{p}_k, \mathbf{M}_k) \subseteq \square \mathcal{Z}(\mathbf{p}_k, \mathbf{M}_k) = \mathbf{p}_k + \mathbf{D}_{M_k} \mathcal{B}^s \tag{21}$$

Thus, based on (21) one can state that the upper and lower bounds, of the actual state vector  $\mathbf{x}_k$ , computed in lines 4 and 5 of Algorithm 1 guarantee the double inequality (16). This ends the proof.  $\square$

In addition, it is worth pointing out that if the matrix  $\mathbf{A}$  is Schur stable, the proposed reachability method guarantees the convergence of the size of the zonotopes  $\mathcal{Z}_k(\mathbf{p}_k, \mathbf{M}_k)$  for a sufficiently large  $k$ .

**Proposition 2.** *If the matrix  $\mathbf{A}$  in (9) is Schur stable, then*

$$\lim_{k \rightarrow +\infty} \text{size}(\mathcal{Z}_k(\mathbf{p}_k, \mathbf{M}_k)) \leq \beta \tag{22}$$

where  $\beta$  is a positive real number.

**Proof 2.** By direct computation, we have:

$$\begin{aligned} \lim_{k \rightarrow +\infty} \text{size}(\mathcal{Z}_{k+1}(\mathbf{p}_{k+1}, \mathbf{M}_{k+1})) &\leq \lim_{k \rightarrow +\infty} \|\mathbf{A}^{k+1}\|_{\infty} \text{size}(\mathcal{Z}_0(\mathbf{p}_0, \mathbf{M}_0)) \\ &+ \lim_{k \rightarrow +\infty} \text{size}(\mathcal{R}_k(\mathbf{0}, \mathbf{S}_k)) \end{aligned} \quad (23)$$

where  $\|\cdot\|_{\infty}$  stands for the subordinate matrix infinity norm. On the other hand, since  $\mathbf{A}$  is assumed to be Schur stable, we obtain

$$\lim_{k \rightarrow +\infty} \|\mathbf{A}^k\|_{\infty} = 0 \quad (24)$$

Thus, from (23) and (24) we can state that:

$$\lim_{k \rightarrow +\infty} \text{size}(\mathcal{Z}_{k+1}(\mathbf{p}_{k+1}, \mathbf{M}_{k+1})) \leq \lim_{k \rightarrow +\infty} \text{size}(\mathcal{R}_k(\mathbf{0}, \mathbf{S}_k)) \quad (25)$$

Now, based on the same argument, we can show that for  $k \rightarrow +\infty$  we have:

$$\mathcal{R}_{k+1}(\mathbf{0}, \mathbf{S}_{k+1}) = \mathcal{R}_k(\mathbf{0}, \mathbf{S}_k) \quad (26)$$

which leads to conclude that the zonotopic sequence generated by the second equation in (17) has a fixed set or an invariant set. That is, at the steady state, the estimation error reaches a bounded set and stays in it. This allows one to claim that there exists a positive real number  $\beta$  such that:

$$\lim_{k \rightarrow +\infty} \text{size}(\mathcal{R}_k(\mathbf{0}, \mathbf{S}_k)) \leq \beta \quad (27)$$

This completes the proof.  $\square$

### 2.3. Comparison throughout a numerical example

To emphasize the effectiveness of the explicit reachability method to face the pessimism propagation, its performance is compared on a numerical example to that obtained by the iterative reachability method. Consider a discrete-time linear system (9), with

$$\mathbf{A} = \begin{pmatrix} 0.9 & 0.63 \\ -0.05 & 0.7 \end{pmatrix} \quad \text{and} \quad \mathbf{E} = \begin{pmatrix} 0.02 & -0.01 \\ -0.03 & 0.04 \end{pmatrix} \quad (28)$$

and the state disturbance vector  $\mathbf{w}_k$  is assumed unknown but belongs in the box  $\mathcal{B}^2$ . That is  $\forall k \geq 0$  one has:

$$\mathbf{w}_k^T \in [-1, 1] \times [-1, 1] \quad (29)$$

The initial domain of the state vector of this system is a box defined by,

$$\mathbf{x}_0 \in \mathcal{X}_0 = [-2, 2] \times [-2, 2] \quad (30)$$

To apply the zonotopic reachability methods (12)-(13) implemented in Algorithm 1, we need to define the following two initial zonotopes:

$$\begin{aligned} \mathcal{Z}_0(\mathbf{p}_0, \mathbf{M}_0) &\supseteq \mathcal{X}_0 \\ \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_w) &\ni \mathbf{E}\mathbf{w}_k, \forall k \geq 0 \end{aligned} \quad (31)$$

where  $\mathbf{p}_0 = \mathbf{0}$ ,  $\mathbf{M}_0 = 2\mathbf{I}_2$  and  $\mathbf{M}_w = \mathbf{E}$ . The simulation results are shown in Figure 1. The blue solid lines represent the upper and lower bounds of the state variables computed by the iterative method (12)-(13), while the green dashed lines correspond to the bounds computed by the explicit method, Algorithm 1. Note that these upper and lower bounds are obtained by the projection of the zonotopes based on the interval full method (4)-(6), steps 4 and 5 of Algorithm 1. The dotted lines represent the state trajectory of the system started from  $\mathbf{x}_0 = (1, 2)^T$  and driven by the disturbance vector  $\mathbf{w}_k = (\sin(0.2k), \sin(3k))^T$ . On the other hand, to limit the increasing dimension of the matrices  $\mathbf{M}_k$  and  $\mathbf{S}_k$  the criterion-based reduction method (7)-(8) is applied with  $s = 5$ .

These results show the advantage, in term of accuracy, of the proposed reachability method. In fact, as observed in Figure 1, at each time instant  $k$  the proposed method provides an outer approximation of the reachable set of the system included inside the outer approximation computed by the classical iterative method.

Note that, it is possible to improve the accuracy of the iterative method by increasing the order of the zonotopes. That is, we have to authorize using a larger number of segments  $s$ . For instance, for this example to obtain similar results to that generated by the explicit method, we have to run the iterative method with  $s$  larger than 50.

### 3. Problem formulation

#### 3.1. System description

This work addresses state estimation for discrete-time linear systems with unknown inputs. This class of systems can be modeled by the following state

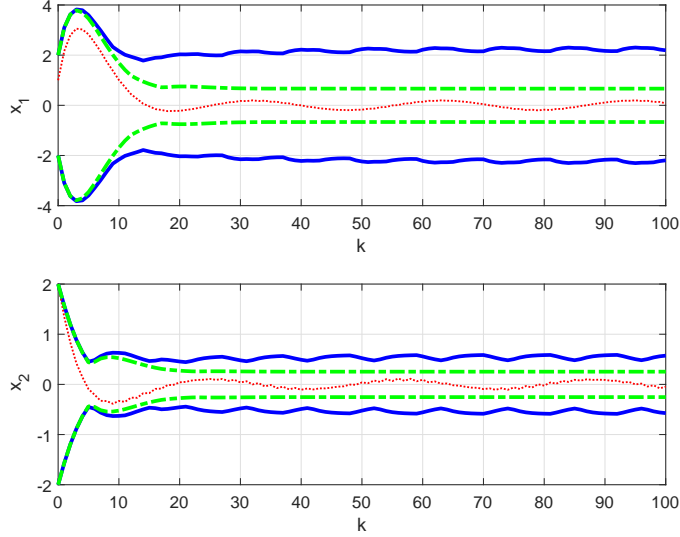


Figure 1: Outer approximation of the reachable set. Top figure shows the state enclosures of first state variable while the bottom figure corresponds to the second state variable.

space representation,

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{E}\mathbf{w}_k + \mathbf{H}\mathbf{d}_k \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \mathbf{F}\mathbf{v}_k \end{aligned} \quad (32)$$

where both state and output equations are considered uncertain. Notice that,  $\mathbf{x}_k \in \mathbb{R}^n$  is the state vector,  $\mathbf{y}_k \in \mathbb{R}^{n_y}$  is the output vector,  $\mathbf{u}_k \in \mathbb{R}^{n_u}$  is the known input vector while  $\mathbf{d}_k \in \mathbb{R}^{n_d}$  is the unknown input vector. The uncertain parts of this system are:

- State disturbance vector  $\mathbf{w}_k \in \mathbb{R}^{n_w}$  which could also include process noises and deterministic modeling errors. The only available information about this signal is its feasible domain  $\mathcal{W} \subset \mathbb{R}^{n_w}$ . That is,

$$\forall k \geq 0, \mathbf{w}_k \in \mathcal{W} \quad (33)$$

- Output error vector  $\mathbf{v}_k \in \mathbb{R}^{n_v}$  which could encompass measurement noises and deterministic sensor imperfections. Apart from its feasible domain  $\mathcal{V} \subset \mathbb{R}^{n_v}$ , there is no additional information about its evolution. That is,

$$\forall k \geq 0, \mathbf{v}_k \in \mathcal{V} \quad (34)$$

- The initial state  $\mathbf{x}_0$  is unknown but belongs into a given set  $\mathcal{X}_0$ . That is for  $k = 0$ ,

$$\mathbf{x}_0 \in \mathcal{X}_0 \quad (35)$$

The matrices  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ ,  $\mathbf{E}$ ,  $\mathbf{F}$  and  $\mathbf{H}$  are of appropriate dimensions. In addition, we assume that  $n_y \geq n_d$  and  $\mathbf{C}$  is a full row rank matrix and  $\mathbf{H}$  is a full column rank matrix. Note that, due to the nature of the considered uncertainties in (33)-(35), system (32) can be viewed as a difference inclusion.

### 3.2. State estimation problem

State estimation is an important problem in practice. Many control, diagnosis and monitoring methods are based on the online knowledge of the internal state of the system. This problem is especially problematic for the cases where the considered systems are uncertain, as in (32). That is, when some parts of the real world system are partially or completely unknown. However, in the literature, the concept of unknown input observer (Wang et al., 1978; Wang and Davison, 1978; Chen and Patton, 1999; Valcher, 1999; Hou and Müller, 1992; Maquin et al., 1994; Meditch and Hostetter, 1973; Patton et al., 1989) has been introduced to deal with state estimation in the presence of unknown inputs for linear systems without state disturbances and measurement noises. That is, systems that could be described by the following state space model:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{H}\mathbf{d}_k \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k \end{aligned} \quad (36)$$

This kind of state observers allows one reconstructing the internal state without demanding any information about the unknown input vector  $\mathbf{d}_k$ .

#### 3.2.1. Unknown input observer (ideal case)

For the class of discrete-time linear systems described by (36), the following dynamical system,

$$\begin{aligned} \mathbf{z}_{k+1} &= \mathbf{N}\mathbf{z}_k + \mathbf{G}\mathbf{B}\mathbf{u}_k + \mathbf{L}\mathbf{y}_k \\ \hat{\mathbf{x}}_k &= \mathbf{z}_k + \mathbf{K}\mathbf{y}_k \end{aligned} \quad (37)$$

is a structure for estimating the state vector, despite the presence of the unknown inputs. The design matrices of (37) are gathered in the following list (Chen and Patton (1999), Table 3.1, page 77):

- $\mathbf{K} = \mathbf{H}(\mathbf{C}\mathbf{H})^*$ , where  $(\mathbf{C}\mathbf{H})^*$  stands for the pseudo-inverse matrix of  $\mathbf{C}\mathbf{H}$ .
- $\mathbf{G} = (\mathbf{I}_n - \mathbf{K}\mathbf{C})$ , where  $\mathbf{I}_n$  is the identity matrix of dimension  $n$ .
- $\mathbf{A}_1 = \mathbf{G}\mathbf{A}$ .
- $\mathbf{L}_1$  is a gain matrix that can be computed, for instance, by pole placement techniques, LQE method or  $H_\infty$  synthesis method.
- $\mathbf{N} = \mathbf{A}_1 - \mathbf{L}_1\mathbf{C}$  is a Schur stable matrix.
- $\mathbf{L}_2 = \mathbf{N}\mathbf{K}$  and  $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$ .

As stated in (Chen and Patton (1999), Theorem 3.1, page 74), an unknown input observer (37) can be designed for (36) if and only if the following assumption holds.

**Assumption 1.** *The matrices  $\mathbf{H}$ ,  $\mathbf{C}$  and  $\mathbf{A}_1$  satisfy the following conditions:*

1.  $\text{rank}(\mathbf{C}\mathbf{H}) = \text{rank}(\mathbf{H})$
2.  $(\mathbf{C}, \mathbf{A}_1)$  is a detectable pair

### 3.2.2. The uncertain case

In the following, an unknown input observer is developed for the case of uncertain systems, where the noise and disturbance, as well as the initial state, belong into bounded sets.

**Assumption 2.** *(Boundedness) The sets  $\mathcal{W}$ ,  $\mathcal{V}$  and  $\mathcal{X}_0$  defined in (33), (34) and (35), respectively, are assumed to be bounded with known bounds.*

Thus, based on Assumption 2 the objective is to design an algorithm able to compute, **independently of the unknown input  $\mathbf{d}_k$** , at each time instant  $k$  a tight outer approximation of the **exact reachable set of (32) defined by:**

$$\mathcal{X}_{k+1} := \left\{ \begin{array}{l} \mathbf{x}_{k+1} \in \mathbb{R}^n, \\ \exists \mathbf{x}_k \in \mathcal{X}_k, \exists \mathbf{w}_k \in \mathcal{W}, \exists \mathbf{v}_{k+1} \in \mathcal{V} \\ \quad \quad \quad | \\ \mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k + \mathbf{E}\mathbf{w}_k + \mathbf{H}\mathbf{d}_k \\ \quad \quad \quad \mathbf{C}\mathbf{x}_{k+1} = \mathbf{y}_{k+1} - \mathbf{F}\mathbf{v}_{k+1} \end{array} \right\} \quad (38)$$

More precisely, the sought outer approximation or state enclosure, denoted by  $\mathcal{Z}_k$ , has to ensure these two properties:

- Framing property:

$$\forall k \geq 0, \mathcal{X}_k \subseteq \mathcal{Z}_k \quad (39)$$

That is, no consistent solution of this problem will be lost.

- Converging property:

$$\lim_{k \rightarrow +\infty} \text{size}(\mathcal{Z}_k) \leq c \quad (40)$$

where  $c$  is a positive constant, which can be viewed as an indicator of the tightness of the estimated state enclosure at the steady state.

To solve this problem, we propose a reliable set-membership estimation method. More precisely, based on the structure of the unknown input observer (37), a set-valued algorithm able to compute an enclosure for the set (38) with the framing and converging properties (39) and (40) is designed.

#### 4. Main result: Zonotopic unknown input state estimator

This section introduces the second contribution of this work. Based on Proposition 1 and Proposition 2, a zonotopic extension of the unknown input observer (37) is proposed. This extension allows one to be robust against the unknown but bounded state disturbances and measurements noises. Before introducing the main result, let us introduce some design parameters. By definition the initial estimation error is  $\mathbf{e}_0 = \mathbf{x}_0 - \hat{\mathbf{x}}_0$ , where  $\hat{\mathbf{x}}_0$  is the initial estimate state vector picked inside the zonotope  $\mathcal{Z}_0(\mathbf{p}_0, \mathbf{M}_0)$  that encloses the initial state domain  $\mathcal{X}_0$  of (32). Thus, the zonotope which includes all possible initial estimation errors is defined as follows:

$$\mathcal{E}_0(\mathbf{q}_0, \mathbf{M}_0) = -\hat{\mathbf{x}}_0 \oplus \mathcal{Z}_0(\mathbf{p}_0, \mathbf{M}_0) \quad (41)$$

where  $\mathbf{q}_0 = \mathbf{p}_0 - \hat{\mathbf{x}}_0$  is its center. On the other hand, denote by  $\mathbf{J}$  the following matrix,

$$\mathbf{J} = [(\mathbf{I}_n - \mathbf{K}\mathbf{C})\mathbf{E}, \quad -\mathbf{L}_1\mathbf{F}, \quad -\mathbf{K}\mathbf{F}] \quad (42)$$

and under the boundedness assumption of the signals  $\mathbf{w}_k$  and  $\mathbf{v}_k$ , we define the following zonotope which includes the feasible domain of the following vector:

$$\forall k \geq 0, \mathbf{J}\mathbf{h}_k \in \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_{vw}) \quad (43)$$



where  $\mathbf{h}_k = (\mathbf{w}_k^T, \mathbf{v}_k^T, \mathbf{v}_{k+1}^T)^T$ ,  $\mathbf{M}_{vw} = \mathbf{J}\mathbf{D}_h$  and  $\mathbf{D}_h$  is a diagonal matrix with appropriate diagonal elements.

**Proposition 3.** *Let Assumptions 1 and 2 hold. Then, Algorithm 2 provides the centers  $\mathbf{p}_k$  and the shape matrices  $\mathbf{M}_k$  of zonotopes enclosing the exact reachable set (38) of system (32). That is:*

- for all  $k \geq 0$

$$\mathcal{X}_k \subseteq \mathcal{Z}_k(\mathbf{p}_k, \mathbf{M}_k), \quad (44)$$

- for all  $\mathbf{x}_k \in \mathcal{X}_k$

$$\underline{\mathbf{x}}_k \leq \mathbf{x}_k \leq \bar{\mathbf{x}}_k. \quad (45)$$

In addition, since the matrix  $\mathbf{N}$  is Schur stable, there exists a positive real number  $\alpha$  such that:

$$\lim_{k \rightarrow +\infty} \text{size}(\mathcal{Z}_k(\mathbf{p}_k, \mathbf{M}_k)) \leq \alpha \quad (46)$$

**Proof 3.** *By direct computation we can describe the dynamics of the estimation error  $\mathbf{e}_k$  as follows:*

$$\mathbf{e}_{k+1} = \mathbf{N}\mathbf{e}_k + \mathbf{J}\mathbf{h}_k \quad (47)$$

Thus, based on steps 3, 5 and 6 of Algorithm 2 and on the results of Proposition 1 and Proposition 2, one can claim that the following zonotopic system,

$$\mathcal{E}_{k+1}(\mathbf{q}_{k+1}, \mathbf{M}_{k+1}) = \mathbf{N}^{k+1}\mathcal{E}_0(\mathbf{q}_0, \mathbf{M}_0) \oplus \mathcal{R}_k(\mathbf{0}, \mathbf{S}_k) \quad (48)$$

$$\mathcal{R}_{k+1}(\mathbf{0}, \mathbf{S}_{k+1}) = \mathbf{N}^{k+1}\mathcal{Z}_w(\mathbf{0}, \mathbf{M}_{vw}) \oplus \mathcal{R}_k(\mathbf{0}, \mathbf{S}_k)$$

provides converging zonotope sets that include all possible estimation errors. That is:

$$\forall k \geq 0, \mathbf{e}_k \in \mathcal{E}_k(\mathbf{q}_k, \mathbf{M}_k) \quad (49)$$

where  $\mathbf{q}_k$  and  $\mathbf{M}_k$  are the centers and the shape matrices of these zonotopes, respectively. Hence, based on (49) and the definition of the estimation error one can state that:

$$\forall k \geq 0, \mathbf{x}_k \in \hat{\mathbf{x}}_k + \mathcal{E}_k(\mathbf{q}_k, \mathbf{M}_k) = \mathcal{Z}_k(\mathbf{p}_k, \mathbf{M}_k) \quad (50)$$

---

**Algorithm 2:** Zonotopic-Unknown-Input-State-Estimator

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- **Require:**  $\mathbf{q}_0, \mathbf{M}_0, \mathbf{M}_{vw}, \mathbf{N}, \mathbf{B}, \mathbf{G}, \mathbf{L}, \mathbf{K}, s$
  - **Set:**  $\mathbf{S}_0 := \mathbf{M}_w, \mathbf{z}_0 := \hat{\mathbf{x}}_0 - \mathbf{K}\mathbf{y}_0$
  - **While**  $k \geq 0$  **do**
    1.  $\mathbf{z}_{k+1} := \mathbf{N}\mathbf{z}_k + \mathbf{G}\mathbf{B}\mathbf{u}_k + \mathbf{L}\mathbf{y}_k$
    2.  $\hat{\mathbf{x}}_{k+1} := \mathbf{z}_{k+1} + \mathbf{K}\mathbf{y}_{k+1}$
    3.  $\mathbf{q}_{k+1} := \mathbf{N}^{k+1}\mathbf{q}_0$
    4.  $\mathbf{p}_{k+1} := \hat{\mathbf{x}}_{k+1} + \mathbf{q}_{k+1}$
    5.  $\mathbf{M}_{k+1} := [\mathbf{N}^{k+1}\mathbf{M}_0, \mathbf{S}_{k+1}]$
    6.  $\mathbf{S}_{k+1} := [\mathbf{N}^{k+1}\mathbf{M}_{vw}, \mathbf{S}_{k+1}]$
    7.  $\bar{\mathbf{x}}_{k+1} := \mathbf{p}_{k+1} + \mathbf{D}_{M_{k+1}}\mathbf{1}_s$
    8.  $\underline{\mathbf{x}}_{k+1} := \mathbf{p}_{k+1} - \mathbf{D}_{M_{k+1}}\mathbf{1}_s$
    9. **if**  $\dim(\mathbf{M}_{k+1}) > s$ 
      - $\mathbf{M}_{k+1} := \text{Size-Reduction}(\mathbf{M}_{k+1}, s)$
    10. **end**
    11. **if**  $\dim(\mathbf{S}_{k+1}) > s$ 
      - $\mathbf{S}_{k+1} := \text{Size-Reduction}(\mathbf{S}_{k+1}, s)$
    12. **end**
  - **end**
-

and

$$\forall k \geq 0, \underline{\mathbf{x}}_k \leq \mathbf{x}_k \leq \bar{\mathbf{x}}_k \quad (51)$$

where  $\underline{\mathbf{x}}_k$  and  $\bar{\mathbf{x}}_k$  are computed in lines 7 and 8 of Algorithm 2. On the other hand, since by construction  $\mathbf{N}$  is Schur stable, one can state that:

$$\lim_{k \rightarrow +\infty} \text{size}(\mathcal{Z}_k(\mathbf{p}_k, \mathbf{M}_k)) = \lim_{k \rightarrow +\infty} \text{size}(\mathcal{E}_k(\mathbf{q}_k, \mathbf{M}_k)) \leq \alpha \quad (52)$$

This completes the proof.  $\square$

**Remark 4.** It is possible to improve the performance of the proposed set-valued state estimator by putting some columns of the matrix  $\mathbf{E}$  in the matrix  $\mathbf{H}$  without violating the conditions of Assumption 1. Doing that, allows to manage the effect of some unknown but bounded state disturbance on the estimation error.

## 5. Illustrative example

In this section, the performance of the proposed zonotopic unknown input state estimator is illustrated throughout a numerical example. The considered system has the form described in (32) and its matrices are defined as follows:

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} -0.5 & -0.5 & 0 \\ -1 & 0 & 0.8 \\ 0 & -0.9 & -0.3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \end{pmatrix} \\ \mathbf{H} &= \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix} \end{aligned} \quad (53)$$

The initial state of this system is unknown but belongs to a box of known bounds:

$$\mathbf{x}_0 \in \mathcal{X}_0 = \begin{pmatrix} [-5, 5] \\ [-5, 5] \\ [-5, 5] \end{pmatrix} \quad (54)$$

The process and measurement noises are poorly-known but the endpoints of their feasible domains are known:

$$\forall k \geq 0, \mathbf{w}_k \in [-1, 1] \quad (55)$$

$$\forall k \geq 0, \mathbf{v}_k \in \begin{pmatrix} [-1, 1] \\ [-1, 1] \end{pmatrix} \quad (56)$$

The known input of this system  $\mathbf{u}_k$  is given by the following function:

$$\mathbf{u}_k = 5 \cos(0.5k), \quad k \geq 0 \quad (57)$$

Note that, this system is also subject to a completely unknown input  $\mathbf{d}_k$ .

### 5.1. Design parameters of the zonotopic state estimator

For this example, the numerical values of the four matrices that define the classical unknown input observer (37) are:

$$\mathbf{N} = \begin{pmatrix} -0.35 & 0 & 0 \\ 0 & 0 & 0.0833 \\ 0 & -0.9 & -0.05 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (58)$$

$$\mathbf{L} = \begin{pmatrix} 0 & 0 \\ -1 & 0.7167 \\ 0 & -0.25 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Note that, the tuned eigenvalues for the matrix  $\mathbf{N}$  are:

$$\lambda_1 = -0.35, \quad \lambda_2 = -0.025 + 0.2727i, \quad \lambda_3 = -0.025 - 0.2727i \quad (59)$$

In addition, for this example, the matrix  $\mathbf{J}$  that represents the distribution of the uncertain input on the dynamics of the estimation error (47) is:

$$\mathbf{J} = \begin{pmatrix} 0 & -0.035 & 0 & -0.1 & 0 \\ 0.1 & 0.1 & -0.0717 & 0 & 0 \\ 0.1 & 0 & 0.025 & 0 & 0 \end{pmatrix} \quad (60)$$

On the other hand, the initial domains of the state vector and the feasible domains of the process and measurement noises affecting the dynamic of the estimation error can be represented by the following zonotope sets:

$$\mathcal{X}_0 = \mathcal{Z}_0(\mathbf{0}, \mathbf{M}_0) \quad (61)$$

$$\forall k \geq 0, \mathbf{Jh}_k \in \mathcal{Z}_w(\mathbf{0}, \mathbf{M}_{vw})$$

where  $\mathbf{M}_0 = 5\mathbf{I}_3$  and  $\mathbf{M}_{vw} = \mathbf{J}\mathbf{I}_5$ .

## 5.2. Simulation results

For the simulation purpose, we have assumed that:  $\mathbf{x}_0^T = (1, 1, 1)$ ,  $\mathbf{w}_k = \sin(0.1k)$  and  $\mathbf{v}_k^T = (\sin(100k), \sin(100k + \frac{\pi}{4}))$ . Moreover, the considered unknown input is defined by:

$$\mathbf{d}_k = \begin{cases} 0, & \text{if } k < 20 \\ 10, & \text{if } k \geq 20 \end{cases} \quad (62)$$

On the other hand, we set  $\hat{\mathbf{x}}_0^T = (4.5, -4.5, 4.5)$ . Thus, we obtain:

$$\mathbf{z}_0^T = (3.5, -4.5, 4.5) \quad (63)$$

and

$$\mathbf{q}_0^T = (-4.5, 4.5, -4.5) \quad (64)$$

For this example, the maximal number  $s$  of the authorized segments of the zonotopes is fixed to  $s = 20$ . This allows one to master the computational complexity of the proposed state estimator (Algorithm 2).

The simulation results are depicted in Figure 2. As shown, the proposed zonotopic unknown input state estimator provides a tight enclosure of the actual state vector of the system. The black tiny lines show the evolution of this state vector, whereas the blue solid lines correspond to the estimated state enclosure. The red dashed lines represent the point estimate of the actual state vector. Note that, the upper and lower bounds of the estimated state enclosure are computed by the interval hull projection method, lines 7 and 8 of Algorithm 2, applied at each iteration to the estimated zonotope.

In Figure 3, the width of the estimated intervals of each state variable are plotted. They stand for the distance between the estimated upper and lower state trajectory bound that are computed as follows:

$$e_{i,k} = \bar{x}_{i,k} - \underline{x}_{i,k}, \quad i = 1, 2, 3 \quad (65)$$

where  $\bar{x}_{i,k}$  is the upper bound of the  $i$ th state variable and  $\underline{x}_{i,k}$  is its lower bound. These bounds are computed by the interval hull method defined in (6). The convergence property of the proposed state estimator is well illustrated in Figure 3. In this figure, we observe that the vector of the estimation error converge at the steady state towards its equilibrium point

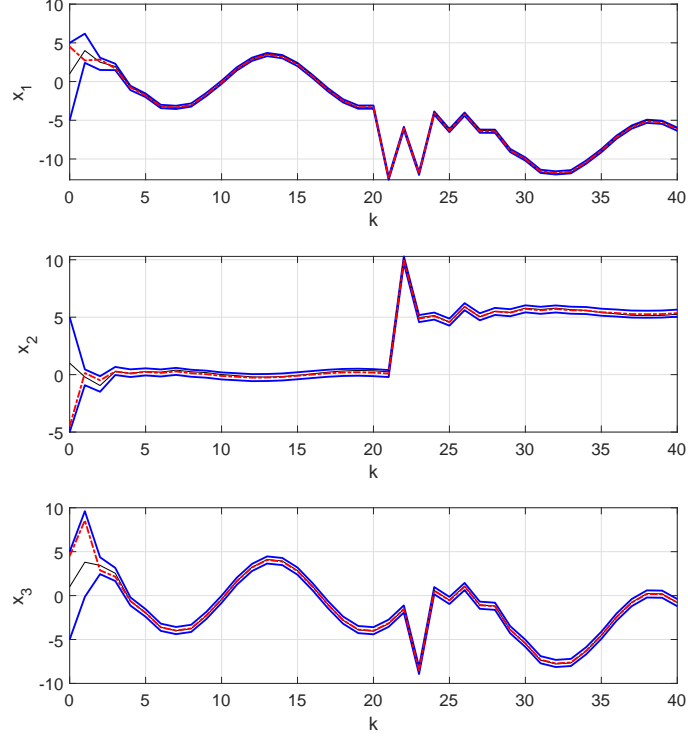


Figure 2: Estimates state enclosure. Solid lines stand for the upper and lower estimate bounds of the actual state vector plotted by the tiny lines. Dashed lines show the point estimate of the actual state vector.

$\mathbf{e}^* = (0.42, 0.61, 0.81)$ . Thus, from this equilibrium point one can determine the  $\alpha$  parameter of Proposition 3 as follows:

$$\alpha = e_1^* \cdot e_2^* \cdot e_3^* = 0.21 \quad (66)$$

Moreover, this figure shows that the estimation error is not impacted by the unknown input. In fact, at  $k = 20$  the unknown input switches from 0 to 10, but no variation is observed on the curves of the estimation error.

## 6. Comparison with some related works

### 6.1. Comparison with an interval observer method:

In this subsection we compare the simulation results of the proposed method with those obtained by the interval observer approach introduced in

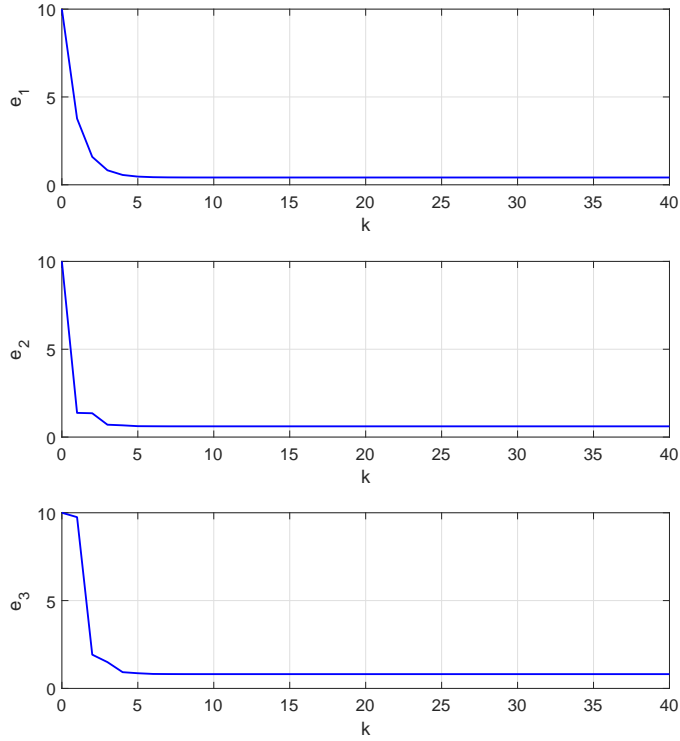


Figure 3: The widths of the estimated intervals of each state variable.

(Meyer et al., 2018). The considered system is borrowed from this reference and described by the following matrices:

$$\mathbf{A} = \begin{pmatrix} 0.3 & -0.7 \\ 0.6 & -0.5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{E} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{H} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{C} = (1 \ 0), \mathbf{F} = 1.$$

The system input vector is defined by  $\mathbf{u}_k = (\sin(0.1k), \cos(0.2k))^T$  and the bounded process and measurement noise are described by,

$$\begin{aligned} w_k &= \delta \sin(0.3k) \\ v_k &= \epsilon \sin(k) \end{aligned}$$

respectively. Parameters  $\delta$  and  $\epsilon$  are assumed unknown but belong into bounded intervals  $\delta \in [-0.5, 0.5]$  and  $\epsilon \in [-0.1, 0.1]$ . The considered unknown input of this system is given by,

$$d_k = \delta \sin(0.5k.x_{2k})$$

where  $x_{2k}$  is the second state variable of the system. In (Meyer et al., 2018)  $\mathbf{K} = (0, 0.56)^T$  is considered as an optimal observer gain. Based on such a gain value, the following matrices are chosen to reproduce the interval observer in (Meyer et al., 2018):

$$\mathbf{L}^+ = \begin{pmatrix} 0.3 & 0 \\ 0.04 & 0 \end{pmatrix}, \mathbf{L}^- = \begin{pmatrix} 0 & -0.7 \\ 0 & -0.5 \end{pmatrix},$$

$$\mathbf{M}^+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{M}^- = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathbf{K}^+ = \begin{pmatrix} 0 \\ 0.56 \end{pmatrix}, \mathbf{K}^- = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The simulation results of both methods are illustrated in Figure 4. As shown

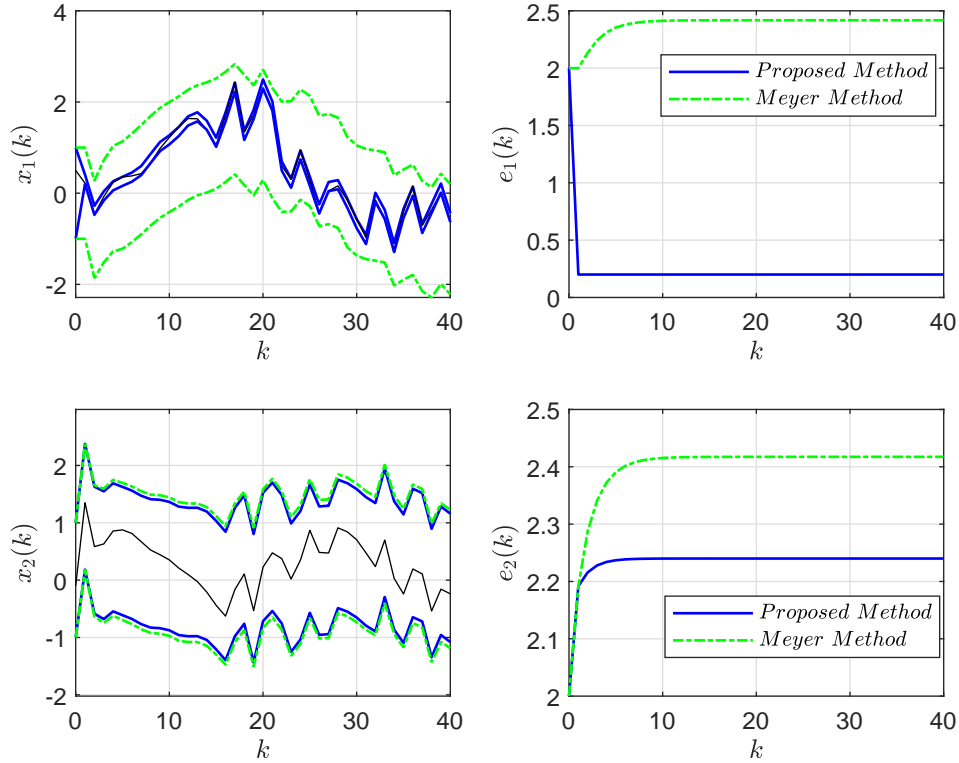


Figure 4: The estimated state enclosure computed by both methods. Continuous lines correspond to the results of the proposed method while dashed lines show those of Meyer's method.



in this figure, in terms of estimation performance the proposed method is better than that introduced in (Meyer et al., 2018). In fact, the interval estimate of each state variable provided by the proposed method is tighter than that computed by the interval observer method. This fact is illustrated in the right column in Figure 1, where the widths of the interval estimates of the state variables are plotted. **Notice that, these curves are considered as key performance indicators of conservatism.** However, in terms of computation cost interval observer method has got better performance. With a processor "Intel(R) Core(TM) i7-8650U CPU @ 1.90GHz" the used computational time is about 0.018s for the proposed method while it is about 0.0045s for the interval observer method.

Now, for completing this comparison, it is worth pointing out two major advantages of the proposed method with respect to interval observer design methods: (i) The restrictive positivity assumption on the dynamics of the estimation error is not needed. Thus, an optimal observer gain in terms of estimation performance can be applied without any restriction; (ii) A point estimate is computed in the same time with the lower and upper bounds of the estimation error.

### 6.2. Comparison with a zonotopic iterative estimator:

First, notice that the set-membership estimator introduced in (Xu et al., 2016) is based on a different structure of the unknown input observer used in the proposed methods. Thus, to be fair in the comparison, we have adapted our method to this different structure of unknown input observers. The considered example is that used in (Xu et al., 2016), where its matrices are defined as follows:

$$\mathbf{A} = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix},$$

$$\mathbf{E} = \begin{pmatrix} 0 & 0.3 & 0 \\ 0.4 & 0 & 0.5 \end{pmatrix}, \mathbf{H} = \begin{pmatrix} 0.5 \\ 0.2 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The input vector of this system is  $\mathbf{u}_k = (10 \sin(0.2k), 10 \sin(0.2k))^T$  and its process and measurement noise vectors are assumed unknown but bounded

signals

$$\mathbf{w}_k \in \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{pmatrix} \mathcal{B}^3$$

$$\mathbf{v}_k \in \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix} \mathcal{B}^2$$

Two simulation tests are performed. In the first one, the considered simulation conditions are taken from (Xu et al., 2016), where the dynamics of the estimation error is described by the real eigenvalues:

$$\lambda_1 = 0.3822, \lambda_2 = -0.0847$$

while in the second test, the case of complex eigenvalues is considered.

**Test 1:** The obtained simulation results are plotted in Figure 5. The width of the state enclosure of each variable provided by the proposed method is slightly lower than that generated by the iterative zonotope method. However, in terms of computational time the iterative method is slightly faster. Indeed, the proposed method spends 0.015s while the iterative method requires 0.01s. Note that, for the two cases, to limit the computational effort we have set  $s = 3$ .

**Test 2:** In this second test, complex eigenvalues for the dynamics of the estimation error are chosen. The considered eigenvalues are:

$$\lambda_1 = 0.5 + 0.3i, \lambda_2 = 0.5 - 0.3i$$

The simulation results are displayed in Figure 6. In this case, the proposed method provides tighter state enclosure than that given by the zonotopic iterative method. To show clearly this advantage we have plotted, in the second column of Figure 6, the width of the estimated interval of each state variable. On the other hand, we have the same remark about the computational time as that made in the first test.

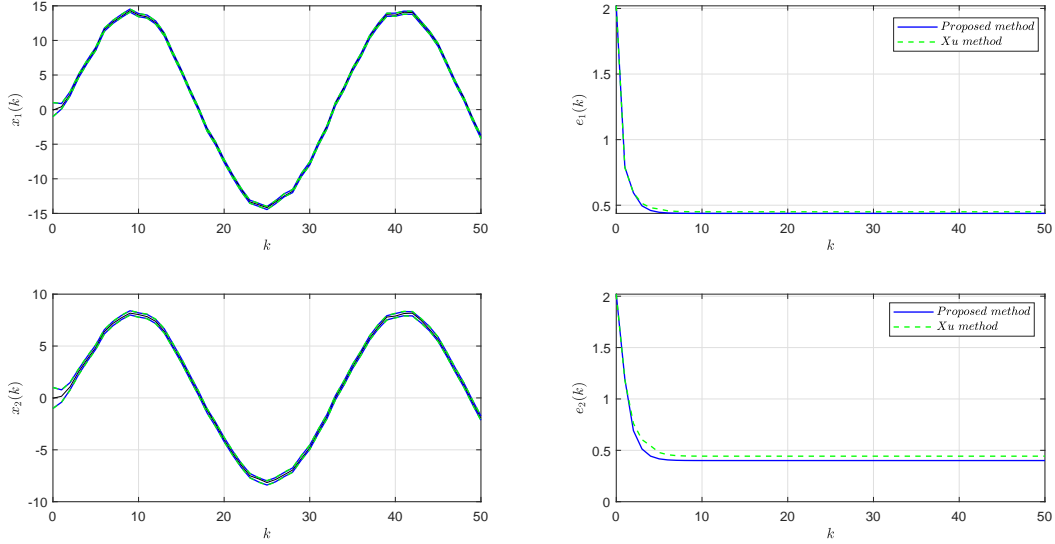


Figure 5: The estimated state enclosure computed by both methods. Continuous lines correspond to the results of the proposed method while dashed lines show those of Xu's method, Test 1.

## 7. Conclusions

A zonotopic extension of the classical unknown input observer for discrete-time linear systems has been proposed in this work. The state estimation problem is tackled in an unknown but bounded error context and a set-valued estimation algorithm is proposed to characterize, in a guaranteed way, all the possible solutions of the problem. This algorithm is based on a non-conservative reachability method applied to the dynamics of the estimation error. In addition, the convergence of this algorithm is shown and its efficiency is illustrated throughout numerical examples borrowed from the literature.

In forthcoming works, we aim at extending, this set-valued estimation methodology to a more complex class of uncertain systems. [For instance, to deal with the case of systems with parametric uncertainties, non-pessimistic algorithms for computing the powers of interval matrices should be designed.](#)

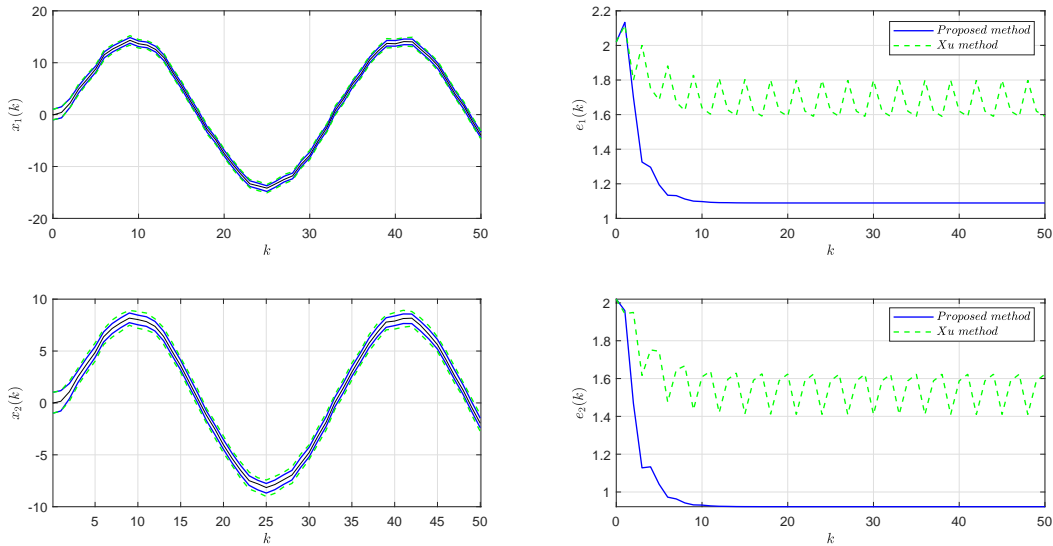


Figure 6: The estimated state enclosure computed by both methods. Continuous lines correspond to the results of the proposed method while dashed lines show those of Xu's method, Test 2.

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