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Inverse moving point source problem for the wave equation

HANIN AL JEBAWY¹, ABDELLATIF ELBADIA¹ AND FAOUZI TRIKI²

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Abstract

In this paper, we consider the problem of identifying a single moving point source for a three-dimensional wave equation from boundary measurements. Precisely, we show that the knowledge of the field generated by the source at six different points of the boundary over a finite time interval is sufficient to determine uniquely its trajectory. We also derive a Lipschitz stability estimate for the inversion.

1 Introduction

Inverse source problems are of importance in several scientific areas including biomedical engineering, antenna synthesis, geology, and medical imaging [1, 2, 3, 4, 5, 8, 9, 10, 14]. In this paper we consider the inverse source problem for the wave equation. Precisely, we study the problem of determining the trajectory of a moving source in a bounded domain from a single boundary measurement. Identification of sources with time-varying locations has many significant applications such as the recovery of mobile pollution sources, or small debris in low-earth orbit, and underwater sonar systems.

We assume here that media surrounding the point source is homogeneous and isotropic, and the measurement of the wave field is provided only on a small part of the boundary of the domain. Let ϕ to be the field generated by a single point source, that is, a solution of the following initial value problem for the three dimensional wave equation

$$\begin{cases} \frac{1}{c^2}\phi_{tt} - \Delta\phi = \lambda\delta(\mathbf{x} - b(t)) & \text{in } \mathbb{R}^3 \times (0, T) \\ \phi(\mathbf{x}, 0) = \phi_t(\mathbf{x}, 0) = 0 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.1)$$

where $T > 0$ is a fixed time, $c > 0$ is the speed of the wave, $\lambda > 0$ is the intensity, and $b \in C^2([0, T]; \mathbb{R}^3)$ is the position of the point source confined within a bounded

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domain $D \subset \mathbb{R}^3$. Let Ω be a smooth bounded domain satisfying $\overline{D} \subset \Omega$ with boundary Γ . Notice that the trajectory of the point source remains away from Γ . Define $G_b(T)$ to be the graph of the function b , that is, $G_b(T) = \{(s, b(s)); s \in [0, T]\}$. We also assume that source is subsonic, in other words, the speed of the source which is the first derivative of b , satisfies

$$\|b'\|_{C([0, T])} \leq c_0 < c. \quad (1.2)$$

The main goal of this paper is to reconstruct the trajectory followed by the source b by measuring ϕ on a part of the surface Γ .

There are few works dealing with inverse moving source problems for the wave equation, and different approaches have been used for solving it. In [7, 16], for example, the authors considered the problem with time-varying point sources, and they applied some algebraic direct methods for reconstructing these sources from a single boundary measurement, based on the concept of the reciprocity gap functional. Later, in [17] the author provided a generalization of this algorithm to the problem of moving sources, yet this method includes some difficulties concerning the assumptions on the trajectories followed by the sources, and no stability estimate has been provided. Another algebraic algorithm for the reconstruction of one moving source was given by [15] using the observed values of the retarded potential and all its derivatives at a single observation point. Other optimization techniques were also used in solving this problem, see for example [6, 12, 13, 18].

In our work, we are interested in identifying the trajectory $b(t)$ of a single moving point source $\delta(\cdot - b(t))$ with a known intensity $\lambda > 0$, from the measurement of the generated field ϕ at six well-chosen points $\mathbf{x}_i, i = 1, \dots, 6$, located on the observation surface Γ .

Throughout the paper we denote \cdot , and $|\cdot|$ the scalar product and the Euclidean norm respectively in \mathbb{R}^3 .

The unique solution $\phi \in C([0, T]; L^2_{\text{Loc}}(\mathbb{R}^3)) \cap C^1([0, T] \times (\mathbb{R}^3 \setminus D))$ of (1.1) is given by [11, 15, 16]

$$\phi(\mathbf{x}, t) = \frac{\lambda}{4\pi c} \frac{Y(r)}{|\mathbf{x} - b(r)| h(\mathbf{x}, r)}, \quad (1.3)$$

where Y denotes the Heaviside function, $r \in C([0, T] \times \overline{\Omega}) \cap C^1((([0, T] \times \overline{\Omega}) \setminus G_b(T)))$ is the unique solution to the equation

$$r(\mathbf{x}, t) = t - \frac{|\mathbf{x} - b(r)|}{c}, \quad (1.4)$$

for each fixed (\mathbf{x}, t) , and

$$h(\mathbf{x}, r) = 1 - \frac{b'(r) \cdot (\mathbf{x} - b(r))}{c|\mathbf{x} - b(r)|}.$$

We note that since $b \in C^2([0, T], \mathbb{R}^3)$ satisfies (1.2), we have

$$h \geq 1 - c^{-1} \|b'\|_{C([0, T])} := h_0 > 0. \quad (1.5)$$

Moreover, differentiating (1.4) in t , we deduce that

$$\frac{\partial r}{\partial t}(\mathbf{x}, t) = \frac{1}{h(\mathbf{x}, r)} > 0, \quad (1.6)$$

and so r is strictly increasing in t .

The objective of our work is to prove that a single observation ϕ on the surface Γ uniquely determines the source term b . Our strategy is to first prove that the boundary observation ϕ uniquely determines r , which in return allows us to reconstruct b using the relation (1.4). Our goal thus is to reconstruct r for different positions \mathbf{x} on Γ . First, we assume that

$$T > \sup_{\mathbf{x} \in \Gamma, \mathbf{y} \in D} \frac{|\mathbf{x} - \mathbf{y}|}{c} := T_0. \quad (1.7)$$

Since the wave field is propagating with a finite speed c , the assumption (1.7) is sufficient to allow the information on the source to arrive to the observation surface Γ .

Now, we take \mathbf{x} on Γ , fixed but arbitrary, and we define the time

$$t_{\mathbf{x}} = \sup\{t > 0; \phi(\mathbf{x}, t) = 0\}. \quad (1.8)$$

We note that from (1.3), and the definition of the Heaviside function, we have $\phi > 0$ for $r > 0$. Due to the initial conditions the set $\{t > 0; \phi(\mathbf{x}, t) = 0\}$ is not empty, and considering (1.4) we remark that $r(\mathbf{x}, t) > 0$ holds for t large enough. Thus, for T satisfying (1.7), we guarantee that $\phi(\mathbf{x}, \cdot)$ is not zero on $(0, T)$, and thus $t_{\mathbf{x}} \in (0, T)$.

On the other hand since r is continuous and strictly increasing in t , we deduce from (1.3) that $r(\mathbf{x}, t_{\mathbf{x}}) = 0$. Therefore, using the relation (1.4), we get

$$|\mathbf{x} - b(0)| = ct_{\mathbf{x}}. \quad (1.9)$$

Repeating the same procedure for different $\mathbf{x} \in \Gamma$, one can estimate the location of $b(0)$.

Our goal now is to reconstruct r for different \mathbf{x} on Γ , which allows us later to reconstruct b using the relation (1.4).

Our paper is organized as follows: First, we provide in section 2 an ODE based method for the reconstruction of r on Γ , then we reconstruct in section 3 the trajectory followed by the source $\{b(t); t \in [0, T]\}$ using the previously calculated values of r and the measurements of ϕ on six well chosen observation points on Γ . The uniqueness of the reconstruction is announced in Theorem 3.1. Finally, stability estimates are derived in section 4. The stability in the recovery of the trajectory is provided in Theorem 4.3.

2 Reconstruction of r

Let $\mathbf{x} \in \Gamma$, fixed but arbitrary. Since r satisfies (1.6), we deduce from (1.3) that for every $t > t_{\mathbf{x}}$, defined in (1.8), r satisfies the following equation

$$\begin{cases} \frac{\partial r}{\partial t}(\mathbf{x}, t) + \frac{4\pi c \phi(\mathbf{x}, t)}{\lambda} r = \frac{4\pi c \phi(\mathbf{x}, t)}{\lambda} t \\ r(\mathbf{x}, t_{\mathbf{x}}) = 0. \end{cases} \quad (2.10)$$

Since $\phi(\mathbf{x}, t)$ is given on the boundary our goal here is to solve (2.10) in r for different \mathbf{x} on Γ .

Remark 2.1 *We note that r is a strictly increasing function in t . Thus, as t varies between $t_{\mathbf{x}}$ and T we have $0 \leq r(\mathbf{x}, t) \leq r(\mathbf{x}, T)$, where $r(\mathbf{x}, T) = T - \frac{|\mathbf{x} - b(r(\mathbf{x}, T))|}{c}$. Therefore, in order to guarantee the reconstruction of $b(t)$ for $t \in (0, T)$, we may assume that our observations continue until a later time*

$$T_{obs} = T + T_0 \geq T + \sup_{\mathbf{x} \in \Gamma} \frac{|\mathbf{x} - b(T)|}{c}, \quad (2.11)$$

where T_0 is defined in (1.7).

For $\mathbf{x} \in \Gamma$ fixed, we consider the system (2.10) with $t \in (t_{\mathbf{x}}, T_{\mathbf{x}})$ where $T_{\mathbf{x}} = T + \frac{|\mathbf{x} - b(T)|}{c}$.

Proposition 2.2 *The system (2.10) has a unique solution $r \in C^1([t_{\mathbf{x}}, T_{\mathbf{x}}]; \mathbb{R}_+)$.*

Proof.

Since $\phi \in C^1([0, T_{obs}] \times (\mathbb{R}^3 \setminus D))$, the result is a direct consequence of Cauchy-Lipschitz Theorem. ■

Now, we are able to reconstruct $|\mathbf{x} - b(r(\mathbf{x}, t))|$ for every $t \in (t_{\mathbf{x}}, T_{\mathbf{x}})$. Since r is smooth and strictly increasing in t , thus we can reconstruct $|\mathbf{x} - b(r)|$ for every $r \in (0, T)$. Repeating the same procedure for finite number of positions \mathbf{x}_i on Γ , one can estimate the location of $b(r)$. This will be explained in details in the next section.

3 Reconstruction of b

The goal of this section is to reconstruct the trajectory followed by the source b using the previously calculated values of r on the observation surface Γ . Our algorithm requires the measurement of ϕ on well chosen observation points $\{\mathbf{x}_i\}_{1 \leq i \leq 6}$ on Γ .

First, we define for every $\mathbf{x}_i \in \Gamma$

$$t_{\mathbf{x}_i} = \frac{|\mathbf{x}_i - b(0)|}{c} \quad \text{and} \quad T_{\mathbf{x}_i} = T + \frac{|\mathbf{x}_i - b(T)|}{c}.$$

We assume in this section that the functions $r(\mathbf{x}_i, t)$ for $t \in (t_{\mathbf{x}_i}, T_{\mathbf{x}_i})$ are previously constructed. Now we take $\tau \in (0, T)$, fixed but arbitrary, and $\mathbf{x}_1 = (x_1, y_1, z_1) \in \Gamma$, then from the regularity of r we deduce that there exists $t_{1, \tau} \in (t_{\mathbf{x}_1}, T_{\mathbf{x}_1})$ such that

$$r(\mathbf{x}_1, t_{1, \tau}) = \tau.$$

Thus, we deduce from (1.4) that $b(\tau)$ satisfies

$$|\mathbf{x}_1 - b(\tau)| = c(t_{1,\tau} - \tau).$$

Therefore, $b(\tau)$ moves on a sphere of center \mathbf{x}_1 and radius $c(t_{1,\tau} - \tau)$, which implies that $b(\tau) = (b_1(\tau), b_2(\tau), b_3(\tau))$ satisfies the equation

$$(b_1(\tau) - x_1)^2 + (b_2(\tau) - y_1)^2 + (b_3(\tau) - z_1)^2 = c^2(t_{1,\tau} - \tau)^2. \quad (3.12)$$

Furthermore, taking $\mathbf{x}_2 = (x_2, y_2, z_2)$ another point on Γ , and repeating the previous procedure, we know that $b(\tau)$ moves on a sphere of center \mathbf{x}_2 and radius $c(t_{2,\tau} - \tau)$ for some $t_{2,\tau} \in (t_{\mathbf{x}_2}, T_{\mathbf{x}_2})$ that satisfies

$$r(\mathbf{x}_2, t_{2,\tau}) = \tau.$$

Therefore, $b(\tau)$ also satisfies

$$(b_1(\tau) - x_2)^2 + (b_2(\tau) - y_2)^2 + (b_3(\tau) - z_2)^2 = c^2(t_{2,\tau} - \tau)^2. \quad (3.13)$$

Subtracting (3.12) and (3.13) we get

$$2(x_2 - x_1)b_1(\tau) + 2(y_2 - y_1)b_2(\tau) + 2(z_2 - z_1)b_3(\tau) = |\mathbf{x}_2|^2 - |\mathbf{x}_1|^2 + c^2(t_{1,\tau} - \tau)^2 - c^2(t_{2,\tau} - \tau)^2. \quad (3.14)$$

Similarly, taking $\mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5$ and $\mathbf{x}_6 \in \Gamma$, we get

$$2(x_4 - x_3)b_1(\tau) + 2(y_4 - y_3)b_2(\tau) + 2(z_4 - z_3)b_3(\tau) = |\mathbf{x}_4|^2 - |\mathbf{x}_3|^2 + c^2(t_{3,\tau} - \tau)^2 - c^2(t_{4,\tau} - \tau)^2, \quad (3.15)$$

and

$$2(x_6 - x_5)b_1(\tau) + 2(y_6 - y_5)b_2(\tau) + 2(z_6 - z_5)b_3(\tau) = |\mathbf{x}_6|^2 - |\mathbf{x}_5|^2 + c^2(t_{5,\tau} - \tau)^2 - c^2(t_{6,\tau} - \tau)^2. \quad (3.16)$$

Equations (3.14), (3.15), and (3.16) can be rewritten in the matrix form

$$XB = \frac{1}{2}A,$$

where

$$X = \begin{pmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_4 - x_3 & y_4 - y_3 & z_4 - z_3 \\ x_6 - x_5 & y_6 - y_5 & z_6 - z_5 \end{pmatrix}, \quad (3.17)$$

$$B = \begin{pmatrix} b_1(\tau) \\ b_2(\tau) \\ b_3(\tau) \end{pmatrix},$$

and

$$A = \begin{pmatrix} A_{12} \\ A_{34} \\ A_{56} \end{pmatrix}, \quad (3.18)$$

with

$$A_{ij} = |\mathbf{x}_j|^2 - |\mathbf{x}_i|^2 + c^2(t_{i,\tau} - \tau)^2 - c^2(t_{j,\tau} - \tau)^2. \quad (3.19)$$

Now, since Γ is a boundary of a connected domain one can choose the observation points $\{\mathbf{x}_i\}_{1 \leq i \leq 6}$ such that X becomes invertible, then B can be reconstructed as

$$B = \frac{1}{2}X^{-1}A. \quad (3.20)$$

Notice that X depends only on Γ . Finally, we give a brief summary of the reconstruction of b that passes through five main steps.

Step 1. Choose $\{\mathbf{x}_i\}_{1 \leq i \leq 6}$ on Γ such that the matrix X defined in (3.17) is invertible.

Step 2. Calculate the values of $t_{\mathbf{x}_i}$ given by

$$t_{\mathbf{x}_i} = \sup\{t > 0; \phi(\mathbf{x}_i, t) = 0\}.$$

Step 3. Construct $r(\mathbf{x}_i, t)$ as solutions of the equations (2.10).

Step 4. For every $\tau \in (0, T)$, evaluate $t_{i,\tau}$ given by

$$r(\mathbf{x}_i, t_{i,\tau}) = \tau.$$

Step 5. Construct the matrix A given by (3.18), and finally evaluate $b(\tau)$ using the relation (3.20).

Then, we have the following result.

Theorem 3.1 *Let $\{\mathbf{x}_i\}_{1 \leq i \leq 6}$ be fixed points on Γ chosen such that the matrix $X = (\mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_4 - \mathbf{x}_3, \mathbf{x}_6 - \mathbf{x}_5)^T$ is invertible. Then the knowledge of $\phi(\mathbf{x}_i, t)$, $i = 1, \dots, 6$ for $t \in [0, T_{obs}]$ where $T_{obs} = T + T_0$, determines uniquely the trajectory of the point source, that is $\{b(t); t \in [0, T]\}$.*

4 Stability estimates

In this section we give a stability estimate for the reconstruction of b from the measurements of ϕ on the observation points $\{\mathbf{x}_i\}_{1 \leq i \leq 6}$. Our work is divided into three steps. In fact, we notice from the previous section that the reconstruction of b passes through three main stages: the reconstruction of $t_{\mathbf{x}_i}$, $r(\mathbf{x}_i, t)$ and finally the trajectory $b(t)$. For this purpose, we first give the stability estimates for $t_{\mathbf{x}_i}$ and r , then we deduce that of the source b .

Recall that $\Omega \subset \mathbb{R}^3$ is a bounded domain of boundary Γ , satisfying $\bar{D} \subset \Omega$. We consider as in the previous section a set of observation points $\{\mathbf{x}_i\}_{1 \leq i \leq 6}$ such that the matrix X defined in (3.17) is invertible, and we suppose that we have two observations ϕ and $\tilde{\phi}$ corresponding respectively to two trajectories b and \tilde{b} . Then, we establish the following stability estimates.

Theorem 4.1 (Stability estimate of $t_{\mathbf{x}_i}$) *Let $\{\mathbf{x}_i\}_{1 \leq i \leq 6} \in \Gamma$ such that the matrix (3.17) is invertible, and take $t_{\mathbf{x}_i}$ and $\tilde{t}_{\mathbf{x}_i}$ defined in (1.8) corresponding to the observations ϕ and $\tilde{\phi}$ respectively. Then, the following estimate holds.*

$$|t_{\mathbf{x}_i} - \tilde{t}_{\mathbf{x}_i}| \leq \frac{8\pi T_{obs} \text{diam}(\Omega)}{\lambda} \|\tilde{\phi}(\mathbf{x}_i, \cdot) - \phi(\mathbf{x}_i, \cdot)\|_{L^\infty(0, T_{obs})}, \quad i = 1, \dots, 6, \quad (4.21)$$

where $T_{obs} = T + T_0$, with T_0 defined in (1.7).

Proof.

We assume without loss of generality that $\tilde{t}_{\mathbf{x}_i} < t_{\mathbf{x}_i}$, then we have

$$\begin{aligned} \int_{\tilde{t}_{\mathbf{x}_i}}^{t_{\mathbf{x}_i}} \tilde{\phi}(\mathbf{x}_i, s) ds &= \int_{\tilde{t}_{\mathbf{x}_i}}^{t_{\mathbf{x}_i}} \tilde{\phi}(\mathbf{x}_i, s) - \phi(\mathbf{x}_i, s) ds \\ &\leq \int_{\tilde{t}_{\mathbf{x}_i}}^{t_{\mathbf{x}_i}} |\tilde{\phi}(\mathbf{x}_i, s) - \phi(\mathbf{x}_i, s)| ds \\ &\leq \int_0^{T_{\text{obs}}} |\tilde{\phi}(\mathbf{x}_i, s) - \phi(\mathbf{x}_i, s)| ds \\ &\leq T_{\text{obs}} \|\tilde{\phi}(\mathbf{x}_i, \cdot) - \phi(\mathbf{x}_i, \cdot)\|_{L^\infty(0, T_{\text{obs}})}. \end{aligned}$$

Therefore,

$$\min \tilde{\phi}(\mathbf{x}_i, \cdot) (t_{\mathbf{x}_i} - \tilde{t}_{\mathbf{x}_i}) \leq T_{\text{obs}} \|\tilde{\phi}(\mathbf{x}_i, \cdot) - \phi(\mathbf{x}_i, \cdot)\|_{L^\infty(0, T_{\text{obs}})}.$$

Furthermore, for every $t > \tilde{t}_{\mathbf{x}_i}$ we have

$$\tilde{\phi} = \frac{\lambda}{4\pi |\mathbf{x}_i - \tilde{b}(\tilde{r})| \tilde{h}(\mathbf{x}_i, \tilde{r})},$$

where, $\tilde{h}(\mathbf{x}_i, \tilde{r}) = 1 - \frac{\tilde{b}'(\tilde{r}) \cdot (\mathbf{x}_i - \tilde{b}(\tilde{r}))}{c|\mathbf{x}_i - \tilde{b}(\tilde{r})|} \leq 1 + \frac{\|\tilde{b}'\|_\infty}{c}$. Moreover, since $\|\tilde{b}'\|_\infty < c$, we deduce that $\tilde{h}(\mathbf{x}_i, \tilde{r}) < 2$. Therefore,

$$\tilde{\phi} \geq \frac{\lambda}{8\pi \text{diam}(\Omega)}.$$

Finally, we obtain

$$(t_{\mathbf{x}_i} - \tilde{t}_{\mathbf{x}_i}) \leq \frac{8\pi T_{\text{obs}} \text{diam}(\Omega)}{\lambda} \|\tilde{\phi}(\mathbf{x}_i, \cdot) - \phi(\mathbf{x}_i, \cdot)\|_{L^\infty(0, T_{\text{obs}})}.$$

■

Theorem 4.2 (Stability estimate for $r(\mathbf{x}_i, \cdot)$) *Let $\{\mathbf{x}_i\}_{1 \leq i \leq 6} \in \Gamma$ such that the matrix (3.17) is invertible, and take $r(\mathbf{x}_i, t)$ and $\tilde{r}(\mathbf{x}_i, t)$ solutions of (2.10) corresponding to the observations ϕ and $\tilde{\phi}$ respectively with $r(\mathbf{x}_i, t_{\mathbf{x}_i}) = 0$ and $\tilde{r}(\mathbf{x}_i, \tilde{t}_{\mathbf{x}_i}) = 0$. Choose $t_{0,i} = \min\{t_{\mathbf{x}_i}, \tilde{t}_{\mathbf{x}_i}\}$, then, the following estimate holds:*

$$\|r(\mathbf{x}_i, \cdot) - \tilde{r}(\mathbf{x}_i, \cdot)\|_{L^\infty(t_{0,i}, T_{\text{obs}})} \leq C \|\phi - \tilde{\phi}\|_{L^\infty(0, T_{\text{obs}})},$$

for some $C = C(\Omega, \lambda, T, c, c_0, D) > 0$.

Proof.

Further C denotes a generic strictly positive constant that depends on $(\Omega, \lambda, T, c, c_0, D)$, and which may be different from line to line.

Let $\mathbf{x}_i \in \Gamma$ for $1 \leq i \leq 6$. Without loss of generality, we assume that $t_{\mathbf{x}_i} \geq \tilde{t}_{\mathbf{x}_i}$, our proof then is divided into two steps. First, we consider the case $t > t_{\mathbf{x}_i}$, then we have

$$\partial_t(r(\mathbf{x}_i, t) - \tilde{r}(\mathbf{x}_i, t)) = \frac{4\pi c}{\lambda} (\phi(\mathbf{x}_i, t) - \tilde{\phi}(\mathbf{x}_i, t))t - \frac{4\pi c}{\lambda} \phi(\mathbf{x}_i, t)r(\mathbf{x}_i, t) + \frac{4\pi c}{\lambda} \tilde{\phi}(\mathbf{x}_i, t)\tilde{r}(\mathbf{x}_i, t).$$

Thus,

$$\partial_t(r(\mathbf{x}_i, t) - \tilde{r}(\mathbf{x}_i, t)) = \frac{4\pi c}{\lambda}(\phi - \tilde{\phi})t - \frac{4\pi c}{\lambda}\phi(r - \tilde{r}) - \frac{4\pi c}{\lambda}(\phi - \tilde{\phi})\tilde{r}.$$

Integrating between $t_{\mathbf{x}_i}$ and t , we get

$$\begin{aligned} r(\mathbf{x}_i, t) - \tilde{r}(\mathbf{x}_i, t) &= \tilde{r}(\mathbf{x}_i, \tilde{t}_{\mathbf{x}_i}) - \tilde{r}(\mathbf{x}_i, t_{\mathbf{x}_i}) \\ &\quad + \frac{4\pi c}{\lambda} \int_{t_{\mathbf{x}_i}}^t (\phi - \tilde{\phi})s \, ds - \frac{4\pi c}{\lambda} \int_{t_{\mathbf{x}_i}}^t \phi(r - \tilde{r}) \, ds - \frac{4\pi c}{\lambda} \int_{t_{\mathbf{x}_i}}^t (\phi - \tilde{\phi})\tilde{r} \, ds. \end{aligned}$$

Therefore,

$$\begin{aligned} |r(\mathbf{x}_i, \mathbf{t}) - \tilde{r}(\mathbf{x}_i, t)| &\leq \|\tilde{r}'\|_{L^\infty(0, T_{\text{obs}})} |\tilde{t}_{\mathbf{x}_i} - t_{\mathbf{x}_i}| + \frac{4\pi c}{\lambda} T_{\text{obs}} \|\phi - \tilde{\phi}\|_{L^\infty(0, T_{\text{obs}})} (t - t_{\mathbf{x}_i}) \\ &\quad + \frac{4\pi c}{\lambda} \|\phi\|_{L^\infty(0, T_{\text{obs}})} \int_{t_{\mathbf{x}_i}}^t r - \tilde{r} \, ds + \frac{4\pi c}{\lambda} \|\tilde{r}\|_{L^\infty(0, T_{\text{obs}})} \|\phi - \tilde{\phi}\|_{L^\infty(0, T_{\text{obs}})} (t - t_{\mathbf{x}_i}). \end{aligned}$$

Using (4.21), we deduce that

$$\begin{aligned} |r(\mathbf{x}_i, \mathbf{t}) - \tilde{r}(\mathbf{x}_i, t)| &\leq \frac{8\pi T_{\text{obs}} \text{diam}(\Omega)}{\lambda h_0} \|\phi - \tilde{\phi}\|_{L^\infty(0, T_{\text{obs}})} \\ &\quad + \frac{8\pi c}{\lambda} T_{\text{obs}}^2 \|\phi - \tilde{\phi}\|_{L^\infty(0, T_{\text{obs}})} + \frac{c}{\text{dist}(\Gamma, D) h_0} \int_{t_{\mathbf{x}_i}}^t r - \tilde{r} \, ds, \end{aligned}$$

where h_0 is defined in (1.5). Therefore,

$$|r(\mathbf{x}_i, \mathbf{t}) - \tilde{r}(\mathbf{x}_i, t)| \leq C \|\phi - \tilde{\phi}\|_{L^\infty(0, T_{\text{obs}})} + \frac{c}{\text{dist}(\Gamma, D) h_0} \int_{t_{\mathbf{x}_i}}^t r - \tilde{r} \, ds.$$

Applying Gronwall's Lemma we deduce that

$$\begin{aligned} |r(\mathbf{x}_i, \mathbf{t}) - \tilde{r}(\mathbf{x}_i, t)| &\leq C \|\phi - \tilde{\phi}\|_{L^\infty(0, T_{\text{obs}})} e^{\frac{c}{\text{dist}(\Gamma, D) h_0} (t - t_{\mathbf{x}_i})} \\ &\leq C e^{\frac{c T_{\text{obs}}}{\text{dist}(\Gamma, D) h_0}} \|\phi - \tilde{\phi}\|_{L^\infty(0, T_{\text{obs}})}. \end{aligned} \tag{4.22}$$

The second case is for $t \in (\tilde{t}_{\mathbf{x}_i}, t_{\mathbf{x}_i})$, then in this case we choose $t^* > t_{\mathbf{x}_i}$ such that

$$|t^* - t| \leq |t_{\mathbf{x}_i} - \tilde{t}_{\mathbf{x}_i}|.$$

Then, we get

$$\begin{aligned} |\tilde{r}(\mathbf{x}_i, \mathbf{t}) - r(\mathbf{x}_i, t)| &\leq |\tilde{r}(\mathbf{x}_i, \mathbf{t}) - \tilde{r}(\mathbf{x}_i, t^*)| + |\tilde{r}(\mathbf{x}_i, t^*) - r(\mathbf{x}_i, t^*)| + |r(\mathbf{x}_i, t^*) - r(\mathbf{x}_i, t)| \\ &\leq \|\tilde{r}'\|_{L^\infty(0, T_{\text{obs}})} |t^* - t| + |\tilde{r}(\mathbf{x}_i, t^*) - r(\mathbf{x}_i, t^*)| + \|\tilde{r}'\|_{L^\infty(0, T_{\text{obs}})} |t^* - t| \\ &\leq \frac{2}{h_0} |t_{\mathbf{x}_i} - \tilde{t}_{\mathbf{x}_i}| + |\tilde{r}(\mathbf{x}_i, t^*) - r(\mathbf{x}_i, t^*)|. \end{aligned}$$

Therefore, following (4.22), and the results of Theorem 4.1, we finally deduce that

$$|\tilde{r}(\mathbf{x}_i, \mathbf{t}) - r(\mathbf{x}_i, t)| \leq \left(\frac{16\pi T_{\text{obs}} \text{diam}(\Omega)}{\lambda h_0} + C e^{\frac{c T_{\text{obs}}}{\text{dist}(\Gamma, D) h_0}} \right) \|\phi - \tilde{\phi}\|_{L^\infty(0, T_{\text{obs}})}.$$

■

Theorem 4.3 (Stability estimate for \mathbf{b}) Consider the two sources trajectories b and \tilde{b} with two different observations ϕ and $\tilde{\phi}$ at the points $\{\mathbf{x}_i\}_{1 \leq i \leq 6} \in \Gamma$, such that the matrix (3.17) is invertible, then the following stability estimate holds

$$\|b - \tilde{b}\|_{L^\infty(0,T)} \leq C \sup_{i=1,3,5} \left(\|\tilde{\phi}(\mathbf{x}_i, \cdot) - \phi(\mathbf{x}_i, \cdot)\|_{L^\infty(0,T_{obs})} + \|\tilde{\phi}(\mathbf{x}_{i+1}, \cdot) - \phi(\mathbf{x}_{i+1}, \cdot)\|_{L^\infty(0,T_{obs})} \right), \quad (4.23)$$

for some $C = C(\Omega, \lambda, T, c, c_0, D) > 0$.

Proof.

C denotes a generic strictly positive constant that depends on $(\Omega, \lambda, T, c, c_0, D)$, and which may be different from line to line.

Let $\tau \in (0, T)$, then following the work done in section 3 we deduce that

$$b(\tau) - \tilde{b}(\tau) = \frac{1}{2} X^{-1} (A - \tilde{A}), \quad (4.24)$$

where X is defined in (3.17), and the matrices A and \tilde{A} are defined in (3.18) with observations ϕ and $\tilde{\phi}$ respectively. Moreover, following (3.19), we obtain

$$\begin{aligned} A_{i \ i+1} - \tilde{A}_{i \ i+1} &= c^2 [(t_{i,\tau} - \tau)^2 - (\tilde{t}_{i,\tau} - \tau)^2] - c^2 [(t_{i+1,\tau} - \tau)^2 - (\tilde{t}_{i+1,\tau} - \tau)^2] \\ &= c^2 (t_{i,\tau} - \tilde{t}_{i,\tau})(t_{i,\tau} + \tilde{t}_{i,\tau} - 2\tau) - c^2 (t_{i+1,\tau} - \tilde{t}_{i+1,\tau})(t_{i+1,\tau} + \tilde{t}_{i+1,\tau} - 2\tau), \end{aligned}$$

for $i = 1, 3, 5$. Furthermore, knowing that

$$r(\mathbf{x}_j, t_{j,\tau}) = \tilde{r}(\mathbf{x}_j, \tilde{t}_{j,\tau}) = \tau \quad \forall 1 \leq j \leq 6,$$

we get

$$\begin{aligned} t_{j,\tau} - \tilde{t}_{j,\tau} &= t_{j,\tau} - \tilde{r}^{-1}(\mathbf{x}_j, \tau) \\ &= t_{j,\tau} - \tilde{r}^{-1}(\mathbf{x}_j, r(\mathbf{x}_j, t_{j,\tau})) \\ &= \tilde{r}^{-1}(\mathbf{x}_j, \tilde{r}(\mathbf{x}_j, t_{j,\tau})) - \tilde{r}^{-1}(\mathbf{x}_j, r(\mathbf{x}_j, t_{j,\tau})) \\ &= \tilde{r}^{-1}(\mathbf{x}_j, \tilde{r}(\mathbf{x}_j, t_{j,\tau}) - r(\mathbf{x}_j, t_{j,\tau})). \end{aligned}$$

Therefore,

$$|t_{j,\tau} - \tilde{t}_{j,\tau}| \leq \|(\tilde{r}^{-1})'\|_{L^\infty(0,T)} |\tilde{r}(\mathbf{x}_j, t_{j,\tau}) - r(\mathbf{x}_j, t_{j,\tau})|.$$

Moreover, we deduce from (1.4) that $(\tilde{r}^{-1})' = \tilde{h}(\mathbf{x}_j, \tilde{r})$, which implies that

$$\begin{aligned} |t_{j,\tau} - \tilde{t}_{j,\tau}| &\leq \|\tilde{h}(\mathbf{x}_j, \cdot)\|_{L^\infty(0,T)} |\tilde{r}(\mathbf{x}_j, t_{j,\tau}) - r(\mathbf{x}_j, t_{j,\tau})| \\ &\leq 2 \|\tilde{r}(\mathbf{x}_j, \cdot) - r(\mathbf{x}_j, \cdot)\|_{L^\infty(0,T_{obs})}. \end{aligned}$$

Following the results of Theorem 4.2, we finally deduce that

$$t_{j,\tau} - \tilde{t}_{j,\tau} \leq C \|\tilde{\phi}(\mathbf{x}_j, \cdot) - \phi(\mathbf{x}_j, \cdot)\|_{L^\infty(0,T_{obs})}.$$

Therefore,

$$A_{i+1} - \tilde{A}_{i+1} \leq 2c^2(T_{\text{obs}} - \tau)C \times \left(\|\tilde{\phi}(\mathbf{x}_i, \cdot) - \phi(\mathbf{x}_i, \cdot)\|_{L^\infty(0, T_{\text{obs}})} + \|\tilde{\phi}(\mathbf{x}_{i+1}, \cdot) - \phi(\mathbf{x}_{i+1}, \cdot)\|_{L^\infty(0, T_{\text{obs}})} \right).$$

Finally, we deduce from (4.24) that $\forall \tau \in (0, T)$

$$|b(\tau) - \tilde{b}(\tau)| \leq c^2(T_{\text{obs}} - \tau)C \|X^{-1}\|_\infty \times \sup_{i=1,3,5} \left(\|\tilde{\phi}(\mathbf{x}_i, \cdot) - \phi(\mathbf{x}_i, \cdot)\|_{L^\infty(0, T_{\text{obs}})} + \|\tilde{\phi}(\mathbf{x}_{i+1}, \cdot) - \phi(\mathbf{x}_{i+1}, \cdot)\|_{L^\infty(0, T_{\text{obs}})} \right),$$

which implies (4.23). ■

5 Conclusion

In this paper we provide a reconstruction procedure of the trajectory of a point source for the wave equation from the knowledge of the field at six well-chosen points on the observation boundary over a finite time interval. We derived a Lipschitz stability estimate for the inversion that shows that the inverse problem is in fact well posed. The method can be easily extended to other dimensions. In this case the reconstruction requires measurement of the field $2d$ well chosen points where d is the dimension of the space. We plan in the future works to numerically implement the method, and to recover simultaneously the intensity of the point source.

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