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COEFFICIENT IDENTIFICATION IN PARABOLIC EQUATIONS WITH FINAL DATA

FAOUZI TRIKI

Abstract. In this work we determine the second-order coefficient in a parabolic équation from the knowledge of a single final data. Under assumptions on the concentration of eigenvalues of the associated elliptic operator, and the initial state, we show the uniqueness of solution, and we derive a Lipschitz stability estimate of the inversion when the final time is large enough. The Lipschitz stability constant grows exponentially with respect to the final time, which makes the inversion ill-posed. The proof of stability estimate is based on a spectral decomposition of the solution to the parabolic equation in terms of the eigenfunctions of the associated elliptic operator, and an ad hoc method to solve a nonlinear stationary transport equation that is itself of interest.

Résumé : Dans ce travail nous déterminons le coefficient du terme de second ordre d'une équation parabolique à partir de la connaissance d'une seule donnée finale. Sous des hypothèses sur la concentration des valeurs propres de l'opérateur associé, nous établirons une estimation de stabilité de type Lipschitz de l'inversion quand le temps final est assez grand. La constante de Lipschitz croît exponentiellement en fonction du temps final, ce qui indique que le problème inverse est mal-posé. La preuve de l'estimation de stabilité est basée sur la décomposition spectrale de la solution de l'équation parabolique, et une méthode originale pour résoudre une équation de transport nonlinéaire qui est intéressante en soi.

1. Introduction and main results

Let Ω be a C^3 bounded domain of $\mathbb{R}^n$, $n = 2, 3$, with a boundary Γ. Let $\nu(x)$ be the outward unitary normal vector at $x \in \Gamma$. For $\mathsf{a}_+ > 1$, a fixed constant, and $\mathsf{a}_0 \in C^1(\Gamma)$, a given function, set

$$\mathsf{A} = \left\{ a \in C^1(\overline{\Omega}) : \quad 1 \leq \mathsf{a}(x); \quad \mathsf{a}|_\Gamma = \mathsf{a}_0; \quad \|\mathsf{a}\|_{C^1(\overline{\Omega})} \leq \mathsf{a}_+ \right\}.$$

Consider, for $u_0 \in L^2(\Omega)$ and $\mathsf{a} \in \mathsf{A}$, the following initial-boundary value problem

$$\left\{ \begin{array}{lll} u_t - \operatorname{div}(\mathsf{a}\nabla u) = 0 & \text{in} & \Omega\times]0,+\infty[, \\ u = 0 & \text{on} & \Omega\times]0,+\infty[, \\ u = u_0 & \text{in} & \Omega \times \{0\}. \end{array} \right. \tag{1}$$

The parabolic system (1) is used to describe a wide variety of time-dependent phenomena, including heat conduction, particle diffusion, and pricing of derivative investment instruments. It is well known that the system (1) has a unique solution $u(x,t) \in C^0\left([0,+\infty[; L^2(\Omega)\right) \cap C^0\left(]0,+\infty[; H^2(\Omega) \cap H_0^1(\Omega)\right)$ [11].

The goal of this work is to study the following inverse problem (P): Given $u(x,T) \in H^2(\Omega)$ for $T > 0$, to find $\mathsf{a} \in \mathsf{A}$ such that u is a solution to the system (1).

This inverse problem finds applications in multi-wave imaging and geophysics [3, 5, 14, 22]. It can be seen as an extension to a non-stationary setting of a well known inverse elliptic problem with interior data, for which uniqueness and stability have been already derived [1, 4, 10]. In such an elliptic context, it can be seen that boundary information on the coefficient a is needed, as well as a unique continuation property of the gradient of solutions. Notice that in dimension one a solution of an inverse problem similar to (P) was

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given under some special assumptions on the boundary data [15]. The inverse problem (P) was recently cited among few other open inverse problems in [2]. Reviews for results concerning other inverse problems for parabolic equations using final datum can be found in the following references [6, 11, 12, 16, 24].

In this paper we show that the inverse problem (P) has a unique solution, and we derive stability estimates for the inversion under some assumptions on the point spectrum distribution of the associated elliptic operator, the initial state u_0, and the observation time T.

It is well known that the unbounded operator $\mathsf{L}_\mathsf{a} : L^2(\Omega) \to L^2(\Omega)$, defined by

$$\mathsf{L}_\mathsf{a} := -\operatorname{div}(\mathsf{a}\nabla\cdot),$$

with a Dirichlet boundary condition on Γ, is self-adjoint, strictly positive operator with a compact resolvent [18]. Its domain is given by $D(\mathsf{L}_\mathsf{a}) = H_0^1(\Omega) \cap H^2(\Omega)$.

The uniqueness and stability estimate presented here depend in an intricate way on the distribution of the eigenvalues of L_a. We denote by $\lambda_k, k \in \mathbb{N}^*$, the eigenvalues of L_a arranged in a non-decreasing order and repeated according to multiplicity. We also introduce the strictly ordered eigenvalues $\hat{\lambda}_k, k \in \mathbb{N}^*$. Notice that the first two values of both sequences coincide.

Definition 1.1. *We say that L_a satisfies the property (G) with constants $\gamma \geq 0$ and $\delta > 0$ if its eigenvalues $\lambda_k, k \in \mathbb{N}^*$, verify the following gap condition:*

$$\hat{\lambda}_{k+1} - \hat{\lambda}_k \quad \geq \quad \delta \hat{\lambda}_k^{-\gamma}, \qquad k \in \mathbb{N}^*. \tag{2}$$

Remark 1.1. *It is well known that under a non-trapping condition the operator L_a, subject to a Dirichlet boundary condition satisfies the property (G) with $\gamma = 0$, and $\delta > 0$ is a constant depending only on a and Ω [8, 9]. The property (G) is also somehow related to the boundary observability problem for Shrödinger and wave equations in control theory [8, 25].*

The obtained stability estimate require that the property (G) be satisfied by the operator L_a. Therefore for $\gamma \geq 0$, and $\delta > 0$ some fixed constants, we introduce the set

$$\mathsf{A}_0 = \{a \in \mathsf{A} : \mathsf{L}_\mathsf{a} \text{ satisfies property (G) with fixed constants } \gamma \geq 0 \text{ and } \delta > 0\}.$$

Let $d_\Omega(x)$ be the distance of $x \in \Omega$ to the boundary Γ. For $v \in L^2(\Omega)$, we further denote $v^+ = \max(v, 0)$ and $v^- = \max(-v, 0)$.

Theorem 1.1. *Let $\mathsf{a}, \tilde{\mathsf{a}} \in \mathsf{A}_0$. Denote $u(x,t)$ and $\tilde{u}(x,t)$ the solutions to the system (1) with respectively coefficients a and $\tilde{\mathsf{a}}$. There exists a constant $\mathfrak{c}_0 > 2$ that depends on A, n and Ω, such that if $v = u_0$ or $v = -u_0$ satisfies*

$$\int_\Omega v_+(x) d_\Omega(x) dx > \mathfrak{c}_0 \int_\Omega v_-(x) d_\Omega(x) dx, \tag{3}$$

then the following stability estimate holds

$$\|\mathsf{a} - \tilde{\mathsf{a}}\|_{L^2(\Omega)} \leq C e^{\mathsf{a}_+ \lambda_1^\Omega T} \|u(\cdot, T) - \tilde{u}(\cdot, T)\|_{H^2(\Omega)}, \tag{4}$$

for all $T > T_0$, where the constants $T_0 > 0$, $C > 0$, depend on A_0, u_0, n, Ω, and λ_1^Ω is the first eigenvalue of the Dirichlet Laplacian on Ω.

Remark 1.2. *The stability estimate implies the uniqueness of the inverse problem (P). The exponential growth of the Lipschitz stability constant (4) shows that the inversion is in general ill-posed. The exponential growth constant $\mathsf{a}_+ \lambda_1^\Omega$ can actually be replaced by $\min(\lambda_1, \tilde{\lambda}_1)$, where $\tilde{\lambda}_1$ is the first eigenvalue of $\mathsf{L}_{\tilde{\mathsf{a}}}$. The required regularity on the right hand side of the stability estimate seems to be optimal, as for the inverse elliptic problem with interior data [1].*

Remark 1.3. *The assumption (3) is fullfiled if $\pm u_0 \geq 0$ on Ω, and it implies that $\int_\Omega u_0(x)\phi_1(x)dx \neq 0$, for all $\mathsf{a} \in \mathsf{A}$, where ϕ_1 is an eigenfunction of L_a associated to λ_1.*

The proof is based on a particular decomposition of $-\partial_t u(x,T)$. The principal idea is to substitute $-\partial_t u(x,T)$ in the parabolic equation of the system (1) by $\hat{\lambda}_1 u(x,T) + \mathsf{F}(\mathsf{a};x,T)$, where $\mathsf{a} \to \mathsf{F}(\mathsf{a};x,T)$ is a Lipschitz non-linear function with Lipschitz constant that decays faster than $u(x,T)$ when T tends towards infinity. Moreover the function $\mathsf{F}(\mathsf{a};x,T)$ is independent of the data $u(x,T)$, and can be entirely recovered from the knowledge a, u_0, and Ω. In this regard, the unknown coefficient a satisfies a nonlinear stationary transport equation

$$\operatorname{div}(\mathsf{a}\nabla u(x,T)) \quad = \quad -\hat{\lambda}_1 u(x,T) + \mathsf{F}(\mathsf{a};x,T), \quad x \in \Omega.$$

Since $\mathsf{F}(\mathsf{a};x,T)$ decays faster than $u(x,T)$ in $L^2(\Omega)$ for large T, the system above can be considered as a nonlinear perturbation of a stationary linear transport equation

$$\operatorname{div}(\mathsf{a}\nabla u(x,T)) \quad = \quad -\hat{\lambda}_1 u(x,T), \quad x \in \Omega.$$

Consequently, by solving the simplified linear equation above, we shall be able to derive the global stability estimate for the nonlinear one using classical perturbation methods. The detailed proof is presented at the end of section 2.

The paper is organized as follows. The first section is dedicated to some useful properties of the solution u of the system (1) including its spectral decomposition. In section 2, we provide the proof of the main Theorem 1.1. In appendix A, we recall some known useful properties of the eigenvalues and eigenfunctions of elliptic operators in a divergence form.

2. Preliminaries results

We first derive some properties of the eigenelements of the unbounded operator L_a. Considering $\tilde{\mathsf{a}}$ as a perturbation of the coefficient a, we derive an upper bound of the perturbation of eigenelements of L_a in terms of $\|\mathsf{a}-\tilde{\mathsf{a}}\|_{L^2(\Omega)}$.

Theorem 2.1. *Let $\mathsf{a},\tilde{\mathsf{a}} \in \mathsf{A}$, and $(\lambda_k)_{k\in\mathbb{N}^*} \subset \mathbb{R}^*$ (resp. $(\tilde{\lambda}_k)_{k\in\mathbb{N}^*} \subset \mathbb{R}^*$) be respectively the increasing sequence of eigenvalues of L_{a} (resp. $\mathsf{L}_{\tilde{\mathsf{a}}}$). Then*

$$|\lambda_k - \tilde{\lambda}_k| \leq C \min(\lambda_k,\tilde{\lambda}_k)^{1+\frac{n}{4}} \|\mathsf{a}-\tilde{\mathsf{a}}\|_{L^2(\Omega)}, \tag{5}$$

where $C>0$ is a constant that depends only on n, A and Ω.

Proof. Without loss of generality we assume that $\lambda_k \geq \tilde{\lambda}_k$.

Denote by $\phi_k, k \in \mathbb{N}^*$ (resp. $\tilde{\phi}_k, k \in \mathbb{N}^*$) the orthonormal sequence of eigenfunctions of L_{a} (resp. $\mathsf{L}_{\tilde{\mathsf{a}}}$) associated to $\lambda_k, k \in \mathbb{N}^*$ (resp. $\lambda_k, k \in \mathbb{N}^*$).

Recall the Min-max characterization of the eigenvalues $(\tilde{\lambda}_k)_{k\in\mathbb{N}^*}$ [20]

$$\tilde{\lambda}_k = \min_{\substack{\Phi_k \subset H_0^1(\Omega) \\ \dim(\Phi_k)=k}} \max_{\phi\in\Phi_k\setminus\{0\}} \frac{\int_\Omega \tilde{\mathsf{a}}|\nabla\phi|^2 dx}{\int_\Omega |\phi|^2 dx} \tag{6}$$

In the expression above the minimum is achieved when Φ_k coincides with the finite dimension space generated by $\widetilde{V}_k := \{\tilde{\phi}_l : l \leq k\}$. Therefore

$$\lambda_k - \tilde{\lambda}_k \leq \max_{\phi\in\widetilde{V}_k\setminus\{0\}} \frac{\int_\Omega \mathsf{a}|\nabla\phi|^2 dx}{\int_\Omega |\phi|^2 dx} - \max_{\phi\in\widetilde{V}_k\setminus\{0\}} \frac{\int_\Omega \tilde{\mathsf{a}}|\nabla\phi|^2 dx}{\int_\Omega |\phi|^2 dx}.$$

Since $\widetilde{V}_k$ is a finite dimension space the first maximum is reached at some vector $\tilde{\psi}_k \in \widetilde{V}_k \setminus \{0\}$, satisfying $\int_\Omega |\tilde{\psi}_k|^2 dx = 1$. Hence

$$\lambda_k - \tilde{\lambda}_k \leq \int_\Omega \mathsf{a}|\nabla\tilde{\psi}_k|^2 dx - \int_\Omega \tilde{\mathsf{a}}|\nabla\tilde{\psi}_k|^2 dx \leq \|\mathsf{a}-\tilde{\mathsf{a}}\|_{L^2(\Omega)} \left(\int_\Omega |\nabla\tilde{\psi}_k|^4 dx\right)^{\frac{1}{2}}. \tag{7}$$

Since $\tilde{\psi}_k \in \widetilde{V}_k \backslash \{0\}$, there exists a real valued sequence $\alpha_l, l \leq k$, satisfying $\sum_{l=1}^k \alpha_l^2 = 1$, and $\tilde{\psi}_k = \sum_{l=1}^k \alpha_l \tilde{\phi}_l$. Therefore, $\tilde{\psi}_k$ verifies the following elliptic equation

$$\mathsf{L}_{\tilde{\mathsf{a}}} \tilde{\psi}_k = \sum_{l=1}^k \alpha_l \tilde{\lambda}_l \tilde{\phi}_l.$$

We then deduce from the classical elliptic regularity (Theorem 8.12 in [18])

$$\|\tilde{\psi}_k\|_{H^2(\Omega)} \leq C_1 \left\| \sum_{l=1}^k \alpha_l \tilde{\lambda}_l \tilde{\phi}_l \right\|_{L^2(\Omega)} \leq C_1 \tilde{\lambda}_k, \tag{8}$$

where $C_1 > 0$ depends only on n, A and Ω.

Classical Sobolev interpolation inequalities for $n = 2, 3$, give [18]

$$\|\nabla \tilde{\psi}_k\|_{L^4(\Omega)} \leq C_2 \|\nabla \tilde{\psi}_k\|_{L^2(\Omega)}^{1-\frac{n}{4}} \|\tilde{\psi}_k\|_{H^2(\Omega)}^{\frac{n}{4}}, \tag{9}$$

where $C_2 > 0$ depends only on n and Ω.

By a simple calculation, and using the fact that $\|\nabla \tilde{\phi}_l\|_{L^2(\Omega)}^2 \leq \tilde{\lambda}_l$, $l \in \mathbb{N}^*$, we obtain

$$\|\nabla \tilde{\psi}_k\|_{L^2(\Omega)} \leq \tilde{\lambda}_k. \tag{10}$$

Combining inequalities (8), (9) and (10), we get

$$\|\nabla \tilde{\psi}_k\|_{L^4(\Omega)} \leq C_3 \tilde{\lambda}_k^{\frac{n+4}{8}},$$

where $C_3 > 0$ depends only on n, A and Ω. We then deduce from (7)

$$\lambda_k - \tilde{\lambda}_k \leq C_3 \|\mathsf{a} - \tilde{\mathsf{a}}\|_{L^2(\Omega)} \tilde{\lambda}_k^{1+\frac{n}{4}},$$

which achieves the proof of the theorem.

□

Remark 2.1. *The estimate (5) may not be optimal. The objective here was to obtain an inequality with an uniform constant for all functions* $\mathsf{a}, \tilde{\mathsf{a}} \in \mathsf{A}$.

Theorem 2.2. *Let* $\mathsf{a}, \tilde{\mathsf{a}} \in \mathsf{A}_0$. *Let* P_k *(resp.* $\widetilde{P}_k$*) be the orthogonal projection onto the eigenspace of* L_{a} *(resp.* $\mathsf{L}_{\tilde{\mathsf{a}}}$*) corresponding to the eigenvalue* $\hat{\lambda}_k$ *(resp.* $\widehat{\tilde{\lambda}_k}$*). There exist constants* $\eta > 0$ *and* $C > 0$ *that depend only on* n, A_0, *and* Ω, *such that if*

$$\|\mathsf{a} - \tilde{\mathsf{a}}\|_{L^2(\Omega)} \leq \eta \max(\hat{\lambda}_k, \widehat{\tilde{\lambda}}_k)^{-(1+\gamma+\frac{n}{4})}, \tag{11}$$

then, the following estimate

$$\|P_k - \widetilde{P}_k\|_{\mathcal{L}^2(\Omega)} \leq C(\max(\hat{\lambda}_k, \widehat{\tilde{\lambda}}_k)^{\gamma+1} + 1)^2 \|\mathsf{a} - \tilde{\mathsf{a}}\|_{L^2(\Omega)}, \tag{12}$$

holds.

Proof. In the proof $C > 0$ denotes a generic constant depending n, A_0, and Ω. Without loss of generality we further assume that $\hat{\lambda}_k \geq \widehat{\tilde{\lambda}}_k$.

Since $\mathsf{a}, \tilde{\mathsf{a}} \in \mathsf{A}_0$, the gap condition (2) implies

$$B_{\rho_k}(\hat{\lambda}_k) \cap \{\hat{\lambda}_l,\ l \in \mathbb{N}^*\} = \{\hat{\lambda}_k\}, \qquad B_{\rho_k}(\widehat{\tilde{\lambda}}_k) \cap \{(\widehat{\tilde{\lambda}}_l,\ l \in \mathbb{N}^*\} = \{\widehat{\tilde{\lambda}}_k\}, \tag{13}$$

where $B_{\rho_k}(z)$ is the complex disc of center $z \in \mathbb{C}$, and radius $\rho_k = \frac{\delta}{4\hat{\lambda}_k^\gamma}$.

On the other hand estimate (5) leads to

$$|\hat{\lambda}_k - \widehat{\tilde{\lambda}}_k| \leq C \widehat{\tilde{\lambda}}_k^{1+\frac{n}{4}} \|\mathsf{a} - \tilde{\mathsf{a}}\|_{L^2(\Omega)}. \tag{14}$$

Now, combining inequalities (14) and (11), we have

$$|\hat{\lambda}_k - \widehat{\tilde{\lambda}}_k| \leq C\eta\delta^{-1}\rho_k. \tag{15}$$

Choosing $\eta > 0$ small enough such that $C\eta < 1$, we obtain

$$\widehat{\tilde{\lambda}}_k \in B_{\rho_k}(\hat{\lambda}_k). \tag{16}$$

Therefore, we also have

$$B_{\rho_k}(\hat{\lambda}_k) \cap \{\widehat{\tilde{\lambda}}_l,\ l \in \mathbb{N}^*\} = \{\widehat{\tilde{\lambda}}_k\}. \tag{17}$$

Consequently, the resolvents $(\lambda I - \mathsf{L}_{\mathsf{a}})^{-1}$ and $(\lambda I - \mathsf{L}_{\tilde{\mathsf{a}}})^{-1}$ are well defined as operators from $L^2(\Omega)$ onto $H_0^1(\Omega) \cap H^2(\Omega)$ for all $\lambda \in \partial B_{\rho_k}(\hat{\lambda}_k)$. In addition, by the well-known Riesz formula, we get [17]

$$P_k = -\frac{1}{2\iota\pi}\int_{|\lambda-\hat{\lambda}_k|=\rho_k} (\lambda I - \mathsf{L}_{\mathsf{a}})^{-1} d\lambda, \qquad \widetilde{P}_k = -\frac{1}{2\iota\pi}\int_{|\lambda-\hat{\lambda}_k|=\rho_k} (\lambda I - \mathsf{L}_{\mathsf{a}})^{-1} d\lambda,$$

where $\iota \in \mathbb{C}$ is the imaginary complex number, and I is the identity operator.

Hence

$$P_k - \widetilde{P}_k = \frac{1}{2\iota\pi}\int_{|\lambda-\hat{\lambda}_k|=\rho_k} (\lambda I - \mathsf{L}_{\mathsf{a}})^{-1}(\mathsf{L}_{\mathsf{a}} - \mathsf{L}_{\tilde{\mathsf{a}}})(\lambda I - \mathsf{L}_{\tilde{\mathsf{a}}})^{-1} d\lambda.$$

Since P_k and $\widetilde{P}_k$ are orthogonal projections, $P_k - \widetilde{P}_k$ is a self-adjoint bounded operator from $L^2(\Omega)$ to itself.

Consequently

$$\begin{aligned}
\|P_k - \widetilde{P}_k\|_{\mathcal{L}^2(\Omega)} &= (2\pi)^{-1} \sup_{\lambda\in\partial B_{\rho_k}(\hat{\lambda}_k),\, f\in W_k,\, \|f\|_{L^2(\Omega)}=1} |\left\langle (\mathsf{L}_{\mathsf{a}} - \mathsf{L}_{\tilde{\mathsf{a}}})\tilde{u}_f^\lambda, u_f^\lambda \right\rangle_{L^2(\Omega)}|, \\
&= (2\pi)^{-1} \sup_{\lambda\in\partial B_{\rho_k}(\hat{\lambda}_k),\, f\in W_k,\, \|f\|_{L^2(\Omega)}=1} \left|\int_\Omega (\mathsf{a} - \tilde{\mathsf{a}})\nabla u_f^\lambda \nabla \tilde{u}_f^\lambda dx\right|, \\
&\leq (2\pi)^{-1}\|\mathsf{a} - \tilde{\mathsf{a}}\|_{L^2(\Omega)} \sup_{\lambda\in\partial B_{\rho_k}(\hat{\lambda}_k),\, f\in W_k,\, \|f\|_{L^2(\Omega)}=1} \|\nabla u_f^\lambda\|_{L^4(\Omega)}\|\nabla \tilde{u}_f^\lambda\|_{L^4(\Omega)},
\end{aligned} \tag{18}$$

where $u_f^\lambda = (\lambda I - \mathsf{L}_{\mathsf{a}})^{-1} f$, $\tilde{u}_f^\lambda = (\lambda I - \mathsf{L}_{\tilde{\mathsf{a}}})^{-1} f$, and W_k is the finite dimension vector space in $L^2(\Omega)$, spanned by the eigenfunctions associated to $\hat{\lambda}_k$, and $\widehat{\tilde{\lambda}}_k$.

By construction, we have

$$\|u_f^\lambda\|_{L^2(\Omega)}, \|\tilde{u}_f^\lambda\|_{L^2(\Omega)} \leq \frac{1}{\rho_k} \tag{19}$$

Similar to the proof of Theorem 2.1, we deduce from the classical elliptic regularity

$$\|u_f^\lambda\|_{H^2(\Omega)} \leq C(\lambda\|u_f^\lambda\|_{L^2(\Omega)} + 1), \quad \|\tilde{u}_f^\lambda\|_{H^2(\Omega)} \leq C(\lambda\|u_f^\lambda\|_{L^2(\Omega)} + 1),$$

which associated to inequalities (19), provide

$$\|u_f^\lambda\|_{H^2(\Omega)}, \|\tilde{u}_f^\lambda\|_{H^2(\Omega)} \leq C(\frac{\lambda}{\rho_k} + 1). \tag{20}$$

Sobolev embedding Theorem gives [18]

$$\|\nabla u_f^\lambda\|_{L^4(\Omega)} \leq C\|u_f^\lambda\|_{H^2(\Omega)}, \quad \|\nabla \tilde{u}_f^\lambda\|_{L^4(\Omega)} \leq C\|\tilde{u}_f^\lambda\|_{H^2(\Omega)}. \tag{21}$$

Combining estimates (20) and (21), we finally obtain

$$\|\nabla u_f^\lambda\|_{L^4(\Omega)}, \|\nabla \tilde{u}_f^\lambda\|_{L^4(\Omega)} \leq C(\frac{\lambda}{\rho_k} + 1). \tag{22}$$

We infer from (18)

$$\|P_k - \widetilde{P}_k\|_{\mathcal{L}^2(\Omega)} \quad \leq \quad C(\hat{\lambda}_k^{\gamma+1}+1)^2\|\mathsf{a}-\tilde{\mathsf{a}}\|_{L^2(\Omega)}.$$

□

3. Proof of Theorem 1.1

We first introduce the nonlinear function $\mathsf{F}(\mathsf{a};x,T)$, and show that its Lipschitz continuous modulus with respect to a, decays faster than $u(x,T)$ for large T.

We first derive the following auxiliary result.

Proposition 3.1. *There exists a constant $\mathfrak{c}_0 > 2$ depending on A, n and Ω, such that if $v = u_0$ or $v = -u_0$ satisfies (3), then*

$$\left|\int_\Omega u_0(x)\phi_1(x)dx\right| > \frac{1}{\sqrt{2\mathfrak{c}_0}}\left|\int_\Omega u_0(x)d_\Omega(x)dx\right| > 0,$$

for all $\mathsf{a} \in \mathsf{A}$.

Proof. Let $\mathfrak{c}_0 = 2C^{-2}$ where $C \in (0,1)$ is the constant of Lemma A.2 which depends on A, n and Ω. Without loss of generality we further assume that $v = u_0$ satisfies (3). Since $\mathfrak{c}_0 > 2$, and $v = v_+ - v_-$, we have $\int_\Omega v(x)d_\Omega(x)dx > 0$.

Combining inequalities (3) and (56) gives

$$\int_\Omega v_+(x)\phi_1(x)dx > 2\int_\Omega v_-(x)\phi_1(x)dx.$$

Consequently

$$2\int_\Omega v(x)\phi_1(x)dx > \int_\Omega v_+(x)\phi_1(x)dx.$$

Applying again inequality (56) in Lemma A.2, we obtain

$$\int_\Omega v(x)\phi_1(x)dx > \frac{C}{2}\int_\Omega v_+(x)d_\Omega(x)dx \geq \frac{C}{2}\int_\Omega v(x)d_\Omega(x)dx,$$

which finishes the proof.

□

Define for $\mathsf{a} \in \mathsf{A}_0$ and $T > 0$, the nonlinear function $\mathsf{F}(\mathsf{a};x,T) \in L^2(\Omega)$, by

$$\mathsf{F}(\mathsf{a};x,T) \quad = \quad \partial_t u(x,T) + \hat{\lambda}_1 u(x,T), \qquad x \in \Omega, \tag{23}$$

where u is the unique solution of the system (1). Without loss of generality we further assume that $v = u_0$ satisfies (3).

Theorem 3.1. *Let $\mathsf{a}, \tilde{\mathsf{a}} \in \mathsf{A}_0$. Then there exists a constant $C > 0$ that depends only on θ, Ω, n, u_0 and A_0, such that the inequality*

$$\|\mathsf{F}(\mathsf{a};x,T) - \mathsf{F}(\tilde{\mathsf{a}};x,T)\|_{L^2(\Omega)} \leq Ce^{-\min(\hat{\lambda}_2,\widehat{\tilde{\lambda}}_2)T}\|\mathsf{a}-\tilde{\mathsf{a}}\|_{L^2(\Omega)}, \tag{24}$$

is valid for all $T \geq 1$.

Proof. In the proof $C > 0$ denotes a generic constant depending on n, A_0, u_0, and Ω.

We start the proof by writing the decomposition of $\mathsf{F}(\mathsf{a};x,T)$ (resp. $\mathsf{F}(\tilde{\mathsf{a}};x,T)$) in terms of the eigenfunctions of the elliptic operator L_a (resp. $\mathsf{L}_{\tilde{\mathsf{a}}}$). Recall P_k (resp. $\widetilde{P}_k$) the orthogonal projection onto the eigenspace of L_a (resp. $\mathsf{L}_{\tilde{\mathsf{a}}}$) associated to the eigenvalue $\hat{\lambda}_k$ (resp. $\widehat{\tilde{\lambda}}_k$).

It is well known that u and $\tilde{u}$ have the following spectral decomposition [11]

$$u(x,t) \;=\; \sum_{k=1}^{\infty} e^{-\hat{\lambda}_k t} P_k u_0(x); \qquad \tilde{u}(x,t) \;=\; \sum_{k=1}^{\infty} e^{-\widehat{\tilde{\lambda}}_k t} \widetilde{P}_k u_0(x)$$

Forward calculations yield

$$\mathsf{F}(\mathsf{a};x,T) = \sum_{k=2}^{\infty} (\hat{\lambda}_k - \hat{\lambda}_1) e^{-\hat{\lambda}_k T} P_k u_0(x); \qquad \tilde{\mathsf{F}}(\mathsf{a};x,T) = \sum_{k=2}^{\infty} (\widehat{\tilde{\lambda}_k} - \widehat{\tilde{\lambda}_1}) e^{-\widehat{\tilde{\lambda}}_k T} \widetilde{P}_k u_0(x).$$

Hence

$$\begin{aligned}\mathsf{F}(\mathsf{a};x,T) - \mathsf{F}(\tilde{\mathsf{a}};x,T) = \\ \sum_{k=2}^{\infty} \left[(\hat{\lambda}_k - \hat{\lambda}_1) e^{-\hat{\lambda}_k T} - (\widehat{\tilde{\lambda}_k} - \widehat{\tilde{\lambda}_1}) e^{-\widehat{\tilde{\lambda}}_k T} \right] \widetilde{P}_k u_0(x) + \sum_{k=2}^{\infty} (\hat{\lambda_k} - \hat{\lambda_1}) e^{-\hat{\lambda}_k T} \left[P_k - \widetilde{P}_k \right] u_0(x) \\ = \mathsf{F}_1 + \mathsf{F}_2.\end{aligned}$$

Let $\beta_k, k \in \mathbb{N} \setminus \{0,1\}$, defined by

$$\beta_k = \min(\hat{\lambda}_k, \widehat{\tilde{\lambda}_k}).$$

Then

$$\|\mathsf{F}_1\|^2_{L^2(\Omega)} \leq \sum_{k=2}^{\infty} \left[|\hat{\lambda}_k - \widehat{\tilde{\lambda}}_k| (1 + \beta_k T) + |\hat{\lambda}_1 - \widehat{\tilde{\lambda}}_1| \right]^2 e^{-2\beta_k T} \|u_0\|^2_{L^2(\Omega)}.$$

Using results of Theorem 2.1 and Lemma A.2, we obtain

$$\begin{aligned}\|\mathsf{F}_1\|^2_{L^2(\Omega)} &\leq C \sum_{k=2}^{\infty} \left[\beta_k^{1+\frac{n}{4}} (1 + \beta_k T) + \beta_1^{1+\frac{n}{4}} \right]^2 e^{-2\beta_k T} \|u_0\|^2_{L^2(\Omega)} \|\mathsf{a} - \tilde{\mathsf{a}}\|^2_{L^2(\Omega)}, \\ &\leq C \sum_{k=2}^{\infty} \beta_k^{2+\frac{n}{2}} e^{-\beta_k T} \|u_0\|^2_{L^2(\Omega)} \|\mathsf{a} - \tilde{\mathsf{a}}\|^2_{L^2(\Omega)} \leq C e^{-\beta_2 T} \sum_{k=2}^{\infty} \beta_k^{2+\frac{n}{2}} e^{-(\beta_k - \beta_2)} \|u_0\|^2_{L^2(\Omega)} \|\mathsf{a} - \tilde{\mathsf{a}}\|^2_{L^2(\Omega)}.\end{aligned}$$

Note that by Weyl's asymptotic formula, we have $\lambda_k \sim C k^{\frac{2}{n}}$ for large k, which guarantees the convergence of the series above [19, 20].

The results of Lemma A.3 imply

$$\|\mathsf{F}_1\|_{L^2(\Omega)} \quad \leq \quad C e^{-\beta_2 T} \|\mathsf{a} - \tilde{\mathsf{a}}\|_{L^2(\Omega)}. \tag{25}$$

Since the orthogonality of the terms of the series F_2 is no longer true, and the fact that the perturbation does not affect uniformly the eigenfunctions, deriving an upper bound for $\|\mathsf{F}_2\|_{L^2(\Omega)}$ is more involved.

Recall that the sequences $\hat{\lambda}_k$ and $\widehat{\tilde{\lambda}}_k$ are strictly increasing. Let $N \in \mathbb{N}^*$ be the smallest integer satisfying

$$\|\mathsf{a} - \tilde{\mathsf{a}}\|_{L^2(\Omega)} > \eta \max(\hat{\lambda}_k, \widehat{\tilde{\lambda}}_k)^{-(1+\gamma+\frac{n}{4})} \geq \eta \beta_k^{-(1+\gamma+\frac{n}{4})}, \qquad \forall k \geq N, \tag{26}$$

where $\eta > 0$ is the constant introduced in Theorem 2.2.

Next, we split F_2 into two parts:

$$\mathsf{F}_2 = \sum_{k=2}^{N-1} (\hat{\lambda}_k - \hat{\lambda}_1) e^{-\hat{\lambda}_k T} \left[P_k - \widetilde{P}_k \right] u_0(x) + \sum_{k=N}^{\infty} (\hat{\lambda}_k - \hat{\lambda}_1) e^{-\hat{\lambda}_k T} \left[P_k - \widetilde{P}_k \right] u_0(x) = \mathsf{F}_{21} + \mathsf{F}_{22}, \tag{27}$$

with the convention that the first sum $F_{21} = 0$ when $N = 2$.

We deduce from (26) the following estimate

$$\begin{aligned}\|\mathsf{F}_{22}\|^2_{L^2(\Omega)} &\leq \eta^{-2}\sum_{k=N}^{\infty}(\hat\lambda_k-\hat\lambda_1)^2\beta_k^{2(1+\gamma+\frac{n}{4})}e^{-2\beta_k T}\|u_0\|^2_{L^2(\Omega)}\|\mathsf{a}-\tilde{\mathsf{a}}\|^2_{L^2(\Omega)},\\ &\leq \eta^{-2}e^{-2\beta_k T}\sum_{k=2}^{\infty}(\hat\lambda_k-\hat\lambda_1)^2\beta_2^{2(1+\gamma+\frac{n}{4})}e^{-2(\beta_k-\beta_2)}\|u_0\|^2_{L^2(\Omega)}\|\mathsf{a}-\tilde{\mathsf{a}}\|^2_{L^2(\Omega)}.\end{aligned}$$

Again using the upper and lower bounds derived in Lemma A.3, we obtain

$$\|\mathsf{F}_{22}\|_{L^2(\Omega)} \leq Ce^{-\beta_2 T}\|\mathsf{a}-\tilde{\mathsf{a}}\|_{L^2(\Omega)}. \tag{28}$$

On the other hand, we have

$$\|\mathsf{F}_{21}\|^2_{L^2(\Omega)} \leq \left\|\sum_{k=2}^{N-1}(\hat\lambda_k-\hat\lambda_1)e^{-\hat\lambda_k T}\left|\left[P_k-\widetilde{P}_k\right]u_0\right|\right\|^2_{L^2(\Omega)}.$$

Cauchy-Shwartz inequality gives

$$\|\mathsf{F}_{21}\|^2_{L^2(\Omega)} \leq \left(\sum_{k=2}^{N-1}(\hat\lambda_k-\hat\lambda_1)^2e^{-\hat\lambda_k T}\right)\left(\sum_{k=2}^{N-1}e^{-\hat\lambda_k T}\left\|P_k-\widetilde{P}_k\right\|^2_{\mathcal{L}(L^2(\Omega))}\right)\|u_0\|^2_{L^2(\Omega)}.$$

By construction, we have

$$\|\mathsf{a}-\tilde{\mathsf{a}}\|_{L^2(\Omega)} \leq \eta\max(\hat\lambda_k,\widehat{\tilde\lambda}_k)^{-(1+\gamma+\frac{n}{4})}, \qquad \forall k\leq N-1. \tag{29}$$

Using the results of Theorem 2.2, leads to

$$\begin{aligned}&\|\mathsf{F}_{21}\|^2_{L^2(\Omega)}\\ &\leq C\left(\sum_{k=2}^{N-1}(\hat\lambda_k-\hat\lambda_1)^2e^{-\hat\lambda_k T}\right)\left(\sum_{k=2}^{N-1}(\max(\hat\lambda_k,\widehat{\tilde\lambda}_k)^{\gamma+1}+1)^4e^{-\hat\lambda_k T}\right)\|u_0\|^2_{L^2(\Omega)}\|\mathsf{a}-\tilde{\mathsf{a}}\|^2_{L^2(\Omega)},\\ \leq Ce^{-2\beta_2 T}&\left(\sum_{k=2}^{\infty}(\hat\lambda_k-\hat\lambda_1)^2e^{-(\hat\lambda_k-\hat\lambda_2)}\right)\left(\sum_{k=2}^{\infty}(\max(\hat\lambda_k,\widehat{\tilde\lambda}_k)^{\gamma+1}+1)^4e^{-(\hat\lambda_k-\hat\lambda_2)}\right)\|u_0\|^2_{L^2(\Omega)}\|\mathsf{a}-\tilde{\mathsf{a}}\|^2_{L^2(\Omega)}.\end{aligned}$$

Applying again the bounds in Lemma A.3, we get

$$\|\mathsf{F}_{21}\|_{L^2(\Omega)} \leq Ce^{-\beta_2 T}\|\mathsf{a}-\tilde{\mathsf{a}}\|_{L^2(\Omega)}. \tag{30}$$

Finally, combining inequalities (25), (28), and (30), conducts to the desired estimate. □

We next study the decay behavior of $\partial_t u(x,T)$ and $|\nabla u(x,T)|^2$ as T tends towards infinity.

Theorem 3.2. *Let $\mathsf{a}\in\mathsf{A}_0$, and u be the unique solution to the system (1). Then there exist $T_1>0$, $\varepsilon_0>0$, and $C>0$ depending only on A_0, Ω, n and u_0 such that the following inequalities*

$$u(x,T) \geq Ce^{-\lambda_1 T}\phi_1(x), \qquad \forall x\in\Omega, \tag{31}$$

$$-\partial_t u(x,T) \geq Ce^{-\lambda_1 T}\phi_1(x), \qquad \forall x\in\Omega, \tag{32}$$

$$|\nabla u(x,T)|^2 \geq Ce^{-2\lambda_1 T}|\nabla\phi_1(x)|^2, \qquad \forall x\in\Omega_\varepsilon, \tag{33}$$

hold for all $T\geq T_1$, and $0<\varepsilon<\varepsilon_0$, with $\Omega_\varepsilon=\{x\in\Omega:\ d_\Omega(x)<\varepsilon\}$.

Proof. In the sequel $C>0$ denotes a generic constant that depends only on A_0, Ω, n and u_0. We further assume that $T\geq 1$.

The proof is based on the following decomposition of u in terms of the eigenfunctions of L_a:

$$u(x,t) = \sum_{k=1}^{\infty}e^{-\hat\lambda_k t}P_k u_0(x), \qquad \forall t>0,\ x\in\Omega.$$

For $T \geq 1$, we have

$$\begin{aligned} \nabla u(x,T) &= \sum_{k=1}^{\infty} e^{-\hat{\lambda}_k T} \nabla P_k u_0(x), \qquad \forall x \in \Omega, \\ -\partial_t u(x,T) &= \sum_{k=1}^{\infty} \lambda_k e^{-\hat{\lambda}_k T} \nabla P_k u_0(x), \qquad \forall x \in \Omega. \end{aligned}$$

Therefore

$$-\partial_t u(x,T) \geq \hat{\lambda}_1 e^{-\lambda_1 T} |P_1 u_0(x)| - \sum_{k=2}^{\infty} \hat{\lambda}_k e^{-\hat{\lambda}_k T} |P_k u_0(x)|, \tag{34}$$

$$|\nabla u(x,T)|^2 \geq \frac{\lambda_1^2}{2} e^{-2\lambda_1 T} |\nabla P_1 u_0(x)|^2 - \left(\sum_{k=2}^{\infty} e^{-\hat{\lambda}_k T} |\nabla P_k u_0(x))|^2 \right)^{\frac{1}{2}} \left(\sum_{k=2}^{\infty} e^{-\hat{\lambda}_k T} \right)^{\frac{1}{2}}, \tag{35}$$

for all $x \in \Omega$.

Next, we derive the first inequality of the lemma. Using inequalities (58), we obtain

$$\begin{aligned} -\partial_t u(x,T) &\geq \left[\hat{\lambda}_1 e^{-\hat{\lambda}_1 T} \|P_1 u_0(x)\|_{L^2(\Omega)} - \sum_{k=2}^{\infty} \hat{\lambda}_k^{\frac{3}{2}+\frac{n}{4}} e^{-\hat{\lambda}_k T} \|u_0\|_{L^2(\Omega)} \right] \phi_1(x), \\ &\geq \left[\hat{\lambda}_1 e^{-\hat{\lambda}_1 T} \|P_1 u_0(x)\|_{L^2(\Omega)} - e^{-\hat{\lambda}_2 T} \sum_{k=2}^{\infty} \hat{\lambda}_k^{\frac{3}{2}+\frac{n}{4}} e^{-(\hat{\lambda}_k - \hat{\lambda}_2)} \|u_0\|_{L^2(\Omega)} \right] \phi_1(x). \end{aligned}$$

Since $a \in \mathsf{A}_0$, the following gap condition (2), holds

$$\hat{\lambda}_2 - \hat{\lambda}_1 \geq \frac{\delta}{\hat{\lambda}_1}.$$

Notice that the gap condition between the two first eigenvalues is always fulfilled [19].

We deduce from Lemma A.3

$$\hat{\lambda}_2 - \hat{\lambda}_1 \geq \frac{\delta}{a_+ \lambda_1^{\Omega}}. \tag{36}$$

Hence

$$\partial_t u(x,T) \geq e^{-\hat{\lambda}_1 T} \left[\hat{\lambda}_1 \|P_1 u_0(x)\|_{L^2(\Omega)} - e^{-\frac{\delta}{a_+ \lambda_1^{\Omega}} T} \sum_{k=2}^{\infty} \hat{\lambda}_k^{\frac{3}{2}+\frac{n}{4}} e^{-(\hat{\lambda}_k - \hat{\lambda}_2)} \|u_0\|_{L^2(\Omega)} \right] \phi_1(x).$$

Again using Lemma A.3 leads to

$$|\partial_t u(x,T)| \geq e^{-\hat{\lambda}_1 T} \left[\lambda_1^{\Omega} \|P_1 u_0(x)\|_{L^2(\Omega)} - C e^{-\frac{\delta}{a_+ \lambda_1^{\Omega}} T} \|u_0\|_{L^2(\Omega)} \right] \phi_1(x).$$

On the other hand, we deduce from Proposition 3.1

$$\|P_1 u_0(x)\|_{L^2(\Omega)} = \int_{\Omega} u_0(x) \phi_1(x) dx > \frac{1}{\sqrt{2\mathfrak{c}_0}} \int_{\Omega} u_0(x) d_{\Omega}(x) dx > 0.$$

Hence there exists a unique $T_{11} \in \mathbb{R}$ solution to the equation

$$\|P_1 u_0(x)\|_{L^2(\Omega)} - C e^{-\frac{\delta}{a_+ \lambda_1^{\Omega}} T_{11}} \|u_0\|_{L^2(\Omega)} = 0.$$

Obviously T_{11} depends only on A_0, Ω, n and u_0.

Consequently

$$-\partial_t u(x,T) \quad \geq \quad Ce^{-\hat{\lambda}_1 T}\phi_1(x), \qquad \forall x\in\Omega,\ t\in]T_{11},+\infty[.$$

Similar analysis on $u(x,T)$, leads to

$$u(x,T) \quad \geq \quad Ce^{-\hat{\lambda}_1 T}\phi_1(x), \qquad \forall x\in\Omega,\ t\in]T_{12},+\infty[,$$

where $T_{12}\in\mathbb{R}$.

Now, we shall focus on the second inequality. To do so we need to estimate $\|\nabla\phi_k\|_{L^\infty(\Omega)}$. There are many works dealing with optimal increasing rate of $\|\phi_k\|_{L^\infty(\Omega)}$ and $\|\nabla\phi_k\|_{L^\infty(\Omega)}$ in terms of λ_k, when k tends to infinity (see for instance [21, 23] and references therein). Most existing results considered the Laplacian operator or did not pay attention to the regularity of the elliptic coefficients. Since the optimal increasing rate is out of the focus of this work, we prefer here deriving similar estimates using classical elliptic regularity combined with results of Lemma A.4.

We deduce from elliptic regularity (Theorem 9.12 in [18])

$$\|\phi_k\|_{W^{2,n+\frac{1}{n}}(\Omega)} \leq C(1+\hat{\lambda}_k)\|\phi_k\|_{L^{n+\frac{1}{n}}(\Omega)}.$$

By Sobolev embedding Theorem, we have

$$\|\nabla\phi_k\|_{C^{\frac{1}{n^2+1}}(\overline{\Omega})} \leq C(1+\hat{\lambda}_k)\|\phi_k\|_{L^\infty(\Omega)}.$$

Lemma A.4 yields

$$\|\nabla\phi_k\|_{C^{\frac{1}{n^2+1}}(\overline{\Omega})} \leq C(1+\hat{\lambda}_k)\hat{\lambda}_k^{\frac{1}{2}+\frac{n}{4}}. \tag{37}$$

Combining (37) with (35), we get

$$|\nabla u(x,T)|^2 \geq \frac{\hat{\lambda}_1^2}{2}e^{-2\hat{\lambda}_1 T}\|P_1u_0(x)\|_{L^2(\Omega)}|\nabla\phi_1(x)|^2 - \left(\sum_{k=2}^{\infty}(1+\hat{\lambda}_k)^2\hat{\lambda}_k^{1+\frac{n}{2}}e^{-2\hat{\lambda}_k T}\|u_0\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\left(\sum_{k=2}^{\infty}e^{-\hat{\lambda}_k T}\right)^{\frac{1}{2}},$$

for all $x\in\Omega$.

Consequently

$$|\nabla u(x,T)|^2 \geq \frac{\hat{\lambda}_1^2}{2}e^{-2\hat{\lambda}_1 T}\|P_1u_0(x)\|_{L^2(\Omega)}|\nabla\phi_1(x)|^2 - e^{-2\hat{\lambda}_2 T}\left(\sum_{k=2}^{\infty}(1+\hat{\lambda}_k)^2\hat{\lambda}_k^{1+\frac{n}{2}}e^{-2(\hat{\lambda}_k-\hat{\lambda}_2)}\|u_0\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\left(\sum_{k=2}^{\infty}e^{-(\hat{\lambda}_k-\hat{\lambda}_2)}\right)^{\frac{1}{2}},$$

for all $x\in\Omega$, and $T\geq 1$.

We deduce from Lemma A.3 and inequality (36), the following estimate

$$|\nabla u(x,T)|^2 \ \geq \ e^{-2\hat{\lambda}_1 T}\left[\frac{(\lambda_1^\Omega)^2}{2}\|P_1u_0(x)\|_{L^2(\Omega)}|\nabla\phi_1(x)|^2 - Ce^{-\frac{\delta}{2a_+\lambda_1^\Omega}T}\|u_0\|_{L^2(\Omega)}\right], \quad \forall x\in\Omega,\ T\geq 1. \tag{38}$$

Proposition 3.2. *There exist constants $\varepsilon_0>0$ and $C_0>0$ that depend only on Ω, n, and A, such that the following inequality*

$$|\nabla\phi_1(x)| \quad \geq \quad C_0, \qquad \forall x\in\Omega_\varepsilon, \tag{39}$$

holds for all $0<\varepsilon<\varepsilon_0$.

Proof. For $\varepsilon > 0$ small enough, Ω_ε becomes a tubular domain, and can be parametrized by

$$\Omega_\varepsilon = \{x + s\nu(x) : \quad x \in \Gamma,\ 0 < s < \varepsilon\}.$$

We deduce from (37), the following estimate

$$|-\nabla\phi_1(x + s\nu(x)) \cdot \nu(x) + \partial_\nu \phi_1(x)| \leq C(1 + \hat{\lambda}_1)\hat{\lambda}_1^{\frac{1}{2}+\frac{n}{4}} \varepsilon^{\frac{1}{n^2+1}}, \qquad \forall s \in]0, \varepsilon[,\ x \in \Gamma.$$

Hence

$$|\nabla\phi_1(x + s\nu(x))| \geq -\nabla\phi_1(x + s\nu(x)) \cdot \nu(x)| \geq -\partial_\nu\phi_1(x) - C(1 + \hat{\lambda}_1)\hat{\lambda}_1^{\frac{1}{2}+\frac{n}{4}} \varepsilon^{\frac{1}{n^2+1}}, \quad \forall s \in]0, \varepsilon[,\ x \in \Gamma.$$

Recall that the constants $\varepsilon_0 > 0$ and C_0 in inequality (39) depend only on Ω, n, and A. Since inequality (39) is valid for all $\varepsilon \in]0, \varepsilon_0[$, we deduce from Lemmata A.2, and A.3, the desired result. □

Let $T_{13} \in \mathbb{R}$ be the unique solution to the following equation

$$\frac{1}{2}\|P_1 u_0(x)\|_{L^2(\Omega)} C_0 = Ce^{-\frac{\delta}{2a_+ + \lambda_1^\Omega} T_{13}} \|u_0\|_{L^2(\Omega)},$$

where $C_0 > 0$ is the constant of inequality (39).

Applying the results of Proposition (3.2) to the inequality (38), we obtain

$$|\nabla u(x, T)|^2 \quad \geq \quad e^{-2\hat{\lambda}_1 T} \frac{(\lambda_1^\Omega)^2}{4} \|P_1 u_0\|_{L^2(\Omega)} |\nabla\phi_1(x)|^2, \tag{40}$$

for all $T \geq T_{12}$.

By taking $T_1 = \max(1, T_{11}, T_{12}, T_{1,3})$, we achieve the proof of the theorem. □

Now, we are ready to prove the main Theorem 1.1. Without loss of generality, we assume that $\hat{\lambda}_2 = \min(\hat{\lambda}_2, \widehat{\tilde{\lambda}_2})$. Further $C > 0$ denotes a generic constant that depending on A_0, Ω, n and u_0.

Since u satisfies (1), a verifies the following nonlinear transport equation

$$\operatorname{div}(\frac{\mathsf{a}}{\hat{\lambda}_1} \nabla u(x, T)) = -u(x, T) + \frac{1}{\hat{\lambda}_1} \mathsf{F}(\mathsf{a}; x, T), \quad x \in \Omega. \tag{41}$$

Similarly, $\tilde{\mathsf{a}}$ satisfies the following nonlinear transport equation

$$\operatorname{div}(\frac{\tilde{\mathsf{a}}}{\widehat{\tilde{\lambda}_1}} \nabla \tilde{u}(x, T)) = -\tilde{u}(x, T) + \frac{1}{\widehat{\tilde{\lambda}_1}} \mathsf{F}(\tilde{\mathsf{a}}; x, T), \quad x \in \Omega. \tag{42}$$

Taking the difference between the two equations (41) and (42), we get

$$\operatorname{div}((\frac{\mathsf{a}}{\hat{\lambda}_1} - \frac{\tilde{\mathsf{a}}}{\widehat{\tilde{\lambda}}}) \nabla u(x, T)) = \tag{43}$$

$$\operatorname{div}(\frac{\tilde{\mathsf{a}}}{\widehat{\tilde{\lambda}_1}} \nabla(u(x, T) - \tilde{u}(x, T))) + \tilde{u}(x, T) - u(x, T) + \frac{1}{\hat{\lambda}_1} \mathsf{F}(\mathsf{a}; x, T) - \frac{1}{\widehat{\tilde{\lambda}_1}} \mathsf{F}(\tilde{\mathsf{a}}; x, T), \quad x \in \Omega.$$

Proposition 3.3. *There exist constants $T_2 > 0$, $C > 0$ and $\varepsilon_0 > 0$ that depend only on Ω, n, A_0 and u_0 such that the following inequalities*

$$|\frac{1}{\hat{\lambda}_1} - \frac{1}{\widehat{\tilde{\lambda}_1}}| \quad \leq \quad C\left[e^{\hat{\lambda}_1 T} \|u - \tilde{u}\|_{H^2(\Omega)} + e^{-(\hat{\lambda}_2 - \hat{\lambda}_1)T} \|\mathsf{a} - \tilde{\mathsf{a}}\|_{L^2(\Omega)}\right], \tag{44}$$

and

$$\int_{\Omega_\varepsilon} |\frac{\mathsf{a}}{\hat{\lambda}_1} - \frac{\tilde{\mathsf{a}}}{\widehat{\tilde{\lambda}_1}}|^2 |\nabla\phi_1|^2 dx + \int_{\Omega} |\frac{\mathsf{a}}{\hat{\lambda}_1} - \frac{\tilde{\mathsf{a}}}{\widehat{\tilde{\lambda}_1}}|^2 \phi_1^2 dx \tag{45}$$
$$\leq C\left[e^{\hat{\lambda}_1 T}\|u-\tilde{u}\|_{H^2(\Omega)} + e^{-(\hat{\lambda}_2-\hat{\lambda}_1)T}\left(\|\mathsf{a}-\tilde{\mathsf{a}}\|_{L^2(\Omega)} + |\frac{1}{\hat{\lambda}_1} - \frac{1}{\widehat{\tilde{\lambda}_1}}|\right)\right]\left\|\frac{\mathsf{a}}{\hat{\lambda}_1} - \frac{\tilde{\mathsf{a}}}{\widehat{\tilde{\lambda}_1}}\right\|_{L^2(\Omega)},$$

hold for all $T \geq T_2$, *and* $\varepsilon \in]0, \varepsilon_0[$.

Proof. Multiplying equation (43) by 1, and integrating over Ω, we obtain

$$|\frac{1}{\hat{\lambda}_1} - \frac{1}{\widehat{\tilde{\lambda}_1}}|\left|\int_\Gamma \mathsf{a}_0 \partial_\nu u(.,T) ds(x)\right| \tag{46}$$
$$\leq C\left[\|u-\tilde{u}\|_{H^2(\Omega)} + |\frac{1}{\hat{\lambda}_1} - \frac{1}{\widehat{\tilde{\lambda}_1}}|\|\mathsf{F}(\mathsf{a};.,T)\|_{L^2(\Omega)} + \frac{1}{\widehat{\tilde{\lambda}_1}}\|\mathsf{F}(\mathsf{a};.,T) - \mathsf{F}(\tilde{\mathsf{a}};.,T)\|_{L^2(\Omega)}\right].$$

Recall

$$\begin{aligned}
\|\mathsf{F}(\mathsf{a};.,T)\|^2_{L^2(\Omega)} &= \sum_{k=2}^{\infty}(\hat{\lambda}_k - \hat{\lambda}_1)^2 e^{-2\hat{\lambda}_k T}\|P_k u_0\|^2_{L^2(\Omega)}, \\
&\leq \sum_{k=2}^{\infty}(\hat{\lambda}_k - \hat{\lambda}_1)^2 e^{-2\hat{\lambda}_k T}\|u_0\|^2_{L^2(\Omega)} \\
&\leq \sum_{k=2}^{\infty}(\hat{\lambda}_k - \hat{\lambda}_1)^2 e^{-2(\hat{\lambda}_k - \hat{\lambda}_2)}\|u_0\|^2_{L^2(\Omega)} e^{-2\hat{\lambda}_2 T},
\end{aligned}$$

for all $T \geq 1$.

Consequently

$$\|\mathsf{F}(\mathsf{a};.,T)\|_{L^2(\Omega)} \leq Ce^{-\hat{\lambda}_2 T}, \qquad \forall T \geq 1. \tag{47}$$

Applying inequalities (57), (24) and (47) to the relation (46), yields

$$|\frac{1}{\hat{\lambda}_1} - \frac{1}{\widehat{\tilde{\lambda}_1}}|\left|\int_\Gamma \mathsf{a}_0 \partial_\nu u(.,T) ds(x)\right| \leq C\left[\|u-\tilde{u}\|_{H^2(\Omega)} + e^{-\hat{\lambda}_2 T}\left(|\frac{1}{\hat{\lambda}_1} - \frac{1}{\widehat{\tilde{\lambda}_1}}| + \|\mathsf{a}-\tilde{\mathsf{a}}\|_{L^2(\Omega)}\right)\right].$$

We deduce from inequalities (56), (33), and the fact that $\mathsf{a}_0 \geq 1$, the following estimate

$$e^{-\hat{\lambda}_1 T}|\frac{1}{\hat{\lambda}_1} - \frac{1}{\widehat{\tilde{\lambda}_1}}| \leq C\left[\|u-\tilde{u}\|_{H^2(\Omega)} + e^{-\hat{\lambda}_2 T}\left(|\frac{1}{\hat{\lambda}_1} - \frac{1}{\widehat{\tilde{\lambda}_1}}| + \|\mathsf{a}-\tilde{\mathsf{a}}\|_{L^2(\Omega)}\right)\right].$$

Inequality (36) gives

$$(1 - Ce^{-\theta T})|\frac{1}{\hat{\lambda}_1} - \frac{1}{\widehat{\tilde{\lambda}_1}}| \leq C\left[e^{\hat{\lambda}_1 T}\|u-\tilde{u}\|_{H^2(\Omega)} + e^{-(\hat{\lambda}_2-\hat{\lambda}_1)T}\|\mathsf{a}-\tilde{\mathsf{a}}\|_{L^2(\Omega)}\right],$$

with $\theta = \frac{\delta}{a_+ \lambda_1^\Omega}$. Finally, taking $T \geq \max(T_{21}, 1)$, where $T_{21} \in \mathbb{R}$ verifies $e^{-\theta T_{21}} = \frac{1}{2C}$, provides the first inequality (44).

Now, we shall focus on the second inequality. Let

$$\zeta = \frac{1}{\mathsf{a}}\left(\frac{\mathsf{a}}{\hat{\lambda}_1} - \frac{\tilde{\mathsf{a}}}{\widehat{\tilde{\lambda}_1}}\right).$$

Multiplying again the equation (43) by $\zeta(x)u(x,T)$, and integrating over Ω, we obtain

$$-\frac{1}{2}\int_\Omega \mathsf{a}u(\cdot,T)\nabla u(\cdot,T)\cdot\nabla\zeta^2 dx-\int_\Omega \zeta^2\mathsf{a}|\nabla u(\cdot,T)|^2dx+\int_\Gamma \mathsf{a}\zeta^2u(\cdot,T)\partial_\nu u(\cdot,T)ds(x)$$
$$=\int_\Omega \mathrm{div}(\frac{\tilde{\mathsf{a}}}{\widetilde{\lambda}_1}\nabla(u(\cdot,T)-\tilde{u}(\cdot,T)))\zeta u(\cdot,T)dx+\int_\Omega(\tilde{u}(\cdot,T)-u(\cdot,T))\zeta u(\cdot,T)dx$$
$$+\int_\Omega\left(\frac{1}{\hat{\lambda}_1}\mathsf{F}(\mathsf{a};\cdot,T)-\frac{1}{\widetilde{\lambda}_1}\mathsf{F}(\tilde{\mathsf{a}};\cdot,T)\right)\zeta u(\cdot,T)dx.$$

Integrating by parts the first term on the left hand side, leads to

$$-\frac{1}{2}\int_\Omega\zeta^2u(\cdot,T)\partial_t u(\cdot,T)dx+\frac{1}{2}\int_\Omega\zeta^2\mathsf{a}|\nabla u(\cdot,T)|^2dx-\int_\Gamma\mathsf{a}\zeta^2u(\cdot,T)\partial_\nu u(\cdot,T)ds(x)$$
$$\leq C\left[\|u-\tilde{u}\|_{H^2(\Omega)}+|\frac{1}{\hat{\lambda}_1}-\frac{1}{\widetilde{\lambda}_1}|\|\mathsf{F}(\mathsf{a};\cdot,T)\|_{L^2(\Omega)}+\frac{1}{\widetilde{\lambda}_1}\|\mathsf{F}(\mathsf{a};\cdot,T)-\mathsf{F}(\tilde{\mathsf{a}};\cdot,T)\|_{L^2(\Omega)}\right]\|u(\cdot,T)\|_{L^2(\Omega)}\|\zeta\|_{L^2(\Omega)}.$$

Since the third term on the left side is positive for $T\geq T_1$, we deduce from inequalities (47), (57), (24), the following estimate

$$-\frac{1}{2}\int_\Omega\zeta^2u(\cdot,T)\partial_t u(\cdot,T)dx+\frac{1}{2}\int_\Omega\zeta^2\mathsf{a}|\nabla u(\cdot,T)|^2dx \tag{48}$$
$$\leq C\left[\|u-\tilde{u}\|_{H^2(\Omega)}+e^{-\hat{\lambda}_2T}\left(|\frac{1}{\hat{\lambda}_1}-\frac{1}{\widetilde{\lambda}_1}|+\|\mathsf{a}-\tilde{\mathsf{a}}\|_{L^2(\Omega)}\right)\right]\|u(\cdot,T)\|_{L^2(\Omega)}\|\zeta\|_{L^2(\Omega)},\quad\forall T\geq T_1.$$

On the other hand, we have

$$\|u\|^2_{L^2(\Omega)}=\sum_{k=1}^\infty\hat{\lambda}_k^2e^{-2\hat{\lambda}_kT}\|P_ku_0\|^2_{L^2(\Omega)}\leq e^{-2\hat{\lambda}_1T}\sum_{k=1}^\infty\hat{\lambda}_k^2e^{-2(\hat{\lambda}_k-1)}\|u_0\|^2_{L^2(\Omega)}.$$

We deduce from inequalities (57)

$$\|u\|_{L^2(\Omega)}\leq Ce^{-\hat{\lambda}_1T},\qquad\forall T\geq 1. \tag{49}$$

Since $\frac{1}{\mathsf{a}}\geq\frac{1}{\mathsf{a}_+}$, we have

$$\|\zeta\|_{L^2(\Omega)}\leq C\left\|\frac{\mathsf{a}}{\hat{\lambda}_1}-\frac{\tilde{\mathsf{a}}}{\widetilde{\lambda}_1}\right\|_{L^2(\Omega)}. \tag{50}$$

Combining inequalities (31), (32), (33), (48), (49), and (50), we finally obtain the second inequality (45) for $T\geq T_2=\max(T_1,T_{21})$.

□

Proposition 3.4. *Let $\mathsf{a}\in\mathsf{A}$, and ϕ_1 be the first eigenfunction of L_a. Then for $\varepsilon\in]0,\varepsilon_0[$, there exists a constant $C>0$ depending only on ε,A,n, and Ω such that*

$$\phi_1^2(x)+\mathbb{1}_{\Omega_\varepsilon}(x)|\nabla\phi_1(x)|^2\geq C,\qquad\forall x\in\Omega, \tag{51}$$

Proof. Since $\varepsilon\in]0,\varepsilon_0[$, inequality (39) in Proposition 3.2 implies

$$|\nabla\phi_1(x)|\geq C_0,\qquad\forall x\in\Omega_\varepsilon. \tag{52}$$

Combining (52) with inequality (56) provides the wanted estimate. □

Back to the proof of Theorem 1.1. Fixing $\varepsilon = \frac{\varepsilon_0}{2}$, we deduce from Propositions 3.3 and 3.4 the following estimate

$$\left\| \frac{\mathsf{a}}{\hat{\lambda}_1} - \frac{\tilde{\mathsf{a}}}{\widetilde{\lambda}_1} \right\|_{L^2(\Omega)} \leq C \left[e^{\hat{\lambda}_1 T} \|u - \tilde{u}\|_{H^2(\Omega)} + e^{-(\hat{\lambda}_2 - \hat{\lambda}_1)T} \left(\|\mathsf{a} - \tilde{\mathsf{a}}\|_{L^2(\Omega)} + |\frac{1}{\hat{\lambda}_1} - \frac{1}{\widetilde{\lambda}_1}| \right) \right], \tag{53}$$

for all $T \geq T_2$.

We further assume that $T \geq T_2$. By a simple calculation, we get

$$\left(\frac{1}{\widetilde{\lambda}_1} \int_\Omega |\mathsf{a} - \tilde{\mathsf{a}}|^2 dx \right)^{\frac{1}{2}} \leq \left\| \frac{\mathsf{a}}{\hat{\lambda}_1} - \frac{\tilde{\mathsf{a}}}{\widetilde{\lambda}_1} \right\|_{L^2(\Omega)} + \left| \frac{1}{\hat{\lambda}_1} - \frac{1}{\widetilde{\lambda}_1} \right| \|\tilde{\mathsf{a}}\|_{L^2(\Omega)}.$$

We deduce from inequalities (57), the following estimate

$$\|\mathsf{a} - \tilde{\mathsf{a}}\|_{L^2(\Omega)} \leq C \left[\left\| \frac{\mathsf{a}}{\hat{\lambda}_1} - \frac{\tilde{\mathsf{a}}}{\widetilde{\lambda}_1} \right\|_{L^2(\Omega)} + \left| \frac{1}{\hat{\lambda}_1} - \frac{1}{\widetilde{\lambda}_1} \right| \right]. \tag{54}$$

Combining estimates (54), (53), and (44), yields

$$\|\mathsf{a} - \tilde{\mathsf{a}}\|_{L^2(\Omega)} \leq C \left[e^{\hat{\lambda}_1 T} \|u - \tilde{u}\|_{H^2(\Omega)} + e^{-(\hat{\lambda}_2 - \hat{\lambda}_1)T} \|\mathsf{a} - \tilde{\mathsf{a}}\|_{L^2(\Omega)} \right]. \tag{55}$$

The gap condition (36) gives

$$(1 - Ce^{-\theta T}) \|\mathsf{a} - \tilde{\mathsf{a}}\|_{L^2(\Omega)} \leq Ce^{\hat{\lambda}_1 T} \|u - \tilde{u}\|_{L^2(\Omega)}$$

with $\theta = \frac{\delta}{a_+ \lambda_1^\Omega}$. Finally, taking $T \geq \max(T_2, T_3)$, where $T_3 \in \mathbb{R}$ verifies $e^{-\theta T_3} = \frac{1}{2C}$, provides the main estimate (4) of Theorem 1.1.

Appendix A.

We recall some known properties of the eigenelements of the unbounded operator L_a.

Lemma A.1. *The eigenvalue λ_1 is simple, and has a strictly positive eigenfunction $\phi_1 \in C^1(\overline{\Omega})$.*

Proof. The proof can be found in many references [18,19]. Since it is too short and for the sake of completeness we give it here.

We can recover the second result by using the Min-max principle. It is well known that the smallest eigenvalue λ_1 is the minimizer of the Rayleigh quotient [19]

$$\lambda_1 = \min_{\phi \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega a |\nabla \phi|^2 dx}{\int_\Omega |\phi|^2 dx}.$$

Since $\phi_1 \in H_0^1(\Omega)$ we also have $|\phi_1| \in H^1(\Omega)$ and $\nabla |\phi_1| = \text{sign}(\phi_1) \nabla \phi_1$, we see that $|\phi_1|$ and ϕ_1 has the similar Rayleigh quotient. Therefore, $|\phi_1|$ is also a minimizer of the Rayleigh quotient and, therefore, an eigenfunction associated to λ_1. By Harnack inequality for elliptic operators, $|\phi_1|$ does not vanish in Ω. Since two functions having contant signs can not be orthogonal in $L^2(\Omega)$, λ_1 is simple. We also deduce from elliptic regularity that ϕ_1 is $C^1(\overline{\Omega})$. □

The proof of the lower bound in the following Lemma can be found in Proposition 2.1 and Lemma 2.1 in [7] or Lemma 4.6.1 in [13]. The upper bound can be derived easily using a Harnack inequality inside Ω, and the regularity of both Γ and the first eigenfunction ϕ_1 near the boundary.

Lemma A.2. *Let* $\mathsf{a} \in \mathsf{A}$*, and* ϕ_1 *be the first eigenfunction of* L_a*. Then there exists a constant* $C \in (0,1)$ *that depends only on* A, n *and* Ω *such that*

$$\frac{1}{C} d_\Omega(x) \geq \phi_1(x) \geq C d_\Omega(x) \qquad \forall x \in \Omega; \qquad -\partial_\nu \phi_1(x) > C \qquad \forall x \in \Gamma. \tag{56}$$

The proof of the following lemma based on the Min-max principle is forward.

Lemma A.3. *Let* $\mathsf{a} \in \mathsf{A}$*, and let* λ_k*,* $k \in \mathbb{N}^*$*, be the increasing eigenvalues of* L_a*. Then*

$$\lambda_k^\Omega \leq \lambda_k \leq a_+ \lambda_k^\Omega, \qquad \forall k \in \mathbb{N}^*, \tag{57}$$

where $\lambda_k^\Omega, k \in \mathbb{N}^*$*, are the increasing Dirichlet eigenvalues of the Laplacian* $-\Delta$ *in* Ω*.*

The proof of the following lemma is based on the analysis of the rate of decay of the heat kernel, and can be found in (Corollary 4.6.3 of [13]).

Lemma A.4. *Let* $\mathsf{a} \in \mathsf{A}$*, and let* λ_k*,* $k \in \mathbb{N}^*$*, be the increasing eigenvalues of* L_a*, and* ϕ_k*,* $k \in \mathbb{N}^*$*, be corresponding orthonormal sequence of eigenfunctions. Then there exists a constant* $C > 0$ *depending on* A, n *and* Ω *such that*

$$|\phi_k(x)| \leq C \lambda_k^{\frac{1}{2}+\frac{n}{4}} \phi_1(x), \qquad \forall x \in \Omega,\ k \in \mathbb{N}^*. \tag{58}$$

References

[1] G. Alessandrini, *An identification problem for an elliptic equation in two variables,* Ann. Mat. Pura e Appl., 4 (1986), 265-296. 1, 2

[2] G. Alessandrini. *A small collection of open problems.* arXiv preprint arXiv:2003.11441 (2020). 2

[3] G. Alessandrini, and S. Vessella. *Error estimates in an identification problem for a parabolic equation,* Boll. Un. Mat. Ital. C(6) 4 (1985), 183-203. 1

[4] G. Alessandrini, M. Di Cristo, E. Francini, and S. Vessella. *Stability for quantitative photoacoustic tomography with well-chosen illuminations,* Ann. Mat. Pura e Appl., 2017, 196, (2) 395-406. 1

[5] H. Ammari, J. Garnier, H. Kang, L. Nguyen, and L. Seppecher. *Multi-Wave Medical Imaging. Modelling and Simulation in Medical Imaging*, Volume 2, World Scientific, London, 2017, 1

[6] H. Ammari, and H. Kang. *Reconstruction of Small Inhomogeneities from Boundary Measurements, Lecture Notes in Mathematics,* Volume 1846, Springer-Verlag, Berlin 2004. 2

[7] K. Ammari, M. Choulli, and F.Triki. *Hölder stability in determining the potential and the damping coefficient in a wave equation.* Journal of Evolution Equations 19.2 (2019): 305-319. 14

[8] K. Ammari, and F. Triki. *On Weak Observability for Evolution Systems with Skew-Adjoint Generators.* SIAM Journal on Mathematical Analysis 52 (2), 1884-1902 (2020). 2

[9] C. Bardos, G. Lebeau, and J. Rauch. *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary,* SIAM J. Control Optim., **30** (1992), 1024-1065. 2

[10] E. Bonnetier, M. Choulli, and F. Triki. *Hölder stability for the qualitative photoacoustic tomography.* Submitted (2019). 1

[11] M. Choulli. *Une introduction aux problèmes inverses elliptiques et paraboliques,* Mathématiques et Applications, 65, Springer, 2009. 1, 2, 7

[12] J. Cheng, and L. Jijun. *An inverse source problem for parabolic equations with local measurements.* Applied Mathematics Letters 103 (2020): 106213. 2

[13] E. B. Davies. *Heat kernels and spectral theory.* Cambridge university press. (Vol. 92) 1990. 14, 15

[14] L. Duckstein, and S. Yakowitz. *Instability in aquifer identification: Theory and case studies,* Water Resources Research,16, 6, 1980, 1045-1064. 1

[15] V. Isakov. *Inverse parabolic problems with the final overdetermination,* Comm. Pure Appl. Math. 44 (1991), no. 2, 185-209. 2

[16] V. Isakov. *Inverse problems for partial differential equations.* Vol. 127. New York: Springer, 2006. 2

[17] T. Kato. *Perturbation theory for linear operators.* Vol. 132. Springer Science & Business Media, 2013. 5

[18] D. Gilbarg, and N. S. Trudinger. *Elliptic partial differential equations of second order.* springer, 2015. 2, 4, 5, 10, 14

[19] A. Henrot. *Extremum problems for eigenvalues of elliptic operators.* Springer Science & Business Media, 2006. 7, 9, 14

[20] M. Reed, and B. Simon. *Methods of Modern Mathematical Physics IV: Analysis of Operators,* Academic Press, 1977. 3, 7

[21] C. D. Sogge. *Concerning the Lp norm of spectral clusters for second-order elliptic operators on compact manifolds.* J. Funct. Anal. 77 (1988), 123-134. 10

[22] A. Tikhonov. *Théorèmes d'unicité pour l'équation de la chaleur,* Mat. Sbor. 42, 1935, pp. 199- 216. 1

[23] X. Xu. *Gradient estimates for the eigenfunctions on compact manifolds with boundary and Hörmander multiplier theorem.* Forum Mathematicum. Vol. 21. No. 3. De Gruyter, 2009. 10

[24] M. Yamamoto. *Carleman estimates for parabolic equations and applications.* Inverse problems 25.12 (2009): 123013. 2

[25] E. Zuazua. *Controllability and observability of partial differential equations: some results and open problems.* Handbook of differential equations: evolutionary equations. Vol. 3. North-Holland, 2007. 527-621. 2

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