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ON ADF GOODNESS-OF-FIT TEST WITH PARAMETRIC HYPOTHESES FOR ERGODIC DIFFUSION PROCESS USING A MINIMUM DISTANCE ESTIMATOR

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A problem of the construction of the goodness-of-fit test for a continuous time ergodic diffusion process is considered. The basic hypothesis is parametric and we use a minimum distance estimator based on the local time estimator (empirical density) of the invariant density. We construct an asymptotically distribution free test using two linear transformations applied to the normalized deviation of the empirical density.

1. Introduction. Goodness-of-fit (GoF) tests play an important role in theoretical and applied statistics. They allow to verify the correspondence between the proposed theoretical models and real data. The goal of this work is the construction of the GoF test in the case of observations of ergodic diffusion process under the parametric basic hypothesis. We propose an asymptotically distribution free test, which is based on two linear transformations of the normalized deviation of the empirical density. Note that the test with the limit distribution not depending on the underlying model is called *asymptotically distribution free* (ADF).

First of all we remind the problem of the construction of the Cramér-von Mises test for the simple basic hypothesis in the case of the i.i.d. classical model. We shall mention that this problem was widely studied by many authors, e.g., [6], [7], [1] and [21]. We have n i.i.d. observations $X^n = (X_1, \dots, X_n)$ with distribution function $F(x)$ and the null hypothesis is simple \mathcal{H}_0 against any fixed alternative \mathcal{H}_1 :

$$\mathcal{H}_0 : F(x) = F_0(x), \quad \mathcal{H}_1 : F(x) \neq F_0(x), \quad x \in \mathbb{R},$$

where $F_0(\cdot)$ is a known continuous distribution function. We have to construct the Cramér-von Mises test Ψ_n of asymptotic size $\alpha \in (0, 1)$, i.e.,

$$\lim_{n \rightarrow \infty} \mathbf{E}_0 \Psi_n = \alpha.$$

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Here Ψ_n is the probability to reject the hypothesis \mathcal{H}_0 and \mathbf{E}_0 is the mathematical expectation under \mathcal{H}_0 . Many GoF tests are based on the following convergence (under \mathcal{H}_0) :

$$\sqrt{n} \left(\hat{F}_n(x) - F_0(x) \right) \Longrightarrow B(F_0(x)), \quad \hat{F}_n(x) = \frac{1}{n} \sum_{j=1}^n \mathbb{I}_{\{X_j < x\}},$$

where $B(\cdot)$ is a Brownian bridge and $\hat{F}_n(x)$ is the empirical distribution function. Therefore, we have the convergence for the Cramér-von Mises statistic (with the change of variable $s = F_0(x)$)

$$\begin{aligned} \phi_n &= n \int_{-\infty}^{\infty} \left(\hat{F}_n(x) - F_0(x) \right)^2 dF_0(x) \\ &\Longrightarrow \int_{-\infty}^{\infty} B(F_0(x))^2 dF_0(x) = \int_0^1 B(s)^2 ds \equiv \phi. \end{aligned}$$

The limit process ϕ does not depend on the model (here $F_0(\cdot)$). Hence the Cramér-von Mises test $\Psi_n = \mathbb{I}_{\{\phi_n > c_\alpha\}}$ is ADF, of asymptotic size α and consistent under alternative \mathcal{H}_1 , i.e.,

$$\mathbf{P}_F \{ \phi_n > c_\alpha \} \rightarrow 1, \quad \text{as } n \rightarrow +\infty.$$

Indeed, the threshold c_α which is solution of the equation

$$\mathbf{P}_0 \{ \phi > c_\alpha \} = \alpha, \quad \alpha \in (0, 1)$$

is the same for all possible $F_0(\cdot)$ (see, e.g., [21]).

Let us recall what happens in the case of the parametric null hypothesis

$$\mathcal{H}_0 : F(\cdot) \in \{F(\vartheta, \cdot), \vartheta \in \Theta = (a, b)\},$$

where $0 < a < b < \infty$ and $F(\vartheta, x)$ is known smooth function of ϑ (unknown and one-dimensional) and x . We have to construct the GoF test $\hat{\Psi}_n$ of asymptotic size α , i.e.,

$$\lim_{n \rightarrow \infty} \mathbf{E}_\vartheta \hat{\Psi}_n = \alpha, \quad \text{for all } \vartheta \in \Theta,$$

where \mathbf{E}_ϑ is the mathematical expectation under \mathcal{H}_0 .

Introduce the Cramér-von Mises type statistic

$$(1.1) \quad \hat{\phi}_n = \int_{-\infty}^{+\infty} U_n(x)^2 dF(\hat{\vartheta}_n, x), \quad U_n(x) = \sqrt{n} \left(\hat{F}_n(x) - F(\hat{\vartheta}_n, x) \right),$$

Here Ψ_n is the probability to reject the hypothesis \mathcal{H}_0 and \mathbf{E}_0 is the mathematical expectation under \mathcal{H}_0 . Many GoF tests are based on the following convergence (under \mathcal{H}_0) :

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$$\lim_{n \rightarrow \infty} \mathbf{E}_\vartheta \hat{\Psi}_n = \alpha, \quad \text{for all } \vartheta \in \Theta,$$

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Based on the above linear transformation applied to $U_n(\cdot)$ the "empirical version" of $U(\cdot)$, we introduce the statistic

$$\hat{v}_n = \int_{-\infty}^{\infty} [L[U_n](x)]^2 dF(\hat{\vartheta}_n, x)$$

and we have the convergence (under \mathcal{H}_0)

$$\hat{v}_n \Rightarrow \int_{-\infty}^{\infty} W(F(\vartheta, x))^2 dF(\vartheta, x) = \int_0^1 W(t)^2 dt \equiv \hat{v}.$$

The limit statistic \hat{v} does not depend on $F(\cdot, \cdot)$ and the unknown parameter ϑ . Therefore the test

$$\hat{\Phi}_n = \mathbb{I}_{\{\hat{v}_n > k_\alpha\}}, \quad \mathbf{P}_\vartheta \{\hat{v} > k_\alpha\} = \alpha$$

is ADF and of asymptotique size α . Note that the linear transformation $L[\cdot]$ defined by (1.4) was proposed by Khmaladze [14] and is based on two strong results. One of Shepp [28] (equivalence of Gaussian measures) and the second of Hitsuda [12] (representation of Gaussian processes equivalent to Wiener process). Another (direct) proof of this result was obtained recently by Kleptsyna & Kutoyants [15] using the solution of Fredholm equation of the second kind with degenerated kernel.

We have to emphasize that in these papers and many other works, e.g., Maglapheridze et al. [22], the estimator used was always the MLE and this is important for the construction of the linear transformation $L[\cdot]$ (1.4) and ADF tests and in our work it is the minimum distance estimator (MDE).

There are at least two reasons to use this approach. The first one : the calculation of the MLE involves the calculation of the large number of stochastic integrals. Its calculation in real problems where the driving process is not exactly Wiener process can be an unstable problem. This problem does not exists in the calculation of the MDE. Therefore the algorithms are more robust. The second reason is of theoretical order. In the large majority of the known GoF tests to obtain the ADF test we are obliged to use the MLE [14] and the theoretical study of the use of non-MLE is rather poor.

In this case, the limit expression for the underlying statistics is as follows:

$$(1.5) \quad U(t) = B(t) - \int_0^1 \tilde{g}(\vartheta, s) dB(s) \int_0^t \tilde{h}(\vartheta, s) ds, \quad \int_0^1 \tilde{g}(\vartheta, s)^2 ds = 1,$$

but with two different functions $\tilde{g}(\cdot, \cdot)$ and $\tilde{h}(\cdot, \cdot)$. Note that this representation can be obtained using the same arguments as it was done in obtention

in the equation (1.3), i.e., as in the case of the MLE. We don't give the details here because the representation (1.5) will not be used in the paper. Now, if we use the relation $B(t) = W(t) - W(1)t$, where $W(t), 0 \leq t \leq 1$ is a Wiener process, then we obtain

$$U(t) = W(t) - \sum_{l=1}^3 \int_0^1 g_l(\vartheta, s) dW(s) \int_0^t h_l(\vartheta, s) ds,$$

where $h_1(\vartheta, s) = g_1(\vartheta, s) = 1$, $h_2(\vartheta, s) = h_3(\vartheta, s) = \tilde{h}(\vartheta, s)$, $g_2(\vartheta, s) = \tilde{g}(\vartheta, s)$ and $g_3(\vartheta, s) = -\int_0^1 \tilde{g}(\vartheta, s) ds$. This will increase the dimension of the functions

$$\tilde{g}(\vartheta, s) = (g_1(\vartheta, s), g_2(\vartheta, s), g_3(\vartheta, s))$$

and

$$\tilde{h}(\vartheta, s) = (h_1(\vartheta, s), h_2(\vartheta, s), h_3(\vartheta, s)).$$

Even if the estimated parameter is one-dimensional, the form of the linear transformation (1.4) of the process (1.5) will become much more complicated. This is probably the reason why this problem was not considered till now.

Recently, the problem of the construction of the linear transformation in the case of other estimators (i.e., if we have not MLE) with limit Gaussian process

$$(1.6) \quad U(t) = W(t) - \int_0^1 g(\vartheta, s) dW(s) \int_0^t h(\vartheta, s) ds$$

was studied in [3]. Note that this representation can be obtained for a large class of estimators in the case of the stochastic processes models. The linear transformation is based on the solution of Fredholm equation of the second kind with degenerated kernel. The several steps of the proof can be found in [3]. We have to note that this linear transformation is rather cumbersome and we understand that this result is in some sense "negative" and says that if we have no MLE it is better to seek another method to obtain the ADF GoF test. In our problem (see (1.6)), the main difference with the i.i.d. case (see (1.5)) is due to the Wiener process. Therefore, it is possible to construct a linear transformation which transforms the Gaussian process (1.6) into Wiener process. Indeed, this property allows us to construct ADF GoF tests for several stochastic processes models (see [3] for the continuous time diffusion process with "small noise" and [2] for the continuous time inhomogeneous Poisson process).

In the present work, we consider a similar problem of the construction of the GoF test for the continuous time ergodic diffusion process with parametric basic hypothesis. We propose an ADF GoF test, which is based on two linear transformations applied to the normalized deviation of the empirical density. The first one was introduced in [17] and the second one was constructed in [3].

First we remind what happens in the case of the simple basic hypothesis for this model of observations (see [17] for more details):

\mathcal{H}_0 : the observed diffusion process satisfies the stochastic differential

$$dX_t = S_0(X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T,$$

where W_t is a Wiener process, $S_0(\cdot)$ is some known function and X_0 is the initial value of X_t .

In this statement, the diffusion process is supposed to be ergodic with the density of invariant law $f_{S_0}(x)$ (can be found, e.g., in [16]). We denote by $F_{S_0}(x)$ the corresponding distribution function. Introduce the *local time estimator* (*empirical density*) of the invariant density

$$(1.7) \quad \hat{f}_T(x) = \frac{2\Lambda_T(x)}{\sigma(x)^2 T},$$

where $\Lambda_T(x)$ is the *local time* of the observed diffusion process

$$\Lambda_T(x) = \frac{|X_T - x| - |X_0 - x|}{2} - \frac{1}{2} \int_0^T \operatorname{sgn}(X_t - x) dX_t.$$

The above definition can be found, e.g., in [16] Chapter 1, Section 1.1.3.

The Cramér-von Mises type statistic based on the *empirical density*

$$\hat{\delta}_T = \int_{-\infty}^{\infty} \zeta_T(x)^2 dF_{S_0}(x), \quad \zeta_T(x) = \sqrt{T} \left(\hat{f}_T(x) - f_{S_0}(x) \right)$$

admits the following limit (under the hypothesis \mathcal{H}_0)

$$\hat{\delta}(S_0) = \int_{-\infty}^{\infty} \zeta(S_0, x)^2 dF_{S_0}(x).$$

Here

$$\zeta(S_0, x) = 2 f_{S_0}(x) \int_{-\infty}^{\infty} \frac{F_{S_0}(y) - \mathbb{1}_{\{y>x\}}}{\sigma(y) \sqrt{f_{S_0}(y)}} dW(y),$$

where $W(\cdot)$ is double-sided Wiener process and the limit process $\hat{\delta}(S_0)$ depends strongly on the model (here $S_0(\cdot)$ and $\sigma(\cdot)$) (the proof can be found in [16]).

To avoid this problem and obtain the ADF test, a linear transformation $L_1[\zeta_T](\cdot)$ of the random function $\zeta_T(\cdot)$ was proposed in [17]

$$(1.8) \quad L_1[\zeta_T](x) = \int_{-\infty}^x \sigma(y) f_{S_0}(y) d \left[\frac{\zeta_T(y)}{2f_{S_0}(y)} \right].$$

Therefore,

$$(1.9) \quad \tilde{\delta}_T = \int_{-\infty}^{\infty} [L_1[\zeta_T](x)]^2 dF_{S_0}(x) \implies \int_0^1 w_s^2 ds,$$

where $w_s, 0 \leq s \leq 1$ is a Wiener process. Then, after transformation, the test statistic $\tilde{\delta}_T$ becomes as follows:

$$\tilde{\delta}_T^* = \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{T}} \int_0^T \frac{\mathbb{I}_{\{X_s \leq x\}}}{\sigma(X_s)} [dX_s - S_0(X_s) ds] \right]^2 dF_{S_0}(x)$$

with the same limit (1.9). Note that this statistic was introduced in the similar problem in [23] to construct the Kolmogorov-Smirnov type ADF test. Therefore, the test

$$\phi_T^* = \mathbb{I}_{\{\tilde{\delta}_T^* > c_\alpha\}}, \quad \mathbf{P} \left(\int_0^1 w_s^2 ds > c_\alpha \right) = \alpha$$

is ADF under the simple basic hypothesis \mathcal{H}_0 (see [17] for more details).

The same problem was studied in [15] in the case of the parametric null hypothesis \mathcal{H}_0 that

$$dX_t = S(\vartheta, X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T, \quad \vartheta \in \Theta = (a, b).$$

The trend coefficient $S(\vartheta, x)$ is some known (smooth) function which depends on the unknown parameter ϑ . Let the invariant density $f(\vartheta, x)$ be given by

$$(1.10) \quad f(\vartheta, x) = \frac{1}{G(\vartheta) \sigma(x)^2} \exp \left\{ 2 \int_0^x \frac{S(\vartheta, y)}{\sigma(y)^2} dy \right\}, \quad x \in \mathbb{R},$$

where $G(\vartheta)$ is the normalizing constant defined by

$$(1.11) \quad G(\vartheta) = \int_{-\infty}^{+\infty} \sigma(x)^2 \exp \left\{ 2 \int_0^x \frac{S(\vartheta, y)}{\sigma(y)^2} dy \right\} dx.$$

We denote by $F(\vartheta, x)$ the corresponding distribution function.

Introduce the statistic

$$\hat{\Delta}_T = \int_{-\infty}^{\infty} \hat{\mu}_T(\hat{\vartheta}_T, x)^2 dF(\hat{\vartheta}_T, x), \quad \hat{\mu}_T(\hat{\vartheta}_T, x) = \sqrt{T} \left(\hat{f}_T(x) - f(\hat{\vartheta}_T, x) \right),$$

where $\hat{\vartheta}_T$ is the MLE of the parameter ϑ . Unfortunately, the immediate use of the test $\hat{\varphi}_T = \mathbb{I}_{\{\hat{\Delta}_T > d_\alpha\}}$ leads to the same problem as in the i.i.d. case. Indeed, the limit distribution of this statistic (as $T \rightarrow \infty$) under \mathcal{H}_0 depends on the model, i.e., $S(\cdot, \cdot)$, $\sigma(\cdot)$ and the unknown parameter ϑ .

In order to obtain the ADF test, two linear transformations which transform the limit statistic into Wiener process were introduced. It was shown that the first linear transformation (1.8) of the statistic $\hat{\mu}_T(\cdot, \cdot)$ gives us statistic which is asymptotically equivalent to the statistic

$$\hat{\xi}_T(\hat{\vartheta}_T, x) = \frac{1}{\sqrt{T}} \int_0^T \frac{\mathbb{I}_{\{X_s < x\}}}{\sigma(X_s)} \left[dX_s - S(\hat{\vartheta}_T, X_s) ds \right].$$

Both statistics converge to the same limit Gaussian process (as $T \rightarrow \infty$)

$$(1.12) \quad U(t) = W(t) - \int_0^1 \bar{h}(\vartheta, s) dW(s) \int_0^t \bar{h}(\vartheta, s) ds, \quad 0 \leq t \leq 1,$$

where

$$\bar{h}(\vartheta, s) = I(\vartheta)^{-\frac{1}{2}} \frac{\dot{S}(\vartheta, F^{-1}(\vartheta, s))}{\sigma(F^{-1}(\vartheta, s))}, \quad \int_0^1 \bar{h}(\vartheta, s)^2 ds = 1.$$

Here $F^{-1}(\vartheta, s)$ is the inverse function of $F(\vartheta, y)$, i.e., y is the solution of the equation $s = F(\vartheta, y)$, $I(\vartheta)$ is the Fisher information and $W(s)$, $0 \leq s \leq 1$ is a Wiener process. Note that the process (1.12) is in some sense a "universal limit" which appears in the problems of goodness-of-fit testing for stochastic processes. For example, the same limit is obtained in the cases of diffusion process with "small noise" and inhomogeneous Poisson process (see, e.g., [18]). The main difference with the i.i.d. case is due to the Wiener process here (see (1.12)), while in the i.i.d. case the Brownian bridge $B(t)$, $0 \leq t \leq 1$ appears (see (1.3)).

The limit Gaussian process $U(\cdot)$ is not distribution free. Hence, the ADF test was based on a second linear transformation $L[\cdot]$ given by (1.4) such that $L[U](t) = w_t$, where w_t , $0 \leq t \leq 1$ is a Wiener process. The convergence

$$\hat{\eta}_T = \int_{-\infty}^{\infty} \left[L \left[\hat{\xi}_T(\hat{\vartheta}_T, \cdot) \right] (x) \right]^2 dF(\hat{\vartheta}_T, x) \implies \int_0^1 w_t^2 dt$$

was proved. Then, the ADF test was defined by $\hat{\phi}_T = \mathbb{I}_{\{\hat{\eta}_T > d_\alpha\}}$ (see [15] for more details).

There are several GoF tests for the continuous time diffusion and inhomogeneous Poisson processes proposed, e.g, in the works [27], [19], [18] and [4]. See also [5] where some GoF tests for diffusion and inhomogeneous Poisson processes with simple basic hypothesis were studied. It was shown that these tests are ADF. The Kolmogorov-Smirnov test for ergodic diffusion process was studied, e.g., in [9] and [10]. In [16], Section 5.4, the author discusses some possibilities of the construction of the Kolmogorov-Smirnov and the Cramér-von Mises tests when the ergodic diffusion process is observed on continuous time. The same problem was considered in [11] but for the hypothesis with sign-type trend coefficient. In this case, two Cramér-von Mises GoF tests based on the empirical distribution function and the empirical density were studied. More about GoF tests for the ergodic diffusion process can be found, e.g., in Kleptsyna & Kutoyants [15] for the parametric hypotheses and Negri & Nishiyama [23] for the simple hypotheses.

We shall mention that the general case of ergodic diffusion process with unknown shift (one dimensional) parameter was introduced in [26]. The authors showed that the limit distribution of the Cramér-von Mises type statistic does not depend on the unknown parameter. Therefore, the test based on this statistic is *asymptotically parameter free* (APF). Moreover, the APF Kolmogorov-Smirnov type tests were studied in [29]. Similar results for the APF tests of the Cramér-von Mises type statistic have been obtained by Kutoyants [17]. The author studied the case of the composite basic hypothesis with a parametric class of diffusion processes including the Ornstein-Uhlenbeck and simple switching processes.

More problems of the construction of the GoF tests for the discrete time observations were widely studied by many authors, e.g., Negri & Nishiyama [25] in the case of the small diffusion process. In [24], the proposed ADF GoF tests were based on continuous time observations, on discrete time observations and on the so-called tick time sampled data for ergodic diffusion processes.

We need some notations else to introduce the linear transformation proposed in [3]. Denote

$$(1.13) \quad \begin{aligned} I_1 &= \int_0^s g(\vartheta, q)^2 dq, \quad I_2 = \int_0^s h(\vartheta, q) g(\vartheta, q) dq, \quad I_3 = \int_0^s h(\vartheta, q) dq, \\ I_4 &= \int_0^s h(\vartheta, q)^2 dq, \quad I_5 = \int_0^s g(\vartheta, q) dq, \end{aligned}$$

where the functions $h(\vartheta, q)$ and $g(\vartheta, q)$ will be defined later in Section 4.

Below we omit ϑ and s for simplicity and put $g = g(\vartheta, s)$ and $h = h(\vartheta, s)$. Introduce the functions

(1.14)

$$\begin{aligned} \varphi_1(s) = & g - h - 3I_2g + I_5hg + I_3g^2 + 2I_2h - 2I_2I_3g^2 + I_1I_2^2h + I_4I_5g^2 \\ & - I_2I_4g + 3I_2^2g + I_2I_5h^2 - 2I_2I_5hg - 2I_1I_2h + I_2^2I_5hg + I_1^2I_3h^2 - I_4h \\ & + 2I_1I_4h - I_1I_4g + I_1I_2I_4g + I_1I_4I_5hg - I_1^2I_4h + I_1h + 2I_2I_3hg + I_4g \\ & - I_2I_4I_5g^2 - I_5h^2 + 2I_1I_3hg - 2I_1I_2I_3hg - 2I_1I_3h^2 - I_2^2h + I_3h^2 \\ & - 2I_3hg - I_4I_5hg - I_1I_2I_5h^2 + I_2^2I_3g^2 + I_1I_5h^2 - I_2^3g, \end{aligned}$$

(1.15)

$$\begin{aligned} \varphi_2(s) = & 1 + I_5h - 3I_2I_5h + I_1I_3h + I_3g - 3I_2I_3g + I_4I_5g - I_3h - I_1I_4^2I_5g \\ & + 3I_2^2I_3g - 2I_2I_4I_5g + 2I_2I_3h - I_2^3I_5h + I_1I_2^2I_3h - I_2^3I_3g + 3I_2^2I_5h \\ & + I_4I_5h - I_2I_4I_5h + 2I_1I_3I_4h + I_3I_4g - I_2I_3I_4g + I_4^2 + 2I_2^2I_4 + I_2^4 \\ & + I_1I_2I_4I_5h - I_1^2I_3I_4h - I_1I_3I_4g + I_1I_2I_3I_4g - I_2^2I_3h - 2I_1I_2^2I_4 \\ & - 2I_1I_4^2 + I_1^2I_4^2 + 2I_4 - 2I_1I_4 - 4I_2I_4 + 4I_1I_2I_4 - 4I_2 + I_4^2I_5g \\ & + I_2^2I_4I_5g - 2I_1I_2I_3h - I_3I_4h + 6I_2^2 - 4I_2^3 - I_1I_4I_5h \end{aligned}$$

and

(1.16)

$$\begin{aligned} \psi_2(s) = & h + I_3hg - 2I_2I_4g + I_5h^2 - 3I_2h - 2I_2I_3hg - I_1I_2I_3h^2 + I_4g \\ & + I_2^2I_4g + I_2^2I_5h^2 - I_2^3h - I_3I_4hg + I_4^2g + I_4h - I_2I_4h - 2I_2I_5h^2 \\ & - I_1I_4h + I_1I_2I_4h + I_1I_3h^2 + I_3I_4g^2 + I_2I_3h^2 - I_3h^2 + I_1I_3I_4hg \\ & - I_2I_3I_4g^2 - 2I_2I_4I_5hg + 2I_4I_5hg + I_2^2I_3hg - I_1I_4^2g + I_4^2I_5g^2 + 3I_2^2h. \end{aligned}$$

Let the linear transformation be given by

$$(1.17) \quad L_2[U](t) = U(t) + \int_0^t \int_0^s \frac{\varphi_1(s)h(\vartheta, q) + \psi_2(s)g(\vartheta, q)}{\varphi_2(s)} dU(q) ds = w_t.$$

Here w_t , $0 \leq t \leq 1$ is a Wiener process, the Gaussian process $U(\cdot)$ is given by (1.6) and we suppose that $\varphi_2(s)$ is a strictly positive function on $[0, 1)$.

In our work the basic model is a continuous time ergodic diffusion process $X^T = \{X_t, 0 \leq t \leq T\}$ solution of the stochastic differential equation

$$dX_t = S(X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T,$$

where W_t , $0 \leq t \leq T$ is a Wiener process. The trend coefficient $S(\cdot)$ is an unknown function, the diffusion coefficient $\sigma(x)^2 > 0$ is known and X_0 is the initial value of X_t . We consider the parametric null hypothesis

$$\mathcal{H}_0 : S(\cdot) \in \{S(\vartheta, \cdot), \vartheta \in \Theta = (a, b)\},$$

i.e., the process X^T is solution of the equation

$$dX_t = S(\vartheta, X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T,$$

where $S(\vartheta, x)$ is known smooth function. We suppose that this process has ergodic properties with invariant density (1.10). In Section 3, we introduce for the unknown parameter ϑ the MDE based on the invariant density $f(\vartheta, x)$ and the *local time estimator* (1.7) (see [16])

$$\vartheta_T^* = \arg \inf_{\vartheta \in \Theta} \int_{\mathbb{R}} [\hat{f}_T(x) - f(\vartheta, x)]^2 dx.$$

In Section 4, we consider the Cramér-von Mises type statistic

$$\delta_T = \int_{-\infty}^{\infty} \eta_T^*(\vartheta_T^*, x)^2 dF(\vartheta_T^*, x), \quad \eta_T^*(\vartheta_T^*, x) = \sqrt{T} \left(\hat{f}_T(x) - f(\vartheta_T^*, x) \right)$$

and under \mathcal{H}_0 , we obtain the convergence

$$\eta_T^*(\vartheta_T^*, x) \implies \eta(\vartheta, x),$$

where

$$(1.18) \quad \eta(\vartheta, x) = 2 f(\vartheta, x) \int_{-\infty}^{+\infty} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y) \sqrt{f(\vartheta, y)}} dW(y) - 2J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y) \sqrt{f(\vartheta, y)}} f(\vartheta, x) \dot{f}(\vartheta, x) dx dW(y) \dot{f}(\vartheta, x).$$

Our goal is to transform the limit process (1.18) into Wiener process in order to construct the ADF test. We do it in two steps. We apply the first linear transformation (see (1.8)) to the process $\eta(\cdot, \cdot)$

$$L_1[\eta](x) = \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[\frac{\eta(\vartheta, y)}{2f(\vartheta, y)} \right],$$

which admits the representation $U(\cdot)$ (1.6) and the functions $g(\cdot, \cdot)$ and $h(\cdot, \cdot)$ will be defined in Section 4 below.

In order to obtain the Wiener process, we introduce the second linear transformation $L_2 [\cdot]$ (1.17) of the Gaussian process $U(\cdot)$ such that

$$L_2 [U] (t) = w_t, \quad 0 \leq t \leq 1.$$

First, we show the asymptotic equivalence between $L_1 [\eta_T^*] (\cdot)$ and the process

$$(1.19) \quad \xi_T (\vartheta_T^*, x) = \frac{1}{\sqrt{T}} \int_0^T \frac{\mathbb{I}_{\{X_s < x\}}}{\sigma(X_s)} [dX_s - S(\vartheta_T^*, X_s) ds],$$

i.e., both statistics converge to the same process (1.6). We shall mention that the statistic (1.19) was introduced in [15] where the MLE was used for the unknown parameter.

Now, we have to realize the similar transformation $L_2 [\cdot]$ (1.17) with the process (1.19) which is much easier to calculate and then we show in Section 5 the convergence in distribution

$$L_2 [\xi_T] (x) \implies L_2 [U] (t) = w_t, \quad 0 \leq t \leq 1.$$

Hence, we have

$$\delta_T^* = \int_{-\infty}^{\infty} [L_2 [\xi_T] (x)]^2 dF(\vartheta_T^*, x) \implies \int_0^1 w_t^2 dt.$$

Therefore, we present in Theorem 2 below the test $\psi_T^* = \mathbb{I}_{\{\delta_T^* > c_\alpha\}}$ which is ADF because the limit distribution of δ_T^* does not depend on $S(\cdot, \cdot)$, $\sigma(\cdot)$ and ϑ .

2. Preliminaries. Suppose that we have continuous time observations $X^T = \{X_t, 0 \leq t \leq T\}$ of a diffusion process, which is solution of the stochastic differential equation

$$dX_t = S(X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T,$$

where W_t , $0 \leq t \leq T$ is a Wiener process. The trend coefficient $S(\cdot)$ is an unknown function, the diffusion coefficient $\sigma(x)^2 > 0$ is known and X_0 is the initial value of X_t .

We consider the problem of testing the parametric (composite) hypothesis

$$\mathcal{H}_0 \quad : \quad S(\cdot) \in \{S(\vartheta, \cdot), \vartheta \in \Theta = (a, b)\},$$

where $0 < a < b < \infty$, i.e., the process X^T is solution of the equation

$$(2.1) \quad dX_t = S(\vartheta, X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T.$$

Here $S(\vartheta, x)$ is known smooth function depending on the unknown parameter ϑ .

We assume that the trend coefficient $S(\vartheta, x)$, $\vartheta \in \Theta$ and the diffusion coefficient $\sigma(x)^2$ satisfy the condition \mathcal{ES} (existence of the solution) :

\mathcal{ES} . The function $S(\vartheta, x)$ is locally bounded, the function $\sigma(x)^2 > 0$ is continuous and for some $C > 0$, the condition

$$xS(\vartheta, x) + \sigma(x)^2 \leq C(1 + x^2)$$

holds.

By this condition, the stochastic differential equation (2.1) has a unique weak solution for all $\vartheta \in \Theta$ (see for instance [8]).

Denote by \mathcal{P} the class of functions with polynomial majorants ($p > 0$)

$$\mathcal{P} = \{h(\cdot) : |h(x)| \leq C(1 + |x|^p)\}.$$

REMARK 1. The functions $S(\vartheta, \cdot)$ and $\sigma(\cdot)^2$ are such that for all $\vartheta \in \Theta$

$$V(\vartheta, x) = \int_0^x \exp \left\{ -2 \int_0^y \frac{S(\vartheta, z)}{\sigma(z)^2} dz \right\} dy \rightarrow \pm\infty, \quad \text{as } x \rightarrow \pm\infty$$

and

$$G(\vartheta) < \infty,$$

where $G(\vartheta)$ is the normalizing constant given by (1.11).

Let us introduce the following condition (ergodicity):

A_0 . The functions $S(\vartheta, \cdot)$, $\sigma(\cdot)^{\pm 1} \in \mathcal{P}$ and for all ϑ

$$\overline{\lim}_{|x| \rightarrow \infty} \sup_{\vartheta \in \Theta} \operatorname{sgn}(x) \frac{S(\vartheta, x)}{\sigma(x)^2} < 0.$$

If this condition is fulfilled then the stochastic process X^T is recurrent positive, i.e., it has ergodic properties. Then there exists an invariant distribution with the density function (1.10) (see, e.g., [16] for more details).

Introduce the regularity conditions \mathcal{R} . (can be found in Kutoyants [16]) \mathcal{R}_1 . The invariant density function $f(\vartheta, x)$ is differentiable w.r.t. ϑ for all $x \in \mathbb{R}$ and the derivative $\dot{f}(\vartheta, x) \in \mathcal{P}$ is uniformly continuous in the following sense: for any $\vartheta_0 \in \Theta$ and $\nu > 0$

$$\lim_{\nu \rightarrow 0} \sup_{\vartheta_0 \in \Theta} \sup_{|\vartheta - \vartheta_0| \leq \nu} \left\| \dot{f}(\vartheta, \cdot) - \dot{f}(\vartheta_0, \cdot) \right\| = 0.$$

\mathcal{R}_2 . The identifiability condition is

$$\inf_{|\vartheta - \vartheta_0| > \nu} \|f(\vartheta, \cdot) - f(\vartheta_0, \cdot)\| > 0.$$

\mathcal{R}_3 . The function $J(\vartheta) = \|\dot{f}(\vartheta, \cdot)\|^2$ is positive uniformly in ϑ . Here $\|\cdot\|$ is the \mathcal{L}^2 -norm:

$$(2.2) \quad \|\dot{f}(\vartheta, \cdot)\|^2 = \int_{\mathbb{R}} \dot{f}(\vartheta, x)^2 dx.$$

In the presentation below we suppose that these conditions are always fulfilled.

3. Minimum distance estimator. The MDE is based on the invariant density (1.10) and the empirical density (1.7). It is defined as a solution of the problem

$$\|\hat{f}_T(\cdot) - f(\vartheta_T^*, \cdot)\| = \inf_{\vartheta \in \Theta} \|\hat{f}_T(\cdot) - f(\vartheta, \cdot)\|,$$

where $\vartheta \in \Theta = (a, b)$ is the one-dimensional (unknown) parameter. The properties and the asymptotic ($T \rightarrow \infty$) behavior of this estimator for this model were studied under regularity conditions \mathcal{R} and \mathcal{A}_0 in [16].

The MDE ϑ_T^* can be written as well as follows:

$$(3.1) \quad \vartheta_T^* = \arg \inf_{\vartheta \in \Theta} \int_{\mathbb{R}} (\hat{f}_T(x) - f(\vartheta, x))^2 dx$$

and it satisfies the *minimum distance equation* (MDEq) (can be found in [16])

$$\int_{\mathbb{R}} (\hat{f}_T(x) - f(\vartheta_T^*, x)) \dot{f}(\vartheta_T^*, x) dx = 0.$$

Let us put $u_T^* = \sqrt{T}(\vartheta_T^* - \vartheta)$. Then, by Taylor's formula, $f(\vartheta_T^*, x) = f(\vartheta, x) + (\vartheta_T^* - \vartheta) \dot{f}(\tilde{\vartheta}, x)$, where $|\tilde{\vartheta} - \vartheta| \leq |\vartheta_T^* - \vartheta|$, the MDEq becomes as follows:

$$\int_{\mathbb{R}} \left(\sqrt{T} (\hat{f}_T(x) - f(\vartheta, x)) - u_T^* \dot{f}(\tilde{\vartheta}, x) \right) \dot{f}(\vartheta_T^*, x) dx = 0.$$

Now, using the consistency of the MDE, we can write

$$u_T^* = \left(\int_{\mathbb{R}} \dot{f}(\vartheta, x)^2 dx \right)^{-1} \int_{\mathbb{R}} \sqrt{T} (\hat{f}_T(x) - f(\vartheta, x)) \dot{f}(\vartheta, x) dx + o(1).$$

Remind that we have the convergence

$$(3.2) \quad \sqrt{T} \left(\hat{f}_T(x) - f(\vartheta, x) \right) \Longrightarrow 2 f(\vartheta, x) \int_{-\infty}^{\infty} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} dW(y),$$

where $W(\cdot)$ is double-sided Wiener process. For the proof see [16].

Therefore, by Fubini's theorem, ϑ_T^* admits the following representation

$$(3.3) \quad \sqrt{T}(\vartheta_T^* - \vartheta) \Longrightarrow 2J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} f(\vartheta, x) \dot{f}(\vartheta, x) dx dW(y).$$

Here $J(\vartheta)$ is given by (2.2).

Moreover, under regularity conditions \mathcal{R} and \mathcal{A}_0 , the MDE ϑ_T^* is consistent and asymptotically normal (for details see Chapter 2, Section 2.2 in [16])

$$\mathcal{L}_{\vartheta}\{\sqrt{T}(\vartheta_T^* - \vartheta)\} \Longrightarrow \mathcal{L}\{\xi\} = \mathcal{N}(0, \sigma^2(\vartheta)),$$

where

$$\sigma^2(\vartheta) = J(\vartheta)^{-2} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\vartheta, x) \dot{f}(\vartheta, x) D_{\vartheta}(x, y) f(\vartheta, y) \dot{f}(\vartheta, y) dx dy.$$

Here

$$D_{\vartheta}(x, y) = 4 \mathbf{E}_{\vartheta} \left(\frac{[F(\vartheta, \xi) - \mathbb{I}_{\{\xi>x\}}][F(\vartheta, \xi) - \mathbb{I}_{\{\xi>y\}}]}{\sigma(\xi)^2 f(\vartheta, \xi)^2} \right).$$

4. Cramér-von Mises type statistic. We are going to study the ADF GoF test based on the Cramér-von Mises type statistic

$$\delta_T = \int_{-\infty}^{\infty} \eta_T^*(\vartheta_T^*, x)^2 dF(\vartheta_T^*, x),$$

where

$$(4.1) \quad \eta_T^*(\vartheta_T^*, x) = \sqrt{T} \left(\hat{f}_T(x) - f(\vartheta_T^*, x) \right)$$

and ϑ_T^* is the MDE defined by (3.1). We will show that we have the convergence (the proof will be given later in Theorem 1)

$$\delta_T \Longrightarrow \delta \equiv \int_{-\infty}^{\infty} \eta(\vartheta, x)^2 dF(\vartheta, x),$$

where $\eta(\cdot, \cdot)$ is given by (1.18).

Remark that the limit statistic δ depends strongly on the model, i.e., $f(\cdot, \cdot)$ and the unknown parameter ϑ . Therefore the test based on this statistic is not ADF. To avoid this difficulty, we have to transform the limit process (1.18) into Wiener process. Indeed, we propose a first transformation (1.8) for the process $\eta(\cdot, \cdot)$

$$(4.2) \quad L_1[\eta](x) = \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[\frac{\eta(\vartheta, y)}{2f(\vartheta, y)} \right] = U(F(\vartheta, x)).$$

Remind that the above transformation was introduced in the problem of the simple basic hypothesis in [17]. Further, let us define the functions

$$(4.3) \quad h(\vartheta, s) = 2 \tilde{J}(\vartheta)^{-1} C(\vartheta)^{\frac{1}{2}} \frac{\dot{S}(\vartheta, F^{-1}(\vartheta, s))}{\sigma(F^{-1}(\vartheta, s))}$$

and

$$(4.4) \quad g(\vartheta, s) = C(\vartheta)^{-\frac{1}{2}} \int_0^1 \frac{[s - \mathbb{I}_{\{s>t\}}]}{a(\vartheta, s)b(\vartheta, s)} \dot{f}(\vartheta, F^{-1}(\vartheta, t)) dt,$$

where $a(\vartheta, s) = \sigma(F^{-1}(\vartheta, s))$, $b(\vartheta, s) = f(\vartheta, F^{-1}(\vartheta, s))$,

$$(4.5) \quad C(\vartheta) = \int_0^1 \left[\int_0^1 \frac{[v - \mathbb{I}_{\{v>t\}}]}{a(\vartheta, v)b(\vartheta, v)} \dot{f}(\vartheta, F^{-1}(\vartheta, t)) dt \right]^2 dv$$

and

$$\tilde{J}(\vartheta) = \int_0^1 \frac{\dot{f}(\vartheta, F^{-1}(\vartheta, s))^2}{f(\vartheta, F^{-1}(\vartheta, s))} ds, \quad \int_0^1 g(\vartheta, s)^2 ds = 1.$$

Here $F^{-1}(\vartheta, s)$ is the inverse function of $F(\vartheta, y)$, i.e., y is solution of the equation $s = F(\vartheta, y)$.

Then, we show in Lemma 1 that the transformation (4.2) has the following representation

$$(4.6) \quad U(t) = w(t) - \int_0^1 g(\vartheta, s) dw(s) \int_0^t h(\vartheta, s) ds, \quad 0 \leq t \leq 1,$$

where $w(s)$, $0 \leq s \leq 1$ is some Wiener process.

Moreover, we show that the transformation $L_1[\cdot]$ (4.2) of $\eta_T^*(\cdot, \cdot)$ (4.1) gives us a statistic which is asymptotically equivalent to the process $\xi_T(\cdot, \cdot)$ (1.19). This means that both statistics have the same limit Gaussian process (4.6). Therefore, our ADF test will be based on the statistic $\xi_T(\cdot, \cdot)$ instead of $L_1[\eta_T^*(\cdot)]$ for simplicity of calculus.

The last step in the construction of the ADF GoF test is to apply the second linear transformation $L_2[\cdot]$ (1.17) to the limit Gaussian process $U(\cdot)$ (4.6) and to obtain the Wiener process

$$L_2[U](t) = w_t, \quad 0 \leq t \leq 1.$$

Below we realize this program. We have the following result.

THEOREM 1. *Let the conditions \mathcal{R} , \mathcal{ES} and \mathcal{A}_0 be fulfilled, then*

$$(4.7) \quad \eta_T^*(\vartheta_T^*, x) \implies \eta(\vartheta, x), \quad \xi_T(\vartheta_T^*, x) \implies U(F(\vartheta, x))$$

and

$$U(F(\vartheta, x)) = \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[\frac{\eta(\vartheta, y)}{2f(\vartheta, y)} \right].$$

Due to Theorem 1, it is easily seen that

$$\begin{aligned} \Delta_T &= \int_{-\infty}^{\infty} [L_1[\eta_T^*](x)]^2 dF(\vartheta_T^*, x) \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^x \sigma(y) f(\vartheta_T^*, y) d \left[\frac{\eta_T^*(\vartheta_T^*, y)}{2f(\vartheta_T^*, y)} \right] \right)^2 dF(\vartheta_T^*, x) \end{aligned}$$

and the statistic

$$\Delta_T^* = \int_{-\infty}^{\infty} \xi_T(\vartheta_T^*, x)^2 dF(\vartheta_T^*, x)$$

have the same limit process

$$\Delta = \int_{-\infty}^{\infty} \left(\int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[\frac{\eta(\vartheta, y)}{2f(\vartheta, y)} \right] \right)^2 dF(\vartheta, x) = \int_0^1 U(t)^2 dt.$$

Here the Gaussian process $U(\cdot)$ is defined by (4.6). Then, it is obvious to see that the test

$$\psi_T = \mathbb{I}_{\{\Delta_T^* > c_\alpha\}}, \quad \mathbf{P}(\Delta > c_\alpha) = \alpha, \quad \alpha \in (0, 1)$$

is not ADF. So we introduce the linear transformation $L_2[\cdot]$ given by (1.17) such that

$$(4.8) \quad \int_0^1 [L_2[U](t)]^2 dt = \int_0^1 w_t^2 dt, \quad 0 \leq t \leq 1.$$

Based on $L_2[\cdot]$, we construct the ADF GoF test in next Section using the "empirical version" of the test statistic with the same limit (4.8).

5. ADF GoF Test. In this Section, our purpose is to construct the ADF GoF test. Recall that the starting statistic

$$\eta_T^*(\vartheta_T^*, x) = \sqrt{T} \left(\hat{f}_T(x) - f(\vartheta_T^*, x) \right)$$

converges to the random function $\eta(\cdot, \cdot)$ defined by (1.18). In order to obtain the distribution free limit statistic, we transform $\eta(\cdot, \cdot)$ into Wiener process in two steps. Indeed, the first linear transformation $L_1[\eta](\cdot)$ given by (4.2) leads to the Gaussian process $U(\cdot)$ defined by (4.6) which is the same limit of the process

$$\xi_T(\vartheta_T^*, x) = \frac{1}{\sqrt{T}} \int_0^T \frac{\mathbb{I}_{\{X_s < x\}}}{\sigma(X_s)} [dX_s - S(\vartheta_T^*, X_s) ds].$$

Then, we apply the second linear transformation (1.17) such that

$$L_2[U](t) = w_t, \quad 0 \leq t \leq 1.$$

Now, we have to realize the similar transformation with the process $\xi_T(\cdot, \cdot)$, which is the "empirical version" of the Gaussian process $U(\cdot)$. To finish, we have to show the convergence in distribution to the Wiener process

$$L_2[\xi_T](x) \implies w_t, \quad 0 \leq t \leq 1.$$

Therefore, the test $\psi_T^* = \mathbb{I}_{\{\delta_T^* > c_\alpha\}}$ with

$$\delta_T^* = \int_{-\infty}^{\infty} [L_2[\xi_T](x)]^2 dF(\vartheta_T^*, x) \implies \int_0^1 w_t^2 dt$$

will be ADF because the limit distribution of δ_T^* does not depend on $S(\cdot, \cdot)$, $\sigma(\cdot)$ and ϑ .

Let us realize this program. Denote the functions

$$(5.1) \quad \hat{h}(\vartheta, z) = 2 J(\vartheta)^{-1} \hat{C}(\vartheta)^{\frac{1}{2}} \frac{\dot{S}(\vartheta, z)}{\sigma(z)}$$

and

$$(5.2) \quad \hat{g}(\vartheta, z) = \hat{C}(\vartheta)^{-\frac{1}{2}} \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z > y\}}}{\sigma(z)f(\vartheta, z)} f(\vartheta, y) \dot{f}(\vartheta, y) dy.$$

Here $J(\vartheta)$ is defined by (2.2) and

$$(5.3) \quad \hat{C}(\vartheta) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z > y\}}}{\sigma(z)f(\vartheta, z)} f(\vartheta, y) \dot{f}(\vartheta, y) dy \right]^2 f(\vartheta, z) dz.$$

Introduce

$$\hat{I}_1 = \int_{-\infty}^y \hat{g}(\vartheta, z)^2 f(\vartheta, z) dz, \quad \hat{I}_2 = \int_{-\infty}^y \hat{h}(\vartheta, z) \hat{g}(\vartheta, z) f(\vartheta, z) dz,$$

$$\hat{I}_3 = \int_{-\infty}^y \hat{h}(\vartheta, z) f(\vartheta, z) dz, \quad \hat{I}_4 = \int_{-\infty}^y \hat{h}(\vartheta, z)^2 f(\vartheta, z) dz$$

and put

$$\hat{I}_5 = \int_{-\infty}^y \hat{g}(\vartheta, z) f(\vartheta, z) dz.$$

Further, we define the functions $\hat{\varphi}_1(\cdot, \cdot)$, $\hat{\varphi}_2(\cdot, \cdot)$ and $\hat{\psi}_2(\cdot, \cdot)$ given by (1.14), (1.15) and (1.16), respectively, where we replace g and h by $\hat{g} = \hat{g}(\vartheta, y)$ and $\hat{h} = \hat{h}(\vartheta, y)$, respectively. Then, we introduce the statistic

(5.4)

$$w_{F(\vartheta, x)} = U(F(\vartheta, x)) + \int_{-\infty}^x \int_{-\infty}^y \frac{[\hat{\varphi}_1(\vartheta, y) \hat{h}(\vartheta, z) + \hat{\psi}_2(\vartheta, y) \hat{g}(\vartheta, z)]}{\hat{\varphi}_2(\vartheta, y)} dU(F(\vartheta, z)) f(\vartheta, y) dy.$$

To construct the test, we have to replace $U(F(\vartheta, x))$ by $\xi_T(\vartheta_T^*, x)$ and the functions $\hat{h}(\cdot, \cdot)$, $\hat{g}(\cdot, \cdot)$, $\hat{\varphi}_1(\cdot, \cdot)$, $\hat{\varphi}_2(\cdot, \cdot)$ and $\hat{\psi}_2(\cdot, \cdot)$ by their "empirical versions" based only on the observations. In fact, we insert the MDE ϑ_T^* instead of the unknown parameter ϑ in the expressions of the above functions.

In the construction of the test we have to introduce two conditions else.

\mathcal{R}_4 . We suppose that $\varphi_2(s)$ defined by (1.15) is strictly positive function on $[0, 1)$.

\mathcal{R}_5 . The function $S(\vartheta, x)$ has two continuous derivatives w.r.t. ϑ such that

$$\dot{S}(\vartheta, x), \ddot{S}(\vartheta, x) \in \mathcal{P}$$

and the functions $\dot{S}(\vartheta, x)$ and $\sigma(x)$ have continuous derivatives w.r.t. x such that

$$\dot{S}'(\vartheta, x), \sigma'(x) \in \mathcal{P}.$$

Moreover, due to the consistency of the MDE, we have (as $T \rightarrow \infty$)

$$\hat{\varphi}_2(\vartheta_T^*, y) - \hat{\varphi}_2(\vartheta, y) \rightarrow 0.$$

Hence, we can introduce the function

$$\hat{\varphi}_2^+(\vartheta_T^*, y) = \begin{cases} \hat{\varphi}_2(\vartheta_T^*, y)^{-1}, & \text{if } \hat{\varphi}_2(\vartheta_T^*, y) > 0, \\ 0, & \text{otherwise,} \end{cases}$$

which asymptotically coincides with $\hat{\varphi}_2(\vartheta, y)^{-1}$ and therefore the limit distribution does not change.

We construct the ADF GoF test based on the following statistic

$$(5.5) \quad W_T^*(x) = \xi_T(\vartheta_T^*, x) + \int_{-\infty}^x \int_{-\infty}^y \hat{\varphi}_2^+(\vartheta_T^*, y) \left[\hat{\varphi}_1(\vartheta_T^*, y) \hat{h}(\vartheta_T^*, z) + \hat{\psi}_2(\vartheta_T^*, y) \hat{g}(\vartheta_T^*, z) \right] d\xi_T(\vartheta_T^*, z) dF(\vartheta_T^*, y),$$

where $\xi_T(\cdot, \cdot)$ is defined by (1.19). Then, we have to show the convergence in distribution

$$W_T^*(x) \implies L_2[U](t) = w_t.$$

The main technical problem in the realization of this program is the definition of the following stochastic integrals:

$$(5.6) \quad N(\vartheta_T^*, y) = \int_{-\infty}^y \hat{h}(\vartheta_T^*, z) d\xi_T(\vartheta_T^*, z)$$

and

$$(5.7) \quad M(\vartheta_T^*, y) = \int_{-\infty}^y \hat{g}(\vartheta_T^*, z) d\xi_T(\vartheta_T^*, z).$$

Remark that we cannot compute these integrals directly because the integrands contain the MDE ϑ_T^* which depends on the whole trajectory $X^T = \{X_t, 0 \leq t \leq T\}$. Therefore, the corresponding stochastic integrals (5.6) and (5.7) are not well defined.

To avoid this problem, we express the stochastic integrals in terms of ordinary integrals using Itô's formula. We shall mention that this approach was introduced in a similar problem in [15] and [19].

Let us suppose that $\hat{h}(\cdot, \cdot)$ and $\hat{g}(\cdot, \cdot)$ are piece-wise continuous functions and consider the calculation of the integrals

$$\int_a^b \hat{h}(\vartheta, z) d\xi_T(\vartheta, z), \quad \int_a^b \hat{g}(\vartheta, z) d\xi_T(\vartheta, z).$$

For any partition $a = z_1 < z_2 < \dots < z_m = b$, we can write

$$\begin{aligned} & \sum_{k=1}^{m-1} \hat{h}(\vartheta, \tilde{z}_k) [\xi_T(\vartheta, z_{k+1}) - \xi_T(\vartheta, z_k)] \\ &= \frac{1}{\sqrt{T}} \int_0^T \sum_{k=1}^{m-1} \frac{\hat{h}(\vartheta, \tilde{z}_k)}{\sigma(X_s)} \mathbb{1}_{\{z_k \leq X_s < z_{k+1}\}} dX_s \end{aligned}$$

$$-\frac{1}{\sqrt{T}} \int_0^T \sum_{k=1}^{m-1} \frac{\hat{h}(\vartheta, \tilde{z}_k)}{\sigma(X_s)} \mathbb{I}_{\{z_k \leq X_s < z_{k+1}\}} S(\vartheta, X_s) ds.$$

Therefore as $\max |z_{k+1} - z_k| \rightarrow 0$, we obtain the limit

$$\begin{aligned} & \lim_{m \rightarrow \infty} \sum_{k=1}^{m-1} \hat{h}(\vartheta, \tilde{z}_k) [\xi_T(\vartheta, z_{k+1}) - \xi_T(\vartheta, z_k)] \\ &= \frac{1}{\sqrt{T}} \int_0^T \frac{\hat{h}(\vartheta, X_s)}{\sigma(X_s)} \mathbb{I}_{\{a \leq X_s < b\}} dX_s \\ & \quad - \frac{1}{\sqrt{T}} \int_0^T \frac{\hat{h}(\vartheta, X_s)}{\sigma(X_s)} \mathbb{I}_{\{a \leq X_s < b\}} S(\vartheta, X_s) ds \end{aligned}$$

and for $a = -\infty$ and $b = y$ (our case), we have the equality

$$\begin{aligned} \int_{-\infty}^y \hat{h}(\vartheta, z) d\xi_T(\vartheta, z) &= \frac{1}{\sqrt{T}} \int_0^T \frac{\hat{h}(\vartheta, X_s)}{\sigma(X_s)} \mathbb{I}_{\{X_s < y\}} dX_s \\ & \quad - \frac{1}{\sqrt{T}} \int_0^T \frac{\hat{h}(\vartheta, X_s)}{\sigma(X_s)} \mathbb{I}_{\{X_s < y\}} S(\vartheta, X_s) ds. \end{aligned}$$

By using the same argument, we obtain

$$(5.8) \quad \begin{aligned} \int_{-\infty}^y \hat{g}(\vartheta, z) d\xi_T(\vartheta, z) &= \frac{1}{\sqrt{T}} \int_0^T \frac{\hat{g}(\vartheta, X_s)}{\sigma(X_s)} \mathbb{I}_{\{X_s < y\}} dX_s \\ & \quad - \frac{1}{\sqrt{T}} \int_0^T \frac{\hat{g}(\vartheta, X_s)}{\sigma(X_s)} \mathbb{I}_{\{X_s < y\}} S(\vartheta, X_s) ds. \end{aligned}$$

Now, we introduce the function

$$F(\vartheta, y, x) = \int_{X_0}^x R(\vartheta, y, z) dz,$$

where

$$R(\vartheta, y, z) = 2 J(\vartheta)^{-1} \hat{C}(\vartheta)^{\frac{1}{2}} \frac{\dot{S}(\vartheta, z)}{\sigma(z)^2} \mathbb{I}_{\{z < y\}}.$$

Here $\hat{C}(\vartheta)$ and $J(\vartheta)$ are given by (5.3) and (2.2), respectively. By the Itô formula,

$$dF(\vartheta, y, X_s) = F'_x(\vartheta, y, X_s) dX_s - \frac{1}{2} F''_{xx}(\vartheta, y, X_s) \sigma(X_s)^2 ds,$$

we can write

$$\int_0^T R(\vartheta, y, X_s) dX_s = \int_{X_0}^{X_T} R(\vartheta, y, z) dz - \frac{1}{2} \int_0^T R'_x(\vartheta, y, X_s) \sigma(X_s)^2 ds.$$

Indeed, we obtain

$$\int_{-\infty}^y \hat{h}(\vartheta, z) d\xi_T(\vartheta, z) = \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta, y, z) dz - \frac{1}{\sqrt{T}} \int_0^T \left[R(\vartheta, y, X_s) S(\vartheta, X_s) + \frac{1}{2} R'_x(\vartheta, y, X_s) \sigma(X_s)^2 \right] ds.$$

Hence, we have no more stochastic integral and we can substitute the MDE ϑ_T^* . Now, the process (5.6) is well defined and has the following expression

$$(5.9) \quad N(\vartheta_T^*, y) = \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz - \frac{1}{\sqrt{T}} \int_0^T \left[R(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds.$$

Using once more the Itô formula, the representation (5.8) becomes as follows:

$$\int_{-\infty}^y \hat{g}(\vartheta, z) d\xi_T(\vartheta, z) = \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta, y, z) dz - \frac{1}{\sqrt{T}} \int_0^T \left[Q(\vartheta, y, X_s) S(\vartheta, X_s) + \frac{1}{2} Q'_x(\vartheta, y, X_s) \sigma(X_s)^2 \right] ds.$$

Here

$$Q(\vartheta, y, z) = \frac{\widehat{C}(\vartheta)^{-\frac{1}{2}}}{\sigma(z)} \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z > y\}}}{\sigma(z)f(\vartheta, z)} f(\vartheta, y) \dot{f}(\vartheta, y) dy \mathbb{I}_{\{z < y\}}.$$

Therefore, the process (5.7) is well defined and has the following representation

$$(5.10) \quad M(\vartheta_T^*, y) = \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) dz - \frac{1}{\sqrt{T}} \int_0^T \left[Q(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds.$$

Now, using the representations (5.9) and (5.10), the statistic (5.5) becomes

as follows:

(5.11)

$$\begin{aligned}
 W_T^{**}(x) = & \xi_T(\vartheta_T^*, x) + \int_{-\infty}^x \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\varphi}_1(\vartheta_T^*, y) \left[\frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \right. \\
 & - \left. \frac{1}{\sqrt{T}} \int_0^T \left[R(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \right] dF(\vartheta_T^*, y) \\
 & + \int_{-\infty}^x \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\psi}_2(\vartheta_T^*, y) \left[\frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) dz \right. \\
 & - \left. \frac{1}{\sqrt{T}} \int_0^T \left[Q(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \right] dF(\vartheta_T^*, y).
 \end{aligned}$$

Let us put

$$\delta_T^* = \int_{-\infty}^{\infty} W_T^{**}(x)^2 dF(\vartheta_T^*, x).$$

The main result of this paper is given in the following Theorem.

THEOREM 2. *Suppose that the conditions $\mathcal{E}\mathcal{S}$, \mathcal{A}_0 , \mathcal{R} , \mathcal{R}_4 and \mathcal{R}_5 are fulfilled, then the test*

$$\psi_T^* = \mathbb{I}_{\{\delta_T^* > c_\alpha\}}, \quad \mathbf{P} \left(\int_0^1 w_t^2 dt > c_\alpha \right) = \alpha$$

is ADF and of asymptotic size $\alpha \in (0, 1)$.

6. Appendix.

6.1. *Proof of Theorem 1.* By Taylor's formula, the statistic $\eta_T^*(\cdot, \cdot)$ defined by (4.1) has the following representation

$$\eta_T^*(\vartheta_T^*, x) = \sqrt{T} \left(\hat{f}_T(x) - f(\vartheta, x) \right) - \sqrt{T} (\vartheta_T^* - \vartheta) \dot{f}(\vartheta, x) + o(1).$$

Remind that the normalized difference $\sqrt{T}(\hat{f}_T(x) - f(\vartheta, x))$ has the asymptotic behavior (3.2) and the MDE ϑ_T^* admits the representation (3.3). Therefore, the following convergence ($T \rightarrow \infty$)

$$\eta_T^*(\vartheta_T^*, x) \implies \eta(\vartheta, x) = 2 f(\vartheta, x) \int_{-\infty}^{+\infty} \frac{F(\vartheta, y) - \mathbb{I}_{\{y > x\}}}{\sigma(y) \sqrt{f(\vartheta, y)}} dW(y)$$

$$-2J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} f(\vartheta, x) \dot{f}(\vartheta, x) dx dW(y) \dot{f}(\vartheta, x) \quad (11.6)$$

holds (under hypothesis \mathcal{H}_0). Here $\dot{f}(\vartheta, x)$ is the derivative of the invariant density (1.10) w.r.t. ϑ

$$\dot{f}(\vartheta, x) = 2 f(\vartheta, x) \left(-\frac{\dot{G}(\vartheta)}{2G(\vartheta)} + \int_0^x \frac{\dot{S}(\vartheta, y)}{\sigma(y)^2} dy \right).$$

In the following Lemma, we show that the transformation (4.2) admits the representation (4.6).

LEMMA 1. *Let the conditions $\mathcal{R}, \mathcal{ES}$ and \mathcal{A}_0 be fulfilled, then we have the equality*

$$U(t) = w(t) - \int_0^1 g(\vartheta, s) dw(s) \int_0^t h(\vartheta, s) ds,$$

where $w(t), 0 \leq t \leq 1$ is a Wiener process.

Proof. Following (1.18) and (4.2), we have

$$\begin{aligned} U(F(\vartheta, x)) &= \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[\frac{\eta(\vartheta, y)}{2f(\vartheta, y)} \right] \\ &= \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[\int_{-\infty}^{\infty} \frac{F(\vartheta, z) - \mathbb{I}_{\{z>y\}}}{\sigma(z)\sqrt{f(\vartheta, z)}} dW(z) \right] \\ &\quad - \frac{2}{J(\vartheta)} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, y) - \mathbb{I}_{\{y>x\}}}{\sigma(y)\sqrt{f(\vartheta, y)}} \dot{f}(\vartheta, x) dF(\vartheta, x) dW(y) \int_{-\infty}^x \frac{\dot{S}(\vartheta, y)}{\sigma(y)} dF(\vartheta, y). \end{aligned}$$

Here $W(\cdot)$ is double-sided Wiener process. Further,

$$\begin{aligned} &\int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[\int_{-\infty}^{\infty} \frac{F(\vartheta, z) - \mathbb{I}_{\{z>y\}}}{\sigma(z)\sqrt{f(\vartheta, z)}} dW(z) \right] \\ &= \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[\int_{-\infty}^y \frac{F(\vartheta, z)}{\sigma(z)\sqrt{f(\vartheta, z)}} dW(z) \right] \\ &\quad + \int_{-\infty}^x \sigma(y) f(\vartheta, y) d \left[\int_y^{\infty} \frac{F(\vartheta, z) - 1}{\sigma(z)\sqrt{f(\vartheta, z)}} dW(z) \right] \\ &= \int_{-\infty}^x \sqrt{f(\vartheta, y)} dW(y). \end{aligned}$$

Therefore, we obtain

$$U(F(\vartheta, x)) = \int_{-\infty}^x \sqrt{f(\vartheta, y)} dW(y) - \frac{2}{J(\vartheta)} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, y) - \mathbb{I}_{\{y > x\}}}{\sigma(y) \sqrt{f(\vartheta, y)}} \dot{f}(\vartheta, x) dF(\vartheta, x) dW(y) \int_{-\infty}^x \frac{\dot{S}(\vartheta, y)}{\sigma(y)} dF(\vartheta, y).$$

Let us change the variables $t = F(\vartheta, x)$, $s = F(\vartheta, y)$, $a(\vartheta, s) = \sigma(F^{-1}(\vartheta, s))$, $b(\vartheta, s) = f(\vartheta, F^{-1}(\vartheta, s))$ and

$$w(s) = \int_{-\infty}^{F^{-1}(\vartheta, s)} \sqrt{f(\vartheta, y)} dW(y), \quad 0 \leq s \leq 1$$

is a Wiener process. Hence, we can write ($0 \leq t \leq 1$)

$$\begin{aligned} U(t) &= w(t) - \frac{2}{\tilde{J}(\vartheta)} \int_0^1 \int_0^1 \frac{[s - \mathbb{I}_{\{s > t\}}]}{a(\vartheta, s)b(\vartheta, s)} \dot{f}(\vartheta, F^{-1}(\vartheta, t)) dt dw(s) \int_0^t \frac{\dot{S}(\vartheta, F^{-1}(\vartheta, s))}{\sigma(F^{-1}(\vartheta, s))} ds \\ &= w(t) - \int_0^1 g(\vartheta, s) dw(s) \int_0^t h(\vartheta, s) ds \end{aligned}$$

and the Lemma is proved.

Below, we show that the linear transformation (4.2) of $\eta_T^*(\cdot, \cdot)$ gives us a statistic which is asymptotically equivalent to (1.19). Indeed, by Taylor's formula, we have (under \mathcal{H}_0)

$$\begin{aligned} \xi_T(\vartheta_T^*, x) &= \frac{1}{\sqrt{T}} \int_0^T \frac{\mathbb{I}_{\{X_s < x\}}}{\sigma(X_s)} [dX_s - S(\vartheta, X_s) ds] \\ &\quad - \frac{1}{\sqrt{T}} \int_0^T \frac{\mathbb{I}_{\{X_s < x\}}}{\sigma(X_s)} [S(\vartheta_T^*, X_s) - S(\vartheta, X_s)] ds \\ &= \frac{1}{\sqrt{T}} \int_0^T \mathbb{I}_{\{X_s < x\}} dW_s \\ &\quad - \sqrt{T}(\vartheta_T^* - \vartheta) \frac{1}{T} \int_0^T \frac{\mathbb{I}_{\{X_s < x\}} \dot{S}(\tilde{\vartheta}, X_s)}{\sigma(X_s)} ds + o(1). \end{aligned}$$

Using central limit theorem for stochastic integrals, the first stochastic integral is asymptotically normal (as $T \rightarrow \infty$)

$$\frac{1}{\sqrt{T}} \int_0^T \mathbb{I}_{\{X_s < x\}} dW_s \implies \int_{-\infty}^x \sqrt{f(\vartheta, y)} dW(y) \sim \mathcal{N}(0, F(\vartheta, x))$$

and by the law of large numbers, we have for the second ordinary integral (as $T \rightarrow \infty$)

$$\frac{1}{T} \int_0^T \frac{\mathbb{I}_{\{X_s < x\}} \dot{S}(\vartheta, X_s)}{\sigma(X_s)} ds \rightarrow \int_{-\infty}^x \frac{\dot{S}(\vartheta, y)}{\sigma(y)} dF(\vartheta, y).$$

Therefore, using the above convergence results and (3.3), one can write

$$\begin{aligned} \xi_T(\vartheta_T^*, x) &\implies \int_{-\infty}^x \sqrt{f(\vartheta, y)} dW(y) \\ &- \frac{2}{J(\vartheta)} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, y) - \mathbb{I}_{\{y > x\}}}{\sigma(y) \sqrt{f(\vartheta, y)}} \dot{f}(\vartheta, x) dF(\vartheta, x) dW(y) \int_{-\infty}^x \frac{\dot{S}(\vartheta, y)}{\sigma(y)} dF(\vartheta, y) \\ &= U(F(\vartheta, x)). \end{aligned}$$

Hence, we have the asymptotic equivalence between the linear transformation (4.2) of $\eta_T^*(\cdot, \cdot)$ and the statistic $\xi_T(\cdot, \cdot)$, which proves the Theorem.

6.2. *Proof of Theorem 2.* It is enough to verify the convergence

$$(6.1) \quad \delta_T^* = \int_{-\infty}^{\infty} W_T^{**}(x)^2 dF(\vartheta_T^*, x) \implies \int_{-\infty}^{\infty} w_{F(\vartheta, x)}^2 dF(\vartheta, x) \equiv \delta^*$$

under hypothesis \mathcal{H}_0 . Here $w_{F(\vartheta, x)}$ and $W_T^{**}(x)$ are given by (5.4) and (5.11), respectively.

Note that due to the consistency of the MDE ϑ_T^* , we have the convergence (as $T \rightarrow \infty$)

$$\begin{aligned} \hat{h}(\vartheta_T^*, z) &\longrightarrow \hat{h}(\vartheta, z), & \hat{g}(\vartheta_T^*, z) &\longrightarrow \hat{g}(\vartheta, z), \\ J(\vartheta_T^*) &\longrightarrow J(\vartheta), & \hat{\psi}_2(\vartheta_T^*, z) &\longrightarrow \hat{\psi}_2(\vartheta, z) \end{aligned}$$

and

$$\hat{\varphi}_1(\vartheta_T^*, z) \longrightarrow \hat{\varphi}_1(\vartheta, z), \quad \hat{\varphi}_2(\vartheta_T^*, z) \longrightarrow \hat{\varphi}_2(\vartheta, z).$$

Remind that we proved the convergence (in Theorem 1)

$$(6.2) \quad \xi_T(\vartheta_T^*, x) \implies U(F(\vartheta, x)).$$

Then, we have to show that (under \mathcal{H}_0)

$$(6.3) \quad N(\vartheta_T^*, y) \implies N(\vartheta, y),$$

where $N(\vartheta_T^*, y)$ is defined by (5.9) and

$$\begin{aligned}
 N(\vartheta, y) &= \int_{-\infty}^y \hat{h}(\vartheta, z) dU(F(\vartheta, z)) \\
 &= J(\vartheta)^{-1} \widehat{C}(\vartheta)^{\frac{1}{2}} \int_{-\infty}^y \frac{\dot{S}(\vartheta, z)}{\sigma(z)} \sqrt{f(\vartheta, z)} dW(z) \\
 (6.4) \quad &- \frac{4\widehat{C}(\vartheta)^{\frac{1}{2}}}{J(\vartheta)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z > y\}}}{\sigma(z) \sqrt{f(\vartheta, z)}} f(\vartheta, y) \dot{f}(\vartheta, y) dy dW(z) \\
 &\quad \int_{-\infty}^y \frac{\dot{S}(\vartheta, z)^2}{\sigma(z)^2} f(\vartheta, z) dz.
 \end{aligned}$$

Here the function $\hat{h}(\cdot, \cdot)$ is given by (5.1). The process $N(\vartheta_T^*, y)$ has the following representation

$$\begin{aligned}
 N(\vartheta_T^*, y) &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \\
 &- \frac{1}{\sqrt{T}} \int_0^T \left[R(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \\
 &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \\
 &- \frac{1}{\sqrt{T}} \int_0^T \left[R(\vartheta_T^*, y, X_s) S(\vartheta, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \\
 &- \frac{1}{\sqrt{T}} \int_0^T R(\vartheta_T^*, y, X_s) [S(\vartheta_T^*, X_s) - S(\vartheta, X_s)] ds = I_T^*(\vartheta_T^*, y) - K_T^*(\vartheta_T^*, y).
 \end{aligned}$$

Combining Itô's formula and the central limit theorem for stochastic integrals, we obtain

$$\begin{aligned}
 I_T^*(\vartheta_T^*, y) &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \\
 &- \frac{1}{\sqrt{T}} \int_0^T \left[R(\vartheta_T^*, y, X_s) S(\vartheta, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \\
 &= \frac{1}{\sqrt{T}} \int_0^T R(\vartheta, y, X_s) \sigma(X_s) dW_s + o(1) \\
 &\Rightarrow \int_{-\infty}^{\infty} R(\vartheta, y, z) \sigma(z) \sqrt{f(\vartheta, z)} dW(z),
 \end{aligned}$$

where $W(\cdot)$ is double-sided Wiener process. Further, by Taylor's formula,

the law of large numbers and the representation (3.3), we can write

$$\begin{aligned}
 K_T^*(\vartheta_T^*, y) &= \frac{1}{\sqrt{T}} \int_0^T R(\vartheta_T^*, y, X_s) [S(\vartheta_T^*, X_s) - S(\vartheta, X_s)] ds \\
 &= \sqrt{T} (\vartheta_T^* - \vartheta) \frac{1}{T} \int_0^T R(\vartheta_T^*, y, X_s) \dot{S}(\tilde{\vartheta}, X_s) ds \\
 &\implies 2 J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z > y\}}}{\sigma(z) \sqrt{f(\vartheta, z)}} f(\vartheta, y) \dot{f}(\vartheta, y) dy dW(z) \\
 &\quad \int_{-\infty}^{\infty} R(\vartheta, y, z) \dot{S}(\vartheta, z) f(\vartheta, z) dz.
 \end{aligned}$$

Therefore, we obtain $N(\vartheta_T^*, y) \implies N(\vartheta, y)$, where $N(\vartheta, y)$ is given by (6.4).

In the same manner, we show that we have the convergence (under \mathcal{H}_0)

$$(6.5) \quad M(\vartheta_T^*, y) \implies M(\vartheta, y),$$

where $M(\vartheta_T^*, y)$ is defined by (5.10) and

(6.6)

$$\begin{aligned}
 M(\vartheta, y) &= \int_{-\infty}^y \hat{g}(\vartheta, z) dU(F(\vartheta, z)) \\
 &= \hat{C}(\vartheta)^{-\frac{1}{2}} \int_{-\infty}^y \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z > y\}}}{\sigma(z) f(\vartheta, z)} f(\vartheta, y) \dot{f}(\vartheta, y) dy \sqrt{f(\vartheta, z)} dW(z) \\
 &\quad - 4 J(\vartheta)^{-1} \hat{C}(\vartheta)^{-\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z > y\}}}{\sigma(z) \sqrt{f(\vartheta, z)}} f(\vartheta, y) \dot{f}(\vartheta, y) dy dW(z) \\
 &\quad \int_{-\infty}^y \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z > y\}}}{\sigma(z)^2 f(\vartheta, z)} f(\vartheta, y) \dot{f}(\vartheta, y) dy \dot{S}(\vartheta, z) f(\vartheta, z) dz.
 \end{aligned}$$

Here the function $\hat{g}(\cdot, \cdot)$ is given by (5.2). For the process $M(\vartheta_T^*, y)$, we have the following representation

$$\begin{aligned}
 M(\vartheta_T^*, y) &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) dz \\
 &\quad - \frac{1}{\sqrt{T}} \int_0^T \left[Q(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \\
 &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) dz \\
 &\quad - \frac{1}{\sqrt{T}} \int_0^T \left[Q(\vartheta_T^*, y, X_s) S(\vartheta, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds
 \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{\sqrt{T}} \int_0^T Q(\vartheta_T^*, y, X_s) [S(\vartheta_T^*, X_s) - S(\vartheta, X_s)] ds \\
& = \tilde{I}_T^*(\vartheta_T^*, y) - \tilde{K}_T^*(\vartheta_T^*, y).
\end{aligned}$$

Due to the Itô formula and the central limit theorem for stochastic integrals, we can write

$$\begin{aligned}
\tilde{I}_T^*(\vartheta_T^*, y) &= \frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) dz \\
& - \frac{1}{\sqrt{T}} \int_0^T \left[Q(\vartheta_T^*, y, X_s) S(\vartheta, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2 \right] ds \\
& = \frac{1}{\sqrt{T}} \int_0^T Q(\vartheta, y, X_s) \sigma(X_s) dW_s + o(1) \\
& \Rightarrow \int_{-\infty}^{\infty} Q(\vartheta, y, z) \sigma(z) \sqrt{f(\vartheta, z)} dW(z).
\end{aligned}$$

Further, we have

$$\begin{aligned}
\tilde{K}_T^*(\vartheta_T^*, y) &= \frac{1}{\sqrt{T}} \int_0^T Q(\vartheta_T^*, y, X_s) [S(\vartheta_T^*, X_s) - S(\vartheta, X_s)] ds \\
& = \sqrt{T}(\vartheta_T^* - \vartheta) \frac{1}{T} \int_0^T Q(\vartheta_T^*, y, X_s) \dot{S}(\tilde{\vartheta}, X_s) ds \\
& \Rightarrow 2 J(\vartheta)^{-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F(\vartheta, z) - \mathbb{I}_{\{z > y\}}}{\sigma(z) \sqrt{f(\vartheta, z)}} f(\vartheta, y) \dot{f}(\vartheta, y) dy dW(z) \\
& \int_{-\infty}^{\infty} Q(\vartheta, y, z) \dot{S}(\vartheta, z) f(\vartheta, z) dz.
\end{aligned}$$

Therefore $M(\vartheta_T^*, y) \Rightarrow M(\vartheta, y)$, where $M(\vartheta, y)$ is defined by (6.6). Finally, (6.2), (6.3) and (6.5) give us the convergence

$$W_T^{**}(x) \Rightarrow w_{F(\vartheta, x)}.$$

Now, to obtain (6.1), we will prove the convergence of the integrals by checking the following three conditions: (see [13] and [15])

1. For any x_1, \dots, x_k

$$(W_T^{**}(x_1), \dots, W_T^{**}(x_k)) \Rightarrow (w_{F(\vartheta, x_1)}, \dots, w_{F(\vartheta, x_k)}),$$

where $w_{F(\vartheta, x)}$ and $W_T^{**}(x)$ are defined by (5.4) and (5.11), respectively.

2. For $|x_i| < L, i = 1, 2$

$$(6.7) \quad \mathbf{E}_\vartheta |W_T^{**}(x_1) - W_T^{**}(x_2)|^2 \leq C |x_1 - x_2|^{\frac{1}{2}},$$

where C and L are some positive constants.

3. For any $\kappa > 0$, there exist $L > 0$ such that

$$(6.8) \quad \int_{|x|>L} \mathbf{E}_\vartheta W_T^{**}(x)^2 f(\vartheta_T^*, x) dx < \kappa.$$

The first convergence of finite-dimensional distributions follows from (6.2), (6.3) and (6.5) for any x_1, \dots, x_k .

Then, in order to obtain the estimate (6.7), we write

$$\begin{aligned} \mathbf{E}_\vartheta |W_T^{**}(x_1) - W_T^{**}(x_2)|^2 &\leq 3 \mathbf{E}_\vartheta |\xi_T(\vartheta_T^*, x_1) - \xi_T(\vartheta_T^*, x_2)|^2 \\ &+ 3 \mathbf{E}_\vartheta \left| \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\varphi}_1(\vartheta_T^*, y) \left[\frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \right. \right. \\ &- \left. \frac{1}{\sqrt{T}} \int_0^T [R(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2] ds \right] dF(\vartheta_T^*, y) \Big|^2 \\ &+ 3 \mathbf{E}_\vartheta \left| \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\psi}_2(\vartheta_T^*, y) \left[\frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) dz \right. \right. \\ &- \left. \frac{1}{\sqrt{T}} \int_0^T [Q(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2] ds \right] dF(\vartheta_T^*, y) \Big|^2. \end{aligned}$$

Further, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \mathbf{E}_\vartheta |\xi_T(\vartheta_T^*, x_1) - \xi_T(\vartheta_T^*, x_2)|^2 &\leq 2 \mathbf{E}_\vartheta \left| \frac{1}{\sqrt{T}} \int_0^T \mathbb{1}_{\{x_2 < X_s < x_1\}} dW_s \right|^2 \\ &+ 2 \mathbf{E}_\vartheta \left| \sqrt{T}(\vartheta_T^* - \vartheta) \frac{1}{T} \int_0^T \frac{\dot{S}(\tilde{\vartheta}, X_s)}{\sigma(X_s)} \mathbb{1}_{\{x_2 < X_s < x_1\}} ds \right|^2 \\ &\leq 2 \int_{x_2}^{x_1} f(\vartheta, y) dy + 2 \left(\mathbf{E}_\vartheta \left| \sqrt{T}(\vartheta_T^* - \vartheta) \right|^4 \right)^{\frac{1}{2}} \left(\mathbf{E}_\vartheta \left| \int_{x_2}^{x_1} \frac{\dot{S}(\tilde{\vartheta}, y)}{\sigma(y)} \hat{f}_T(y) dy \right|^4 \right)^{\frac{1}{2}} \\ &\leq 2 \int_{x_2}^{x_1} f(\vartheta, y) dy + C \left(\int_{x_2}^{x_1} P(y) f(\vartheta, y) dy \right)^{\frac{1}{2}}. \end{aligned}$$

Here for any measurable function $l(\cdot)$, we use the following relation (can be found in [16], Section 1.1.3)

$$\frac{1}{T} \int_0^T l(X_t) dt = \int_{-\infty}^{\infty} l(x) \hat{f}_T(x) dx,$$

where $\hat{f}_T(x)$ is the empirical density given by (1.7).

Remind that by the conditions \mathcal{A}_0 and \mathcal{R}_5 , the functions $\dot{S}(\vartheta, \cdot)$ and $\sigma(\cdot)^{\pm 1}$ have polynomial majorants $P(y)$. In addition, the invariant density (1.10) has exponentially decreasing tails by condition \mathcal{A}_0 , i.e., there exist the constants $C_1 > 0$ and $\lambda > 0$ such that

$$f(\vartheta, x) \leq C_1 e^{-\lambda|x|}.$$

Therefore,

$$\mathbf{E}_\vartheta |\xi_T(\vartheta_T^*, x_1) - \xi_T(\vartheta_T^*, x_2)|^2 \leq C|x_1 - x_2| + C|x_1 - x_2|^{\frac{1}{2}} \leq C|x_1 - x_2|^{\frac{1}{2}}.$$

Then, we have

$$\begin{aligned} & \mathbf{E}_\vartheta \left| \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\varphi}_1(\vartheta_T^*, y) \left[\frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \right. \right. \\ & \quad \left. \left. - \frac{1}{\sqrt{T}} \int_0^T [R(\vartheta_T^*, y, X_s)S(\vartheta_T^*, X_s) + \frac{1}{2}R'_x(\vartheta_T^*, y, X_s)\sigma(X_s)^2] ds \right] dF(\vartheta_T^*, y) \right|^2 \\ & \leq 2 \mathbf{E}_\vartheta \left| \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\varphi}_1(\vartheta_T^*, y) \left[\frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta, y, z) dz \right. \right. \\ & \quad \left. \left. - \frac{1}{\sqrt{T}} \int_0^T [R(\vartheta, y, X_s)S(\vartheta_T^*, X_s) + \frac{1}{2}R'_x(\vartheta, y, X_s)\sigma(X_s)^2] ds \right] dF(\vartheta_T^*, y) \right|^2 \\ & \quad + 2 \mathbf{E}_\vartheta \left| \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\varphi}_1(\vartheta_T^*, y) \sqrt{T}(\vartheta_T^* - \vartheta) \left[\frac{1}{T} \int_{X_0}^{X_T} \dot{R}(\tilde{\vartheta}, y, z) dz \right. \right. \\ & \quad \left. \left. - \frac{1}{T} \int_0^T [\dot{R}(\tilde{\vartheta}, y, X_s)S(\vartheta_T^*, X_s) + \frac{1}{2}\dot{R}'_x(\tilde{\vartheta}, y, X_s)\sigma(X_s)^2] ds \right] dF(\vartheta_T^*, y) \right|^2 \\ & \leq 2 \mathbf{E}_\vartheta \left| \frac{1}{\sqrt{T}} \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) \int_0^T R(\vartheta, y, X_s) dW_s dF(\vartheta, y) \right|^2 \\ & \quad + 2 \mathbf{E}_\vartheta \left| \sqrt{T}(\vartheta_T^* - \vartheta) \frac{1}{T} \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) \int_0^T \dot{R}(\vartheta, y, X_s) dW_s dF(\vartheta, y) \right|^2 \\ & \quad + o(1). \end{aligned}$$

Indeed, here we used the Itô formula

$$\begin{aligned} & \int_{X_0}^{X_T} R(\vartheta, y, z) dz - \int_0^T [R(\vartheta, y, X_s)S(\vartheta, X_s) + \frac{1}{2}R'_x(\vartheta, y, X_s)\sigma(X_s)^2] ds \\ & = \int_0^T R(\vartheta, y, X_s) dW_s \end{aligned}$$

and

$$\begin{aligned} & \int_{X_0}^{X_T} \dot{R}(\tilde{\vartheta}, y, z) dz - \int_0^T [\dot{R}(\tilde{\vartheta}, y, X_s) S(\vartheta, X_s) + \frac{1}{2} \dot{R}'_x(\tilde{\vartheta}, y, X_s) \sigma(X_s)^2] ds \\ &= \int_0^T \dot{R}(\tilde{\vartheta}, y, X_s) dW_s. \end{aligned}$$

Additionally, by Fubini's theorem and the Cauchy-Schwarz inequality, we write

$$\begin{aligned} & \mathbf{E}_\vartheta \left| \frac{1}{\sqrt{T}} \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) \int_0^T R(\vartheta, y, X_s) dW_s dF(\vartheta, y) \right|^2 \\ & \leq \mathbf{E}_\vartheta \left| \frac{1}{\sqrt{T}} \int_0^T \int_{x_2}^{x_1} R(\vartheta, y, X_s) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) dW_s \right|^2 \\ & \leq \mathbf{E}_\vartheta \frac{1}{T} \int_0^T \left| \int_{x_2}^{x_1} R(\vartheta, y, X_s) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) \right|^2 ds \\ & \leq \mathbf{E}_\vartheta \int_{-\infty}^{\infty} \left| \int_{x_2}^{x_1} R(\vartheta, y, z) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) \right|^2 \hat{f}_T(z) dz \\ & \leq [F(\vartheta, x_1) - F(\vartheta, x_2)] \\ & \quad \int_{x_2}^{x_1} \int_{-\infty}^{\infty} [R(\vartheta, y, z) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y)]^2 f(\vartheta, z) dz dF(\vartheta, y). \end{aligned}$$

Therefore, using the conditions \mathcal{A}_0 and \mathcal{R}_5 , we obtain

$$\begin{aligned} & \mathbf{E}_\vartheta \left| \frac{1}{\sqrt{T}} \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) \int_0^T R(\vartheta, y, X_s) dW_s dF(\vartheta, y) \right|^2 \\ & \leq C |x_1 - x_2|. \end{aligned}$$

Then, we have

$$\begin{aligned} & \mathbf{E}_\vartheta \left| \sqrt{T} (\vartheta_T^* - \vartheta) \frac{1}{T} \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) \int_0^T \dot{R}(\vartheta, y, X_s) dW_s dF(\vartheta, y) \right|^2 \\ & \leq \mathbf{E}_\vartheta \left| \sqrt{T} (\vartheta_T^* - \vartheta) \frac{1}{T} \int_0^T \int_{x_2}^{x_1} \dot{R}(\vartheta, y, X_s) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) dW_s \right|^2 \\ & \leq \mathbf{E}_\vartheta \left| \sqrt{T} (\vartheta_T^* - \vartheta) \right|^2 \left| \frac{1}{T} \int_0^T \int_{x_2}^{x_1} \dot{R}(\vartheta, y, X_s) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) dW_s \right|^2 \\ & \leq \left(\mathbf{E}_\vartheta \left| \sqrt{T} (\vartheta_T^* - \vartheta) \right|^4 \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
& \frac{1}{T^2} \left(\mathbf{E}_\vartheta \left| \int_0^T \int_{x_2}^{x_1} \dot{R}(\vartheta, y, X_s) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) dW_s \right|^4 \right)^{\frac{1}{2}} \\
& \leq \frac{C}{T^2} \left(T \mathbf{E}_\vartheta \int_0^T \left| \int_{x_2}^{x_1} \dot{R}(\vartheta, y, X_s) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) \right|^4 ds \right)^{\frac{1}{2}} \\
& \leq \frac{C}{T} \left(\mathbf{E}_\vartheta \int_{-\infty}^{\infty} \left| \int_{x_2}^{x_1} \dot{R}(\vartheta, y, z) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) dF(\vartheta, y) \right|^4 \hat{f}_T(z) dz \right)^{\frac{1}{2}} \\
& \leq \frac{C}{T} \left([F(\vartheta, x_1) - F(\vartheta, x_2)]^3 \right. \\
& \quad \left. \int_{x_2}^{x_1} \int_{-\infty}^{\infty} \left| \dot{R}(\vartheta, y, z) \hat{\varphi}_2^+(\vartheta, y) \hat{\varphi}_1(\vartheta, y) \right|^4 f(\vartheta, z) dz dF(\vartheta, y) \right)^{\frac{1}{2}} \leq \frac{C}{T}.
\end{aligned}$$

Similarly, we obtain the following estimate

$$\begin{aligned}
& \mathbf{E}_\vartheta \left| \int_{x_2}^{x_1} \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\psi}_2(\vartheta_T^*, y) \left[\frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) dz \right. \right. \\
& \quad \left. \left. - \frac{1}{\sqrt{T}} \int_0^T [Q(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2] ds \right] dF(\vartheta_T^*, y) \right|^2 \\
& \leq C |x_1 - x_2|.
\end{aligned}$$

Now, we have to check the condition (6.8). Indeed,

$$\begin{aligned}
& \mathbf{E}_\vartheta |W_T^{**}(x)|^2 \leq 3 \mathbf{E}_\vartheta |\xi_T(\vartheta_T^*, x)|^2 \\
& + 3 \mathbf{E}_\vartheta \left| \int_{-\infty}^x \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\varphi}_1(\vartheta_T^*, y) \left[\frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \right. \right. \\
& \quad \left. \left. - \frac{1}{\sqrt{T}} \int_0^T [R(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2] ds \right] dF(\vartheta_T^*, y) \right|^2 \\
& + 3 \mathbf{E}_\vartheta \left| \int_{-\infty}^x \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\psi}_2(\vartheta_T^*, y) \left[\frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) dz \right. \right. \\
& \quad \left. \left. - \frac{1}{\sqrt{T}} \int_0^T [Q(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2] ds \right] dF(\vartheta_T^*, y) \right|^2.
\end{aligned}$$

We can write

$$\mathbf{E}_\vartheta |\xi_T(\vartheta_T^*, x)|^2 \leq 2 \mathbf{E}_\vartheta \left| \frac{1}{\sqrt{T}} \int_0^T \mathbb{I}_{\{X_s < x\}} dW_s \right|^2$$

$$\begin{aligned}
& + 2 \mathbf{E}_\vartheta \left| \sqrt{T} (\vartheta_T^* - \vartheta) \frac{1}{T} \int_0^T \frac{\dot{S}(\tilde{\vartheta}, X_s)}{\sigma(X_s)} \mathbb{I}_{\{X_s < x\}} ds \right|^2 \\
& \leq 2F(\vartheta, x) + 2 \left(\mathbf{E}_\vartheta |\sqrt{T}(\vartheta_T^* - \vartheta)|^4 \right)^{\frac{1}{2}} \left(\mathbf{E}_\vartheta \left| \frac{1}{T} \int_0^T \frac{\dot{S}(\tilde{\vartheta}, X_s)}{\sigma(X_s)} \mathbb{I}_{\{X_s < x\}} ds \right|^4 \right)^{\frac{1}{2}} \\
& \leq 2F(\vartheta, x) + C \left(\int_{-\infty}^x P(y) f(\vartheta, y) dy \right)^{\frac{1}{2}} \leq C.
\end{aligned}$$

Moreover, using the same representation as above, we have

$$\begin{aligned}
& \mathbf{E}_\vartheta \left| \int_{-\infty}^x \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\varphi}_1(\vartheta_T^*, y) \left[\frac{1}{\sqrt{T}} \int_{X_0}^{X_T} R(\vartheta_T^*, y, z) dz \right. \right. \\
& \quad \left. \left. - \frac{1}{\sqrt{T}} \int_0^T [R(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} R'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2] ds \right] dF(\vartheta_T^*, y) \right|^2 \\
& \leq C
\end{aligned}$$

and

$$\begin{aligned}
& \mathbf{E}_\vartheta \left| \int_{-\infty}^x \hat{\varphi}_2^+(\vartheta_T^*, y) \hat{\psi}_2(\vartheta_T^*, y) \left[\frac{1}{\sqrt{T}} \int_{X_0}^{X_T} Q(\vartheta_T^*, y, z) dz \right. \right. \\
& \quad \left. \left. - \frac{1}{\sqrt{T}} \int_0^T [Q(\vartheta_T^*, y, X_s) S(\vartheta_T^*, X_s) + \frac{1}{2} Q'_x(\vartheta_T^*, y, X_s) \sigma(X_s)^2] ds \right] dF(\vartheta_T^*, y) \right|^2 \\
& \leq C.
\end{aligned}$$

Finally, the convergence (6.1) is proved and the test ψ_T^* is ADF.

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