Hitting probabilities for a non-convex lattice with obstacles

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Dedicated to Professor Marius Stoka

Abstract

In this paper we consider a lattice with cell represented in fig. 1 and we determine the probability that a segment of constant length and random direction, according to the exponential distribution and \( \gamma(2) \) distribution, intersects a side of the lattice.


1 Preliminary Results

Let \( \mathcal{R}(a; m) \) the lattice with fundamental cell \( C_0 \) represented in fig. 1
where \( m < a/2 \). By this figure we have

\[
\text{area}C_0 = 3a^2 - 7m^2
\]  

(1)

We want to compute the probability that a segment \( s \) of constant length \( l < \frac{a}{2} - m \) and random direction intersects a side of lattice \( \mathcal{R} \), i.e. the probability \( P_{\text{int}} \) that \( s \) intersects a side of fundamental cell \( C_0 \).

The position of segment \( s \) is determined by its center and by angle \( \varphi \) that \( s \) forms with the line \( BC \) (or \( AF \)).

To compute the probability \( P_{\text{int}} \) we consider the limiting position, for a fixed angle \( \varphi \) in the cell \( C_0 \). So we obtain the fig. 2.
and the formula

\[ \text{area}\tilde{C}_0(\varphi) = \text{area}C_0 - \sum_{i=1}^{22} \text{area}a_i(\varphi). \]  

(2)

By fig. 2 we have

\[ |AA_5| = l \sin \varphi, \quad |AA_6| = l \cos \varphi, \]  

(3)

then

\[ \text{area}a_1(\varphi) + \text{area}a_2(\varphi) = \text{area}a_7(\varphi) + \text{area}a_8(\varphi) = \frac{l^2}{4} \sin 2\varphi - m^2. \]  

(4)

The fig. 2 and first formula (3) give us

\[ h_3 = \frac{l}{2} \cos \varphi, \quad |A_5B_1| = 2a - m - |AA_5| = 2a - m - l \sin \varphi, \]  

then
areaa_3(\varphi) = \left(a - \frac{m}{2}\right)l \cos \varphi - \frac{l^2}{4} \sin 2\varphi. \quad (5)

By fig. 2 we have

\[ h_4 = m \sin \varphi, \quad |B_3B_5| = \frac{l}{2}. \]

Then

\[
\begin{align*}
\text{areaa}_4(\varphi) &= \text{areaa}_{10}(\varphi) = \text{areaa}_{14}(\varphi) = \\
\text{areaa}_{16}(\varphi) &= \text{areaa}_{20}(\varphi) = \frac{ml}{2} \sin \varphi.
\end{align*}
\]

(6)

In the same we have

\[ h_5 = m \cos \varphi, \quad |B_2B_6| = \frac{l}{2}, \]

so that

\[
\begin{align*}
\text{areaa}_5(\varphi) &= \text{areaa}_{11}(\varphi) = \text{areaa}_{15}(\varphi) = \\
\text{areaa}_{17}(\varphi) &= \text{areaa}_{21}(\varphi) = \frac{ml}{2} \cos \varphi.
\end{align*}
\]

(7)

The fig. 2 and second formula (3) give us

\[ h_6 = \frac{l}{2} \sin \varphi, \quad |B_2C_6| = 2a - m - |AA_6| = 2a - m - l \cos \varphi, \]

then

\[
\begin{align*}
\text{areaa}_6(\varphi) &= \left(a - \frac{m}{2}\right)l \sin \varphi - \frac{l^2}{4} \sin 2\varphi.
\end{align*}
\]

(8)

By fig. 2 and by first formula (3) we have

\[ h_9 = \frac{l}{2} \cos \varphi, \quad |C_5D_1| = a - m - |AA_5| = a - m - l \sin \varphi, \]

i.e.

\[
\begin{align*}
\text{areaa}_9(\varphi) &= \frac{(a - m)}{2}l \cos \varphi - \frac{l^2}{4} \sin 2\varphi.
\end{align*}
\]

(9)

The fig. 2 give us

\[ |E_4E_5| = m, \quad |E_4E_{15}| = l \cos \varphi - m, \]

(10)
therefore

\[ \text{areaa}_{13}(\varphi) = \frac{m^2}{2} \cos \varphi. \]  

(11)

Considering the fig. 2 and second formula (10) we can write

\[ h_{12} = \frac{l}{2} \cos \varphi, \quad |D_2D_{15}| = a - 2m - |E_4E_{15}| = a - m - l \cos \varphi, \]

then

\[ \text{areaa}_{12}(\varphi) = \frac{(a - m)l}{2} \sin \varphi - \frac{l^2}{4} \sin 2\varphi. \]  

(12)

By fig. 2 follows \( a_{18} = a_2, a_{13} = a_1 \). Then, considering formulas (4) and (11), we have

\[ \text{areaa}_{18}(\varphi) = \text{areaa}_1(\varphi) + \text{areaa}_2(\varphi) - \text{areaa}_{13}(\varphi), \]

i.e.

\[ \text{areaa}_{18}(\varphi) = \frac{l^2}{4} \sin 2\varphi - m^2 - \frac{m^2}{2} \cos \varphi. \]  

(13)

The fig. 2 and relation (3) give us

\[ |E_1E_7| = |A_1A_5| = l \sin \varphi - m, \quad |E_7F_1| = a - 2m - |E_1E_7| = a - m - l \sin \varphi, \]

\[ h_{19} = \frac{l}{2} \cos \varphi. \]

Therefore

\[ \text{areaa}_{19}(\varphi) = \frac{(a - m)l}{2} \cos \varphi - \frac{l^2}{4} \sin 2\varphi. \]  

(14)

In the end by fig. 2 and by formula (3) we have

\[ h_{22} = \frac{l}{2} \sin \varphi, \quad |A_6F_2| = a - m - |A_{A_6}| = a - m - l \cos \varphi, \]

then

\[ \text{areaa}_{22}(\varphi) = \frac{(a - m)l}{2} \sin \varphi - \frac{l^2}{4} \sin 2\varphi. \]  

(15)

Considering the relations (4), (5), ..., (9), (11), (12), ..., (15) we obtain

\[ A(\varphi) = \sum_{i=1}^{22} \text{areaa}_i(\varphi) = l(2a + m)(\sin \varphi + \cos \varphi) - \frac{l^2}{2} \sin 2\varphi - 3m^2. \]  

(16)

Replacing this expression in (2) follows
\[ \text{area} \tilde{C}_0(\varphi) = \text{area} C_0 - A(\varphi). \] (17)

Denoting with \( M \) the set of segments \( s \) that have the center in the cell \( C_0 \) and with \( N \) the set of segments \( s \) entirely contained in the cell \( C_0 \) we have [2]:

\[ P_{\text{int}} = 1 - \frac{\mu(N)}{\mu(M)}, \] (18)

where \( \mu \) is Lebesgue measure in the Euclidean plane.

To compute the measures \( \mu(M) \) and \( \mu(N) \) we use the cinematic measure of Poincaré [1]:

\[ dk = dx \land dy \land d\varphi, \]

where \( x, y \) are the coordinates of the center of \( s \) and \( \varphi \) the fixed angle.

Assuming that the direction of line support of the segment \( s \) is a random variable with probability density \( f(\varphi) \) and, for \( \varphi \in [0, \pi/2] \), we have:

\[
\mu(M) = \int_0^{\pi/2} f(\varphi) d\varphi \int_0^{\pi/2} dx dy = \int_0^{\pi/2} (\text{area} C_0) f(\varphi) d\varphi = (\text{area} C_0) \int_0^{\pi/2} f(\varphi) d\varphi. \] (19)

Then, considering formula (17), we can write

\[
\mu(N) = \int_0^{\pi/2} f(\varphi) d\varphi \int_0^{\pi/2} dx dy = \int_0^{\pi/2} \left[ \text{area} \tilde{C}_0(\varphi) \right] f(\varphi) d\varphi = \int_0^{\pi/2} \left( \text{area} C_0 - A(\varphi) \right) f(\varphi) d\varphi = (\text{area} C_0) \int_0^{\pi/2} f(\varphi) d\varphi - \int_0^{\pi/2} A(\varphi) f(\varphi) d\varphi. \] (20)

The (18), (19) and (20) we have that

\[
P_{\text{int}} = \frac{1}{(\text{area} C_0) \int_0^{\pi/2} f(\varphi) d\varphi} \int_0^{\pi/2} A(\varphi) f(\varphi) d\varphi,
\]
i.e., with (1) and (16),

\[
P_{\text{int}} = \frac{1}{(3a^2 - 7m^2) \int_0^{\pi/2} f(\varphi) d\varphi} \int_0^{\pi/2} \left[ l(2a + m)(\sin \varphi + \cos \varphi) - \frac{l^2}{2} \sin 2\varphi - 3m^2 \right] f(\varphi) d\varphi. \] (21)

Now we consider the exponential distribution and \( \gamma \) (2) distribution.
2 Exponential distribution

We have \( f(\varphi) = e^{-\varphi}, \ (\varphi \geq 0) \).

From paper [3] we have the formulas:

\[
\int_0^{\pi/2} e^{-\varphi} d\varphi = 1 - e^{-\frac{\pi}{2}}, \quad \int_0^{\pi/2} e^{-\varphi} \sin \varphi d\varphi = \frac{1}{2} \left(1 - e^{-\frac{\pi}{2}}\right),
\]

\[
\int_0^{\pi/2} e^{-\varphi} \cos \varphi d\varphi = \frac{1}{2} \left(1 + e^{-\frac{\pi}{2}}\right), \quad \int_0^{\pi/2} e^{-\varphi} \sin 2\varphi d\varphi = \frac{2}{5} \left(1 + e^{-\frac{\pi}{2}}\right).
\]

Replacing these values in the (21) we obtain

\[
P_{int} = \frac{1}{(3a^2 - 7m^2) \left(1 - e^{-\frac{\pi}{2}}\right)} \left[ (2a + m) l - \frac{l^2}{5} \left(1 + e^{-\frac{\pi}{2}}\right) - 3m^2 \left(1 - e^{-\frac{\pi}{2}}\right) \right].
\]

3 Distribution \( \gamma(2) \)

We have \( f(\varphi) = \varphi e^{-\varphi}, \ (\varphi \geq 0) \).

From [3] we have

\[
\int_0^{\pi/2} \varphi e^{-\varphi} d\varphi = 1 - \left(\frac{\pi}{2} + 1\right) e^{-\frac{\pi}{2}}, \quad \int_0^{\pi/2} \varphi e^{-\varphi} \sin \varphi d\varphi = \frac{1}{2} - \frac{\pi}{4} e^{-\frac{\pi}{2}},
\]

\[
\int_0^{\pi/2} \varphi e^{-\varphi} \cos \varphi d\varphi = \frac{1}{2} e^{-\frac{\pi}{2}} \left(1 + \frac{\pi}{2}\right), \quad \int_0^{\pi/2} e^{-\varphi} \sin 2\varphi d\varphi = \frac{4}{25} + \left(\frac{\pi}{5} + \frac{4}{25}\right) e^{-\frac{\pi}{2}}.
\]

Replacing this one in the (21) we have

\[
P_{int} = \frac{1}{2 (3a^2 - 7m^2) \left[1 - \left(\frac{\pi}{2} + 1\right) e^{-\frac{\pi}{2}}\right]}
\]

\[
\left\{ (2a + m) l \left(1 + e^{-\frac{\pi}{2}}\right) - \frac{l^2}{25} \left[4 + (5\pi + 4) e^{-\frac{\pi}{2}}\right] - 6m^2 \left[1 - \left(\frac{\pi}{2} + 1\right) e^{-\frac{\pi}{2}}\right] \right\}.
\]

References