TOWARDS A L 2 COHOMOLOGY THEORY FOR HODGE MODULES ON INFINITE COVERING SPACES: L 2 CONSTRUCTIBLE COHOMOLOGY AND L 2 DE RHAM COHOMOLOGY FOR COHERENT D-MODULES

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TOWARDS A L_2 COHOMOLOGY THEORY FOR HODGE MODULES ON INFINITE COVERING SPACES: L_2 CONSTRUCTIBLE COHOMOLOGY AND L_2 DE RHAM COHOMOLOGY FOR COHERENT $\mathcal{D}$-MODULES.

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Abstract. This article constructs Von Neumann invariants for constructible complexes and coherent $\mathcal{D}$-modules on compact complex manifolds, generalizing the work of the author on coherent L_2-cohomology. We formulate a conjectural generalization of Dingoyan's L_2-Mixed Hodge structures in terms of Saito's Mixed Hodge Modules and give partial results in this direction.
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This article is an extension of Dingoyan's L_2-Mixed Hodge theory [17] and a first step towards a version of Gromov's influential article on Kähler-hyperbolic manifolds [26] that would apply to singular Kähler varieties.

Gromov found a way to use the L_2-De Rham theory of an infinite Galois covering space of a compact Kähler manifold X and obtain algebro-geometric restrictions if X is Kähler-hyperbolic, for instance a compact complex submanifold of a neat quotient of a bounded symmetric domain. This inspired the influential works of Campana [6, 7] and Kollár [33, 34] masterfully exploiting in Kähler geometry the striking ideas of [1] to study compact Kähler manifolds with infinite fundamental group. Gromov's ideas were also extended in [19] to polarized Variations of Hodge Structures (actually to harmonic bundles) on a *compact* Kähler manifold X. They were also extended in [8, 22] to a theory of coherent L_2-cohomology in Complex Analytic Geometry. Some applications were given, say in [20, 21, 53], and an extremely striking one was recently found [4].

With applications in mind, the author is interested in further extending the theory to Mixed Hodge Modules [43].

Let X be a compact complex manifold. Let $\pi : \widetilde{X} \to X$ be an infinite Galois covering space with $\mathrm{Deck}(\widetilde{X}/X) = \Gamma$.

If $F^\bullet$ is a bounded complex of $\mathbb{C}$-vector spaces with constructible cohomology on X we construct, using a classical observation of Kashiwara, cohomology groups $\mathbb{H}^\bullet_{(2)}(\widetilde{X}, F^\bullet)$ that coïncide in the case $F^\bullet = \mathbb{C}_X$ to the L_2-cohomology of $\widetilde{X}$, see [35]. They obey Atiyah's L_2 index theorem, Poincaré-Verdier duality and are compatible with proper morphisms of complex analytic spaces.

If $\mathcal{M}$ is a coherent $\mathcal{D}$-module on X, we construct, using the construction of [22], cohomology groups $\mathbb{H}^\bullet_{DR,(2)}(\widetilde{X}, \mathcal{M})$ that coïncide in the case $\mathcal{M} = \mathcal{O}_X$ to the L_2-De Rham cohomology of $\widetilde{X}$ with respect to a Riemannian metric pulled back fom X and if $\mathcal{M} = \mathcal{D} \otimes_{\mathcal{O}_X} \mathcal{F}$, $\mathcal{F}$ being a coherent analytic sheaf with the L_2 cohomology groups $H^\bullet_2(\widetilde{X}, \mathcal{F} \otimes \omega_X^n)$ constructed in [22]. They obey Atiyah's L_2 index theorem.

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We did not check except in the simplest cases whether they are compatible with proper holomorphic mappings and did not study Verdier duality.

When an isomorphism in the derived category of sheaves $rh : F^\bullet \cong DR(\mathcal{M})$ is given, $\mathcal{M}$ being holonomic, we construct a natural isomorphism $rh_{(2)} : \mathbb{H}^\bullet_{(2)}(\widetilde{X}, F^\bullet) \cong \mathbb{H}^\bullet_{DR,(2)}(\widetilde{X}, \mathcal{M})$.

These cohomology groups are typically infinite dimensional quotients of Hilbertian Γ-modules by non necessarily closed submodules. They are also modules over $\mathcal{N}(\Gamma)$ the Von Neumann algebra of Γ. But one can be much more precise.

Given Γ a discrete countable group, the exact category of finite type projective Hibert Γ-modules naturally embeds in a rather simple abelian category $E_f(\Gamma)$ due to Farber [23] and Lück [35] endowed with a faithful functor to $Mod(\mathcal{N}(\Gamma))$. This abelian category has projective dimension one, its projective objects being finite type projective Hilbert Γ-modules. The preceding L_2 cohomology groups are in the essential image of the forgetful functor and the isomorphism $rh_{(2)}$ lifts too.

Theorem 1. *Let X be a compact complex manifold and $\widetilde{X} \to X$ be a Galois covering with Galois group Γ. Let $MD(X)$ be the abelian category whose objects are triples*

$$\mathbb{M} = (\mathcal{M} = \mathbb{M}^{DR}, P = \mathbb{M}^{Betti}, \alpha)$$

where $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module admitting a good filtration, P is a perverse sheaf of $\mathbb{R}$-vector spaces and $\alpha : P \otimes_{\mathbb{R}} \mathbb{C} \to DR(\mathcal{M})$ is an isomorphism in the derived category of sheaves and whose morphisms are the obvious ones.

There is a ∂-functor which, on the Betti side, is compatible with proper direct images, satisfies Atiyah's L_2 index theorem and Poincaré-Verdier duality:

$$L_2dR : D^bMD(X) \to D^bE_f(\Gamma)$$

and for each $\mathbb{M} \in MD(X)$ and $q \in \mathbb{Z}$ functorial isomorphisms in $E_f(\Gamma)$

$$H^q(L_2dR(\mathbb{M})) \cong \mathbb{H}^q_{(2)}(\widetilde{X}, \mathbb{M}^{Betti}) \cong \mathbb{H}^q_{DR,(2)}(\widetilde{X}, \mathbb{M}^{DR}).$$

If X is a projective algebraic manifold every coherent $\mathcal{D}_X$-module admits a global good filtration - a fact the author has learned from talks given by B. Malgrange. The author does not believe admitting a good filtration is an essential restriction here.

For applications, it seems to be useful to consider the case X is only a compact complex-analytic space such that one can embed X in a complex manifold Z'. In that situation, one can construct, taking a regular neighborhood Z of X, an infinite Galois covering space with $\mathrm{Deck}(\widetilde{Z}/Z) = \Gamma$ and a Γ-equivariant embedding $\widetilde{X} \to \widetilde{Z}$ covering the closed embedding $X \to Z$. Theorem 1 extends to this situation restricting one's attention to modules on Z whose support is contained in X.

Saito's category of Mixed Hodge Modules $MHM(X)$ [43] is an abelian subcategory of $MD(X)$.

Corollary 2. *Let X be a compact Kähler manifold and $\widetilde{X} \to X$ be a Galois covering with Galois group Γ.*

There is a ∂-functor which, on the Betti side, is compatible with proper direct images, satisfies Atiyah's L_2 index theorem and Poincaré-Verdier duality:

$$L_2dR : D^bMHM(X) \to D^bE_f(\Gamma)$$

and for each $\mathbb{M} \in MHM(X)$ and $q \in \mathbb{Z}$ functorial isomorphisms in $E_f(\Gamma)$

$$H^q(L_2dR(\mathbb{M})) \cong \mathbb{H}^q_{(2)}(\widetilde{X}, \mathbb{M}^{Betti}) \cong \mathbb{H}^q_{DR,(2)}(\widetilde{X}, \mathbb{M}^{DR}).$$

These cohomology groups are endowed with a real structure, a real filtration W coming from the weight filtration on $\mathbb{M}^{Betti}$ and a complex filtration F coming from

Saito's Hodge filtration on $\mathbb{M}^{DR}$. These filtrations and real structures are compatible with morphisms of Mixed Hodge Modules.

It is not clear whether these filtrations define a Mixed Hodge Structure. It seems difficult not to pass to reduced L^2-cohomology. Using an idea of Dingoyan [17], we conjecture:

Conjecture 3. *Let $\mathcal{N}(\Gamma) \subset \mathcal{U}(\Gamma)$ be the algebra of affiliated operators. Let $\mathbb{M}$ be a Mixed Hodge module (resp. a pure Hodge Module).*

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H^q(\widetilde{X}, L_2 dR(\mathbb{M}))$$

carries a Mixed (resp. a pure) Hodge structure in the abelian category of real $\mathcal{U}(\Gamma)$-modules with finite Γ-dimension. The restrictions on the Hodge numbers are as in the compact case.

We will use the notation $H^q(L_2 dR(\mathbb{M})) \stackrel{not.}{=} H^q(\widetilde{X}, L_2 dR(\mathbb{M}))$ whenever it is necessary to emphasize that $\mathbb{M}$ lives on X and that we are considering the covering space $\widetilde{X} \to X$.

We do not understand Saito's theory well enough to conjecture a similar statement for the derived category of $MHM(X)$.

Theorem 4. *Conjecture 3 is true in the following cases:*

- *There is a closed complex submanfold $i : Z \hookrightarrow X$ and a smooth polarized $\mathbb{Q}$-VSH $(Z, \mathbb{V}, F, S)$ on Z such that $\mathbb{M} = \mathbb{M}_i(\mathbb{V})$ is the corresponding Hodge Module on X.*
- *There is an open embedding $j : U \hookrightarrow X$ such that $X \setminus U$ is a divisor with simple normal crossings and a smooth $\mathbb{Q}$-VSH $(X, \mathbb{V}, F, S)$ on X such that $\mathbb{M} = Rj_* j^{-1} \mathbb{M}_X(\mathbb{V})$.*
- *There is an open embedding $j : U \hookrightarrow X$ such that $X \setminus U$ is a divisor with simple normal crossings and a smooth $\mathbb{Q}$-VSH $(X, \mathbb{V}, F, S)$ on X such that $\mathbb{M} = Rj_! j^{-1} \mathbb{M}_X(\mathbb{V})$.*

The first case follows easily from [19]. The second item in case $\mathbb{V} = \mathbb{Q}_X$ follows from [17] and Theorem 4 could be proved with a slight variation on Dingoyan's approach. We nevertheless felt it was helpful to recast Dingoyan's results in our language. The third case does not follow from [17]. A more general result holds, it is enough that the Gr_W of the Mixed Hodge module is a direct sum of modules of the form $\mathbb{M}_i(\mathbb{V}^\alpha)$.

The general case requires only to be able to settle the case of pure polarizable Hodge modules. The author believes one can settle this in case $\dim(X) = 1$. The author hopes the general case will be doable when a proof of the coincidence of the algebraic and the analytic definition of the Hodge filtration will be available.

The author believes one can endow the reduced L^2-cohomology of an infinite Galois cover of a projective algebraic variety with a functorial $\mathcal{U}(\Gamma)$-Mixed Hodge structure using techniques developped here and cohomological descent [14] and hopes to come back to this question in a future work. Subsuming Deligne's approach into Saito's is not straightforward [45] and one really needs to exercise some more care to dare draw this conclusion.

The recent preprint [18] suggests an extension of the theory for twistor $\mathcal{D}_X$-modules might be possible.

The article is organized as follows. The first section constructs L_p-constructible cohomology. The second section constructs L_p De Rham cohomology for coherent $\mathcal{D}$-module on complex manfolds. The third section reviews some facts on the homological algebra for $\mathcal{N}(\Gamma)$-modules and about $\mathcal{U}(\Gamma)$. The fourth section lifts the L_2-cohomology theory to $E_f(\Gamma)$ and finishes the proof of Theorem 1. It gives a

statement of a refined form of Conjecture 3 in terms of the reduced L^2 cohomology of Mixed Hodge Modules and a brief treatment of the singular case. The fifth section studies analytic L^2 Hodge decomposition in the Kähler case. The sixth section gives a proof of Theorem 4. An appendix gives more details on some technical facts the author prefered not to include in the text. A final section indicates briefly how using the algebra of affiliated operators simplifies the theory in [22] and gives a version of Theorem 1 valid without good filtrations. The reader is referred to [19, 17] for examples.

The article is a write-up of a project that was started 18 years ago with the definition of constructible L_2-cohomology in $E_f(\Gamma)$. After [17] appeared, the scope of the project was extended to include Mixed Hodge Modules. The author has given a handful of seminar and conference talks on this project during these years and wishes to apologize for not having made a text available. At some point, it was a work in collaboration with P. Dingoyan, who withdrew from the projet. The author would like to address special thanks to him for many enlightening discussions.

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1. Constructible L_p-cohomology

In the following $1 \leq p < +\infty$ will be a real number. No applicable results will be lost if one restricts oneself to the case $p = 2$. We also let $\mathbb{Q} \subset K \subset \mathbb{C}$ be a subfield of the complex numbers.

1.1. Equivariant constructible sheaves on Γ-simplicial complexes.

In this section, we recall basic well-known definitions, cf [32], chap. VIII.

Let Γ be a discrete countable group. Let T be a paracompact topological space endowed with an action of Γ (by homeomorphisms). We denote by $\mathbf{Mod}_\Gamma(K_T)$, the category of Γ-equivariant sheaves of K-vector spaces [1]. Let A be an abelian category, we also call $D^b(A)$ its bounded derived category[2]. We use the shorter notation $D^b_{K,\Gamma}(T) := D^b\mathbf{Mod}_\Gamma(K_T)$. We shall drop dependance on K when $K = \mathbb{C}$.

A Γ-simplicial complex $\mathbb{S}$ is a locally finite simplicial complex endowed with a proper left action of Γ, i.e. $\mathbb{S} = (S, \Delta, i)$ where S is a non-empty set endowed with an action of Γ $i : \Gamma \to \mathfrak{S}(S)$ and Δ is a set of non-empty finite subsets of S, the *simplices* of $\mathbb{S}$ such that:

- For every element s of S, the singleton $\{s\}$ belongs to Δ.
- For every element σ of Δ, any non-empty subset τ of σ belongs to Δ.
- For every element s of S, the subset of Δ consisting in the simplices containing s is finite.
- Γ preserves Δ.
- Γ acts on S with finite stabilizers.

Obviously, Γ acts on Δ with finite stabilizers and Γ acts properly on the topological realization $|\mathbb{S}|$ of $\mathbb{S}$. $|\mathbb{S}|$ is a closed subspace of $\mathbb{R}^S$ (endowed with the product topology) decomposed as $|\mathbb{S}| = \cup_{\sigma\in\Delta}|\sigma|$ where

$$|\sigma| = \{x \in \mathbb{R}^S | x(p) = 0 \text{ if } p \notin \sigma,\ x(p) > 0 \text{ if } p \in \sigma,\ \sum_{p\in\sigma} x(p) = 1\}.$$

Say $\mathbb{S}$ is finite dimensional if $\sup_{\sigma\in\Delta} \mathrm{Card}(\sigma) < \infty$. Say $\mathbb{S}$ is cocompact if it is finite dimensional and $\Gamma\backslash S$ is finite. In this case, $\Gamma\backslash\Delta$ is finite and $\Gamma\backslash|\mathbb{S}|$ is compact.

[1] A compatible action of Γ on a sheaf $\mathcal{S}$ is a continuous action on $Et(\mathcal{S})$ the espace étalé of $\mathcal{S}$ such that the canonical local homeomorphism $Et(\mathcal{S}) \to T$ is Γ-equivariant.

[2] There is no need to restrict to the bounded derived category until section 4.2, but we will not pursue more generality.

A Γ-equivariant sheaf of K-vector spaces F on $|\mathbb{S}|$ is *weakly* $\mathbb{S}$-*constructible* sheaf, resp. $\mathbb{S}$-*constructible*, if for every simplex σ, $i_{|\sigma|}^{-1}F$ is a constant sheaf[3], resp. and for every x in $|\mathbb{S}|$, F_x is of finite dimension. The abelian category of $\mathbb{S}$-constructible (resp. weakly $\mathbb{S}$-constructible) equivariant sheaves will be denoted by $\mathbf{Cons}_{K,\Gamma}(\mathbb{S})$ (resp. $w\mathbf{Cons}_{K,\Gamma}(\mathbb{S})$) A complex of Γ-equivariant sheaves $F^\bullet$ with bounded cohomology (i.e.: an object of $D^b_\Gamma(|\mathbb{S}|)$) is called $\mathbb{S}$-constructible (resp. weakly $\mathbb{S}$-constructible) if its cohomology sheaves $\mathcal{H}^j(F^\bullet)$ are $\mathbb{S}$-constructible (resp. weakly $\mathbb{S}$-constructible). $\mathbb{S}$-constructible complexes (resp. weakly $\mathbb{S}$-constructible complexes) are the objects of a full thick triangulated subcategory $D^b_{\mathbb{S}-c,K,\Gamma}(|\mathbb{S}|)$ (resp. $D^b_{w-\mathbb{S}-c,K,\Gamma}(|\mathbb{S}|)$) of $D^b_{K,\Gamma}(|\mathbb{S}|)$.

Proposition 1.1.1. *Let $\mathbb{S}$ be a finite dimensional Γ-simplicial complex. Then the natural functors*

$$D^b(w\mathbf{Cons}_{K,\Gamma}(\mathbb{S})) \to D^b_{w-\mathbb{S}-c,K,\Gamma}(|\mathbb{S}|),\ D^b(\mathbf{Cons}_{K,\Gamma}(\mathbb{S})) \to D^b_{\mathbb{S}-c,K,\Gamma}(|\mathbb{S}|)$$

are equivalences of triangulated categories.

Proof: The proof of Theorems 8.1.10 and 8.1.11 p.326 in [32] (which is the special case where Γ is the trivial group) applies here mutatis mutandis. Actually if the action of Γ is free we can use the natural equivalence of categories between the various categories of Γ-equivariant sheaves on $|\mathbb{S}|$ and of sheaves on $\Gamma\backslash|\mathbb{S}|$ to formally reduce the statement to [32, Chapter VIII]. □

1.2. **L_p-cohomology for equivariant constructible sheaves. The case of simplicial complexes.** Let $\mathbb{S}$ be a finite dimensional Γ-simplicial complex. Consider the natural quotient map $\pi : |\mathbb{S}| \to \Gamma\backslash|\mathbb{S}|$. It is easy to see that $\pi_!$ the direct image with proper support is exact on $\mathbb{S}$-constructible sheaves [4]. On the category of equivariant $\mathbb{S}$-constructible sheaves, $\pi_!$ factorizes through the category of sheaves of left $K\Gamma$-modules on $\Gamma\backslash|\mathbb{S}|$.

The left and right regular representations, denoted by λ and ρ, on the set $l_p\Gamma$ of complex valued functions $(a_\gamma)_{\gamma\in\Gamma}$ defined on Γ such that $\sum_{\gamma\in\Gamma}|a_\gamma|^p < \infty$ defines a bimodule over $\mathbb{C}\Gamma$. We call $Rl_p\Gamma$ the right Γ-module attached to ρ. In particular, given a sheaf F of left $K\Gamma$-modules on a topological space T, the tensor product $Rl_p\Gamma \otimes_{K\Gamma} F$ is a sheaf of left $\mathbb{C}\Gamma$-modules. It is actually a sheaf of $W_{l,p}(\Gamma)$-modules where $W_{l,p}(\Gamma)$ is the bicommutant of $\lambda(\mathbb{Z}\Gamma)$ in the algebra of continuous linear endomorphisms of $l_p\Gamma$.

Lemma 1.2.1. *The functor $F \mapsto Rl_p\Gamma \otimes_{K\Gamma} \pi_!F$ is exact on $\mathbf{Cons}_{K,\Gamma}(\mathbb{S})$.*

Proof: Since $\mathbb{C}$ is a K-vector space it is flat over K and $\pi_!$ commutes with $\otimes_K\mathbb{C}$. Hence, it is enough to prove exactness of $F_\mathbb{C} \mapsto Rl_p\Gamma \otimes_{\mathbb{C}\Gamma} \pi_!F_\mathbb{C}$ on $\mathbb{S}$-constructible equivariant sheaves of $\mathbb{C}$-vector spaces. Since the stalk at $p \in \Gamma\backslash|\mathbb{S}|$ of $\pi_!\mathcal{F}_\mathbb{C}$ is isomorphic to $\mathbb{C}[\Gamma/H_{\tilde{p}}]^{\oplus n}$ where n is a nonnegative integer and $H_{\tilde{p}}$ is the stabilizer of some lift $\tilde{p} \in |\mathbb{S}|$ of p, it follows that it is a projective module over $\mathbb{C}\Gamma$. Indeed, whenever H is a finite subgroup of Γ, the pull-back injection $i : \mathbb{C}[\Gamma/H] \to \mathbb{C}\Gamma$ has a right inverse $\pi((a_\gamma)_{\gamma\in\Gamma})_g = \frac{1}{\mathrm{Card}(H)}\sum_{h\in H} a_{gh}$ which is equivariant for the left action of Γ. Exactness follows from the facts that stalks of tensor products are computed stalkwise, that $\pi_!$ is exact and that a short exact sequence of projective modules splits. □

Lemma 1.2.2. *The functor $F \mapsto Rl_p\Gamma \otimes_{K\Gamma} \pi_!F$ is exact on $\mathbf{Mod}_\Gamma(K_{|\mathbb{S}|})$.*

[3]Whenever Z is a locally closed subset of X, we denote by $i_Z : Z \to X$ the resulting embedding.

[4]It is a direct consequence of [32] Proposition 8.1.4 p. 323 in case the action is free. The general case is easily taken care of by a barycentric subdivision argument.

Proof: The above proof also works with a minor modification for all sheaves of K-vector spaces. □

Definition 1.2.3. *Let $F^\bullet$ be an object of $D^b(\mathbf{Cons}_{K,\Gamma}(\mathbb{S}))$. Its kth L_p- hypercohomology groups is the $W_{l,p}(\Gamma)$-module*

$$\mathbb{H}^k_{(p)}(|\mathbb{S}|, F^\bullet) := \mathbb{H}^k(\Gamma\backslash|\mathbb{S}|, Rl_p\Gamma \otimes_{K\Gamma} \pi_! F^\bullet).$$

Lemma 1.2.4. *The composition of the derived functor of $H^0(|\mathbb{S}|, -)$ and of $Rl_p\Gamma \otimes_{K\Gamma} -$ gives a ∂-functor [5] of triangulated categories*

$$\mathbb{H}^\bullet_{(p)}(|\mathbb{S}|, -) : D^b(\mathbf{Cons}_{K,\Gamma}(\mathbb{S})) \to D^b(Mod_{W_{l,p}(\Gamma)}),$$

where $Mod_{W_{l,p}(\Gamma)}$ stands for the category of left $W_{l,p}(\Gamma)$-modules such that the L_p-hypercohomology groups are its cohomology objects.

This definition gives rise to the long exact sequence attached to a short exact sequence and to various spectral sequences generalizing it.

Thanks to lemma 1.2.2, we also get:

Lemma 1.2.5. *The same formula as in definition 1.2.3 defines an extension of $\mathbb{H}^*_{(p)}(|\mathbb{S}|, -)$ to a ∂-functor*

$$\mathbb{H}^\bullet_{(p)}(|\mathbb{S}|, -) : D^b_{K,\Gamma}(|\mathbb{S}|) \to D^b(Mod_{W_{l,p}(\Gamma)}).$$

We don't use a different notation hoping this will not cause any confusion.

1.3. **The subanalytic case.**

1.3.1. *Subanalytic stratifications and constructible sheaves.* A subanalytic[6] Γ-space is a Γ-space that can be realized as a locally closed Γ-invariant subanalytic subset of a real analytic manifold endowed with a proper real analytic action of Γ. A stratified subanalytic space $\mathbb{X}$ is a subanalytic space $\widetilde{X} := |\mathbb{X}|$ endowed with a locally finite partition $\widetilde{X} = \cup_i X_i$ in disjoint subanalytic submanifolds satisfying $X_i \cap \bar{X}_j \neq \emptyset \Longrightarrow X_i \subset \bar{X}_j$. A stratified subanalytic Γ-space $\mathbb{X}$ is a proper analytic action of Γ on $\widetilde{X} = |\mathbb{X}|$ such that for every $g \in \Gamma$ and every point $x \in |\mathbb{X}|$ the germ at x of the stratification is carried by g to the germ at gx of the stratification. When the stratification comes from a Γ-simplicial complex, one calls it a triangulation. Proposition 8.2.5 in [32] implies that, in the cocompact case, any Γ-stratification may be refined to a Γ-triangulation and that the Γ-triangulations form a cofinal system with respect to refinement.

The obvious extension of the definitions and notations of section 1.1 will be left to the reader, the only change being that $\mathbb{X}$-constructible sheaves on $|\mathbb{X}|$ are now assumed to be locally constant along the strata of $\mathbb{X}$.

Let $\widetilde{X}$ be a subanalytic Γ-space. A sheaf of K-vector spaces F on $\widetilde{X}$ is called constructible if it is constructible with respect to some subanalytic stratification of $\widetilde{X}$. We denote by $\mathbb{R}\mathbf{Cons}_{K,\Gamma}(\widetilde{X})$ the category of equivariant constructible sheaves on $\widetilde{X}$ and by $D^b_{K,\mathbb{R}c,\Gamma}(\widetilde{X})$ the thick full subcategory of $D^b_{K,\Gamma}(\widetilde{X})$ consisting of complexes with bounded constructible cohomology. Theorem 8.4.5 in [32, p. 339] is easily generalized to

Proposition 1.3.1. *The natural functor $D^b(\mathbb{R}\mathbf{Cons}_{K,\Gamma}(\widetilde{X})) \to D^b_{\mathbb{R}c,K,\Gamma}(\widetilde{X})$ is an equivalence of triangulated categories if $\widetilde{X}$ is cocompact.*

[5] As in [28, p. 22], a ∂-functor of triangulated categories is an additive functor which commutes with the translation functor and respects distinguished triangles.

[6] Actually, 'definable in a o-minimal structure' is the natural hypothesis.

1.3.2. *Constructible L_p-cohomology.*

Proposition 1.3.2. *Let $\widetilde{X}$ be a cocompact subanalytic Γ-space. Let $\mathcal{F}^\bullet$ be an object of $D^b_{\mathbb{R}-c,K,\Gamma}(\widetilde{X})$. Its kth L_p-hypercohomology groups is the $W_{l,p(\Gamma)}$-module*

$$\mathbb{H}^k_{(p)}(\widetilde{X},\mathcal{F}^\bullet) := \mathbb{H}^k(\Gamma\backslash\widetilde{X}, Rl_p\Gamma\otimes_{K\Gamma}\pi_!\mathcal{F}^\bullet).$$

There is a ∂-functor of triangulated categories

$$\mathbb{H}^\bullet_{(p)}(\widetilde{X},-) : D^b_{\mathbb{R}-c,K,\Gamma}(\widetilde{X}) \to D^b(Mod_{W_{l,p}(\Gamma)}),$$

where $Mod_{W_{l,p}(\Gamma)}$ stands for the category of left $W_{l,p}(\Gamma)$-modules such that

$$\mathbb{H}^k_{(p)}(\widetilde{X},\mathcal{F}^\bullet) = H^k(\mathbb{H}^k_{(p)}(\widetilde{X},\mathcal{F}^\bullet)).$$

Proof: We can replace $D^b_{\mathbb{R}-c,K,\Gamma}(\widetilde{X})$ by $D^b\mathbb{R}\mathbf{Cons}_{K,\Gamma}(\widetilde{X})$ since the natural functor is an equivalence by Proposition 1.1.1 and $Rl_p\Gamma\otimes_{K\Gamma}\pi_!\mathbb{C}\otimes -$ is exact by Lemma 1.2.5.

Since $D^b\mathbb{R}\mathbf{Cons}_{K,\Gamma}(\widetilde{X})$ is the limit of its full subcategories $D^b\mathbf{Cons}_{K,\Gamma}(\mathbb{X})$, $\mathbb{X}$ running through all subanalytic Γ-triangulations, this follows from definition 1.2.3. □

1.4. **Complex Analytic case.** We assume here $K = \mathbb{C}$ and drop K from the notation.

Assume from now on that $\widetilde{X}$ is a cocompact complex Γ-space. The relevant stratifications are complex analytic stratification (by definition, a subanalytic stratification is complex analytic if so are the closures of the strata) and we say that an equivariant sheaf is constructible if is constructible with respect to some complex analytic stratification and that a complex of equivariant sheaves is constructible if so are its cohomology sheaves. Then $\mathbf{Cons}_\Gamma(\widetilde{X})$ is a full abelian subcategory of $\mathbb{R}\mathbf{Cons}_\Gamma(\widetilde{X})$ stable by extensions, $D^b_{c,\Gamma}(\widetilde{X})$, the full subcategory of $D^b_{\mathbb{R}c,\Gamma}(\widetilde{X})$ whose cohomology objects are in $\mathbf{Cons}_\Gamma(\widetilde{X})$, is a thick triangulated subcategory and we have a natural ∂-functor

$$D^b\mathbf{Cons}_\Gamma(\widetilde{X}) \to D^b_{c,\Gamma}(\widetilde{X}).$$

Remark 1.4.1. *This functor is an equivalence of categories if $\widetilde{X}$ is a Galois topological covering space of the analytization of a complex projective variety using GAGA and* [39].

Definition 1.4.2. *We can restrict $\mathbb{H}^\bullet_{(p)}(\widetilde{X},-)$ to $D^b_{c,\Gamma}(\widetilde{X})$ to get the* constructible L_p-cohomology functor*:*

$$\mathbb{H}^\bullet_{(p)}(\widetilde{X},-) : D^b_{c,\Gamma}(\widetilde{X}) \to D^b(Mod_{W_{l,p}(\Gamma)}).$$

An important special case is L_p-intersection cohomology.

Definition 1.4.3. *Let Z be a singular compact complex space and $\pi : \widetilde{Z} \to Z$ its universal covering space. Its k-th intersection L_p cohomology is the $W_{l,p}(\Gamma)$-module $\mathbb{H}^k_{(p)}(\widetilde{Z}, \pi^{-1}\mathcal{IC}^\bullet_Z)$ where $\mathcal{IC}^\bullet_Z$ is the intersection cohomology sheaf of Z* [2].

The initial impetus for this work was to formulate the following:

Conjecture 1.4.4. *If $p = 2$, and Z is a closed analytic subset of a compact Kähler hyperbolic manifold then $\mathbb{H}^k_{(2)}(\widetilde{Z}, \pi^{-1}\mathcal{IC}^\bullet_Z) = 0$ for $k \neq \dim(Z)$.*

1.5. **Real structures.** The algebra $W_{l,p}(\Gamma)$ carry a real structure, namely a conjugate linear algebra involutive automorphism $\dagger$, and a real structure on a module over $W_{l,p}(\Gamma)$ is just a conjugate linear automorphism on the underlying $\mathbb{C}$-vector space compatible with $\dagger$. For instance $l_p\Gamma$ has a real structure.

Modules with real structures form a $\mathbb{R}$-linear abelian category $Mod_{W_{l,p}(\Gamma),\mathbb{R}}$ which has an exact faithful forgetful functor to $Mod_{W_{l,p}(\Gamma),\mathbb{R}}$ and if $K \subset \mathbb{R}$ the functors of Proposition 1.3.2 and Definition 1.4.2 lift to $D^b(Mod_{W_{l,p}(\Gamma),\mathbb{R}})$.

2. L^p-cohomology and differential operators for coherent sheaves

2.1. **Coherent L^p-cohomology.** Let X be a complex manifold and let $\mathcal{O}_X$ (resp. $\mathcal{D}_X$) denote its structure sheaf (resp. the sheaf of holomorphic differential operators). Let $\pi : \widetilde{X} \to X$ be a Galois topological covering space and $\Gamma = \mathrm{Gal}(\widetilde{X}/X)$ be its Galois group, acting on $\widetilde{X}$ on the left.

If $\mathcal{R}$ is a sheaf of rings on X, denote by $\mathbf{Mod}(\mathcal{R})$ the abelian category of sheaves of left $\mathcal{R}$-modules by $Hom_{\mathcal{R}}$ its group of morphisms and by $\underline{Hom}_{\mathcal{R}}$ the internal Hom bifunctor on $\mathbf{Mod}(\mathcal{R})$. When considering right $\mathcal{R}$-modules, we use the notations $\mathbf{Mod}(\mathcal{R}^o)$, $Hom_{\mathcal{R}^o}$, $\underline{Hom}_{\mathcal{R}^o}$. If R is a ring then we denote by R_X the sheaf of rings of locally constant functions with values in R.

Denote by $\mathbf{Coh}(\mathcal{O}_X)$ the full abelian subcategory of $\mathbf{Mod}(\mathcal{O}_X)$ whose objects are the coherent analytic sheaves of X. Denote by $\mathbf{Coh}(\mathcal{D}_X)$ (resp. $\mathbf{Hol}(\mathcal{D}_X)$) its the full abelian subcategory of $\mathbf{Mod}(\mathcal{D}_X)$ whose objects are the coherent (resp. holonomic) $\mathcal{D}_X$-modules. A $\mathcal{O}_X$-module is quasi coherent if it is locally the limit of its coherent submodules.

For every $p \in [1, +\infty[$ and $\mathcal{F}$ a coherent analytic sheaf on X, [22] (see also [8]) constructs a subsheaf $l^p\pi_*\mathcal{F} \subset \pi_*\pi^{-1}\mathcal{F}$ which can be described locally as follows. Choose $\varphi : \mathcal{O}_X^{\oplus N} \to \mathcal{F}$ a presentation of $\mathcal{F}$ on a Stein open subset U such that $\pi^{-1}(U) = \Gamma \times U$ then:

$$\begin{aligned} l^p\pi_*\mathcal{F}(U) &= \{(s_\gamma)_{\gamma\in\Gamma} \in \mathcal{F}(U)^\Gamma,\ \exists s'_\gamma \in \mathcal{O}_X^{\oplus N}(U)^\Gamma,\ \varphi(s'_\gamma) = s_\gamma \text{ and} \\ &\quad \forall K \Subset U \quad \sum_{\gamma\in\Gamma} \int_K |s_\gamma|^p < +\infty\}. \end{aligned}$$

The independance on φ is checked in [22]. Given $\phi : \mathcal{F} \to \mathcal{F}'$ a $\mathcal{O}_X$-linear morphism of coherent sheaves

$$\pi_*\pi^{-1}\phi : \pi_*\pi^{-1}\mathcal{F} \to \pi_*\pi^{-1}\mathcal{F}'$$

maps $l^p\pi_*\mathcal{F}$ into $l^p\pi_*\mathcal{F}'$. Denote by $l^p\pi_*\phi : l^p\pi_*\mathcal{F} \to l^p\pi_*\mathcal{F}'$ the restriction of $\pi_*\pi^{-1}\phi$. The resulting functor $l^p\pi_* : \mathbf{Coh}(\mathcal{O}_X) \to \mathbf{Mod}(W_{l,p}(\Gamma) \otimes_{\mathbb{C}} \mathcal{O}_X)$ is exact [22] and one can define

$$H^\bullet_{L^p}(\widetilde{X}, \mathcal{F}) := H^\bullet(X, l^p\pi_*\mathcal{F}).$$

Since $H^q_{L^p}(\widetilde{X}, \mathcal{F}) = 0$ for $q > \dim_{\mathbb{C}}(X)$ (at least when X is compact) this yields a good cohomology theory on $\mathbf{Coh}(\mathcal{O}_X)$, indeed a a ∂-functor

$$D^b\mathbf{Coh}(\mathcal{O}_X) \to D^b Mod_{W_{l,p}(\Gamma)}.$$

Observe that if $\mathcal{F}$ is coherent $l^p\pi_*\mathcal{F} = l^p\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F}$. Hence, the functor $l^p\pi_*$ extends to $D^b(\mathbf{Mod}(\mathcal{O}_X))$ setting $l^p\pi_*\mathcal{L} := l^p\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L}$ thanks to:

Lemma 2.1.1. *The functor $l^p\pi_* = l^p\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_X}$ is exact on $\mathbf{Mod}(\mathcal{O}_X)$.*

Proof: The problem is local. Since this functor is exact on $\mathbf{Coh}(\mathcal{O}_X)$, it follows from the fact that tensor products of sheaves commute with taking the stalks [24, p. 137] that $Tor_1^{\mathcal{O}_{X,x}}((l^p\pi_*\mathcal{O}_X)_x, \mathcal{O}_{X,x}/I_x) = 0$ for every (finitely generated) ideal of $\mathcal{O}_{X,x}$. Hence $(l^p\pi_*\mathcal{O}_X)_x$ is a flat $\mathcal{O}_{X,x}$-module and exactness follows applying [24, p. 137] once more. $\square$

It should however be noted that if a sheaf $\mathcal{F}$ has two different $\mathcal{O}_X$-module structures, say $\mathcal{F}_1$ and $\mathcal{F}_2$, it may be the case that $l^p\pi_*\mathcal{F}_1 \neq l^p\pi_*\mathcal{F}_2$ as subsheaves of $\pi_*\pi^{-1}\mathcal{F}_1 = \pi_*\pi^{-1}\mathcal{F}_2 = \pi_*\pi^{-1}\mathcal{F}$.

Remark 2.1.2. *There is a natural structure of left $\mathcal{D}_X$-module on $l^p\pi_*\mathcal{O}_X$ hence $l^p\pi_*$ gives rise to an exact endofunctor of $\mathbf{Mod}(\mathcal{D}_X)$. When Γ is infinite, it does not preserve the full subcategory of coherent or quasicoherent modules.*

2.2. **Differential operators.** We need to check that differential operators between quasicoherent analytic sheaves preserve $l^p\pi_*$.

Recall from [42] that for $\mathcal{L}, \mathcal{L}'$ two $\mathcal{O}_X$-modules $\mathrm{Diff}_X(\mathcal{L}, \mathcal{L}')$ is the image of the natural injective morphism:

$$Hom_{\mathcal{D}_X^o}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \to Hom_{\mathbb{C}_X}(\mathcal{L}, \mathcal{L}')$$

given by the composition of the natural adjunction

$$Hom_{\mathcal{D}_X^o}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \xrightarrow{\sim} Hom_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X),$$

$\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X$ being endowed with the **right** $\mathcal{O}_X$-module structure, with left composition by the natural $\mathbb{C}$-linear (actually left $\mathcal{O}_X$-linear) morphism

$$\nu_{\mathcal{L}'} : \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \mathcal{L}'$$

which maps $\ell \otimes P$ to $P(1)\ell$. One has $\nu_{\mathcal{L}'} = \mathcal{L}' \otimes_{\mathcal{O}_X} \nu_{\mathcal{O}_X}$ where $\nu_{\mathcal{O}_X} : \mathcal{D}_X \to \mathcal{O}_X$ is the naturel **left** $\mathcal{D}_X$-linear (hence left $\mathcal{O}_X$-linear) morphism mapping $P \in \mathcal{D}_X$ to $P(1) \in \mathcal{O}_X$.

Lemma 2.2.1. *Assume $\mathcal{L}'$ is quasicoherent. Let $(\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X)_l$ resp. $(\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X)_r$ the left resp. the right $\mathcal{O}_X$-modules structures of $\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X$. Then:*

$$l^p\pi_*(\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X)_l = l^p\pi_*(\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X)_r \subset \pi_*\pi^{-1}\mathcal{L}' \otimes \mathcal{D}_X.$$

Proof: Let us begin by treating the case where $\mathcal{L}' = \mathcal{O}_X$. Then both $l^p(\pi_*(\mathcal{D}_X))_\sharp$, $\sharp = l, r$ are the increasing union of the subsheaves $l^p\pi_*(F_k\mathcal{D}_X)_\sharp$ where $F_k\mathcal{D}_X$ is the sub-$\mathcal{O}_X$-bimodule consisting of the holomorphic differential operators of degree $\leq k$. Hence it is enough to show that $l^p\pi_*(F_k\mathcal{D}_X)_l = l^p\pi_*(F_k\mathcal{D}_X)_r$. The problem being local assume we have a coordinate system on an open set U such that $\pi^{-1}(U) \simeq \Gamma \times U$. A section of $\pi_*\pi^{-1}F_k\mathcal{D}_X$ of the form $(\sum_{|\alpha|\leq k} f_{\alpha,\gamma}\partial^\alpha)$ is in $l^p(\pi_*(\mathcal{D}_X))_l$ iff, for all $K \Subset U$, $\sum_\gamma \int_K \sum_\alpha |f_{\alpha,\gamma}|^p < +\infty$ whereas a section $\pi_*\pi^{-1}F_k\mathcal{D}_X$ of the form $(\sum_{|\alpha|\leq k} \partial^\alpha g_{\alpha,\gamma})$ is in $l^p(\pi_*(\mathcal{D}_X))_r$ iff, for all $K \Subset U$,

$$\sum_\gamma \int_K \sum_\alpha |g_{\alpha,\gamma}|^p < +\infty.$$

Since $\partial^\alpha g = g\partial^\alpha + \sum_{\beta<\alpha} P_{\beta,\alpha}(g)\partial^\beta$ where $P_{\beta,\alpha}$ is a universal differential operator, the Cauchy inequality gives

$$\int_K \sum_\alpha |f_{\alpha,\gamma}|^p \leq C_{K,K'} \int_{K'} \sum_\alpha |g_{\alpha,\gamma}|^p$$

if $K' \Subset U$ is a compact neighborhood of K. Whence the inclusion $l^p\pi_*(F_k\mathcal{D}_X)_l \subset l^p\pi_*(F_k\mathcal{D}_X)_r$. The reverse inclusion follows by the same token.

This implies the lemma for $\mathcal{L}'$ a free $\mathcal{O}_X$-module of possibly infinite rank.

Now, for the general case. The statement being local, we may choose $\varphi : \mathcal{O}_X^N \to \mathcal{L}'$ a presentation, N being some cardinal. The definition implies that $l^p\pi_*(\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X)_\sharp \in \pi_*\pi^{-1}\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is the image by $\pi_*\pi^{-1}\varphi$ of $l^p\pi_*(\mathcal{O}_X^N \otimes_{\mathcal{O}_X} \mathcal{D}_X)_\sharp$ in $\pi_*\pi^{-1}\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X$. The lemma follows. □

Lemma 2.2.2. *Under the hypothesis of Lemma 2.2.1, let* $P \in \mathrm{Diff_X}(\mathcal{L}, \mathcal{L}')$ *and let* $p \in Hom_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X)$ *be the unique right* $\mathcal{O}_X$*-linear morphism such that* $P = \nu_{\mathcal{L}'} \circ p$*. Then*

$$l^p\pi_* P := \nu_{l^p\pi_*\mathcal{L}'} \circ l^p\pi_* p : l^p\pi_*\mathcal{L} \to l^p\pi_*\mathcal{L}'$$

is the restriction of $\pi_*\pi^{-1}P$ *and defines a* $W_{l,p}(\Gamma) \otimes_{\mathbb{C}} \mathcal{O}_X$*-linear morphism of sheaves.*

The assignement $P \mapsto l^p\pi_* P$ *defines an additive functor*

$$l^p\pi_* : \mathbf{Qcoh}(\mathcal{O}_X, \mathrm{Diff_X}) \longrightarrow \mathbf{Mod}(\underline{W_{l,p}(\Gamma)}_X)$$

where $\underline{W_{l,p}(\Gamma)}_X$ *is the constant sheaf with constant value* $W_{l,p}(\Gamma)$ *and* $\mathbf{Qcoh}(\mathcal{O}_X, \mathrm{Diff_X})$ *is the additive category whose objects are quasi coherent* $\mathcal{O}_X$*-modules and whose morphisms are differential operators.*

Proof: The definition makes sense thanks to lemma 2.2.1. The statement is thus an easy consequence of the definition and of the properties of $l^p\pi_*$ described above. □

2.3. L_p **De Rham cohomology.** Let $\mathcal{M}$ be a (quasi) coherent $\mathcal{D}_X$-module viewed as a $\mathcal{O}_X$-module endowed with a flat connection $\nabla : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_X$. The De Rham complex of $\mathcal{M}$ defined as:

$$DR(\mathcal{M}) = (\mathcal{M} \overset{\nabla}{\to} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_X \overset{\nabla}{\to} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^2_X \to \ldots)[\dim X]$$

is a complex in $\mathbf{Qcoh}(\mathcal{O}_X, \mathrm{Diff_X})$. Applying the functor $l^p\pi_*$ we define the L^p De Rham complex $l^p\pi_* DR(\mathcal{M})$ and the L^p De Rham cohomology:

$$H^\bullet_{DR,L^p}(\tilde{X}, \mathcal{M}) := H^\bullet(X, l^p\pi_* DR(\mathcal{M})).$$

We will not try to put more structure than the natural $W_{l,p}(\Gamma)$-module structure on these general L^p cohomology groups.

The L^p De Rham constructible cohomology groups come from a ∂-functor

$$H^\bullet_{DR,L^p,} : D^b(\mathbf{Coh}(\mathcal{D}_X)) \to D^b(Mod_{W_{l,p}(\Gamma)}).$$

Example 2.3.1. *If* $\mathcal{F}$ *is a quasi coherent* $\mathcal{O}_X$*-module,*

$$H^\bullet_{DR,L^p}(\widetilde{X}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}) = H^\bullet_{L^p}(\widetilde{X}, \mathcal{F} \otimes \omega_X).$$

Proof: The natural augmentation $\epsilon : \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \otimes_{\mathcal{O}_X} \omega_X \to \mathcal{F} \otimes_{\mathcal{O}_X} \omega_X$ gives rise to a quasi-isomorphism $DR(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}) \overset{\epsilon}{\to} \mathcal{F} \otimes_{\mathcal{O}_X} \omega_X$. Locally it is a Koszul complex for the regular sequence $(\partial_{x_1}, \ldots, \partial_{x_n})$. The same is actually true for its $l^p\pi_*$ and we get a quasi-isomorphism $l^p\pi_* DR(\mathcal{F}) \overset{l^p\pi_*\epsilon}{\longrightarrow} l^p\pi_*\mathcal{F} \otimes_{\mathcal{O}_X} \omega_X$. □

Example 2.3.2. *Denote by* $\ell^p\pi_*\mathbb{C}_{\widetilde{X}} \subset \pi_*\mathbb{C}_{\widetilde{X}}$ *the locally constant sheaf of* $W_{l,p}(\Gamma)$*-modules attached to the right regular representation of* Γ *in* $L^p\Gamma$*. Let* $\mathbb{V}$ *be a finite rank complex local system on* X *and* $\mathcal{V}$ *be the* $\mathcal{D}_X$*-module whose underlying finite rank locally free* $\mathcal{O}_X$*-module is* $\mathbb{V} \otimes_{C_X} \mathcal{O}_X$ *and holomorphic connection* ∇ *so that the natural morphism* $\sigma : \mathbb{V} \to \mathcal{V}$ *represents* $\ker(\nabla)$*.*

Then $H^\bullet_{DR,L^p}(\widetilde{X}, \mathcal{V}) = H^{\bullet+\dim(X)}(X, l^p\pi_*\mathbb{C}_{\widetilde{X}} \otimes_{\mathbb{C}_X} \mathbb{V}) = \mathbb{H}^{\bullet+\dim(X)}_{(p)}(\tilde{X}, \mathbb{V})$.

Proof: Left to the reader. □

Remark 2.3.3. *With the notation of Remark 2.1.2,* $l^p\pi_* DR(\mathcal{M}) = DR(l^p\pi_*\mathcal{M})$.

2.4. **Compatibility to the Riemann-Hilbert correspondance.**

Remark 2.4.1. *The natural sheaf monomorphism $i_\pi : \ell^p\pi_*\mathbb{C}_{\widetilde{X}} \otimes_{\mathbb{C}_X} \mathcal{O}_X \to l^p\pi_*\mathcal{O}_X$ is not an epimorphism. One would need a completed tensor product of sheaves in locally convex topological vector spaces we will not try and discuss.*

Proposition 2.4.2. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then the natural map $Rl^p\Gamma \otimes_{\mathbb{C}\Gamma} \pi_!\pi^{-1}DR(\mathcal{M}) \to l^p\pi_*DR(\mathcal{M})$ is a quasi-isomorphism.*

Proof: The problem is local. Thus, we can assume $\mathcal{M}$ has a good filtration $F_.$. Then $F_.DR(\mathcal{M})$ is a filtration of $DR(\mathcal{M})$ by differential complexes of coherent sheaves. For $q \gg 0$ $F_qDR(M) \to DR(\mathcal{M})$ is a quasi-isomorphism [42, Lemma 1.14] (see also [3, Lemma 1.5.6 p. 31]). Hence it is enough to prove that $Rl^p\Gamma \otimes_{\mathbb{C}\Gamma} \pi_!\pi^{-1}F_qDR(\mathcal{M}) \to l^p\pi_*F_qDR(\mathcal{M})$ is a quasi-isomorphism for such a $q \gg 0$.

Thanks to the Kashiwara constructibility theorem[7] [30, 36], the cohomology of $F_qDR(\mathcal{M})$ is constructible. Choosing U appropriately such that it is Stein and

$$H^i(U, \mathcal{H}^j(DR(\mathcal{M}))) = 0$$

for $i > 0$. Note that the kernel and images of the differentials in $F_qDR(\mathcal{M})$ have also vanishing cohomology in positive degree on U.

We have to show that every element z of $Ker(d) : F_{q+k}\mathcal{M} \otimes \Omega^k(U) \to F_{q+k+1}\mathcal{M} \otimes \Omega^{k+1}(U)$ can be decomposed as a sum $z = dt + g(h(z))$ where

$$g : \mathcal{H}^{k-n}(F_qDR(\mathcal{M})) \to Ker(d) \subset F_{q+k}\mathcal{M}$$

is a section over U of the morphism of sheaves

$$h : Ker(d : F_{q+k}\mathcal{M} \otimes \Omega^k \to F_{q+k+1}\mathcal{M} \otimes \Omega^{k+1}) \to \mathcal{H}^{k-n}(DR(\mathcal{M}))$$

and t a section over U of $F_{q+k-1}\mathcal{M} \otimes \Omega^{k-1}$ with local L^p-estimates.

This means the following. The Fréchet structure of $\mathcal{G}(U)$ where $\mathcal{G}$ is coherent is given by an inverse limit of a countable family of L^p norms $(\| - \|_n)_{n\in\mathbb{N}}$ defined by integration on an exhaustive family of compact subsets of U if the sheaf is locally free, of quotient norms of such L_p norms in a locally presentation of the sheaf in general [22]. A local L^p estimate is then, for all $n \in \mathbb{N}$, a series of estimates of the form:

$$\|t\|_n \leq C_n.\|z\|_{n'}$$

for some $n' \in \mathbb{N}$.

This follows from the continuity of d for this Fréchet structure, the fact that a continuous operator of Fréchet spaces has closed range if it has a finite dimensional kernel and the open mapping theorem for Fréchet spaces using a standard argument (cf. e.g. [22, pp. 534-535]). □

Corollary 2.4.3. *The natural map i_π induces a natural invertible transformation of functors on* $\mathbf{Hol}(\mathcal{D}_X)$*:*

$$rh_{(p)} : H^\bullet_{DR,L^p}(\widetilde{X}, \_) \xleftarrow{\simeq} \mathbb{H}^{\bullet+\dim_{\mathbb{C}}(\tilde{X}}_{(p)}(\widetilde{X}, DR(\_)).$$

Remark 2.4.4. *As in section 1 we may work in the more general set-up of a proper action of Γ on a complex manifold $\widetilde{X}$ with cocompact quotient or even restrict our attention to cocompactly supported equivariant coherent $\mathcal{D}_X$ or $\mathcal{O}_X$-modules.*

[7] Which is used implicitely in the statement of the proposition.

3. Farber's abelian category and its localisation

Up to this point, we were working with L^p-cohomology. Now, it is time to admit that unless $p = 2$ the objects we constructed are out of control.

We will change our notations and define $\mathcal{N}(\Gamma) = W_{l,2}(\Gamma)$ and survey some relevant homological algebraic aspects of modules over this operator algebra.

3.1. **Hilbert Γ-modules.** Let us first briefly review a very nice construction due to Farber and Lück [23]. For a longer review adapted to our purposes, see [22, pp.539-544]. For a complete review including applications in topology and algebra, see the bible of the subject [35].

Definition 3.1.1. *A Hilbert Γ-module (resp. of finite type, resp. separable) is a topological $\mathbb{C}$-vector space with a continuous Γ-action which can be realized as a closed Γ-invariant subspace of $l_2\Gamma\hat{\otimes}H$ where H is a Hilbert space (resp. of finite dimension, resp. separable).*

Lemma 3.1.2. *The action of $\mathbb{C}\Gamma$ on a Hilbert Γ-module E extends uniquely to an action of the C^*-algebra $\mathcal{N}(\Gamma)$ in such a way that the image of $\mathcal{N}(\Gamma)$ is strongly closed in $\mathfrak{B}(E)$.*

Proposition 3.1.3. *The following categories $E_f(\Gamma) \subset E(\Gamma)$:*

- *Objects of $E(\Gamma)$ are triples (E_1, E_2, e) where E_1 et E_2 are Hilbert Γ-module and e continuous Γ-equivariant linear map.*
- *$Hom_{E(\Gamma)}((E_1, F_2, e), (F_1, F_2, f))$ is the set of pairs $(\phi_1 : E_1 \to F_1, \phi_2 : E_2 \to F_2)$ of continuous Γ-equivariant linear maps such that $\phi_2 e = f\phi_1$ under the equivalence relation $(\phi_1, \phi_2) \sim (\phi_1', \phi_2') \Leftrightarrow \exists T \in L_\Gamma(E_2, F_1),\ \phi_2' - \phi_2 = fT$.*
- *$E_f(\Gamma)$ is the full subcategory of $E(\Gamma)$ whose objects (E_1, E_2, e) have the property that E_2 is of finite type (E_1 is then also of finite type).*

are abelian categories of projective dimension one. The forgetful functor Φ from $E(\Gamma)$ to the category of $\mathcal{N}(\Gamma)$-modules defined by $\Phi((E_1, E_2, e)) := E_2/e(E_1)$ is faithful, respects direct sums, kernels and cokernels and is conservative.

Proof: See [22]. The main point is that the proof in [23] does not require finite type. □

Remark 3.1.4. *It is not clear to the author whether the forgetful functor Φ is fully faithful on $E_f(\Gamma)$. It is fully faithful on the full subcategory of projective modules thanks to* [25]. *Fully faithfulness would follow if $\Phi(E)$ was a projective $\mathcal{N}(\Gamma)$-module whenever E is a finite type Hilbert Γ-module but it doesn't seem to be true.*

The following corollary greatly simplifies our treatment:

Corollary 3.1.5. *If $f^\bullet : K^\bullet \to L^\bullet$ is a continuous morphism of complexes of Hilbert Γ-modules whose terms are in $E(\Gamma)$, hence a morphism of complexes in $E(\Gamma)$, $f^\bullet$ induces a isomorphism in $D(E(\Gamma))$ if and only if $\Phi(f^\bullet)$ induces an algebraic isomorphism in cohomology.*

An object $X = (E_1, E_2, e)$ of $E_f(\Gamma)$ has two basic invariants. Its Von Neumann dimension $\dim_\Gamma X \in \mathbb{R}$ depends only on $P(X) = E_2/\overline{eE_1}$ and has properties similar to the dimension function of ordinary linear algebra and its Novikov-Shubin invariant $NoSh(X) = (E_1, \overline{eE_1}, e)$.

Remark 3.1.6. *(Tapia) This construction of an abelian category is a special case of* [2, pp. 20, 40-41]. *Actually Hilbert Γ-modules form an exact category, even a quasi abelian one as follows from* [48, section 3.2], *which satisfies the conditions in* [2]. *The same holds with $\mathcal{N}(\Gamma)$-Fréchet modules.*

3.2. **Γ-Fredholm Complexes.** The main nice property complexes of Hilbert Γ-modules can have in general is being Γ-Fredholm.

Definition 3.2.1. *A bounded complex of Hilbert Γ-modules (with a positive inner product) (C^k, d_k) is Γ-Fredholm if and only if the spectral family $E_\lambda^{dd^*+d^*d}$ satisfies $\exists \lambda > 0$ such that the image of E_λ has finite Γ-dimension.*

This notion depends on the notion of a Fredholm operator given in [35, Definition 1.20, p. 26]. It is invariant by quasi-isomorphisms in $E(\Gamma)$ thanks to [35, Theorem 2.19 p. 83] . There is a stronger notion.

Definition 3.2.2. *A bounded complex of Hilbert Γ-modules (with a positive inner product) $(\bar{C}^k, \bar{d}_k)$ is strongly Γ-Fredholm if and only if it is quasi-isomorphic as a complex in $E(\Gamma)$ to another complex (C^k, d_k) whose spectral family $E_\lambda^{dd^*+d^*d}$ satisfies $\exists \lambda > 0$ such that the image of E_λ is a finitely generated Hilbert Γ-module.*

This is a stronger notion since a finite Γ-dimensional Hilbert module need not be finitely generated (e.g. for $\Gamma = \mathbb{Z}$).

Question 3.2.3. *Can one drop the quasi-isomorphism? Perhaps the proof of* [35, Theorem 2.19 p. 83] *can be modified using the center-valued trace.*

Lemma 3.2.4. *The homotopy category of bounded strongly Γ-Fredholm is equivalent to $D^b(E_f(\Gamma))$.*

Proof: Since the full abelian subcategory $E_f(\Gamma) \subset E(\Gamma)$ has enough $E(\Gamma)$ projective and both have finite projective dimension $\psi : D^b(E_f(\Gamma)) \to D^b_{E_f(\Gamma)}(E(\Gamma))$ is an equivalence.

Certainly a strongly Fredholm complex has its cohomology in $E_f(\Gamma)$.

Quasi-isomorphisms in the homotopy category of complexes in $E(\Gamma)$ are exactly the homotopy classes of morphisms of complexes that are algebraic quasi isomorphisms thanks to the exactness of the faithful forgetful functor $E(\Gamma) \to Mod_{\mathcal{N}(\Gamma)}$.

Since strongly Fredholm complexes are complex of projective objects in $E(\Gamma)$, the functor ψ' from the homotopy category of bounded above strongly Γ-Fredholm complexes to the derived category $D^bE(\Gamma)$ is fully faithful and takes its values in $D^b_{E_f(\Gamma)}(E(\Gamma))$. Since ψ is an equivalence, whose image is contained in the image of ψ', ψ' is essentially surjective.

□

3.3. **An equivalence of categories.** There is a more algebraic approach to $E_f(\Gamma)$ [35, p. 288]. $\mathcal{N}(\Gamma)$ is a semihereditary [35, Theorem 6.7 p. 239] hence coherent ring. It turns out that $E_f(\Gamma)$ is equivalent to the abelian category of finitely presentable $\mathcal{N}(\Gamma)$-modules. But the equivalence in question, denote it by ν, is not given by Φ. Indeed it is constructed using the equivalence given by the functor on finite rank free $\mathcal{N}(\Gamma)$-modules defined by $M \mapsto l^2\Gamma \otimes_{\mathcal{N}(\Gamma)} M$. It is not obvious that it is an equivalence.

There is also a dimension theory for arbitrary $\mathcal{N}(\Gamma)$-modules which generalizes $\dim_\Gamma$ and more or less reduces the theory of L_2-Betti numbers and Novikov-Shubin invariants to algebra. However non zero $\mathcal{N}(\Gamma)$-modules of dimension 0 may exist in sharp contreast with projective Hilbert Γ-modules.

3.4. **Affiliated Operators.** The algebra of affiliated operators $\mathcal{U}(\Gamma)$ is a flat extension $\mathcal{N}(\Gamma) \subset \mathcal{U}(\Gamma)$ [35, Theorem 8.2.2] such that $\mathcal{U}(\Gamma) \otimes NoSh(X) = 0$ whenever X is an object in $E_f(\Gamma)$. It is a coherent ring, even a Von Neumann regular one. So that finitely presented $\mathcal{U}(\Gamma)$-modules form an abelian category of projective dimension 0 (all objects are projective!). Furthermore $\dim_\Gamma$ extends to $\mathcal{U}(\Gamma)$-modules (no topological structure needed) in such a way that objects of the form $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} E$

where E is a Hilbert Γ-module of finite type are finite dimensionnal, $\dim_\Gamma$ being preserved. In particular, for a complex $K^\bullet$ in $E_f(\Gamma)$,

$$H^q(\mathcal{U}(\Gamma)\otimes K^\bullet)=\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}\bar{H}^q(K^\bullet)=\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}P(H^q(K^\bullet)).$$

The author does not know how to use in Complex Analytic Geometry the torsion information lost in that process. So forgetting about it seems to be appropriate from a pragmatic point of view.

The relevance of this localization process for Hodge theory is one of the main ideas of [17]. There, intermediate abelian categories fitting in a succession of exact functors of abelian categories

$$E_f(\Gamma)\overset{faith.}{\longrightarrow}Mod(\mathcal{N}(\Gamma))\to Mod(\mathcal{N}(\Gamma))/\tau\to Mod(\mathcal{U}(\Gamma))$$

are introduced where τ is a torsion theory (or an appropriate Serre subcategory). Here we will only consider the case $\tau=\tau_{\mathcal{U}(\Gamma)}$ in the notations of loc.cit.: when it is possible we will work in $E_f(\Gamma)$ and when it becomes necessary we will apply the functor $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}$.

However, the reason [35] introduces affiliated operators is his $[0,+\infty]$-valued dimension theory for $\mathcal{U}(\Gamma)$-modules which enables him to make the theory of L^2-Betti numbers more or less algebraic. This enables one to look at non-locally finite simplicial complexes like $K(\Gamma,1)$ simplifying Cheeger-Gromov's article [10]. One has however to do a minimal amount of functional analysis to prove the $\mathcal{U}(\Gamma)$-modules we encounter are finite dimensional or finitely generated projective. This is why we will not just apply the functor $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}$ to the construction of the two preceding sections with $p=2$ although it is extremely tempting.

We conclude with the following lemmas:

Lemma 3.4.1. *$\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}\nu$ is naturally equivalent to $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}$ on $E_f(\Gamma)$.*

Proof: This follows from the construction of ν and of the relation $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}l^2\Gamma=\mathcal{U}(\Gamma)$ which in turn follows from the realization of $\mathcal{U}(\Gamma)$ as an Ore localization of $\mathcal{N}(\Gamma)$ [35]. □

Lemma 3.4.2. *Let E be an object of $E_f(\Gamma)$ endowed with 3 filtrations W, F, G. $\bar{F}$ and $\bar{G}$ are n-opposed on $Gr_{\bar{W}^n}P(E)$ if and only if $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}F$ and $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}G$ are n-opposed on $Gr_{\mathcal{U}(\Gamma)\otimes_{\mathcal{N}g}W^n\mathcal{U}(\Gamma)}\otimes_{\mathcal{N}(\Gamma)}E$*

Proof: Left to the reader. □

3.5. **A natural question.** The construction of a completely satisfying L_2 Mixed Hodge Theory might be eased by the use of further results from the theory of operators algebras. A saliant feature of $\mathcal{U}(\Gamma)$ is that it is self injective which is exactly, according to a remark in [35], what is needed for neat duality statements. It would be helpful if the following question had a positive answer:

Question 3.5.1. *Assume we have a complex of separable Hilbert Γ-modules (or Fréchet $\mathcal{N}(\Gamma)$-modules) whose cohomology is isomorphic as a $\mathcal{N}(\Gamma)$-module to the $\mathcal{N}(\Gamma)$-module underlying an object of $E_f(\Gamma)$. Is the complex strongly Γ-Fredholm?*

3.6. **Real Structures.** The $*$-algebras $\mathcal{N}(\Gamma)$ and $\mathcal{U}(\Gamma)$ carry a real structure, namely a conjugate linear algebra involutive automorphism $\dagger$ commuting with the conjugate linear algebra involutive anti-automorphism $*$, and a real structure on a module over these algebras is just a conjugate linear automorphism on the underlying vector space compatible with $\dagger$. For instance $l_2\Gamma$ has a real structure. We will denote by $\mathbb{R}E_f(\Gamma)$ the category of formal quotients of real Hilbert Γ-modules.

4. Refined L^2-cohomology

4.1. Finiteness theorem for L^2 constructible cohomology.

Proposition 4.1.1. *Let $\mathbb{S}$ be a cocompact Γ-simplicial complex. There is a ∂-functor $\mathbb{H}_2^*(|\mathbb{S}|,-) : D^b(\mathbf{Cons}_{K,\Gamma}(\mathbb{S})) \to D^b(E_f(\Gamma))$ such that, upon composition with the forgetful functor to the derived category of $\mathcal{N}(\Gamma)$-modules $\Phi(\mathbb{H}_2^*(|\mathbb{S}|,-)) = \mathbb{H}_{(2)}^*(|\mathbb{S}|,-)$ (cf. Definition 1.2.3).*

Proof: First assume that $\mathbb{S}$ satisfies the following technical assumption: for every pair of distinct adjacent vertices $p, q \in S$, $\psi(p) \neq \psi(q)$ where $\psi : S \to \Gamma\backslash S$ is the quotient map. In particular, the set of vertices of a given simplex maps injectively to $\Gamma\backslash S$. Choose a well-ordering of $\Gamma\backslash S$. This provides each simplex σ with an order $<_\sigma$ on its vertices such that $<_{\gamma.\sigma} = \gamma(<_\sigma)$ and if $\tau \subset \sigma$, $<_\sigma |_\tau = <_\tau$. This defines a sign $\epsilon_{\tau,\sigma}$ for every pair of simplices $\tau \subset \sigma$ such that $\mathrm{Card}(\sigma) = \mathrm{Card}(\tau)+1$, namely $\epsilon_{\tau,\sigma} = (-1)^\nu$ where $(\sigma, <_\sigma) = p_0 < p_1 < \ldots < p_{\mathrm{Card}(\sigma)-1}$ and $\sigma - \tau = \{p_\nu\}$.

Let $\mathcal{F}$ be an object of $\mathbf{Cons}_\Gamma(\mathbb{S})$. For every simplex σ, set $U_\sigma := \cup_{\sigma\subset\tau}|\tau|$ and $F_\sigma := H^0(U_\sigma, \mathcal{F})$. For $\tau \subset \sigma$, the sheaf structure gives a map $\rho_{\tau,\sigma} : F_\tau \to F_\sigma$. Set $C_c^p(\mathbb{S},\mathcal{F}) := \oplus_{\mathrm{Card}(\sigma)=p+1} F_\sigma$ and for $f_\tau \in \mathcal{F}_\tau$,

$$df_\tau = \sum_{\tau\subset\sigma, \mathrm{Card}(\sigma)=\mathrm{Card}(\tau)-1} \epsilon_{\tau,\sigma}\rho_{\tau,\sigma}(f_\tau).$$

This defines a complex of Γ-modules $C_c^\bullet(\mathbb{S},\mathcal{F})$.

This complex is actually the Čech complex of $\pi_!\mathcal{F}$ in the covering $(U'_q)_{q\in\Gamma\backslash S}$ of $\Gamma\backslash|\mathbb{S}|$ where we define $U'_q = \pi(U(p))$ where $p \in S$ satisfies $\pi(p) = q$. Using Proposition 8.1.4 p.323 in [32], we see that $l_p\Gamma\otimes_{K\Gamma} C_c^\bullet(\mathbb{S},\mathcal{F})$ computes $H_{(p)}^\bullet(|\mathbb{S}|,\mathcal{F})$. In case $p = 2$; this complex is in fact a complex in $E_f(\Gamma)$.

This construction is obviously functorial, and taking the simple complex associated to a double complex one would construct the sought-for ∂-functor. The technical assumption on $\mathbb{S}$ is not always satisfied, but it holds for the barycentric subdvision $\beta\mathbb{S}$. We certainly have a fully faithful forgetful functor $\mathbf{Cons}_{K,\Gamma}(\mathbb{S}) \to \mathbf{Cons}_{K,\Gamma}(\beta\mathbb{S})$ and we define a functor between categories of complexes to be $s(l_2\Gamma\otimes_{K\Gamma} C^\bullet(\beta\mathbb{S},-))$. Passing to derived categories, it descends to $\mathbb{H}_2^*(|\mathbb{S}|,-)$. □

Proposition 4.1.2. *Let $\widetilde{X}$ be a cocompact subanalytic Γ-space.*

There is a ∂-functor $\mathbb{H}_2^*(\widetilde{X},-) : D^b_{\mathbb{R}-c,\Gamma}(\widetilde{X})) \to D^b(E_f(\Gamma))$ such that one has $\Phi(\mathbb{H}_2^*(X,-)) = \mathbb{H}_{(2)}^*(X,-)$.

This functor enjoys the following properties:

- *(Leray spectral sequence) For every proper Γ-equivariant morphism $f : \widetilde{X} \to \widetilde{Y}$ $\mathbb{H}_2(\widetilde{X},-)$ and $\mathbb{H}_2(\widetilde{Y},-)\circ Rf_*$ are naturally isomorphic functors.*
- *(Atiyah's L_2 index theorem) If Γ is fixed point free on $\widetilde{X}$*

$$\sum_i (-1)^i \dim_\Gamma \mathbb{H}_2^i(\widetilde{X},\mathcal{F}^\bullet) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathbb{H}^i(\Gamma\backslash\widetilde{X}, \Gamma\backslash\mathcal{F}^\bullet).$$

Proof: As in the proof of Theorem 1.3.2, the first part derives from Proposition 4.1.1. The proof of the additionnal statements is simpler than the proof of similar statements for coherent cohomology in [22] and will not be given in detail. □

If $\widetilde{X}$ is complex analytic, we will restrict $\mathbb{H}_2^*(\widetilde{X},-)$ to $D^b(\mathbf{Cons}_\Gamma(\widetilde{X}))$. We will also denote by $\overline{\mathbb{H}}_2^k(\widetilde{X},-)$ the k-th reduced cohomology functor

$$\overline{\mathbb{H}}_2^k(\widetilde{X},-) = P(H^k(\mathbb{H}_2^*(\widetilde{X},-)).$$

It is a projective Hilbert Γ-module and one has $\dim_\Gamma \mathbb{H}_2^i(\widetilde{X},\mathcal{F}^\bullet) = \dim_\Gamma \overline{\mathbb{H}}_2^i(\widetilde{X},\mathcal{F}^\bullet)$.

4.2. **Finiteness theorem for L^2 coherent De Rham cohomology.** Assume in this subsection that X be a compact complex manifold. The L^2 coherent cohomology functor

$$H^\bullet_{L^2}(\widetilde{X},-) : D^b\mathbf{Coh}(\mathcal{O}_X) \to D^b(E_f(\Gamma))$$

defined in [22, Théorème 5.3.8] comes from a $\mathbb{C}$-linear functor denoted by $\mathcal{C}$ from a category A of coherent $\mathcal{O}_X$-modules endowed with local locally free presentations taking values in the so-called qhtf complexes of an additive category $MontMod$ of so-called Montelian modules which contain $E(\Gamma)$ as a full subcategory.

The homotopy category of qhtf complexes of Montelian modules localized with respect to a specific class of quasi-isomorphisms is naturally equivalent to $D^b(E_f(\Gamma))$.

The resulting functor $C^b(A) \to D^b(E_f(\Gamma))$ factors through $D^b(\mathbf{Coh}(\mathcal{O}_X))$.

$\mathcal{C}$ is essentially the Čech cohomology of $l^2\pi_*$ with respect to some Stein covering $\mathfrak{U}_0$ of X. Actually, we can choose an appropriate germ at $t=0$ of an increasing family of coverings $\mathfrak{U}_t$ defined for $t \geq 0$ such that $\mathfrak{U} = \mathfrak{U}_0$ and:

$$C_*(t) = \mathcal{C}^\bullet(\mathfrak{U}_t, l^2\pi_*\mathcal{F}) \to \mathcal{C}^\bullet(\mathfrak{U}_{t'}, l^2\pi_*\mathcal{F})$$

is a quasi-isomorphism for $t' \leq t$. Then $\mathcal{C}(\mathcal{F}) = (C_*(t))_{t\geq 0}$ up to some inessential auxiliary data. We relegate to an appendix the more detailed discussion of these auxiliary data.

Lemma 4.2.1. *The functor $\mathcal{C}'$ extends to $C^b\mathbf{Coh}(\mathcal{O}_X, \mathrm{Diff_X})$ the full subcategory of $C^b\mathbf{M}(\mathcal{O}_X, \mathrm{Diff_X})$ whose objects are differential complexes of coherent analytic sheaves as an additive functor.*

Proof: Since differential operators between coherent analytic sheaves act continuously on the Fréchet space of their sections, the arguments of [22] apply. □

Recall from [42] that the correct notion of quasi isomorphism in the triangulated category $K^b\mathbf{M}(\mathcal{O}_X, \mathrm{Diff_X})$ of homotopy classes of bounded complexes in $\mathbf{M}(\mathcal{O}_X, \mathrm{Diff_X})$ are the differential quasi isomorphisms. We denote by *dqi* this localizing class which is a priori smaller than the class *qi* of sheaf-theoretic quasi-isomorphisms. The class *dqi* is needed to invert the De Rham functor and one has a ∂-functor

$$\nu : K^b\mathbf{Coh}(\mathcal{O}_X, \mathrm{Diff_X})_{dqi} \to K^b\mathbf{Coh}(\mathcal{O}_X, \mathrm{Diff_X})_{qi}.$$

Here $K^b\mathbf{Coh}(\mathcal{O}_X, \mathrm{Diff_X})_{dqi}$ stands for the essential image of $C^b\mathbf{Coh}(\mathcal{O}_X, \mathrm{Diff_X})$ (the essential image is a stricly full category).

Corollary 4.2.2. *The functor $H^\bullet_{L^2}(\widetilde{X},-)$ extends to a ∂-functor*

$$H^\bullet_2(\widetilde{X},-) : K^b\mathbf{Coh}(\mathcal{O}_X, \mathrm{Diff_X})_{dqi} \to D^b(E_f(\Gamma))$$

such that $\Phi \circ H^\bullet_{DR,2}(\widetilde{X},-)$ is naturally isomorphic to $H^\bullet(X, l^2\pi_*-)$.

Proof: This follows from the fact that a complex in $E_f(\Gamma)$ is acyclic iff it is algebraically acyclic by Corollary 3.1.5. □

We will from now on make a technical assumption, namely that the coherent $\mathcal{D}_X$-modules we consider admit a global good filtration. A second possibility would be to work on the category $Filt\mathbf{Mod}(\mathcal{D}_X)$.

Lemma 4.2.3. *Let $\mathcal{K}^\bullet \in Ob(K^b\mathbf{Coh}(\mathcal{D}_X))$. Let $F_.$ be a filtration of $K^\bullet$ inducing a good filtration on each term. Then $F_pDR(\mathcal{K}^\bullet) \to DR(\mathcal{K}^\bullet)$ is a quasi isomorphism for $p \gg 0$ and $F_pDR(\mathcal{K}^\bullet)$ is independant of $p \gg 0$ up to a unique differential quasi-isomorphism hence defines unambiguously an object $DR'(\mathcal{K}^\bullet)$ of $K^b\mathbf{Coh}(\mathcal{O}_X, \mathrm{Diff_X})_{dqi}$. This assignement is functorial.*

Proof: See [42], in particular for the functorial behaviour of this construction. □

It is tempting to believe that one could work with local good filtrations (actually with the local presentations inducing them) using simplicial gluing techniques as in [22, Section 6] but we shall refrain from doing so.

We can thus define a ∂-functor

$$tDR : D^b\mathbf{Coh}(\mathcal{D}_X)_{goodfilt} \to K^b\mathbf{Coh}(\mathcal{O}_X, \mathrm{Diff}_X)_{dqi}$$

which is compatible to the restriction to $D^b\mathbf{Coh}(\mathcal{D}_X)$ of Saito's equivalence:

$$DR : D^b\mathbf{Mod}(\mathcal{D}_X) \to K^b\mathbf{M}(\mathcal{O}_X, \mathrm{Diff}_X)_{dqi}$$

Proposition 4.2.4. *The functor*

$$H^\bullet_{DR,2}(\widetilde{X}, -) = H^\bullet_2(\widetilde{X}, -) \circ tDR : D^b\mathbf{Coh}(\mathcal{D}_X)_{goodfilt} \to D^b(E_f(\Gamma))$$

is a ∂-functor such that $\Phi \circ H^\bullet_{DR,2}(\widetilde{X}, -)$ is naturally equivalent to the restriction of the functor $H^\bullet_{DR,L^2}(\widetilde{X}, -)$.

Once again, one can define the reduced L_2 cohomology of $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module admitting a good filtration (or a complex of such):

$$\bar{H}^k_{DR,2}(\widetilde{X}, \mathcal{M}) = P(H^k_{DR,2}(\widetilde{X}, \mathcal{M})).$$

Remark 4.2.5. *When $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, one can form the induced $\mathcal{D}_X$-module $ind(\mathcal{F}) := \mathcal{D}_X \otimes \mathcal{F}$, a coherent $\mathcal{D}_X$-module with a global good filtration, and there is a morphism of complexes of sheaves $DR(ind(\mathcal{F})) \to \mathcal{F} \otimes \omega_X^n$ which is a quasi-isomorphism. We have a natural isomorphism in $D^b(E_f(\Gamma))$*

$$H^\bullet_{DR,2}(\widetilde{X}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}) \simeq H^q_2(\widetilde{X}, \mathcal{F} \otimes \omega_X^n)$$

where we use the notation of [22] *for coherent L^2-cohomology.*

4.3. L^2 **Poincaré-Verdier Duality.** If $\mathbb{S}$ be a cocompact (in particular finite dimensionnal) Γ-simplicial complex $\mathbf{Cons}_{K,\Gamma}(\mathbb{S})$ can described combinatorially in terms of the poset $(\Sigma(\mathbb{S}), \leq)$ where $\Sigma(\mathbb{S})$ is the set of simplices and $\sigma \leq \tau$ if and only if σ is a face of $\overline{\tau}$. The partial order is Γ-equivariant. Then $\mathbf{Cons}_{K,\Gamma}(\mathbb{S})$ is the category of Γ-equivariant functors

$$(\Sigma(\mathbb{S}), \leq) \to \text{Finite} - \text{dimensionnal } K - \text{Vector Spaces.}$$

This is nothing but a reformulation of a part of the construction in the proof of Proposition 4.1.1. We call the maps $F_\tau \to F_\sigma$ when $\tau \leq \sigma$ the corestriction maps of $F \in \mathrm{Ob}(\mathbf{Cons}_{K,\Gamma}(\mathbb{S}))$.

Poincaré-Verdier has an explicit combinatorial formulation [52, 47, 12] which is presented very efficiently in the note [11] and was apparently first observed by A. Shephard in his 1985 unpublished thesis under R. MacPherson's direction.

Proposition 4.3.1. *Let F be a Γ-constructible sheaf. Then its Verdier dual is represented by the complex of injective Γ-constructible sheaves :*

$$\mathbb{D}(F) = \ldots \longrightarrow \bigoplus_{\dim(\sigma)=i} \iota_{\bar{\sigma}*}\underline{K}_{\bar{\sigma}} \otimes F_\sigma^\vee \xrightarrow{\partial} \bigoplus_{\dim(\tau)=i-1} \iota_{\bar{\tau}*}\underline{K}_{\bar{\tau}} \otimes F_\tau^\vee \longrightarrow \ldots$$

where

- *$\iota_\sigma : \bar{\sigma} \to |\mathbb{S}|$ is the closed embedding,*
- *For every $i \in \mathbb{N}$, $\mathbb{D}^i(F) = \bigoplus \iota_{\bar{\sigma}*}\underline{K}_{\bar{\sigma}} \otimes F_\sigma^\vee$ is placed in degree $-i$,*
- *The (τ, σ) matrix component of ∂ is the tensor product of the transpose of the corestriction map $F_\tau \to F_\sigma$ when $\tau \leq \sigma$ and the homology boundary map $\iota_{\bar{\sigma}*}\underline{K}_{\bar{\sigma}} \to \iota_{\bar{\tau}*}\underline{K}_{\bar{\tau}}$ and 0 if τ is not a facet of $\bar{\sigma}$.*
- *$V \mapsto V^\vee$ is the usual duality functor on finite-dimensionnal K - Vector Spaces,*

If $F^\bullet$ is a bounded complex of Γ-constructible sheaves $\mathbb{D}(F^\bullet)$ is represented by the totalisation of the double complex obtained by applying $\mathbb{D}$.

This proposition gives a lift of Verdier Duality to the category of complexes, before taking the derived category.

Now if F is a Γ-constructible sheaf, we may construct $F^\vee$ by applying the usual duality functor and get a Γ-constructible cosheaf, namely a *contravariant* functor

$$(\Sigma(\mathbb{S}), \leq) \to \text{Finite} - \text{dimensionnal } K - \text{Vector Spaces}.$$

We may construct its homology chain complex $C_\bullet(\mathbb{S}, F^\vee)$, concentrated in positive degrees, and view it as a *cochain* complex $C_{-\bullet}(\mathbb{S}, F^\vee)$ concentrated in negative degrees. These are complexes of projective $K\Gamma$-modules.

Lemma 4.3.2. *There is a functorial monomorphism of complexes of $K\Gamma$-modules:*

$$C_{-\bullet}(\mathbb{S}, F^\vee) \to C_c^\bullet(\mathbb{S}, \mathbb{D}(F))$$

such that the quotient complex is acyclic.

Proof. The complex $C_c^\bullet(\mathbb{S}, \mathbb{D}^i(F))$ is the same as

$$\bigoplus_{\dim(\sigma)=i} C^\bullet(\bar{\sigma}) \otimes F_\sigma^\vee$$

where $C^\bullet(\bar{\sigma})$ is the simplicial cochain complex of the closed simples $\bar{\sigma}$.

We have a natural cochain equivalence $K[0] \to C^\bullet(\bar{\sigma})$ sending 1 to the cochain which 1 on all vertices of $\bar{\sigma}$. This enables to construct a Γ-equivariant cochain equivalence:

$$\bigoplus_{\dim(\sigma)=i} F_\sigma^\vee[0] \to \bigoplus_{\dim(\sigma)=i} C^\bullet(\bar{\sigma}) \otimes F_\sigma^\vee$$

which commutes with the natural boundaries by construction. This proves the lemma since $C_c^\bullet(\mathbb{S}, \mathbb{D}(F))$ is the simple complex attached to the double complex $C_c^\bullet(\mathbb{S}, \mathbb{D}^\bullet(F))$. □

Lemma 4.3.3. *Let $F^\bullet$ be a bounded complex of Γ-constructible sheaves, then there is a functorial map of bounded complexes of projective Hilbert Γ-modules obtained by taking the simple complex attached to*

$$l^2\Gamma \otimes_{K\Gamma} C_{-\bullet}(\mathbb{S}, F^{\bullet\vee}) \to l^2\Gamma \otimes_{K\Gamma} C_c^\bullet(\mathbb{S}, \mathbb{D}(F^\bullet)).$$

Proof. Obvious. □

Lemma 4.3.4. *Given $F^\bullet$ an object of $D^b(\mathbf{Cons}_{K,\Gamma}(\mathbb{S})$, there is a functorial perfect duality of projective Hilbert Γ-modules*

$$\overline{\mathbb{H}}_2^{\,i}(|\mathbb{S}|, F^\bullet \otimes_K \mathbb{C}) \otimes \overline{\mathbb{H}}_2^{\,-i}(|\mathbb{S}|, \mathbb{D}(F^\bullet) \otimes_K \mathbb{C}) \to \mathbb{C}$$

which preserves the natural real structure if $K \subset \mathbb{R}$.

Proof. This follows from the previous lemma and [35, Lemma 2.17 (2), p. 82]. □

Proposition 4.3.5. *Notations of Proposition 4.1.2. There is a functorial perfect duality of projective Hilbert Γ-modules*

$$\overline{\mathbb{H}}_2^{\,i}(\widetilde{X}, F^\bullet \otimes_K \mathbb{C}) \otimes \overline{\mathbb{H}}_2^{\,-i}(\widetilde{X}, R\underline{Hom}^\bullet(F^\bullet, \omega^\bullet) \otimes_K \mathbb{C}) \to \mathbb{C}$$

which preserves the natural real structure if $K \subset \mathbb{R}$ where $\omega^\bullet = \mathbb{D}(\underline{K}_{\widetilde{X}})$ is the Verdier dualizing complex.

Proof. This is a restatement the previous lemma. □

Corollary 4.3.6. $\dim_\Gamma \overline{\mathbb{H}}_2^{\,i}(\widetilde{X}, F^\bullet \otimes_K \mathbb{C}) = \dim_\Gamma \overline{\mathbb{H}}_2^{\,-i}(\widetilde{X}, R\underline{Hom}^\bullet(F^\bullet, \omega^\bullet) \otimes_K \mathbb{C})$.

Remark 4.3.7. *It is extremely tempting to conjecture that, with the notations of* [22]*, we have a L_2-Serre Duality theorem for Coherent analytic sheaves on complex spaces stating that there is perfect duality:*

$$\bar{H}_2^q(\widetilde{X},\mathcal{F})\otimes\bar{H}_2^{-q}(\widetilde{X},R\underline{Hom}^\bullet_{\mathcal{O}_X}(\mathcal{F},\Omega_X^\bullet))\to\mathbb{C},$$

where $\Omega_X^\bullet$ is the dualizing complex and that this even holds for coherent $\mathcal{D}_X$-modules if X is smooth. Proving this conjecture is likely to be quite technical and does not seem promising for applications.

4.4. **The comparison isomorphism.** We need to show that the comparison isomorphism which a priori lives is a quasi-isomorphism of complexes of $\mathcal{N}(\Gamma)$-module lifts to a quasi isomorphism of bounded complexes in $E_f(\Gamma)$. This is not completely trivial. However, with the notations in Theorem 1:

Lemma 4.4.1. *If Σ is a triangulation of X refining a stratification S of X and $DR(\mathcal{M})$ has S-constructible cohomology, and if P is a bounded complex of Σ - constructible sheaves of $\mathbb{C}$-vector spaces then one represent the quasi-isomorphism α by a morphism of complexes $\widetilde{\alpha}:P\to F_pDR(\mathcal{M})$ composed with the natural quasi-isomorphism $F_pDR(\mathcal{M})\to DR(\mathcal{M})$ for some $p\gg 1$.*

Proof. This follows from [32, Prop 8.1.9]. □

Corollary 4.4.2. *If $\mathfrak{U}$ is a finite covering by Oka-Weil domains such that:*

- *it refines the covering $\mathfrak{V}$ of X by the stars of the vertices of Σ*
- *the non empty intersections are contractible,*

we have a quasi-isomorphism of $\mathcal{N}(\Gamma)$-Fréchet modules, the leftmost two being in $E_f(\Gamma)$:

$$\mathcal{C}^\bullet(\mathfrak{V},P)\longleftarrow\mathcal{C}^\bullet(\mathfrak{U},P)\xrightarrow{\widetilde{\alpha}}\mathcal{C}^\bullet(\mathfrak{U},F_pDR(\mathcal{M})).$$

Now we have a model of $\mathbb{H}_{2,DR}(\widetilde{X},\mathcal{M}$ which is a bounded complex of finite type projective Hilbert Γ-Modules with a quasi isomorphism:

$$L^\bullet\to\mathcal{C}^\bullet(\mathfrak{U},F_pDR(\mathcal{M})).$$

Since the left hand side underlies a qhtf complex of Montelian modules, by [22, Proposition 4.4.14], one constructs a morphism of complexes of projective Hilbert Γ-module which is a quasi isomorphism

$$\mathcal{C}^\bullet(\mathfrak{V},P)\to M^\bullet$$

and is the promised lift of rh_2 to an isomorphism in $D^b(E_f(\Gamma))$. The functoriality of the construction is left to the reader. This concludes the proof of Theorem 1.

4.5. **L_2-cohomology of Mixed Hodge Modules.** Now, X is a complex projective manifold.

Let $\mathbb{M}$ be a Mixed Hodge Module in the sense of [43]. It is a triple

$$((\mathcal{M},F,W),(\mathbb{M}^B,W^B),\alpha)$$

where:

- $(\mathcal{M},F,W)$ is a bifiltered $\mathcal{D}_X$-module (which is regular holonomic),
- $(\mathcal{M},F)$ is a good filtration,
- $\mathbb{M}^B$ is a perverse sheaf over $\mathbb{Q}$,
- W_B is a filtration of $\mathbb{M}^B$ in the abelian category of perverse sheaves,
- $\alpha:DR(\mathcal{M})\to\mathbb{M}^B\otimes_{\mathbb{Q}}\mathbb{C}$ an isomorphism in $D^b_c(X,\mathbb{C})$, actually a filtered quasi isomorphism if the weight filtrations are taken into account.

all these data satisfying some non-trivial conditions.

Proposition-Definition 4.5.1. *We can define in $E_f(\Gamma)$, a real structure and a real weight filtration W on the $\mathcal{N}(\Gamma)$-module $\mathbb{H}_2^k(\widetilde{X}, \pi^{-1}\mathbb{M}^B)$ by taking the image of the functorial morphism*

$$\mathbb{H}_2^k(\widetilde{X}, \pi^{-1}W_\bullet\mathbb{M}^B) \longrightarrow \mathbb{H}_2^k(\widetilde{X}, \pi^{-1}\mathbb{M}^B)$$

and a filtration $F_{DR,2}$ on $H^k_{DR,2}(\widetilde{X}, \mathcal{M})$ by taking the image of the natural map

$$H_2^k(\widetilde{X}, F_\bullet DR(\mathcal{M})) \to H_2^k(\widetilde{X}, DR(\mathcal{M})).$$

Transporting the $F_{DR,2}$ filtration by the isomorphism rh_2 induced by α between these objects of $E_f\Gamma$, we get a real structure , a real W-filtration, the weight filtration, and a complex filtration F, which we shall call the algebraically defined Hodge filtration, on $H_2^k(\widetilde{X}, \mathbb{M}) := \mathbb{H}_2^k(\widetilde{X}, \pi^{-1}\mathbb{M}^B)$.

There is a perfect duality of the Hilbert Γ-modules $\bar{H}_2^q(\widetilde{X}, \mathbb{M})$ and $\bar{H}_2^{-q}(\widetilde{X}, \mathbb{D}(\mathbb{M}))$.

Proof. This is a direct application of our construction. The last statement follows from Proposition 4.3.5. □

The following implies Conjecture 3 in the introduction.

Conjecture 4.5.2. *After taking reduced cohomology and closure of W, F, the real structure, the weight filtration and the algebraically defined Hodge filtration on $H_2^k(\widetilde{X}, \mathbb{M})$ are the constituents of a functorial graded polarisable Mixed Hodge structure in the abelian category $\mathbb{R}E_f(\Gamma)$.*

The mixed Hodge numbers (namely the dimension of its $I^{p,q}$) obey the same restrictions as in [13, 14, 43].

If $\mathbb{M}$ is pure polarized, L is the cup product by a Hodge class, and S is a polarization defined by the combination of a Saito polarization and L_2-Poincaré Verdier duality, $(\bigoplus_k \overline{H_2^k}(\widetilde{X}, \mathbb{M}), L, S)$ is a polarized Hodge-Lefschetz in $\mathbb{R}E_f(\Gamma)$ in the sense of [40, Part 0, Chapter 3].

In the rest of the article we will see what can be done in that direction using only standard results. To establish the Mixed Hodge structure, it is enough to prove that, after tensoring with $\mathcal{U}(\Gamma)$, F and $F^\dagger$ become n-opposed in Gr_W^n, hence that the tensor product with $\mathcal{U}(\Gamma)$ is a $\mathcal{U}(\Gamma)$-Mixed Hodge Structure thanks to Lemma 3.4.2. The Hodge Lefschetz structure seems to require that the construction of the Hodge filtration is compatible with [9][31]. The duality statement survives after tensoring with $\mathcal{U}(\Gamma)$ thanks to the duality anti-equivalence on finitely generated $\mathcal{U}(\Gamma)$-modules given by $M \mapsto M^\vee = Hom_{\mathcal{U}(\Gamma)}(M, \mathcal{U}(\Gamma))$ (recall that $\mathcal{U}(\Gamma)$ is selfinjective and that all finitely generated $\mathcal{U}(\Gamma)$ modules are projective) the following form:

Lemma 4.5.3. *There is natural isomorphism*

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \overline{H_2^k}(\widetilde{X}, \mathbb{M}) \to (\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \overline{H_2^{-k}}(\widetilde{X}, \mathbb{D}(\mathbb{M}))^\vee.$$

Proof: This follows from Proposition 4.3.5. □

5. Analytical L_2 Hodge Structures

5.1. Complex polarized VHS on complete Kähler manifolds.

Definition 5.1.1. *Let M be a complex manifold. A quadruple $(M, \mathbb{V}, F^\cdot, S)$ is called a complex polarized variation of Hodge structure (a VHS) iff $\mathbb{V}$ is a flat bundle of finite dimensionnal complex vector spaces with flat connection D, $F^\cdot$ a deacreasing filtration by holomorphic subbundles of $\mathbb{V}$ indexed by integers and S a flat non degenerate $(-1)^w$-hermitian pairing such that*

(1) The C^∞ vector bundle V associated to $\mathbb{V}$ decomposes as a direct sum $V = \oplus_{p+q=w} H^{p,q}$ with $F^P = \oplus_{p \geq P} H^{p,q}$

(2) $p \neq r) \Rightarrow S(H^{p,q}, H^{r,s}) = 0$ *and* $(\sqrt{-1})^{p-q}S$ *is positive definite on* $H^{p,q}$
(3) $D^{1,0}F^p \subset F^{p-1} \otimes \Omega^{1,0}_M$

The subbundle $H^{p,q}$ can be given an holomorphic structure by the isomorphism $H^{p,q} \to F^p/F^{p+1}$. Denote by d''_p the corresponding Dolbeault operator and set $d'' = \oplus_p d''_p$. $D^{1,0}$ induces a C^∞-linear map $\nabla'_p : H^{p,q} \to H^{p-1,q+1} \otimes \Omega^1$ called the Gauss-Manin connection and set $\nabla' = \oplus_p \nabla'_p$. The Hermitian metric $H = \oplus_p (\sqrt{-1})^{p-q} S_{H^{p,q}}$ will be called the Hodge metric. The triple $(\mathbb{V}, d'', \nabla')$ is a Higgs bundle.

Following Deligne, we define $E^{P,Q}(\mathbb{V}) = \oplus_{p+r=P, s+q=Q} H^{p,q} \otimes E^{r,s}$ and $D'' = d'' + \nabla'$. One also defines $E^k(\mathbb{V}) = \oplus_{r+s=k} \oplus_{p,q} H^{p,q} \otimes E^{r,s}$. It follows that $D''E^{P,Q}(\mathbb{V}) \subset E^{P,Q+1}(\mathbb{V})$. Then, see [54], given any Kähler metric $\omega_{\widetilde{X}}$ on $M = \widetilde{X}$, taking formal adjoints of differential operators with respect to this Kähler metric and the Hodge metric on $\mathbb{V}$, the usual Kähler identities hold.

If furthermore the metric $\omega_{\widetilde{X}}$ is complete then the Dirac operators $D'' + \mathfrak{d}'', D + \mathfrak{d}, \ldots$ and the Laplace operator $\Delta_D = 2\Delta_{D''} = 2\Delta_{D'}$ are formally self-adjoint unbounded operators on the Hilbert space of L_2 forms with values in $\mathbb{V}$ see [19, Section 5.1] in this case or [5] and the references therein for the general theory. Thanks to [16, Chap. VIII, Theorem 3.2], it also follows that the closure of D, D'', D' is given by the naïve ansatz (namely the domain of D is the space of globally L_2 forms ϕ such that $D\phi$ taken in the sense of distributions is globally L_2), the Hilbert space adjoints of D, D'', D' are given by the naïve adjoints (namely the domain of $\mathfrak{d}$ is the space of globally L_2 forms ϕ such that $\mathfrak{d}\phi$ taken in the sense of distributions is globally L_2) and that the L_2 decomposition theorem holds replacing images of D, D', D'' and their adjoints by their closure , namely we have an orthogonal decomposition:

$$L^2(\widetilde{X}, E^k(\mathbb{V})) = \mathcal{H}^k(\widetilde{X}, \mathbb{V}) \oplus \overline{\mathrm{Im}(D)} \oplus \overline{\mathrm{Im}(\mathfrak{d})},$$

where $\mathcal{H} := \ker(\Delta_D)$ is the space of L_2 harmonic forms and similarly for D''.

The L_2 De Rham complex $L^2DR^\bullet(\widetilde{X}, \mathbb{V})$ (resp. its Dolbeault counterpart) is the complex of bounded linear operators obtained by restriccting D (resp. D'') to its domain. The L_2 de Rham cohomology groups $\ker(D)/DDom(D)$ (resp. their L_2-Dolbeault counterparts) are not represented by harmonic forms but the reduced cohomology groups $\ker(D)/\overline{DDom(D)}$ (resp.) are.

Lemma 5.1.2. *The k-th reduced L^2 cohomology of the complete Kähler manifold $\widetilde{X}$ with coefficients in the VHS $\mathbb{V}$ has a Hodge structure of weight $w + k$.*

Proof. It follows from the fact $\Delta_D = 2\Delta_{D''}$ commutes with the decomposition in (P, Q) type. □

Lemma 5.1.3. *The Hodge-Lefschetz package holds for the reduced L^2 cohomology of the complete Kähler manifold $\widetilde{X}$ with coefficients in the VHS $\mathbb{V}$. More precisely $(\overline{H}^*_2(\widetilde{X}, \mathbb{V}), L)$ is a Hodge-Lefschetz structure polarized by $\int_{\widetilde{X}} S(- \wedge -))$ in the sense of* [40, Part 0, Chapter 3].

Proof. The usual proof applies. □

Since Δ_D is essentially self-adjoint there exists a spectral decomposition

$$\Delta_D = \int_0^\infty \lambda dE_\lambda$$

where $(E_\lambda)_{\lambda>0}$ is the spectral family of Δ_D, an increasing orthonormal projector-valued function on $[0, +\infty[$ converging strongly to Id . The support of this spectral projector valued measure dE_λ is the spectrum of Δ_D. E_0 is the Hilbert space

projector on the closed subspace $\mathcal{H} := \ker(\Delta_D)$ and E_λ is the projector on the space of L_2 forms ϕ such that:

$$\forall n \in \mathbb{N} \quad < \Delta_D^n \phi, \phi > \leq \lambda^n < \phi, \phi > .$$

The E_λ commute with decomposition in (P,Q)-type and actually with all the operators $D, D', D'', \mathfrak{d}, \ldots, L, \Lambda$. The statement that E_λ commutes with a differential operator means in particular that it preserves its domain.

For future use, we record the following more precise notation, for every $\lambda > 0$:

$$E_\lambda^k(\widetilde{X}, \mathbb{V}) = \mathrm{Im}(E_\lambda) \cap L^2 DR^k(\widetilde{X}, \mathbb{V}).$$

This gives a subcomplex of the L^2 De Rham complex:

$$E_\lambda^\bullet(\widetilde{X}, \mathbb{V}) = (\ldots E_\lambda^k(\widetilde{X}, \mathbb{V}) \xrightarrow{D} E_\lambda^{k+1}(\widetilde{X}, \mathbb{V}) \to \ldots).$$

This first order differential operators have closed range if and only if $E_\epsilon = E_0$ for some $\epsilon > 0$ namely if and and only 0 is isolated in the spectrum of the Laplace operator. This fails for instance on the complex line.

The natural analog of the space of smooth forms in the compact case is the following subcomplex of $L^2 DR^\bullet(\widetilde{X}, \mathbb{V})$:

$$L^2 DR_\infty^\bullet(\widetilde{X}, \mathbb{V}) = (\bigoplus_k \bigcap_{n>0} Dom(\Delta_D^n|_{L^2(\widetilde{X}, E^k)}), D).$$

It is a complex of $\mathcal{N}(\Gamma)$-Fréchet spaces and we have $L^2 DR_\infty^\bullet(\widetilde{X}, \mathbb{V}) \subset C^{\infty,\bullet}(\widetilde{X}, \mathbb{V})$ by standard elliptic estimates. The same construction works also for the Dolbeault complex. See [5] for a wider perspective.

Lemma 5.1.4. *Assume $\lambda' > \lambda > 0$. Then the following inclusions of complexes:*

$$E_\lambda^\bullet(\widetilde{X}, \mathbb{V}) \subset E_{\lambda'}^\bullet(\widetilde{X}, \mathbb{V}) \subset L^2 DR_\infty^\bullet(\widetilde{X}, \mathbb{V}) \subset L^2 DR^\bullet(\widetilde{X}, \mathbb{V}).$$

are quasi-isomorphisms. In fact $E_\lambda^\bullet(\widetilde{X}, \mathbb{V})$ is a homotopy retract of the three other complexes.

The same holds for the L^2-Dolbeault complex of a Γ-equivariant holomorphic hermitian vector bundle.

Proof: Define $g = \int_\lambda^\infty \mu^{-1} dE_\mu$. Then g, a continuous linear operator, preserves all the 4 complexes above and so does $h = \mathfrak{d} g$. Now, one has $[D, h] = Id - E_\lambda$. The proof works for the Dolbeault complex too, using the Dolbeault laplacian and $\mathfrak{d}''$. □

Hence $E_\lambda^\bullet(\widetilde{X}, \mathbb{V}) \to L^2 DR^\bullet(\widetilde{X}, \mathbb{V})$ is an isomorphism in the derived category of the abelian category of formal quotients of Hilbert spaces (aka separable Hilbert $\{1\}$-modules). And $E_\lambda^\bullet(\widetilde{X}, \mathbb{V}) \to L^2 DR_\infty^\bullet(\widetilde{X}, \mathbb{V})$ is an isomorphism in the derived category constructed in [48].

We endow the 4 complexes in Lemma 5.1.4 with filtration induced by the Hodge filtration $F^p = \oplus_{P \geq p} E^{P,Q}(\mathbb{V})$ on $L^2(\widetilde{X}, E^k(\mathbb{V}))$. This filtration is in each degree a closed subspace which is furthermore a summand. Actually the first three complexes are bigraded in the usual fashion.

Lemma 5.1.5. *The first two inclusions of Lemma 5.1.4 are filtered quasi-isomorphisms.*

Proof: We have to prove that the maps between the F-exact sequences are isomorphic at the E_1 page. The usual proof does work perfectly well for the first three complexes. Indeed $Gr_F E_\lambda^\bullet(\widetilde{X}, \mathbb{V}) = (E_{2\lambda}(\Delta_{D''}), D'')$ and $Gr_F L^2 DR_\infty^\bullet(\widetilde{X}, \mathbb{V}) = L^2 Dolb_\infty^\bullet(\widetilde{X}, \mathbb{V})$ whose cohomology are isomorphic by the Dolbeault version of Lemma 5.1.4. □

Remark 5.1.6. *For the third one, it seems to be more delicate. One has:*

$$Gr_F L^2DR^\bullet_\infty(\widetilde{X},\mathbb{V}) \subsetneqq Gr_F L^2DR^\bullet(\widetilde{X},\mathbb{V}) \subsetneqq L^2Dolb^\bullet(\widetilde{X},\mathbb{V}).$$

Using $\mathfrak{d}''g$ we get obtain a quasi-isomorphism of the two extreme complexes with $Gr_F E_\lambda(\widetilde{X},\mathbb{V})$, hence the natural inclusion is a quasi isomorphism between them. The problem is that $\mathfrak{d}''g$ does not seem to preserve $Gr_F^P L^2DR^\bullet(\widetilde{X},\mathbb{V})$. This group contains $Dom(D')\cap Dom(D'')$ which is preserved but the inclusion may be strict.

However, the classical case applies without any modification under a strong hypothesis that fails in the simplest case of the universal covering space of a genus one curve:

Lemma 5.1.7. *Zero is isolated in the spectrum of Δ_D if and only if $(E_0,0)\subset(E_\lambda,D)$ is a quasi-isomorphism.*

Lemma 5.1.8. *If zero is isolated in the spectrum of Δ_D then:*

1. *The decomposition theorem is valid without taking the closure of $Im(D)$, $Im(\mathfrak{d})$. Namely, $Im(D)$ and $Im(\mathfrak{d})$ are L^2-closed and:*

$$L^2(\widetilde{X},E^k(\mathbb{V})) = \mathcal{H}^k(\widetilde{X},\mathbb{V})\oplus \mathrm{Im}(D)\oplus \mathrm{Im}(\mathfrak{d}),$$

and also we have an equivariant decomposition as a direct sum of closed Fréchet subspaces:

$$L^2DR^k_\infty(\widetilde{X},\mathbb{V}) = \mathcal{H}^k(\widetilde{X},\mathbb{V})\oplus D(L^2DR^{k-1}_\infty(\widetilde{X},\mathbb{V}))\oplus \mathfrak{d}(L^2DR^{k+1}_\infty(\widetilde{X},\mathbb{V})).$$

2. *The decomposition theorem for the L^2 Dolbeault complex is valid without taking the closure of $Im(D'')$, $Im(\mathfrak{d}'')$.*
3. *The decomposition theorem for the L^2 D' complex is valid without taking the closure of $Im(D')$, $Im(\mathfrak{d}')$.*
4. *The $D'D''$ lemma holds. Namely,*

$$\phi\in L^2DR^k_\infty(\widetilde{X},\mathbb{V})\cap Im(D')\cap Im(D'')\Rightarrow \exists\psi\in L^2DR^{k-2}_\infty(\widetilde{X},\mathbb{V})\ \phi = D'D''\psi.$$

5. *The Hodge to De Rham spectral sequence of $L^2DR^k_\infty(\widetilde{X},\mathbb{V})$ degenerates at E_1 and D is F-strict.*
6. *The Hodge to De Rham spectral sequence of $E^\bullet_\lambda(\widetilde{X},\mathbb{V})$ degenerates at E_1 and D is F-strict.*

5.2. **Polarized VHS on Galois covering spaces of compact Kähler manifolds.** Let X be a compact Kähler manifold and $(X,\mathbb{V},F^\cdot,S)$ be a polarized complex Variation of Hodge Structure of weight w. Assume $\widetilde{X}$ is a Galois covering space of X so that its Galois group Γ acts properly discontinuously by automorphisms on $(\widetilde{X},\pi^{-1}\omega_X,\pi^{-1}\mathbb{V},\pi^{-1}F^\cdot,\pi^{-1}S)$. Then it is easy to see that all the Hilbert spaces considered in the previous section are separable projective Γ-modules and as such are endowed with a $\mathcal{N}(\Gamma)$-module structure. Furthermore if the VHS is real the E_λ and the De Rham L_2 cohomology groups carry a natural real structure. Basic elliptic theory gives:

Proposition 5.2.1. *The L^2-De Rham complex $L^2DR^\bullet(\widetilde{X},\pi^{-1}\mathbb{V})$ is strongly Γ-Fredholm.*

Proof: This is essentially in [1]. One can construct a Γ equivariant parametrix namely a L^2 bounded Γ-equivariant operator

$$L^2DR^\bullet(\widetilde{X},\pi^{-1}\mathbb{V})\to L^2DR^\bullet(\widetilde{X},\pi^{-1}\mathbb{V})[-1]$$

such that $[D,P]=I-S$ where S is a smoothing operator. This also follows from [50] which applies to any elliptic complex (including the case of operators!). For the reader's convenience, we will however give an easy argument.

Let $(\phi_a)_{a\in A}$ be a finite family of smooth real functions such that

$$\sum_{a\in A}\phi_a^2=1$$

and $\mathrm{Supp}(\phi_a)\subset U_a$ where U_a is an open subset of X small enough so that $\pi^{-1}(U_a)\cong\Gamma\times U_a$.

If $\psi\in L^2DR^k(\widetilde{X},\pi^{-1}\mathbb{V})$, $\phi_a\psi$ identifies with an element with compact support in $L^2\Gamma\hat{\otimes}L^2DR^k(U_a,\mathbb{V})$ (Hilbert space tensor product) which we may prolongate by 0 to an element $\overline{\phi_a\psi}$ of $L^2\Gamma\hat{\otimes}L^2DR^k(X,\mathbb{V})$. By construction the map Φ:

$$L^2DR^k(\widetilde{X},\pi^{-1}\mathbb{V})\to(L^2\Gamma)^A\hat{\otimes}L^2DR^k(X,\mathbb{V})\quad\psi\mapsto\Phi(\psi)=(\overline{\phi_a\psi})_{a\in A}$$

is a Γ-equivariant Hilbert space isometric (hence closed) embedding.

Let σ_{D+D^*} be the symbol of the operator $D+D^*$. For every ψ in the domain of $D+D^*$ on $L^2DR^k(\widetilde{X},\pi^{-1}\mathbb{V})$ we have:

$$(D+D^*)\phi_a\psi=\phi_a(D+D^*)\psi+\pi^{-1}\sigma(d\phi_a)\psi.$$

Summing up, we obtain

$$\|(D+D^*)\psi\|+K\|\psi\|\geq\|\mathrm{Id}_{L^2\Gamma^A}\otimes(D+D^*)(\overline{\phi_a\psi})_{a\in A}\|\geq\|(D+D^*)\psi\|-K\|\psi\|$$

where $K=Card(A)\max_{x\in X}\|\sigma_x\|$.

Assume now $\psi\in E^k_{\lambda^2}(\widetilde{X},\pi^{-1}\mathbb{V})$. Then $\|(D+D^*)\psi\|\leq\lambda\|\psi\|$. Hence

$$\|\mathrm{Id}_{pr^k_\mu\circ\Phi}\otimes(D+D^*)(\overline{\phi_a\psi})_{a\in A}\|\leq(\lambda+K)\|\psi\|..$$

Introduce the tensor product by $\mathrm{Id}_{(L^2\Gamma)^A}$ of the spectral projector E_μ of $(v,\mathbb{V}:$

$$pr^k_\mu:(L^2\Gamma)^A\hat{\otimes}L^2DR^k(X,\mathbb{V})\to(L^2\Gamma)^A\hat{\otimes}E^k_\mu(X,\mathbb{V}).$$

Then if $\sqrt{\mu}>\lambda+K$ we have $\|pr^k_\mu\circ\Phi(\psi)\|\geq\epsilon\|\psi\|$. for $\epsilon=\sqrt{\mu-\lambda-K}>0$.

It follows that we have a closed embedding of Hilbert Γ-modules:

$$pr^k_\mu\circ\Phi:E^k_\lambda(\widetilde{X},\pi^{-1}\mathbb{V})\to(L^2\Gamma)^A\hat{\otimes}E^k_\mu(X,\mathbb{V}).$$

Since $E^k_\mu(X,\mathbb{V})$ is a finite dimensional vector space by standard elliptic theory it follows that $E^k_\lambda(\widetilde{X},\pi^{-1}\mathbb{V})$ is a finitely generated Hilbert Γ-module for every $\lambda\geq0$. We conclude using:

Lemma 5.2.2. *The L_2 De Rham and Dolbeault complexes are Γ-Fredholm (resp. strongly) if and only if there exists $\epsilon>0$ such that E_ϵ is a finite Γ-dimensinal (resp. finite type) projective module.*

Proof: We leave this exercise to the reader. See [35]. □

□

In the most general relevant case, Γ need not act in a cocompact fashion, hence we have to add the Γ-Fredholm hypothesis to state the following:

Lemma 5.2.3. *Assume $L^2DR^\bullet(\widetilde{X},\pi^{-1}\mathbb{V})$ is Γ-Fredholm. The following inclusions of complexes:*

$$E^\bullet_\lambda(\widetilde{X},\pi^{-1}\mathbb{V})\subset E_{\lambda'}(\widetilde{X},\pi^{-1}\mathbb{V})\subset L^2DR^\bullet_\infty(\widetilde{X},\pi^{-1}\mathbb{V})\subset L^2DR^\bullet(\widetilde{X},\pi^{-1}\mathbb{V})$$

are quasi-isomorphisms of complexes of Hilbert (resp. Fréchet for the third one) Γ-modules and define the same element of $D^b(E_{sep}(\Gamma))$ (resp. of the derived category of the exact category of $\mathcal{N}(\Gamma)$-Fréchet modules) all of whose cohomology groups have finite Γ-dimension.

Lemma 5.2.4. *Assume $L^2DR^\bullet(\widetilde{X},\pi^{-1}\mathbb{V})$ is Γ-Fredholm.*

(1) *The decomposition theorem is valid. Namely, we have a direct sum decomposition of* $\mathcal{U}(\Gamma)$ *modules*

$$\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)} L^2(\widetilde{X}, E^k(\pi^{-1}\mathbb{V})) = \mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)} \mathcal{H}^k(\widetilde{X}, \pi^{-1}\mathbb{V}) \oplus \mathrm{Im}(D) \oplus \mathrm{Im}(\mathfrak{d}).$$

(2) *The decomposition theorem for the corresponding* $\mathcal{U}(\Gamma)$*-Dolbeault complexes is valid.*

(3) *The decomposition theorem for the* $\mathcal{U}(\Gamma)$*-*D' *complexes is valid.*

(4) $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)} E_0^\bullet(\widetilde{X}, \pi^{-1}\mathbb{V}) \subset \mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)} E_\lambda^\bullet(\widetilde{X}, \pi^{-1}\mathbb{V})$ *is a filtered quasi-isomorphism where* $\lambda > 0$.

(5) *The Hodge to De Rham spectral sequence of* $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)} E_\lambda^\bullet(\widetilde{X}, \pi^{-1}\mathbb{V})$ *degenerates at* E_1 *and* D *is* F*-strict.*

(6) *The Hodge to De Rham spectral sequence of* $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)} L^2DR_\infty^\bullet(\widetilde{X}, \pi^{-1}\mathbb{V})$ *degenerates at* E_1 *and* D *is* F*-strict.*

(7) *Dingoyan's* $D'D''$ *lemma holds. Namely,*

$$\phi \in L^2DR^k(\widetilde{X}, \mathbb{V}) \cap Im(D') \cap Im(D'') \Rightarrow \exists\psi \exists u \in \mathcal{U}(\Gamma)^\times \ u\phi = D'D''\psi.$$

Proof: As in [17], (1), (2), (3) follow from [17, Lemme 2.15] and the fact these complexes are Γ-Fredholm. Observing that the formation of the cohomology of a complex commutes with $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}$, since $\mathcal{N}(\Gamma) \to \mathcal{U}(\Gamma)$ is flat, (4) follows from the fact that we have an isomorphism on cohomology after tensoring with $\mathcal{U}(\Gamma)$, and we also have an isomophism on cohomology after passing to Gr_F since the L^2-Dolbeault complex is Γ-Fredholm too (Γ-Fredholmness means that E_λ has finite Γ-dimension for $\lambda > 0$ small enough). (5) follows from (4) and the fact that the statement is trivially true for $\lambda = 0$ and invariant by filtered quasi-isomorphism. (6) follows in the same way from Lemma 5.1.5 and (5). (7) follows by an easy adaptation of the argument of [17, Lemma 3.13].

□

Theorem 5.2.5. *The* k*-th cohomology group of* $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)} L^2DR_\infty^\bullet(\widetilde{X}, \pi^{-1}\mathbb{V})$ *carries a* $\mathcal{U}(\Gamma)$*-Hodge structure of weight* $k + w$ *which we call the analytic Hodge filtration.*

It gives rise to a weight $w + k$ *real Hodge structure on* $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)} H_2^k(\widetilde{X}, \mathbb{M})$.

Every Kähler class on X *induces a Hodge-Lefschetz isomorphism*

$$L^k : \mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)} H_2^{\dim(X)-k}(\widetilde{X}, \mathbb{V}) \to \mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)} H_2^{\dim(X)+k}(\widetilde{X}, \mathbb{V}).$$

5.3. $\mathcal{U}(\Gamma)$**-Hodge Complex.** Now its is time to compare with the previous construction.

We can construct on X a resolution of $l^2\pi_*\pi^{-1}\mathbb{V}$ by the sheafified L^2 De Rham complex. This is a complex of sheaves $l^2DR^\bullet(\mathbb{V})$ whose value over $U \subset X$ is given in degree k by

$$l^2DR^\bullet(\mathbb{V})(U) = \{\omega \in L^2_{loc}(\pi^{-1}(U), E^k(\pi^{-1}\mathbb{V})) \ \forall K \Subset U \ \int_K \|\omega\|^2 + \|D\omega\|^2 < +\infty\}.$$

One can construct a polarizable Hodge module $\mathbb{M} = \mathbb{M}_X(\mathbb{V})$ such that $\mathbb{M}^B = \mathbb{V}[\dim(X)]$, with a trivial W-filtration, the underlying filtered $\mathcal{D}_X$-module the $\mathcal{V} = \mathbb{V}\otimes_{\mathbb{C}} \mathcal{O}_X$ endowed with the filtration $F^\cdot$ made increasing, the underlying perverse sheaf is $\mathbb{V}[\dim(X)]$ and the comparison morphism α is the usual resolution $\mathbb{V}[\dim(X)] \to DR(\mathcal{V})$.

Proposition 5.3.1. *There is a natural filtered quasi isomorphism of complexes in* $E(\Gamma)$

$$CD_2 : (\mathcal{C}(DR(\mathcal{V}), F_\cdot) \to (L^2DR_\infty^\bullet(\widetilde{X}, \pi^{-1}\mathbb{V}), F_\cdot)$$

such that the composition with the comparison morphism induced by α*:*

$$rh_2 : \mathbb{H}_2^\bullet(\widetilde{X}, \pi^{-1}\mathbb{V}) \to \mathcal{C}(DR(\mathcal{V})) \simeq H_{DR,2}^\bullet(\widetilde{X}, \mathcal{V})$$

is the Čech-De Rham comparison isomorphism.

Proof. First observe that CD_2 indeed maps into $L^2DR^\bullet_\infty$ and that there is indeed a morphism of complexes. Then, it is a routine task to check that these maps are quasi-isomorphisms and that they have the stated compatibility. It is a filtered quasi-isomorphism thanks to the Čech-Dolbeault isomorphism for $Gr^F\mathcal{V}$ described in the appendix.

□

Proposition 5.3.2. *Consider a locally finite covering $\mathfrak{U}$ of X by small enough Oka-Weil domains, we can define a morphism of filtered $\mathcal{N}(\Gamma)$-Fréchet complexes*

$$(\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_* DR(\mathbb{M}_X^{DR}(\mathbb{V}))), F) \to (L^2DR^\bullet_\infty(\widetilde{X}, \pi^{-1}\mathbb{V}), F.)$$

which is a filtered quasi isomorphism of complexes of $\mathcal{N}(\Gamma)$-modules.

Proof. The preceding argument gives also this. □

Using the flatness of $\mathcal{N}(\Gamma) \subset \mathcal{U}(\Gamma)$, we deduce:

Corollary 5.3.3. *Tensoring by $\mathcal{U}(\Gamma)$, we obtain a filtered morphism of complexes of $\mathcal{U}(\Gamma)$-modules*

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} (\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_* DR(\mathbb{M}_X^{DR}(\mathbb{V}))), F) \to \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} (L^2DR^\bullet_\infty(\widetilde{X}, \pi^{-1}\mathbb{V}), F.)$$

which is a filtered quasi isomorphism.

Before summarizing the outcome of the discussion, we need the following definition [14]:

Definition 5.3.4. *A Hodge complex of $\mathcal{U}(\Gamma)$-modules with real structure and weight w is a triple: $(A^\bullet, (B^\bullet, F), \gamma)$ where $A^\bullet$ is a complex of $\mathcal{U}(\Gamma)$-modules with real structures, $\gamma : A^\bullet \to B^\bullet$ an isomorphism in the derived category of $\mathcal{U}(\Gamma)$-modules such that the F-spectral sequence degenerates at E_1 and the pair of filtrations $(F, \overline{F})$ on $H^k(B^\bullet)$ is a Hodge structure of weight $w+k$ in the category of $\mathcal{U}(\Gamma)$-modules.*

We say $(B^\bullet, F)$ underlies a Hodge complex of $\mathcal{U}(\Gamma)$-modules with real structure and weight w if it can be completed to such a triple.

Theorem 5.3.5. *Let X be a compact Kähler manifold and $(X, \mathbb{V}, F^\cdot, S)$ be a polarized complex Variation of Hodge Structure of weight w.*

1. *The F-spectral sequence of $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} (\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_* DR(\mathbb{M}_X^{DR}(\mathbb{V}))), F)$ degenerates at E_1.*
2. *The algebraically defined Hodge filtration gives rise to a weight $w+k$ real Hodge structure on $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H_2^k(\widetilde{X}, \mathbb{V})$ which coincides with the analytically defined one.*
3. *$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} (\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_* DR(\mathbb{M}_X^{DR}(\mathbb{V}))), F)$ underlies a weight w Hodge complex of $\mathcal{U}(\Gamma)$-modules with real structure and finite $\mathcal{U}(\Gamma)$-dimensional cohomology objects.*
4. *Every Kähler class on X induces a Hodge-Lefschetz isomorphism*

$$L^k : \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H_2^{\dim(X)-k}(\widetilde{X}, \mathbb{V}) \to \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H_2^{\dim(X)+k}(\widetilde{X}, \mathbb{V}).$$

Proof. Immediate. We use the natural quasi-isomorphism $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} rh_2$ to constuct the Hodge complex in the third statement. Actually, the cohomology objects are in the abelian category of finitely presented $\mathcal{U}(\Gamma)$-modules.

□

5.4. **Using analytic realisations of Hodge Modules.** Assume now that $U \subset X$ is a Kähler snc compactification and let $\tilde{X} \to X$ be a Galois covering space. Endow U with a Poincaré Kähler metric ω_U [54, 9, 31]. The lift $\omega_{\widetilde{U}}$ of ω_U to $\widetilde{U} := U \times_X \widetilde{X}$ is a complete Kähler metric which is Poincaré with respect to the partial Kähler snc compactification $\widetilde{U} \subset \widetilde{X}$. Consider $(U, \mathbb{V}, F^\cdot, S)$ a (say real) polarized VHS on U with quasi unipotent monodromy. Recall the fundamental result of [9, 31] that the sheafified L^2 De Rham complex $\mathcal{DR}_2^\bullet(U, \mathbb{V})$ with respect to the Poincaré metric on U is a fine model of the perverse sheaf $\mathcal{IC}_X(\mathbb{V})$.

A trivial modification of the definition in [31] replacing $U \to X$ with $\widetilde{U} \to X$ yields sheaves on X we shall denote by $l^2\pi_*\mathcal{DR}_2^k(\widetilde{U}, \pi^{-1}\mathbb{V})$ and the operators D', D'', D betwwen these sheaves on X and we have:

Proposition 5.4.1. *$l^2\pi_*\mathcal{DR}_2^\bullet(\widetilde{U}, \pi^{-1}\mathbb{V})$ is a fine model of $l^2\pi_*\pi^{-1}\mathcal{IC}_X(\mathbb{V})$.*

Proof: Same method as in the proof of Proposition 2.4.2. Left to the reader. □

Corollary 5.4.2. *Under the current assumptions, the k-th cohomology of the L_2 De Rham complex of $\widetilde{U}$ with values in $\pi^{-1}\mathbb{V}$ in the Poincaré metric $\omega_{\widetilde{U}}$ is isomorphic as a $\mathcal{N}(\Gamma)$-module to the $\mathcal{N}(\Gamma)$-module underlying $\mathbb{H}_2^k(\widetilde{X}, \pi^{-1}\mathcal{IC}_X(\mathbb{V}))$.*

Corollary 5.4.3. *If the L^2 De Rham complex is strongly Γ-Fredholm, the reduced k-th cohomology group twisted by $\mathcal{U}(\Gamma)$,*

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} \mathbb{H}_2^k(\widetilde{X}, \pi^{-1}\mathcal{IC}_X(\mathbb{V})) = H^k(X, \mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} l^2\pi_*\pi^{-1}\mathcal{IC}_X(\mathbb{V}))$$

carries a natural real $\mathcal{U}(\Gamma)$-Hodge structure of weight $k + w$.

Proof: Immediate. □

We do not have a proof of the Γ-Fredholmness in this case. It is not clear whether the filtrations of this analytically defined Hodge Structure coincide with the algebraic ones we have constructed in this article.

Remark 5.4.4. *It has been announced in* [51] *that one can construct for every indecomposable Hodge module $\mathbb{M}$ a distinguished complete metric on the regular part U of its strict support. Distinguished means that the sheafified L^2 De Rham complex $\mathcal{DR}_2^\bullet(U, \mathbb{V})$ with respect to the distinguished metric on U is a fine model of the perverse sheaf $\mathbb{M}^{Betti}$. The above considerations apply to distinguished metrics.*

6. Proof of Theorem 4

Let X be a compact Kähler manifold and $\mathfrak{U}$ be a finite covering of X by sufficiently small Oka-Weil domains.

6.1. **Direct image by a closed immersion.** Let $i : Z \to X$ be a closed immersion of a smooth compact complex manifold and $(Z, \mathbb{V}, F^\cdot, S)$ be a polarized complex Variation of Hodge Structure of weight w. The case when $Z = X$ follows from Theorem 5.3.5.

Then $i_*^{MHM}\mathbb{M}_Z(\mathbb{V}) = \mathbb{M}_i(\mathbb{V})$. The filtered $\mathcal{D}_X$-module $(\mathbb{M}_i^{DR}(\mathbb{V}), F)$ can be computed as $i_+(\mathbb{M}_Z^{DR}(\mathbb{V}), F)$. $DR(\mathbb{M}_i^{DR}(\mathbb{V}))$ is not equal to $i_*DR(\mathbb{M}_Z^{DR}(\mathbb{V}))$ except if $Z = X$ where $i_* = Ri_*$ is the ordinary sheaf theoretic direct image. Nevertheless, we can prove:

Lemma 6.1.1. *$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} (R\Gamma(X, l^2\pi_*\mathbb{M}^B), (\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_*\mathbb{M}_i^{DR}(\mathbb{V}), F), rh_2)$ is a $\mathcal{U}(\Gamma)$ Hodge Complex.*

Proof. For $q \gg 1$, there is a filtered quasi isomorphism

$$F_q(DR(\mathbb{M}_i^{DR}(\mathbb{V})), F) \to (DR(\mathbb{M}_i^{DR}(\mathbb{V})), F)$$

and there is a (differential) filtered quasi-isomorphism of bounded differential complexes cohenrent sheaves of

$$i_*DR(\mathbb{M}_Z^{DR}(\mathbb{V})) \to F_q(DR(\mathbb{M}_i^{DR}(\mathbb{V})), F).$$

On the other hand, one does have $i_*\mathbb{V}[\dim(Z)] = \mathbb{M}_i^B(\mathbb{V})$ and the comparison isomorphism satisfies $\alpha_{\mathbb{M}_i(\mathbb{V})} = i_*\alpha \circ \eta$.

Consider a locally finite covering $\mathfrak{U}$ of X by small enough Oka-Weil domains. We have a canonical identification of filtered $\mathcal{N}(\Gamma)$-Fréchet complexes

$$(\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_* i_* DR(\mathbb{M}_Z^{DR}(\mathbb{V})), F) = (\mathcal{C}^\bullet(i^{-1}\mathfrak{U}, l^2\pi_* DR(\mathbb{M}_Z^{DR}(\mathbb{V}))), F)$$

in particular it is a filtered quasi isomorphism.

It follows that there is a filtered quasi isomorphism

$$\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}(\mathcal{C}^\bullet(i^{-1}\mathfrak{U}, l^2\pi_* DR(\mathbb{M}_Z^{DR}(\mathbb{V}))), F) \to \mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)}(\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_* \mathbb{M}_i^{DR}(\mathbb{V}), F).$$

which is compatible with the comparison isomorphism. Hence, the lemma is a consequence of Theorem 5.3.5. □

It follows from the construction that the real Hodge structure on $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} H^q(\widetilde{X}, L^2dR(\mathbb{M}_i^{DR}(\mathbb{V}))$ is the same as the one on $\mathcal{U}(\Gamma)\otimes_{\mathcal{N}(\Gamma)} H^q(\widetilde{Z}, L^2dR(\mathbb{M}_Z^{DR}(\mathbb{V}))$. The first part of Theorem 4 is proved.

6.2. **Comparison with Dingoyan's work.** In this subsection, we finish the proof of the second case of Theorem 4.

Definition 6.2.1 ([14])**.** *A Mixed Hodge complex of $\mathcal{U}(\Gamma)$-modules with real structure is a triple $((A^\bullet, W), (B^\bullet, W, F), \gamma)$ where $A^\bullet$ is a biregular increasingly filtered complex of $\mathcal{U}(\Gamma)$-modules with real structure, $\gamma : (A^\bullet, W) \to (B^\bullet, W)$ an isomorphism in the filtered derived category of $\mathcal{U}(\Gamma)$-modules such that, for all $k \in \mathbb{Z}$,*

$$(Gr_W^k A^\bullet, (Gr_W^k B^\bullet, Gr_W^k F), Gr_W^k \gamma)$$

is Hodge complexes of weight k with real structure.

Lemma 6.2.2 ([14])**.** *If $((A, W), (B, W, F), \gamma)$ is a Mixed Hodge complex of $\mathcal{U}(\Gamma)$-modules with real structure, for all $n \in \mathbb{Z}$*

$$(H^n(A), \mathrm{Im}(H^n(W) \to H^n(A)), H^n(\gamma)^{-1}(F_{H^n(B)}), (H^n(\gamma)^{-1}(F_{H^n(B)}))^\dagger)$$

where $F_{H^n(B)} = \mathrm{Im}(H^n(F) \to H^n(B))$ is a real $\mathcal{U}(\Gamma)$ mixed Hodge structure.

Lemma 6.2.3. *Let X be a compact Kähler manifold and $\mathbb{M}$ such that the Gr_W^k satisfy Conjecture 3. Then $\mathbb{M}$ satisfies Conjecture 3.*

Proof. Under these hypotheses, we see immediately that:

$$\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} (R\Gamma(X, l^2\pi_*\mathbb{M}^B, W), (\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_*\mathbb{M}_i^{DR}(\mathbb{V}), W, F), rh_2)$$

is a $\mathcal{U}(\Gamma)$-Mixed Hodge Complex. □

Thanks to [43] -see [48, Example 5.4] for one smooth divisor- the second case of Theorem 1 follows from the first case and Lemma 6.2.3.

The third one also follows using some of the properties of of Verdier duality on Mixed Modules, see [43]. Indeed $Rj_!j^{-1}\mathbb{M}_X(\mathbb{V}) = \mathbb{D}Rj_*j^{-1}M_X(\mathbb{V}^\vee)$ hence $Rj_!j^{-1}\mathbb{M}_X(\mathbb{V})$ is a Mixed Hodge Module. Furthermore $\mathbb{D}(Gr_W^k\mathbb{M}) = Gr_W^{-k}(\mathbb{D}(\mathbb{M}))$ and $\mathbb{D}(\mathbb{M}_i(\mathbb{V})) = \mathbb{M}_i(\mathbb{V}^\vee)$.

Appendix: Fréchet Sheaves and the functor $\mathcal{C}$

Čech model.

Given B a Banach algebra and X a secound countable locally compact topological space, a B-Fréchet sheaf is just a sheaf taking its values in the category of Fréchet spaces with a continuous action of B. A coherent analytic sheaf on a complex analytic space is a $\mathbb{C}$-Fréchet sheaf [27, Chapter VIII], in fact a sheaf of Fréchet modules over the structure sheaf which is a $\mathbb{C}$-Fréchet sheaf of algebras, see [29, 49].

Given $\mathcal{F}$ a Γ-equivariant coherent analytic sheaf on a proper Γ-complex manifold $\widetilde{X}$, the sheaf $l^2\pi_*\mathcal{F}$ is a Fréchet sheaf of $\mathcal{N}(\Gamma)$-modules as follows from the construction in [8, 22] but it is not Montel in the sense of [27].

The sheaf $l^2\pi_*\mathcal{F}$ is not Montel when Γ is infinite. There must be a good concept of Γ-Montel sheaves (see [22] for the corresponding notion of Γ-compactness and cp. [27, p. 235]) but we don't want to try and develop it. It will be enough for our present purposes to use the ad hoc theory given in [22].

For a locally finite covering $\mathfrak{U}$ of $\Gamma\backslash\widetilde{X}$ by small enough Oka-Weil domains [27, p. 211] we can define $\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_*\mathcal{F})$ the Čech complex of $l^2\pi_*\mathcal{F}$. Here, an open subset Ω is *small enough* if and only if the preimage in $\widetilde{X}$ is a disjoint union of open subsets finite over Ω.

By a standard application of Leray's theorem, $\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_*\mathcal{F})$ computes the cohomology of $l^2\pi_*\mathcal{F}$.

The complex $\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_*\mathcal{F})$ does not depend on $\mathfrak{U}$ in the derived category D of the exact category of $\mathcal{N}(\Gamma)$-Fréchet modules (cf. Remark 3.1.6) hence in the derived category of the abelian category of $\mathcal{N}(\Gamma)$-modules. In [22] it is proved that the functor defined $D^b\mathbf{Coh}(\mathcal{O}_X) \to DMod(\mathcal{N}(\Gamma))$ defined by $\mathcal{F} \mapsto \mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_*\mathcal{F})$ at the level of complexes lifts uniquely and functorially to $D^b(E_f(\Gamma))$ under the natural functor $D^b(E_f(\Gamma)) \to DMod(\mathcal{N}(\Gamma))$ if $\Gamma\backslash\widetilde{X}$ is compact. More generally, this holds if $\Gamma\backslash Supp(\mathcal{F})$ is compact. We denote by $\mathcal{C} : D^b\mathbf{Coh}(\mathcal{O}_X) \to D^b(E_f(\Gamma)))$ the resulting functor.

The main ingredient is a construction of a quasi-isomorphism

$$K^\bullet \to \mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_*\mathcal{F})$$

from a bounded complex of projective finite-type Hilbert Γ-modules $K^\bullet$ representing $\mathcal{C}(\mathcal{F})$.

To this end, one uses that the Γ-Fréchet space $l^2\pi_*\mathcal{F}(\Omega)$ of an open subset $\Omega \subset X$ is the inverse limit of a sequence of Hilbert Γ-modules with Γ-compact transition maps and the fact that Hilbert Γ-modules are projective in $E(\Gamma)$.

Actually one has to do something slightly more complicated, one has a germ at $t = 0$ of an increasing family of coverings $\mathfrak{U}_t$ defined for $t \geq 0$ such that $\mathfrak{U} = \mathfrak{U}_0$ and:

$$C_*(t) = \mathcal{C}^\bullet(\mathfrak{U}_t, l^2\pi_*\mathcal{F}) \to \mathcal{C}^\bullet(\mathfrak{U}_{t'}, l^2\pi_*\mathcal{F})$$

is a quasi-isomorphism for $t' \leq t$ and there there is a germ at $t = 0$ of a decreasing family of Hilbert Γ-modules $(C(t)_{t>0})$ such that $C(t) \to C(t')$ is Γ-compact for $t > t'$ and $C_*(t) = \varprojlim_{t'<t} C(t')$ if $t > 0$. The family $(C(t))_{t>0}$ is an inessential auxiliary datum but it is instrumental to the construction of $K^\bullet$, its essential uniqueness and its functorial properties. At least when $\mathcal{F}$ is locally free, $C(t)$ is the subspace of L_2 Čech cochains in $C_*(t)$, L_2 being measured with respect to a smooth volume form.

Dolbeault model.

In the case where $\mathcal{F}$ is locally free and $\widetilde{X}$ is smooth there is a much better way to proceed. Define

$$Dolb_2^k(\widetilde{X},\mathcal{F}) = \{s \in L^2_{loc}(\widetilde{X},\mathcal{E}^{0,q}(\mathcal{F})), \qquad \int_{\widetilde{X}} \|s\|^2 + \|\bar\partial s\|^2 < +\infty\}$$

where the norms and volume form are computed with respect to a Γ-equivariant hermitian metric on $\mathcal{F}$ and a Γ-equivariant hermitian metric on $\widetilde{X}$

Indeed it is very natural to use the natural map of complexes of $\mathcal{N}(\Gamma)$-Fréchet modules which will be refered to as CD_2, the Čech-Dolbeault comparison map :

$$\mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_*\mathcal{F}) \to Dolb_2^\bullet(\widetilde{X},\mathcal{F})) = (\ldots \to Dolb_2^k(\widetilde{X},\mathcal{F})) \xrightarrow{\bar\partial} Dolb_2^{k+1}(\widetilde{X},\mathcal{F}) \to \ldots)$$

attached to a smooth partition of unity (ϕ_α) subordinate to $\mathfrak{U}$ sending the q-cochain $(s_{\alpha_0\alpha(1)\ldots\alpha(q)})_{|\alpha|=q}$ to the twisted $(0,q$ form:

$$\sum_{|\alpha|=q} s_{\alpha_0\alpha(1)\ldots\alpha(q)}\bar\partial\phi_{\alpha(0)} \wedge \ldots \bar\partial\phi_{\alpha(q-1)}.\phi_{\alpha(q)}.$$

The complex $Dolb_2^\bullet(\widetilde{X},\mathcal{F}))$ is complex of separable Γ-Hilbert modules and the resulting map $K^\bullet \to Dolb_2^\bullet(\widetilde{X},\mathcal{F}))$ is an algebraic isomorphism induced by a continuous maps at level of the representatives hence $Dolb_2^\bullet(\widetilde{X},\mathcal{F})) \in D^b_{E_f(\Gamma)}E_{sep}(\Gamma)$ and is quasi-isomorphic to $\mathcal{C}(\mathcal{F}) \in D^bE_f(\Gamma)$.

Čech-De Rham comparison map

In a similar fashion, if $\mathbb{V}$ is a local system on X, and the intersections of elements of $\mathfrak{U}$ are contractible, we can construct, as in [15], a quasi-isomorphism of separable projective Hilbert Γ-modules in the essential image of $D^bE_f(\Gamma)$

$$CDR_2 : \mathcal{C}^\bullet(\mathfrak{U}, l^2\pi_*\mathbb{V}) \to L^2DR^\bullet(\widetilde{X},\mathcal{V}).$$

Given a smooth partition of unity (ϕ_α) subordinate to $\mathfrak{U}$, it is defined by sending the Čech q-cochain $(s_{\alpha_0\alpha(1)\ldots\alpha(q)})_{|\alpha|=q}$ to the twisted q form:

$$\sum_{|\alpha|=q} s_{\alpha_0\alpha(1)\ldots\alpha(q)}d\phi_{\alpha(0)} \wedge \ldots d\phi_{\alpha(q-1)}.\phi_{\alpha(q)}.$$

Concluding remarks

Assume now X be a complex analytic space that need not be smooth nor reduced. Once one knows that $\dim_{\mathcal{N}(\Gamma)} H_2^q(\widetilde{X}, l^2\pi_*\mathcal{F}) < +\infty$, e.g. that it lies in the essential image of $E_f(\Gamma)$, for all $\mathcal{F} \in \mathbf{Coh}(\mathcal{O}_X)$, two purely algebraic properties, the special model of the Leray spectral sequence used in [22, section 6.1], whose main feature is that it comes from a $E_f(\Gamma)$-spectral sequence, can be replaced with the usual Leray spectral sequence and Lück's theory of $\dim_{\mathcal{N}(\Gamma)}$ to prove the version of Atiyah's L_2-index theorem given in [22, Theorem 6.2.1]. The abstract nonsense in [22] needed to establish the functoriality of the lift to $E_f(\Gamma)$ can also be eliminated.

We conclude by stating a weaker form Theorem 1 in terms of finite dimensional $\mathcal{N}(\Gamma)$-modules, thus losing the information contained in Novikov-Shubin invariants. It follows from the part of the proof of Theorem 1 where global good filtrations are used (subsections 4.2, 4.4):

Theorem 5. *Let X be a complex manifold and $\widetilde{X} \to X$ be a Galois covering with Galois group Γ. Let $MD'(X)$ be the abelian category whose objects are triples*

$$\mathbb{M} = (\mathcal{M} = \mathbb{M}^{DR}, P = \mathbb{M}^{Betti}, \alpha)$$

where $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module supported on a Γ-cocompact analytic subspace, P is a perverse sheaf of $\mathbb{R}$-vector spaces and $\alpha : P \otimes_{\mathbb{R}} \mathbb{C} \to DR(\mathcal{M})$ is an isomorphism in the derived category of sheaves and whose morphisms are the obvious ones.

There is a ∂-functor which, on the Betti side, is compatible with proper direct images, satisfies Atiyah's L_2 index theorem:

$$L_2dR : D^bMD'(X) \to \text{EssentialImage}(D^b(E_f(\Gamma)) \stackrel{\Phi}{\to} D^bMod(\mathcal{N}(\Gamma)))$$

and for each $\mathbb{M} \in MD(X)$ and $q \in \mathbb{Z}$ functorial isomorphisms in $E_f(\Gamma)$

$$H^q(L_2dR(\mathbb{M})) \cong \mathbb{H}^q_{(2)}(\widetilde{X}, \mathbb{M}^{Betti}) \cong \mathbb{H}^q_{DR,(2)}(\widetilde{X}, \mathbb{M}^{DR}).$$

The functor $\mathcal{U}(\Gamma) \otimes_{\mathcal{N}(\Gamma)} L_2dR : D^bMD'(X) \to D^bMod(\mathcal{U}(\Gamma))$ takes its values in the essential image of the the category of complexes of finite type projective $\mathcal{U}(\Gamma)$-modules, is compatible with proper direct images, satisfies Atiyah's L_2 index theorem and Poincaré-Verdier duality.

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