



HAL
open science

PRE-CALABI-YAU ALGEBRAS AND HOMOTOPY DOUBLE POISSON GEBRAS

Johan Leray, Bruno Vallette

► **To cite this version:**

Johan Leray, Bruno Vallette. PRE-CALABI-YAU ALGEBRAS AND HOMOTOPY DOUBLE POISSON GEBRAS. 2022. hal-03605677

HAL Id: hal-03605677

<https://hal.science/hal-03605677v1>

Preprint submitted on 11 Mar 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

PRE-CALABI-YAU ALGEBRAS AND HOMOTOPY DOUBLE POISSON GEBRAS

JOHAN LERAY AND BRUNO VALLETTE

"Le souvenir de ta réciproque" – Roberta Scarsella

ABSTRACT. We prove that the notion of a curved pre-Calabi–Yau algebra is equivalent to the notion of a curved homotopy double Poisson algebra, thereby settling the equivalence between the two ways to define derived noncommutative Poisson structures. We actually prove that the respective differential graded Lie algebras controlling both deformation theories are isomorphic. This allows us to apply the recent developments of the properadic calculus in order to establish the homotopical properties of curved pre-Calabi–Yau algebras: ∞ -morphisms, homotopy transfer theorem, formality, Koszul hierarchy, and twisting procedure.

CONTENTS

Introduction	2
1. Double Poisson algebras up to homotopy	5
1.1. Double Poisson algebra	5
1.2. Properads	6
1.3. Deformation theory of morphisms of properads	9
1.4. Explicit description of the Koszul dual coproperad DPois^1	10
1.5. Homotopy double Poisson algebras	14
1.6. Curved homotopy double Poisson algebras	17
2. Pre-Calabi–Yau algebras	18
2.1. Cyclic non-symmetric operads	19
2.2. Deformation theory of morphisms of cyclic non-symmetric operads	23
2.3. Generalised necklace algebra	25
2.4. Higher Hochschild complex and the main theorem	27
3. Infinity-morphisms	29
3.1. Moduli spaces	29
3.2. Infinity-morphisms of homotopy double Poisson algebras	30
3.3. Infinity-morphisms of curved homotopy double Poisson algebras	33
Appendix A. Combinatorics of the decomposition maps	35
Appendix B. Proof of the main theorem	39
References	42

Date: March 11, 2022.

Key words and phrases. pre-Calabi–Yau algebra, double Poisson algebra, cyclic operad, properad, deformation theory.

2020 Mathematics Subject Classification. Primary 18M85; Secondary 17B, 18M70, 14A22.

The first author was supported by a postdoctoral grant of the DIM Math Innov – Région Île de France and the second author was supported by the Institut Universitaire de France and the project ANR-20-CE40-0016 HighAGT.

INTRODUCTION

Poisson geometry. In commutative geometry, there are two equivalent ways to define the notion of a *Poisson structure*. One can first consider a Lie bracket

$$\{-, -\}: C^\infty(M) \otimes C^\infty(M) \longrightarrow C^\infty(M)$$

satisfying the Leibniz rule with respect to the commutative product of the algebra of smooth functions on a manifold M . Then, one can notice that this data is equivalent to a bivector field satisfying the Maurer–Cartan equation with respect to the Schouten–Nijenhuis bracket of polyvector fields. In general, an algebra made up of a compatible pair of a commutative product and a Lie bracket is called a *Poisson algebra*.

Noncommutative Poisson geometry. How can one develop a noncommutative analogue of a Poisson structure? Let us recall the guiding *Kontsevich–Rosenberg principle of noncommutative geometry* [KR00]: the noncommutative analogue of a type of structures on schemes should be a type of structures on associative algebras A , viewed as noncommutative affine schemes, which induces the original type of structures on the affine schemes of representations of A . M. Van den Bergh solved this question in [VdB08]: the noncommutative analogue of a Poisson structure is a *double Lie bracket*

$$\{\{-, -\}\}: A \otimes A \longrightarrow A \otimes A$$

satisfying a Leibniz relation with respect to the associative product on A . Similarly, one can coin an algebra of noncommutative polyvector fields which induces polyvector fields on representation schemes: it is given by the tensor algebra on derivations from A to $A \otimes A$, equipped with a Schouten–Nijenhuis type (double) bracket. A *noncommutative bivector field* is a bitensor satisfying the associated Maurer–Cartan equation. In the smooth case, these two types of structures were shown to be equivalent in [VdB08, Section 4].

The aforementioned type of bialgebras is called a *double Poisson algebra*; it has already found applications in many fields of mathematics like quantum algebra, low-dimensional geometry, symplectic geometry, integrable systems, and mathematical physics. An exhaustive list of references can be found on the web site [Fai22] of M. Fairon. (In this paper, as in our previous works like [HLV20], we prefer to use the terminology *gebra* to refer to algebraic structure with several inputs and several outputs. We reserve the terminology “algebra” for algebraic structure with exactly one output, like associative or Lie algebras.)

COMMUTATIVE GEOMETRY	NONCOMMUTATIVE GEOMETRY
commutative algebras	associative algebras
symplectic structures	bisymplectic structures
Poisson structures	<i>double Poisson structures</i>

Derived Poisson geometry. The passage to *derived* (algebraic) geometry amounts basically to working now with *differential graded* (or simplicial) commutative algebras. One can use either definition to extend the classical notion of a Poisson structure on this derived level. In the differential graded context, the proper generalization of the notion of a Poisson algebra, satisfying the expected homotopical properties, is that of a *Poisson algebra up to homotopy*. Such an algebraic structure is made up of infinite series of operations with various arities which relax up to homotopy all the relations satisfied by the commutative product and the Lie bracket of a Poisson algebra. Since the combinatorics of these higher operations is quite involved, one enjoys encoding them conceptually using operads: this is rendered possible by the fact that the operad encoding Poisson algebras is Koszul, see [LV12, Section 13.3.7].

In derived geometry, a homotopical version of *shifted polyvector fields*, equipped with a Schouten–Nijenhuis type bracket, was introduced in [PTVV13, CPT⁺17]. A *shifted Poisson structure* is nothing but a shifted bivector field satisfying the associated Maurer–Cartan equation. V. Melani proved in [Mel16] that these two notions of derived Poisson structures are equivalent for a large class of well-behaved derived stacks.

Derived noncommutative Poisson geometry. Again, how can one develop a noncommutative analogue of a derived Poisson structure? According to the Kontsevich–Rosenberg principle, one should consider some structures on some *derived representation schemes* of a differential graded associative algebra. Hopefully this latter notion was developed by Y. Berest, G. Khachatryan, and A. Ramadoss in [BKR13]. So one can try first to come up with a notion of a *double Poisson gebra up to homotopy* such that this kind of structure on A induces a homotopy Poisson algebra structure on the derived representation schemes of A . Here again, the guiding algebraic toolbox already exists: the notion of a double Poisson gebra can be encoded by a properad [Val07], which is a generalisation of the notion of an operad allowing several outputs. This properad has recently been proved to be Koszul by the first named author in [Ler20]. In the first section of this paper (Section 1), we make explicit the notion of a homotopy double Poisson gebra in this way.

Dually, the derived noncommutative space of polyvector fields is provided by the *generalised necklace Lie-admissible algebra* made up symmetric and nonsymmetric tensors of A and A^* and equipped with a Schouten–Nijenhuis type bracket, see Section 2.3. M. Kontsevich and Y. Vlassopoulos defined what should be a “noncommutative shifted bivector” as a Maurer–Cartan element in it and they dubbed it a *pre-Calabi–Yau algebra* structure on A , see [KTV21] and also [TZ07, Sei12] for related structures. W.-K. Yeung proved that this definition of a derived noncommutative Poisson structure satisfies the Kontsevich–Rosenberg principle: any pre-Calabi–Yau algebra structure on A induces a shifted Poisson structure on its derived moduli stack of representations [Yeu19, Corollary 4.78].

DERIVED GEOMETRY	DERIVED NONCOMMUTATIVE GEOMETRY
polyvector fields	higher Hochschild complex
shifted symplectic structures	Calabi–Yau structures
shifted Poisson structures	<i>homotopy double Poisson gebra/pre-Calabi–Yau algebras</i>

In the literature, several authors have already compared pre-Calabi–Yau algebras to some homotopical variations of double Poisson gebra. First, N. Iyudu, M. Kontsevich, and Y. Vlassopoulos have shown that any double Poisson algebra structure induces canonically a pre-Calabi–Yau algebra structure, see [IK18, IK19, IKV21]. Another proof of this result was given by W.-K. Yeung in [Yeu19, Example 1.35]. D. Fernández and E. Herscovich have extended this result in [FH20, FH21] to double Poisson-infinity gebra, as defined by T. Schedler in [Sch09], and to double quasi-Poisson gebra [VdB08].

Present achievements. The first result of the present paper extends *the equivalence between the two definitions of a Poisson structure in classical geometry to the level of derived noncommutative geometry*:

$$\text{curved pre-Calabi–Yau algebras} = \text{curved homotopy double Poisson gebra} .$$

Notice that, for both notions to be strictly equivalent, it is mandatory to consider some mild generalisations including curvatures on both sides. Our main theorem actually holds on the level of the differential graded Lie-admissible algebras encoding both notions.

Theorem (Theorem 2.43). *There exists canonical monomorphisms of differential graded Lie-admissible algebras*

$$\begin{array}{ccc} \text{necf}_A & \hookrightarrow & \text{cnecf}_A \xrightarrow{\cong} \text{hh}_A \cong \text{cD}\mathfrak{Pois}_A, \\ \text{pre-Calabi-Yau algebras} & \hookrightarrow & \text{curved pre-Calabi-Yau algebras} \xrightarrow{\cong} \text{curved homotopy double Poisson gbras}, \end{array}$$

where the second one is an isomorphism if and only if A is degree-wise finite dimensional.

The other interesting aspect of this latter isomorphism lies in the fact that it opens the doors to all the properties of the properadic calculus developed recently in [HLV20, HLV22]. This allows us to settle a suitable notion of an ∞ -morphism of curved pre-Calabi-Yau/homotopy double Poisson algebras, see Section 3. We show that the associated category carries all the required homotopical properties for this type of structures: homotopy transfer theorem (Theorem 3.10), homological invertibility of ∞ -quasi-isomorphisms (Theorem 3.9), and equivalence between zig-zags of quasi-isomorphisms and ∞ -quasi-isomorphisms (Theorem 3.12). (Another notion of an ∞ -morphism for pre-Calabi-Yau algebras is proposed in [KTV21, Definition 25] but whose homotopical properties are still to be established.) In [CV22], we integrate dg Lie-admissible algebras of properadic convolution type into an explicit deformation gauge group. In the present case, its action on Maurer–Cartan elements induces two universal ways to create derived noncommutative Poisson structures. The first one, called the *Koszul hierarchy*, produces a homotopy double Poisson gebra from the data of a shifted Koszul dual double Poisson gebra and a chain complex (Theorem 3.13). The second one, called the *twisting procedure*, perturbs curved pre-Calabi-Yau algebras with the data of just one element (Theorem 3.17). Altogether, this forms the second contribution of the present paper.

Since the definition and the composition of ∞ -morphisms are given by algebraic arguments (coproperadic decomposition maps), their intricate underlying combinatorics remains to be unfolded; we make it explicit in Appendix A. A particular careful attention has been put on signs: throughout this text, we make all of them explicit. Even if they nearly all come from a direct application of the Koszul sign rule and the Koszul sign convention, computing them is often a tedious exercise. The proof of the main result is an illuminating example. In order to preserve the flow of exposition, we postponed it to Appendix B.

Layout. In the first section, we make explicit the notion of a (curved) homotopy double Poisson gebra prescribed by the Koszul duality for properads. The second section is devoted to (curved) pre-Calabi-Yau algebras using the deformation theory of morphisms of cyclic non-symmetric operads. We conclude it with the main result: the two differential graded Lie-admissible algebras encoding these two types of structures are isomorphic. We apply it to settle the homotopy theory of homotopy double Poisson algebras and curved pre-Calabi-Yau algebras in the third section. The first appendix contains a description of the properadic decomposition maps. The proof of the main theorem is given in the second appendix.

Conventions. We work over a field \mathbb{k} of characteristic 0 and its category dgVect of differential graded (dg) vector spaces. Since every object will be differential graded, we will drop the prefix "dg" for simplicity. We use the homological degree convention with the differential of degree -1 . The degree of a homogenous element a will be denoted by $|a|$. The symmetric monoidal category structure on dg vector spaces carries the *Koszul sign rule*

$$(12) \cdot (a \otimes b) := (-1)^{|a||b|} b \otimes a$$

and the *Koszul sign convention* lies in the following definition of the tensor product $f \otimes g$ of two maps

$$(f \otimes g)(a \otimes b) := (-1)^{|a||g|} f(a) \otimes g(b).$$

The linear dual of dg vector space A is defined by $(A^*)_n := \text{Hom}(A_{-n}, \mathbb{k})$ and by $d_{A^*}(f) := -(-1)^{|f|} f \circ d_A$. We denote the homological suspension (resp. desuspension) by s (resp. s^{-1}), i.e. the one-dimensional dg vector space concentrated in degree 1 (resp. -1). Notice that $s^* = s^{-1}$, and thus $s^{-1}s = 1 = -ss^{-1}$.

We denote the symmetric groups by \mathbb{S}_n and the cyclic groups by \mathbb{C}_n , which generating cycle $\tau_n := (12 \dots n)$. We will represent graphically the various cyclic constructions in a planar way, for instance with boxes instead of ringed bands, in order to address the main issue which is to compute signs coming from the Koszul sign rule and convention. We follow the conventions for operads given in [LV12].

Acknowledgements. We would like to acknowledge enlightening discussions with Damien Calaque, Estanislao Herscovich, Eric Hoffbeck, Guillaume Laplante-Anfossi, Geoffrey Powell, Salim Rivière, Victor Roca i Lucio, and Vladimir Rubtsov. We express our deep appreciation to all the participants of the Villaroger 2021 workshop on double Poisson structures. The first author thanks Catherine and Victor for the warm welcomes in Nice and Charles De Clercq for introducing him to the great poetry.

1. DOUBLE POISSON GEBRAS UP TO HOMOTOPY

In this section, we first recall the notion of a *double Poisson gebra* before to make explicit its homotopy version as prescribed by the Koszul duality of properads. To ease the reading, we provide the reader with two subsections recalling the basic properties of properads and the deformation theory of their morphisms. One subsection is devoted to the computation of the Koszul dual (co)properad of double Poisson gebras. Finally, we go one step further by settling the even more general notion of a *curved homotopy double Poisson gebra*.

1.1. Double Poisson gebra.

Definition 1.1 (Double Lie gebra). A *double Lie gebra* amounts to a data $(A, \{\{-, -\}\})$ made up of a dg vector space A and a morphism

$$\{\{-, -\}\}: A \otimes A \longrightarrow A \otimes A,$$

called the *double bracket*, satisfying the following two relations, where we use the Sweedler notation

$$\{\{a, b\}\} = \{\{a, b\}'\} \otimes \{\{a, b\}''\}$$

for any $a, b \in A$.

SKEW-SYMMETRY:

$$\{\{a, b\}\} = -(-1)^{|a||b|+|\{\{a,b\}'\}||\{\{a,b\}''\}|} \{\{b, a\}\}'' \otimes \{\{b, a\}\}'$$

DOUBLE JACOBI RELATION:

$$\{\{a, \{\{b, c\}\}_L\}\}_L + (-1)^{|a|(|b|+|c|)} (123) \cdot \{\{b, \{\{c, a\}\}\}_L\} + (-1)^{|c|(|a|+|b|)} (132) \cdot \{\{c, \{\{a, b\}\}\}_L\} = 0,$$

where $\{\{a, \{\{b, c\}\}\}_L\} := \{\{a, \{\{b, c\}'\}\}\} \otimes \{\{b, c\}''\}$ and where

$$(123) \cdot (u \otimes v \otimes w) := (-1)^{|w|(|u|+|v|)} w \otimes u \otimes v,$$

$$(132) \cdot (u \otimes v \otimes w) := (-1)^{|u|(|v|+|w|)} v \otimes w \otimes u.$$

Definition 1.2 (Double Poisson gebra [VdB08]). A *double Poisson gebra* is a triple $(A, \mu, \{\{-, -\}\})$ made up of a dg associative product $\mu: A \otimes A \rightarrow A$ and a double Lie gebra structure $(A, \{\{-, -\}\})$ satisfying the following compatibility relation.

DERIVATION:

$$\{\{a, \mu(b, c)\}\} = (-1)^{|a||b|} \mu(b, \{\{a, c\}'\}) \otimes \{\{a, c\}''\} + \{\{a, b\}'\} \otimes \mu(\{\{a, b\}''\}, c)$$

EXAMPLE 1.3.

- (1) In [VdB08, Section 6], the author shows that for every finite quiver Q , one can define a double Poisson bracket on the path algebra of the double of Q .

- (2) In [ORS13], the authors classify double Poisson brackets on the free noncommutative associative algebra $\mathbb{C}\langle x, y \rangle$.
- (3) For other significant examples, the interested reader can consult the exhaustive website [Fai22] of M. Fairon, who has listed all the articles about double brackets.

1.2. Properads. In this section, we briefly recall the definitions of a properad and a coproperad, following mainly the presentation of [HLV20, Section 2]. Let Bij be the groupoid of non empty finite sets with bijections.

Definition 1.4 (\mathbb{S} -bimodule). A \mathbb{S} -bimodule is a module over the groupoid $\text{Bij} \times \text{Bij}^{\text{op}}$. The associated category is denoted by $\mathbb{S}\text{-bimod}$.

EXAMPLE 1.5. Let A and B be two dg vector spaces. We define the \mathbb{S} -bimodule End_B^A by

$$\text{End}_B^A(Y, X) := \text{Hom} \left(\bigotimes_{x \in X} A, \bigotimes_{y \in Y} B \right),$$

for any non-empty finite sets X and Y .

The groupoid Bij admits for skeletal category the one made up of the sets $n := \{1, \dots, n\}$ equipped with the symmetric groups \mathbb{S}_n for automorphisms. So the data of a \mathbb{S} -bimodule M is equivalent to a collection $\{M(m, n)\}_{m, n \in \mathbb{N}^*}$ of dg vector spaces equipped with two compatible actions of the symmetric groups, one of \mathbb{S}_m on the left and one of \mathbb{S}_n on the right, under the formula

$$M(Y, X) := \left(\prod_{f \in \text{Bij} \times \text{Bij}^{\text{op}}(m \times n, Y \times X)} M(m, n) \right) / \sim,$$

where $|X| = n, |Y| = m$ and where $(f, \mu) \sim (g, g^{-1}f \cdot \mu)$.

We consider the set G of connected graphs directed by a global flow and the endofunctor $\mathcal{G} : \mathbb{S}\text{-bimod} \rightarrow \mathbb{S}\text{-bimod}$ defined by

$$\mathcal{G}(M)(m, n) := \bigoplus_{g \in G(m, n)} g(M)$$

where $g(M) := \bigotimes_{v \in \text{vert}(g)} M(m(v), n(v))$, where $n(v)$ stands for the number of inputs of the vertex v and where $m(v)$ stands for the number of outputs of the vertex v . The operation of forgetting the nesting of connected graphs in $\mathcal{G}(\mathcal{G}(M))$, produces elements of $\mathcal{G}(M)$ and thus induces a monad structure on \mathcal{G} . We call this monad the *monad of connected graphs*.

Definition 1.6 (Properad). A *properad* is an algebra over the monad \mathcal{G} of connected graphs.

This definition of a properad is actually not the original one. In [Val07], the second author defined a monoidal product \boxtimes called *the connected composition product* on the category $\mathbb{S}\text{-bimod}$, which amounts to composing operations along 2-level connected graphs, as it is illustrated in Figure 1. We refer the reader to [HLV20, Definition 1.13] for a formal definition, see also [Ler19, Definition 3.11]. Its unit I is the \mathbb{S} -bimodule made up of a one-dimension space in arity $(1, 1)$.

Proposition 1.7 ([Val07]). *The category of algebras over the monad \mathcal{G} is isomorphic to the category of monoids with respect to the connected composition product:*

$$\mathcal{G}\text{-alg} \cong \text{Mon}(\mathbb{S}\text{-bimod}, \boxtimes, I).$$

EXAMPLE 1.8. For any dg vector space A , we consider the properad structure on the \mathbb{S} -bimodule $\text{End}_A := \text{End}_A^A$ defined by the composite of maps. It is called the *endomorphism properad* of A . For instance, given two maps $f : A^{\otimes n} \rightarrow A^{\otimes m}$ and $g : A^{\otimes n'} \rightarrow A^{\otimes m'}$, their partial composite $f \circ_i^j g$ along the i th input of f and the j th output of g is organised as follows.

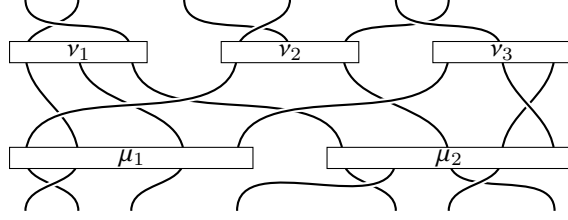


FIGURE 1. An element of $M \boxtimes N$.

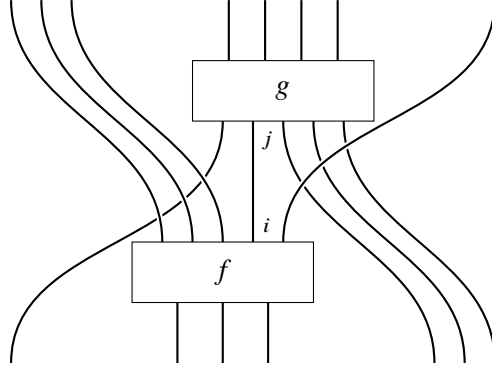


FIGURE 2. The partial composite of two multilinear maps.

The most important examples of properads used in this paper are the following two ones.

Definition 1.9 (DLie and DPois). The properad DLie is defined by the following presentation:

$$\text{DLie} := \frac{\mathcal{G} \left(\begin{array}{c} \begin{array}{c} 1 \quad 2 \\ \text{---} \\ 1 \quad 2 \end{array} = - \begin{array}{c} 2 \quad 1 \\ \text{---} \\ 2 \quad 1 \end{array} \end{array} \right)}{\left(\begin{array}{c} \begin{array}{c} 1 \quad 2 \quad 3 \\ \text{---} \\ 1 \quad 2 \quad 3 \end{array} + \begin{array}{c} 2 \quad 3 \quad 1 \\ \text{---} \\ 2 \quad 3 \quad 1 \end{array} + \begin{array}{c} 3 \quad 1 \quad 2 \\ \text{---} \\ 3 \quad 1 \quad 2 \end{array} \end{array} \right)},$$

where the generator has degree 0. The properad DPois is defined by the following presentation:

$$\text{DPois} := \frac{\mathcal{G} \left(\begin{array}{c} \begin{array}{c} 1 \quad 2 \\ \vee \\ 1 \end{array} ; \begin{array}{c} 1 \quad 2 \\ \text{---} \\ 1 \quad 2 \end{array} = - \begin{array}{c} 2 \quad 1 \\ \text{---} \\ 2 \quad 1 \end{array} \end{array} \right)}{\left(\begin{array}{c} \begin{array}{c} 1 \quad 2 \quad 3 \\ \vee \\ 1 \end{array} - \begin{array}{c} 1 \quad 2 \quad 3 \\ \vee \\ 1 \end{array} ; \begin{array}{c} 1 \quad 2 \quad 3 \\ \text{---} \\ 1 \quad 2 \end{array} - \begin{array}{c} 2 \quad 1 \quad 3 \\ \text{---} \\ 1 \quad 2 \end{array} - \begin{array}{c} 1 \quad 2 \quad 3 \\ \text{---} \\ 1 \quad 2 \end{array} ; \begin{array}{c} 1 \quad 2 \quad 3 \\ \text{---} \\ 1 \quad 2 \quad 3 \end{array} + \begin{array}{c} 2 \quad 3 \quad 1 \\ \text{---} \\ 2 \quad 3 \quad 1 \end{array} + \begin{array}{c} 3 \quad 1 \quad 2 \\ \text{---} \\ 3 \quad 1 \quad 2 \end{array} \end{array} \right)},$$

where the generators have degree 0.

Definition 1.10 (P-gebra). Let A be a dg vector space and let P be a properad. A *structure of P-gebra on A* is a morphism of properads $P \rightarrow \text{End}_A$.

Lemma 1.11. *There is a one-to-one correspondence between DLie-gebra (respectively DPois-gebra) structures and double Lie algebra (respectively double Poisson algebra) structures.*

Proof. This is straightforward. □

The notion and properties of homotopy P-gebra are efficiently encoded by the dual notion of a coproperad.

Definition 1.12 (Coproperad). A *coproperad* is a comonoid in the monoidal category $(\mathbb{S}\text{-bimod}, \boxtimes, I)$.

EXAMPLE 1.13. The Koszul duality theory for properads [Val07] provides us with Koszul dual coproperads. In this article, we will mainly consider the coproperads DLie^i and DPois^i , which are Koszul dual to the abovementioned properads, see Section 1.4.

Dual to the monad \mathcal{G} of connected directed graphs, we consider the *comonad* \mathcal{G}^c of *reduced connected directed graphs* $\overline{G} := G \setminus \{\emptyset\}$:

$$\mathcal{G}^c(M) := \bigoplus_{g \in \overline{G}} g(M),$$

with coproduct given by the sum of all the ways to partition the underlying graph g into connected directed sub-graphs. A coalgebra over this comonad is called a *comonadic coproperad* in [HLV20, Section 2.2]. Adding a coaugmented counit to comonadic coproperad produces coproperads which are called *conilpotent*.

EXAMPLE 1.14. The Koszul dual coproperads DLie^i and DPois^i are conilpotent.

The linear dual of a coproperad carries a canonical structure of a properad. The reverse statement does not hold true in general; one can however recover in this case some interesting coproperadic structure as follows. Recall that the (reduced) graph (co)monad admits a presentation with generators given by the summand made up of graphs with two vertices:

$$M \boxtimes_{(1,1)} M := \mathcal{G}(M)^{(2)} \cong \mathcal{G}^c(M)^{(2)}.$$

Definition 1.15 (Partial coproperad). An *partial coproperad* is an \mathbb{S} -bimodule C equipped with a morphism of \mathbb{S} -bimodules

$$\Delta_{(1,1)}: C \rightarrow C \boxtimes_{(1,1)} C,$$

which is coassociative.

By definition, any (comonadic) coproperad carries a canonical partial coproperad structure. The reverse holds true if and only if the iterations of the infinitesimal decomposition map $\Delta_{(1,1)}$ on any element of C produce finite sums of labelled graphs.

Proposition 1.16. *The linear dual of any arity-wise finite dimensional properad carries a canonical structure of a partial coproperad.*

Proof. This is obtained in a straightforward way by considering the linear dual $P^* \rightarrow P^* \boxtimes_{(1,1)} P^*$ of the infinitesimal composition product $P \boxtimes_{(1,1)} P \rightarrow P$ of the properad. \square

We already refer the reader to Section 1.6 for the main example of partial coproperad considered here and to Section 3.3 for the way its crucial properties are used.

REMARK 1.17. For simplicity, this section is written under the assumption that $n \geq 1$, but everything holds true *mutatis mutandis* for $n \geq 0$. We refer the reader to [HLV20, Section 1] for more details in this latter case. This setting will be mandatory in Section 1.6.

1.3. Deformation theory of morphisms of properads. In this section, we recall the deformation theory of morphisms of properads after [MV09b, Section 2].

Definition 1.18 (Totalisation of S-bimodule). The *totalisation* of an S-bimodule M is the dg vector space defined by

$$\widehat{M} := \prod_{m,n \in \mathbb{N}^*} M(m,n)^{\mathbb{S}_m^{\text{op}} \times \mathbb{S}_n}.$$

Recall that a binary product is called *Lie-admissible* when its commutator satisfies the Jacobi relation, i.e. is a Lie bracket.

Lemma 1.19 ([MV09a, Proposition 6]). *The following assignment defines a functor from the category of properads to the category of Lie-admissible algebras*

$$\begin{aligned} \text{properads} &\longrightarrow \text{Lie-admissible algebras} \\ (P, d_P, \gamma) &\longmapsto (\widehat{P}, d, \star), \end{aligned}$$

where the differential d is induced by the differential d_P and where the binary product \star is defined by

$$\mu \star \nu := \sum_{g \in G^{(2)}} \gamma(g(\mu, \nu)),$$

for any $\mu, \nu \in \widehat{P}$.

Lemma 1.20 ([MV09a, Lemma 2]). *The following assignment defines a functor*

$$\begin{aligned} \text{coproperads}^{\text{op}} \times \text{properads} &\longrightarrow \text{properads} \\ ((C, d_C, \Delta), (P, d_P, \gamma)) &\longmapsto (\text{Hom}(C, P), \partial, \gamma_{\text{Hom}}), \end{aligned}$$

where the symmetric groups act by conjugation, and where the composition

$$\gamma_{\text{Hom}}: \text{Hom}(C, P) \boxtimes \text{Hom}(C, P) \longrightarrow \text{Hom}(C, P)$$

is given by

$$C \xrightarrow{\Delta} C \boxtimes C \xrightarrow{(f_1, \dots, f_k) \boxtimes (g_1, \dots, g_l)} P \boxtimes P \xrightarrow{\gamma} P,$$

for $f_1, \dots, f_k, g_1, \dots, g_l \in \text{Hom}(C, P)$.

Definition 1.21 (Properadic convolution algebra). The composite of the above two functors produces functorially the *properadic convolution Lie-admissible algebra*

$$\widehat{\text{Hom}}(C, P) := \left(\prod_{m,n \in \mathbb{N}^*} \text{Hom}_{\mathbb{S}_m^{\text{op}} \times \mathbb{S}_n}(C(m,n), P(m,n)), \partial, \star \right),$$

where the product $f \star g$ is equal to

$$C \xrightarrow{\Delta_{(1,1)}} C \boxtimes_{(1,1)} C \xrightarrow{f \boxtimes_{(1,1)} g} P \boxtimes_{(1,1)} P \xrightarrow{\gamma} P.$$

REMARK 1.22.

- (1) When the coproperad C is coaugmented, we rather consider its coaugmentation coideal \overline{C} , which is the kernel of its coaugmentation map, in the above definition.
- (2) Since only the infinitesimal decomposition map $\Delta_{(1,1)}$ of the coproperad is involved in this construction, the same statement holds true when C is only a partial coproperad.

Recall that when $P = \mathcal{G}(E)/(R)$ is a quadratic properad, its Koszul dual coproperad P^i is the conilpotent coproperad cogenerated by the suspension sE and the double suspension s^2R , see [Val07] for more details.

EXAMPLE 1.23. One important example of properadic convolution algebra in the present paper is the one associated to the conilpotent coproperad DPois^i , which is Koszul dual to the properad encoding double Poisson gebras, and the endomorphism properad End_A . We denote it by

$$\mathfrak{D}\mathfrak{Pois}_A := \widehat{\text{Hom}}(\text{DPois}^i, \text{End}_A) .$$

Definition 1.24 (Properadic twisting morphism). The solutions to the Maurer–Cartan equation

$$(1) \quad \partial\alpha + \alpha \star \alpha = 0$$

in the properadic convolution algebra $\widehat{\text{Hom}}(C, P)$ are called the *twisting morphisms* from C to P ; their set is denoted by $\text{Tw}(C, P)$.

Definition 1.25 (P_∞ -gebra). When P is a Koszul properad, a gebra over the cobar resolution $\Omega P^i \xrightarrow{\sim} P$ is called a *homotopy P -gebra* or *P_∞ -gebra*.

Proposition 1.26 ([MV09a, Proposition 17]). *Homotopy P -gebra structures on a dg vector space A are in canonical one-to-one correspondence with twisting morphisms $\text{Tw}(P^i, \text{End}_A)$.*

Let us apply this to the case of double Poisson gebras; the most important result of the Ph.D. Thesis of the first author lies in the following statement.

Theorem 1.27 ([Ler20, Theorem 5.11]). *The properads DLie and DPois are Koszul.*

Proof. Let us just mention that the strategy of this proof relies on the introduction and the development of a new notion dubbed *protooperads*, see [Ler19]. \square

Thanks to this result, we get a homotopy meaningful notion of DPois_∞ -gebra as a Maurer–Cartan element in the properadic convolution algebra $\mathfrak{D}\mathfrak{Pois}_A$. It remains to make the former one explicit.

1.4. Explicit description of the Koszul dual coproperad DPois^i . The derivation relation between the double Lie bracket and the associative product in the definition of a double Poisson gebra induces a rewriting rule in the properad DPois between the sub-properad DLie and the sub-operad Ass , viewed as a sub-properad concentrated in output arity one.

Proposition 1.28 ([Ler20, Corollary 5.10]). *This rewriting rule induces a distributive law. This implies that the properad of double Poisson gebra is canonically isomorphic to*

$$(2) \quad \text{DPois} \cong \text{Ass} \boxtimes \text{DLie}$$

and that its Koszul dual coproperad is canonically isomorphic to

$$(3) \quad \text{DPois}^i \cong \text{DLie}^i \boxtimes \text{Ass}^i .$$

Let $P := \mathcal{G}(E)/(R)$ be a quadratic properad. Since it is much easier to work with properads than with coproperads, we consider the arity wise linear dual of the Koszul dual coproperad $P^! := (P^i)^*$, that we call the *Koszul dual properad* of P .

Lemma 1.29 ([Val07, Corollary 7.12]). *When E is finite dimensional, the Koszul dual properad admits the following presentation*

$$(4) \quad P^! \cong \mathcal{G}(s^{-1}E^*) / (s^{-2}R^\perp) .$$

Proof. Notice that we slightly changed the definitions from *loc. cit.*: we do not use any suspension properad here. For more details, we refer to the complete proof of [LV12, Proposition 7.2.1], which applies *mutatis mutandis* to the present case. \square

We start by the description of the Koszul dual of the properad $\text{DLie} = \mathcal{G}(E)/(R)$. The \mathbb{S} -bimodule $s^{-1}E^*$ has dimension 2 with basis depicted by “planar boxes”

$$(5) \quad \begin{array}{c} 1 \quad 2 \\ \hline 1 \quad 2 \end{array} \quad \text{and} \quad \begin{array}{c} 1 \quad 2 \\ \hline 2 \quad 1 \end{array},$$

where the action of (12) on the left-hand side of the first term gives the second term and where the action of (12) on the right-hand side of the first term gives minus the second term. The \mathbb{S} -bimodule $s^{-2}R^\perp$ is generated under the action of the symmetric groups by

$$(6) \quad \begin{array}{c} 2 \quad 3 \\ \hline 1 \quad 2 \end{array} - \begin{array}{c} 3 \quad 1 \\ \hline 2 \quad 3 \end{array} \quad \text{and} \quad \begin{array}{c} 1 \quad 2 \\ \hline 1 \quad 2 \end{array}.$$

REMARK 1.30. We depict the elements of properads \mathbb{P} and their Koszul dual coproperad \mathbb{P}^\dagger by white boxes and we depict the elements of the Koszul dual properad \mathbb{P}^\dagger by gray boxes. In other words, elements of E are in white and elements of E^* are in gray.

Lemma 1.31 ([Ler20, Section 5.1]). *The properad DLie^\dagger is concentrated in arities (n, n) , for $n \geq 1$, where $\text{DLie}^\dagger(n, n)$ is concentrated in degree $1-n$ with a basis labelled by $(\mathbb{S}_n \times \mathbb{S}_n)/\mathbb{C}_n$ given by the following stairways, up to cyclic permutations:*

$$\begin{array}{c} i_1 \quad i_2 \\ \hline j_1 \quad j_2 \quad j_3 \end{array} \begin{array}{c} i_{n-2} \quad i_{n-1} \quad i_n \\ \hline j_{n-1} \quad j_n \end{array} = (-1)^{n-1} \begin{array}{c} i_2 \quad i_3 \\ \hline j_2 \quad j_3 \quad j_4 \end{array} \begin{array}{c} i_{n-1} \quad i_n \quad i_1 \\ \hline j_n \quad j_1 \end{array}.$$

From this result, we deduce the following form of the Koszul dual coproperad DLie^\dagger . We denote the dual basis of the stairways by

$$\begin{array}{c} i_1 \quad i_2 \quad \dots \quad i_n \\ \hline j_1 \quad j_2 \quad \dots \quad j_n \end{array} = (-1)^{n-1} \begin{array}{c} i_2 \quad \dots \quad i_n \quad i_1 \\ \hline j_2 \quad \dots \quad j_n \quad j_1 \end{array}$$

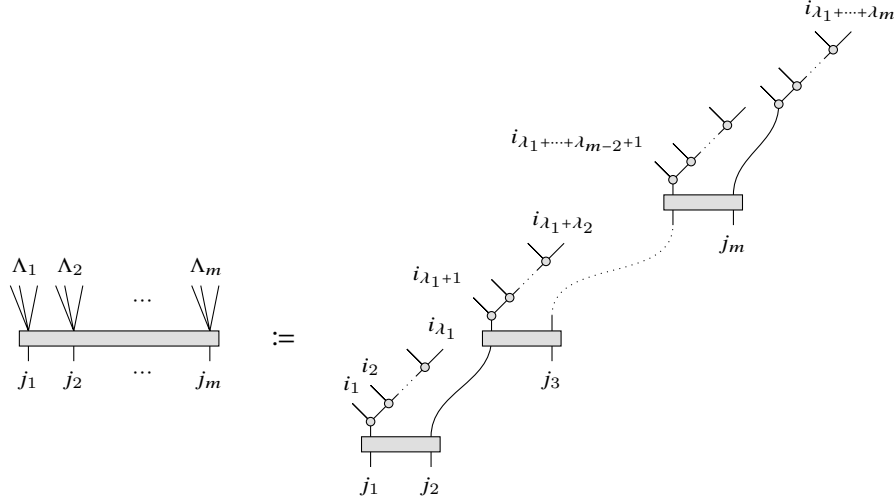
understood to be of degree $n - 1$. Their image under the infinitesimal decomposition map is equal to

$$\Delta_{(1,1)} : \begin{array}{c} 1 \quad 2 \quad \dots \quad n \\ \hline 1 \quad 2 \quad \dots \quad n \end{array} \mapsto \sum_{\substack{2 \leq k \leq n-1 \\ \sigma \in \mathbb{C}_n}} (-1)^{(k-1)(n-k)} \text{sgn}(\sigma) \begin{array}{c} \sigma(1) \quad \sigma(2) \quad \dots \quad \sigma(k) \quad \dots \quad \sigma(n) \\ \hline \sigma(1) \quad \sigma(2) \quad \dots \quad \sigma(k) \quad \dots \quad \sigma(n) \end{array}$$

REMARK 1.32. Since the underlying combinatorics of the decomposition map $\Delta: \text{DLie}^\dagger \rightarrow \text{DLie}^\dagger \boxtimes \text{DLie}^\dagger$ is more involved, we postpone it to Appendix A.

We are now ready to pass to the Koszul dual (co)properad of DPois. Let us denote by $\text{Part}_m(n)$ the set of ordered partitions $\lambda_1 + \dots + \lambda_m = n$ of n into m positive integers.

Lemma 1.33. *The properad $\text{DPois}^!$ is concentrated in arities (m, n) , for $n \geq m \geq 1$, where $\text{DPois}^!(m, n)$ is concentrated in degree $1 - n$ with a basis labelled by $(\mathbb{S}_m \times \mathbb{S}_n \times \text{Part}_m(n))/\mathbb{C}_m$ given by the following tree-wise stairways*



where $\Lambda_1 \sqcup \dots \sqcup \Lambda_m = \{1, \dots, n\}$ is a partition, with $\lambda_l = |\Lambda_l|$ and $\Lambda_l := (i_{\lambda_1+\dots+\lambda_{l-1}+1}, \dots, i_{\lambda_1+\dots+\lambda_l})$, up to cyclic permutations:

$$\begin{array}{c} \Lambda_1 \quad \Lambda_2 \quad \dots \quad \Lambda_m \\ \diagdown \quad \diagdown \quad \dots \quad \diagdown \\ \text{---} \\ j_1 \quad j_2 \quad \dots \quad j_m \end{array} = (-1)^{n\lambda_1+\lambda_m} \begin{array}{c} \Lambda_2 \quad \dots \quad \Lambda_m \quad \Lambda_1 \\ \diagdown \quad \dots \quad \diagdown \quad \diagdown \\ \text{---} \\ j_2 \quad \dots \quad j_m \quad j_1 \end{array} .$$

By convention, there are only binary trees, i.e. no box, for $m = 1$.

Proof. This is a direct corollary of Proposition 1.28 and Lemma 1.31. □

For the Koszul dual coproperad $\text{DPois}^!$, we represent the linear dual basis elements by

$$(7) \quad \begin{array}{c} \Lambda_1 \quad \Lambda_2 \quad \dots \quad \Lambda_m \\ \diagdown \quad \diagdown \quad \dots \quad \diagdown \\ \text{---} \\ j_1 \quad j_2 \quad \dots \quad j_m \end{array} = (-1)^{n\lambda_1+\lambda_m} \begin{array}{c} \Lambda_2 \quad \dots \quad \Lambda_m \quad \Lambda_1 \\ \diagdown \quad \dots \quad \diagdown \quad \diagdown \\ \text{---} \\ j_2 \quad \dots \quad j_m \quad j_1 \end{array} .$$

and we consider the following notation for the most canonical ones:

$$v_{\lambda_1, \dots, \lambda_m} := \begin{array}{c} 1 \quad \dots \quad \lambda_1 \quad \lambda_1+1 \quad \dots \quad \lambda_1+\lambda_2 \quad \dots \quad \lambda_1+\dots+\lambda_m \\ \diagdown \quad \dots \quad \diagdown \quad \dots \quad \diagdown \\ \text{---} \\ 1 \quad 2 \quad \dots \quad m \end{array} .$$

Proposition 1.34. For any partition ordered partition $\lambda_1 + \dots + \lambda_m = n$ of n into m positive integers, the image of the basis element $v_{\lambda_1, \dots, \lambda_m}$ under the infinitesimal decomposition map is equal to

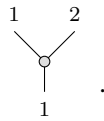
$$\Delta_{(1,1)}(v_{\lambda_1, \dots, \lambda_m}) = \sum_{k=1}^m \sum_{\sigma \in \mathbb{C}_m} \sum_{\substack{0 \leq p < \lambda_{\sigma(m)} \\ 0 \leq q < \lambda_{\sigma(k)}}} (-1)^\theta \text{Diagram} ,$$

where $\bar{\lambda}_l := \{\lambda_1 + \dots + \lambda_{l-1} + 1, \dots, \lambda_1 + \dots + \lambda_l\}$, where \bar{p} is the set made up of the first p terms of $\bar{\lambda}_{\sigma(m)}$, where \bar{q} is the set made up of the last q terms of $\bar{\lambda}_{\sigma(k)}$, and where

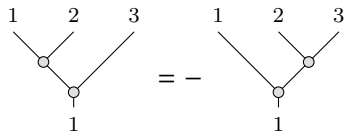
$$\begin{aligned} \theta = & q(\lambda_{\sigma(k)} - 1) + p(\lambda_{\sigma(k)} + \dots + \lambda_{\sigma(m-1)}) + \lambda_m + (n+1)(\lambda_1 + \dots + \lambda_{\sigma(1)-2}) + n\lambda_{\sigma(1)-1} \\ & + (\lambda_{\sigma(1)} + \dots + \lambda_{\sigma(k-1)} + p + q)(\lambda_{\sigma(k)} + \dots + \lambda_{\sigma(m)} - p - q - 1) . \end{aligned}$$

The summand $k = 1$ is understood without a white box on the bottom left side, that is just a corolla with $p + 1 + q$ leaves, with $p + 1 + q \geq 2$. The summand $k = m$ is understood without a white box on the top right side, that is just a corolla whose leaves are labelled by $\bar{\lambda}_{\sigma(m)} \setminus (\bar{p} \cup \bar{q})$, which is required to have at least two elements.

Proof. Lemma 1.29 shows that the Koszul dual properad $\text{DPois}^!$ admits the same generators (5) as the Koszul dual properad $\text{DLie}^!$ plus the “planar” binary product



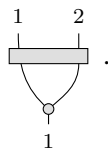
Its relations are that of $\text{DLie}^!$, depicted above in (6), plus the anti-associativity relation



the rewriting rules

$$(8) \quad \text{Diagram 1} = - \text{Diagram 2} \quad \text{and} \quad \text{Diagram 3} = - \text{Diagram 4} ,$$

and the genus vanishing relation



It was already shown in [Ler20, Lemma 5.4] that the composite of any elements in the properad DLie^1 along a graph of positive genus vanishes. Using the aforementioned relations, the same property holds true in the Koszul dual properad DPois^1 . Indeed, given any composite along a graph of positive genus, one can pull up the binary product, unless one encounters a box connected to the two inputs of such a product, in which case it vanishes. Once all the binary products stand above, one finds below a composite of boxes along a graph of positive genus, which vanishes.

So the infinitesimal decomposition map of the Koszul dual coproperad DPois^1 splits any element into two along one edge. In order to understand which ones, we work in the Koszul dual properad DPois^1 . Using the rewriting rule (8), one can see that only the partial composite of two basis elements given in Lemma 1.33 along the 2-vertices graphs given on the right-hand side of the above formula can produce a given basis element. Here, we have been using the cyclic symmetric to place the composed output at the first place and the composed input at the last place.

We conclude with the computation of the sign by systematically applying the Koszul sign rule and convention as follows. The partial composite of these two basis elements in the Koszul dual properad DPois^1 is given by first permuting the right-most q binary operations of the bottom element above the first binary box of the top element using q iterations of the right-most rewriting rule (8): this produces the sign $(-1)^q$. At the first input of the left bottom binary box of the top element, one has now the composite of a right comb with $\lambda_{\sigma(k)} - q - 1$ vertices with a right comb made up of q vertices. Rewriting it a total right comb produces the sign $(-1)^{q(\lambda_{\sigma(k)} - q - 1)}$. All together, they form the first term of θ .

Then, one permutes the remaining right-most p binary operations of the bottom element above all the binary boxes of the top element, using p iterations of the left-most rewriting rule (8), and one also permutes them with all the right-combs of the top element except for the last one: this produces the second term of θ . This way, one gets a basis type element of DPois^1 , that is a stairway with right-combs on each step, that one has to rotate cyclically in order to have one with first input and output label by 1: this produces the next three terms of θ .

Finally, one has to dualise linearly this partial composition product in the Koszul dual properad DPois^1 in order to obtain the infinitesimal decomposition map in the Koszul dual coproperad DPois^1 . This produces an extra sign given by the product of the degrees of the two elements which is the last term of θ . \square

Proposition 1.34 shows that the Koszul dual coproperad DPois^1 is actually a codioperad [Gan03], that is its (infinitesimal) decomposition map produces only genus 0 graphs. This is salient feature of the properad DPois : even if a category of gebras can be encoded by a dioperad, which is the case for double Poisson gebras, nothing ensures *a priori* that its Koszul dual codioperad is big enough to contain all the higher homotopies (operadic syzygies) to resolve the associated properad, see [MV09a, Section 5.6] for a counterexample. This property is actually equivalent to the vanishing of the homology groups of some involved graph complexes concentrated in positive genera [Ler20]. A codioperad of *multi-corollas*, defined by an infinitesimal decomposition map, and its cobar construction was introduced in [KTV21, Section 6.5.3], see also [Yeu22]. It is shown in [KTV21, Proposition 47] to encode pre-Calabi–Yau algebras. By carefully checking the signs, Proposition 1.34 would show that this codioperad is isomorphic to DPois^1 ; this is somehow what we perform in the proof of Theorem 2.43, see Appendix B. However, to procedure further with the homotopical properties of homotopy double Poisson gebras, like the ∞ -morphisms of Section 3, we need the description of the *full* decomposition map $\text{map } \Delta: \text{DPois}^1 \rightarrow \text{DPois}^1 \boxtimes \text{DPois}^1$, that we develop this in Appendix A.

1.5. Homotopy double Poisson gebras. Now we have everything at hand to make explicit the notion of a homotopy double Poisson gebra.

Theorem 1.35. *A homotopy double Poisson algebra is a dg vector space A equipped with a collection of operations*

$$m_{\lambda_1, \dots, \lambda_m} : A^{\otimes n} \longrightarrow A^{\otimes m}$$

of degree $n - 2$, for any ordered partition $\lambda_1 + \dots + \lambda_m = n$ of $n \geq 1$ into positive integers, without the trivial partition of 1, satisfying the following relations.

CYCLIC SKEW SYMMETRY:

$$(9) \quad m_{\lambda_2, \dots, \lambda_m, \lambda_1} = (-1)^{n\lambda_1 + \lambda_m} \tau_m^{-1} \cdot m_{\lambda_1, \lambda_2, \dots, \lambda_m} \cdot \tau_{\lambda_1, \dots, \lambda_m},$$

where $\tau_{\lambda_1, \dots, \lambda_m} \in \mathbb{S}_n$ permutes cyclically the blocks of size $\lambda_1, \dots, \lambda_m$.

HOMOTOPY DOUBLE POISSON RELATIONS:

$$(10) \quad \partial(m_{\lambda_1, \dots, \lambda_m}) = \sum_{k=1}^m \sum_{\sigma \in \mathbb{C}_m} \sum_{\substack{0 \leq p < \lambda_{\sigma(m)} \\ 0 \leq q < \lambda_{\sigma(k)}}} (-1)^\xi \sigma^{-1} \cdot \left(m_{\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(k-1)}, p+1+q} \circ_i^1 m_{\lambda_{\sigma(k)-q}, \lambda_{\sigma(k+1)}, \dots, \lambda_{\sigma(m-1)}, \lambda_{\sigma(m)-p}} \right) \cdot \omega,$$

where $i := \lambda_{\sigma(1)} + \dots + \lambda_{\sigma(k-1)} + p + 1$, where

$$\xi = q(\lambda_{\sigma(k)} - 1) + p(\lambda_{\sigma(k)} + \dots + \lambda_{\sigma(m-1)}) + \lambda_m + (n+1)(\lambda_1 + \dots + \lambda_{\sigma(1)-2}) + n\lambda_{\sigma(1)-1} + (\lambda_{\sigma(1)} + \dots + \lambda_{\sigma(k-1)} + p + q)(\lambda_{\sigma(k)} + \dots + \lambda_{\sigma(m)} - p - q) + 1,$$

and where ω is the permutation sending $(1, \dots, n)$ to

$$\left(\tilde{\lambda}_{\sigma(1)} + 1, \dots, \tilde{\lambda}_{\sigma(k)}, \tilde{\lambda}_{\sigma(m)} + 1, \dots, \tilde{\lambda}_{\sigma(m)} + p, \tilde{\lambda}_{\sigma(k)} + 1, \dots, \tilde{\lambda}_{\sigma(k+1)} - q, \tilde{\lambda}_{\sigma(k+1)} + 1, \dots, \tilde{\lambda}_{\sigma(m)}, \tilde{\lambda}_{\sigma(m)} + p + 1, \dots, \tilde{\lambda}_{\sigma(m+1)}, \tilde{\lambda}_{\sigma(k+1)} - q + 1, \dots, \tilde{\lambda}_{\sigma(k+1)} \right).$$

under the convention $\tilde{\lambda}_l := \lambda_1 + \dots + \lambda_{l-1}$, for $1 < l \leq m$, and $\tilde{\lambda}_1 := 0$.

Graphically, the partial composite appearing on the right-hand side of the homotopy double Poisson relations is represented in Figure 3.

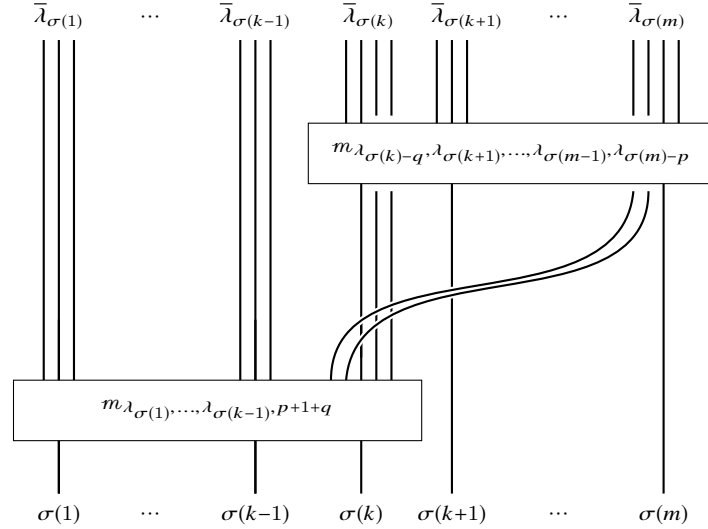


FIGURE 3. Partial composite involved in the homotopy double Poisson relations.

Proof. Recall from Proposition 1.26 that a structure of a homotopy double Poisson gebra on A corresponds to a twisting morphism $\alpha: \text{DPois}^i \rightarrow \text{End}_A$, that is a Maurer–Cartan element of the properadic convolution Lie-admissible algebra

$$\widehat{\text{Hom}}(\text{DPois}^i, \text{End}_A) \cong \left(\prod_{n,m \geq 1} \text{Hom}_{\mathbb{S}_m^{\text{op}} \times \mathbb{S}_n} \left(\overline{\text{DPois}}^i(m, n), \text{Hom}(A^{\otimes n}, A^{\otimes m}) \right), \partial, \star \right).$$

Let us denote by

$$m_{\lambda_1, \dots, \lambda_m} := \alpha(v_{\lambda_1, \dots, \lambda_m})$$

the image of the basis elements. Since these later ones have degree $n-1$ and since a twisting morphism α has degree -1 , the operations $m_{\lambda_1, \dots, \lambda_m}$ have degree $n-2$. The cyclic skew symmetry (7) of the basis elements of DPois^i induces the cyclic skew symmetry (9) of the structural operations $m_{\lambda_1, \dots, \lambda_m}$. The Maurer–Cartan equation (1) satisfied by α , once evaluated on the basis elements $v_{\lambda_1, \dots, \lambda_m}$, produces the homotopy double Poisson relations, after the form of the infinitesimal decomposition map of the Koszul dual coproperad DPois^i given in Proposition 1.34. The only extra sign created here comes from the Koszul convention: one has to permute α with the bottom basis element in order to apply it to the top basis element. \square

REMARK 1.36.

- (1) Since the Koszul dual cooperad Ass^i of the operad encoding associative algebras, embeds into the coproperad DPois^i , any homotopy double Poisson gebra contains an A_∞ -algebra structure made up of the operations m_n , for $n \geq 2$.
- (2) In [Sch09, Definition 4.1], T. Schedler defined the notion of a *double Poisson-infinity (al)gebra*, which is a dg associative algebra made up of a collection of brackets

$$\{-, \dots, -\}_n: A^{\otimes n} \longrightarrow A^{\otimes n}$$

of degree $n-2$, for all $n \in \mathbb{N}^*$, satisfying some identities. Such a structure is a special case of homotopy double Poisson gebra, where the operations $m_{\lambda_1, \dots, \lambda_m} = 0$ vanish, for every partition with at least one $\lambda_i > 1$, and under the identification

$$m_{1, \dots, 1} := \{-, \dots, -\}_n$$

adding extra symmetries.

- (3) In [Yeu19, Pri20], W. Yeung and J.P. Pridham define the notion of an *n-shifted double Poisson algebra*. We will only consider here the case $n = -1$ and we will refer to this structure as *semi homotopy double Poisson gebra*. Such a notion is modelled by the properad shDPois given by

$$\mathcal{G} \left(\begin{array}{c} \begin{array}{c} 1 \quad 2 \\ \diagdown \quad \diagup \\ \vee \\ 1 \end{array}, \quad \begin{array}{c} 1 \quad 2 \quad \dots \quad n \\ \hline 1 \quad 2 \quad \dots \quad n \end{array}, \quad n \geq 2 \end{array} \right), \\ \left(\begin{array}{c} \begin{array}{c} 1 \quad 2 \quad 3 \\ \diagdown \quad \diagup \\ \vee \\ 1 \end{array} - \begin{array}{c} 1 \quad 2 \quad 3 \\ \diagup \quad \diagdown \\ \vee \\ 1 \end{array}, \quad \begin{array}{c} n \quad n+1 \\ \hline 1 \quad 2 \quad \dots \quad n \end{array} - \begin{array}{c} 1 \quad 2 \quad n-1 \quad n+1 \\ \hline 1 \quad 2 \quad n-1 \quad n \end{array} - \begin{array}{c} 1 \quad 2 \quad n \\ \hline 1 \quad 2 \quad \dots \quad n \end{array} \end{array} \right),$$

equipped with the differential of DLie_∞ . Sending the generators of DPois_∞ , which are not present here, to zero, defines a canonical surjection of properads

$$\text{DPois}_\infty \twoheadrightarrow \text{shDP}.$$

This induces a canonical functor from semi homotopy double Poisson gebras to homotopy double Poisson gebras. Notice that the definition given in *loc. cit.* of a semi homotopy double Poisson gebra is in terms of a Maurer–Cartan element in a Lie algebra of noncommutative

vector fields. It would be interesting to compare this latter one with the deformation Lie algebras studied here.

1.6. Curved homotopy double Poisson gebras. The notion a homotopy double Poisson gebra admits a curved generalisation in a way similar to homotopy associative algebras [FOOO09] (that it contains) and homotopy Lie algebras, cf. [DSV22, Chapter 4]. To settle it, we use a properadic extension of the method presented in *loc. cit.* which relies on the principle that “curvature is Koszul dual to unit”.

Definition 1.37 (The properad $\mathfrak{uDPois}^!$). The *unital extension of the properad $\mathfrak{DPois}^!$* is defined by

$$\mathfrak{uDPois}^! := \frac{\mathfrak{DPois}^! \vee u}{(\nu \circ_1 u = -\text{id}; \nu \circ_2 u = \text{id})} = \frac{\mathcal{G} \left(\begin{array}{c} 1 \ 2 \\ \vee \\ 1 \end{array} ; \begin{array}{c} 1 \ 2 \\ \longleftarrow \\ 1 \ 2 \end{array} = - \begin{array}{c} 2 \ 1 \\ \longleftarrow \\ 2 \ 1 \end{array} ; \begin{array}{c} \circ \\ | \\ 1 \end{array} \right)}{\left(\begin{array}{c} 1 \ 2 \ 3 \\ \vee \\ 1 \end{array} + \begin{array}{c} 1 \ 2 \ 3 \\ \vee \\ 1 \end{array} ; \begin{array}{c} \circ \\ | \\ 1 \end{array} + \begin{array}{c} | \\ | \\ 1 \end{array} ; \begin{array}{c} 1 \ 2 \\ \vee \\ 1 \end{array} - \begin{array}{c} | \\ | \\ 1 \end{array} ; \begin{array}{c} 1 \ 2 \\ \cup \\ 1 \end{array} ; \begin{array}{c} 1 \ 2 \\ \longleftarrow \\ 1 \ 2 \end{array} \right)}$$

where $\nu = \begin{array}{c} 1 \ 2 \\ \vee \\ 1 \end{array}$ has degree -1 , $\begin{array}{c} 1 \ 2 \\ \longleftarrow \\ 1 \ 2 \end{array}$ has degree -1 , and where $u = \begin{array}{c} \circ \\ | \\ 1 \end{array}$ has degree 1 .

REMARK 1.38. A $\mathfrak{uDPois}^!$ -algebra structure on the suspension sA of a dg vector space amounts to a unital associative algebra structure on A and a degree -1 skew-symmetric operation $\chi: A^{\otimes 2} \rightarrow A^{\otimes 2}$ satisfying Relation (6), the vanishing of the genus 1 composite of the associative product with χ , and Relations (8) without signs.

Lemma 1.39. *The unital extension of the properad $\mathfrak{DPois}^!$ is given by a distributive law, so it is canonically isomorphic to*

$$\mathfrak{uDPois}^! \cong \text{DLie}^! \boxtimes \text{SuAss},$$

where $S = \text{End}_{\text{sk}}$ is the suspension operad, cf. [LV12, Section 7.2.2].

Proof. This is a direct consequence of Proposition 1.28. □

By Proposition 1.16, the arity-wise linear dual $\mathfrak{cDPois}^i := (\mathfrak{uDPois}^!)^*$ of this properad forms a partial coproperad. (It does not form a coproperad neither a comonadic coproperad due to the infinite series that appear in the iterations of the infinitesimal decomposition maps; it carries a counit which fails to be coaugmented.) Lemma 1.39 implies that its underlying S -bimodule is isomorphic to

$$\mathfrak{cDPois}^i \cong \text{DLie}^i \boxtimes (\mathbb{k}u^* \oplus \text{Ass}^i).$$

Definition 1.40 (Curved homotopy double Poisson gebra). A *curved homotopy double Poisson gebra* structure of a graded vector space A is a Maurer–Cartan element in the convolution algebra

$$\mathfrak{cDPois}_A := \widehat{\text{Hom}}(\mathfrak{cDPois}^i, \text{End}_A).$$

Notice that in the definition of this convolution algebra, we use the full partial coproperad \mathfrak{cDPois}^i , and not any coaugmentation coideal, since this latter one fails to be coaugmented. This explains while one does not start from a dg vector space but just from a graded vector space.

Proposition 1.41. *A curved homotopy double Poisson gebra is a graded vector space A equipped with a collection of operations*

$$m_{\lambda_1, \dots, \lambda_m} : A^{\otimes n} \longrightarrow A^{\otimes m}$$

of degree $n - 2$, for any ordered partition $\lambda_1 + \dots + \lambda_m = n$ of $n \geq 0$ into non-negative integers, satisfying the following relations.

CYCLIC SKEW SYMMETRY:

$$m_{\lambda_2, \dots, \lambda_m, \lambda_1} = (-1)^{\lambda_1(n-m-1)-n} \tau_m^{-1} \cdot m_{\lambda_1, \lambda_2, \dots, \lambda_m} \cdot \tau_{\lambda_1, \dots, \lambda_m},$$

where $\tau_{\lambda_1, \dots, \lambda_m} \in \mathbb{S}_n$ permutes cyclically the blocks of size $\lambda_1, \dots, \lambda_m$.

HOMOTOPY DOUBLE POISSON RELATIONS:

$$\sum_{k=1}^m \sum_{\sigma \in \mathbb{C}_m} \sum_{\substack{0 \leq p \leq \lambda_{\sigma(m)} \\ 0 \leq q \leq \lambda_{\sigma(k)}}} (-1)^\varepsilon \sigma^{-1} \cdot \left(m_{\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(k-1)}, p+1+q} \circ_i^1 m_{\lambda_{\sigma(k)-p}, \lambda_{\sigma(k+1)}, \dots, \lambda_{\sigma(m-1)}, \lambda_{\sigma(m)-q}} \right) \cdot \omega = 0,$$

with the same notations as in (10) for homotopy double Poisson gebras.

In the summand $k = 1$, the left-hand operation is m_{p+1+q} , with $p + 1 + q \geq 1$. In the summand $k = m$, the right-hand operations is $m_{\lambda_{\sigma(m)-p-q}}$, with $p + q \leq \sigma(m)$.

This result shows that a homotopy double Poisson gebra is a curved homotopy double Poisson gebra such that the operations $m_{\lambda_1, \dots, \lambda_m}$ are trivial when at least one of the λ_i is trivial. (In this case, the operation m_1 squares to zero and is considered as the differential of A .) This result is also a direct consequence of the following general arguments. Lemma 1.39 shows that the canonical map

$$\text{DPois}^! \hookrightarrow \text{uDPois}^!$$

is an embedding of properads. So the the linear dual morphism of partial coproperads

$$\text{cDPois}^! \twoheadrightarrow \text{DPois}^!$$

is surjective. This induces an embedding of (dg Lie-admissible) dg Lie algebras

$$\mathfrak{DPois}_A = \widehat{\text{Hom}}(\text{DPois}^!, \text{End}_A) \hookrightarrow \widehat{\text{Hom}}(\text{cDPois}^!, \text{End}_A) = \text{c}\mathfrak{DPois}_A.$$

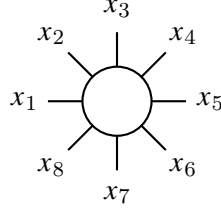
So the deformation theory of the homotopy double Poisson gebras is included inside the deformation theory of curved homotopy double Poisson gebras.

REMARK 1.42. The presentation given in this section tries to go straight to the point but it is surely not the most conceptual one. In order to encode faithfully the notion of curved gebras and to settle their properties in general, one should follow the method introduced by V. Roca i Lucio in [RiL22]: introduce a new notion of a *curved properad* and develop *mutatis mutandis* the associated curved properadic calculus.

2. PRE-CALABI-YAU ALGEBRAS

In this section, we recall the notion of a *pre-Calabi-Yau algebra* [KTV21] via the deformation theory of morphisms of cyclic non-symmetric operads. We provide the literature with a complete treatment of this theory in order to settle carefully all the signs appearing. This leads naturally to the (curved) necklace Lie-admissible algebra which encodes (curved) pre-Calabi-Yau algebra as its Maurer-Cartan elements. The even more general higher Hochschild complex is introduced and stated to be isomorphic to the Lie-admissible algebra encoding curved homotopy double Poisson gebras.

2.1. Cyclic non-symmetric operads. Let Cyc be the groupoid of finite cyclic sets $\langle x_1, \dots, x_n \rangle$ of cardinality $n \geq 1$, i.e. sets equipped with an identification with the edges of an oriented n -gon, modulo rotations. The morphisms are the bijections which respect the respective cyclic orders. We will often represent cyclically ordered sets like gears.



Definition 2.1 (Cyclic module). A *cyclic module* is a module $\mathcal{M} : \text{Cyc}^{\text{op}} \rightarrow \text{dgVect}$ over the groupoid Cyc . The associated category is denoted by CycMod .

The groupoid Cyc admits for skeletal category the one made up of one cyclically ordered set $\langle n \rangle := \langle 1, \dots, n \rangle$, for any $n \geq 1$, equipped with the cyclic groups $\mathbb{C}_n \subset \mathbb{S}_n$ for automorphisms. So the data of a cyclic module \mathcal{M} is equivalent to a collection $\{\mathcal{M}(\langle n \rangle)\}_{n \geq 1}$ of dg vector spaces equipped with actions of the cyclic groups \mathbb{C}_n under the formula

$$\mathcal{M}(X) := \left(\prod_{f \in \text{Cyc}(\langle n \rangle, X)} \mathcal{M}(\langle n \rangle) \right) / \sim,$$

where $|X| = n$ and where $(f, \mu) \sim (g, g^{-1}f \cdot \mu)$. From now on, we will identify both notions, coordinate-free and skeletal, and use the more appropriate description each time.

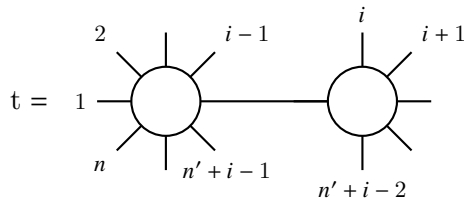
For any cyclic set X , we consider the set $\text{PT}(X)$ of (non-necessarily rooted) planar trees with leaves labeled bijectively and clockwise cyclically by the elements of X . They induce the endofunctor $\text{PT} : \text{CycMod} \rightarrow \text{CycMod}$ defined by

$$\text{PT}(\mathcal{M})(X) := \coprod_{t \in \text{PT}(X)} t(\mathcal{M}),$$

where $t(\mathcal{M}) := \bigotimes_{v \in \text{vert}(t)} \mathcal{M}(\text{in}(v))$, where $\text{in}(v)$ stands for cyclic set of leaves of the vertex v . The operation of forgetting the nesting of planar trees in $\text{PT}(\text{PT}(\mathcal{M}))$, produces elements of $\text{PT}(\mathcal{M})$ and thus induces a monad structure on PT . We call this monad the *monad of planar trees*. We refer the reader to [DV17, Section 1].

Definition 2.2 (Cyclic non-symmetric operad [Mar99]). A *cyclic non-symmetric (ns) operad* is an algebra over the monad PT of planar trees.

The monad of planar trees admits a homogenous quadratic presentation, with 2-vertices planar trees for generators, which simplifies the definition of a cyclic non-symmetric operad. There is a one-to-one correspondence between 2-vertices planar trees



and triples (n, n', i) , with $n \geq 2$, $n' \geq 1$, and $2 \leq i \leq n$.

So the action of the planar tree monad on a cyclic non-symmetric operad \mathcal{P} corresponding to 2-vertices planar trees is equivalent to *partial composition maps*:

$$\circ_i : \mathcal{P}(\langle n \rangle) \otimes \mathcal{P}(\langle n' \rangle) \rightarrow \mathcal{P}(\langle n + n' - 2 \rangle), \text{ for } n \geq 2, n' \geq 1, \text{ and } 2 \leq i \leq n.$$

Proposition 2.3. *A cyclic non-symmetric operad structure on a cyclic module is equivalent to the data of partial composition maps satisfying the parallel-sequential relation*

$$(11) \quad (\mu \circ_i \nu) \circ_j \omega = \begin{cases} (-1)^{|\nu||\omega|} (\mu \circ_j \omega) \circ_{i+n'-2} \nu, & \text{for } 2 \leq j \leq i-1, \\ \mu \circ_i (\nu \circ_{j-i+2} \omega), & \text{for } i \leq j \leq i+n'-2, \\ (-1)^{|\nu||\omega|} (\mu \circ_{j-n'+2} \omega) \circ_i \nu, & \text{for } i+n'-1 \leq j \leq n+n'-2, \end{cases}$$

and the equivariance property

$$(12) \quad \begin{cases} (\mu \circ_i \nu)^{\tau_{n+n'-2}} = (\mu^{\tau_n}) \circ_{i-1} \nu, & \text{for } i \geq 3, \\ (\mu \circ_2 \nu)^{\tau_{n+n'-2}} = (-1)^{|\mu||\nu|} (\nu^{\tau_{n'}}) \circ_{n'} (\mu^{\tau_n}), \end{cases}$$

for any $\mu \in \mathcal{P}(\langle n \rangle), \nu \in \mathcal{P}(\langle n' \rangle), \omega \in \mathcal{P}(\langle n'' \rangle)$.

Proof. This is a non-symmetric version of [GK95, Theorem 2.2]. In one direction, any cyclic non-symmetric operad structure carries partial composition maps satisfying the parallel-sequential axioms and the equivariance property. In the other way round, the data of such partial composition maps allows one to define the action of any planar tree as it can be written as iterating graftings with 2-vertices planar trees. \square

REMARK 2.4. Any cyclic non-symmetric operad induces a structure of a non-symmetric operad on $\mathcal{P}_n := \mathcal{P}(\langle n+1 \rangle)$ by forgetting the cyclic group actions. In the other way round, a cyclic non-symmetric operad structure amounts to a non-symmetric operad structure on a cyclic module satisfying the equivariant property (12), see [MSS02, Page 257].

EXAMPLE 2.5.

- (1) We consider the one-dimensional cyclic module $\mathcal{A}(\langle n \rangle) := \mathbb{k}\mu_n$ with trivial \mathbb{C}_n action and where $|\mu_n| = 0$, for $n \geq 3$, and $\mathcal{A}(\langle 2 \rangle) = \mathcal{A}(\langle 1 \rangle) = 0$. It forms a cyclic non-symmetric operad once equipped with the following partial composition maps

$$\mu_n \circ_i \mu_{n'} := \mu_{n+n'-2}.$$

- (2) Let $(V, d_V, \langle, \rangle)$ be a differential graded vector space equipped with a symmetric bilinear form of degree 0. For instance, this can be given by $V := A \oplus A^*$ equipped with the usual linearity paring $\langle f, x \rangle = f(x)$. Its *endomorphism cyclic non-symmetric operad* $\mathcal{E}nd_V$ is defined on the underlying cyclic module

$$\mathcal{E}nd_V(\langle n \rangle) := V^{\otimes n}$$

by the partial composition map

$$\circ_i (a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_{n'}) := (-1)^{(|b_1|+\cdots+|b_{n'}|)(|a_{i+1}|+\cdots+|a_n|)} \langle a_i, b_1 \rangle a_1 \otimes \cdots \otimes a_{i-1} \otimes b_2 \otimes \cdots \otimes b_{n'} \otimes a_{i+1} \otimes \cdots \otimes a_n.$$

- (3) The genus 0 part of any differential graded modular operad carries a canonical cyclic non-symmetric operad structure, where one retains only the actions of the cyclic groups and the partial composition maps. For instance, the homology $\{H_\bullet(\overline{\mathcal{M}}_{0, \langle n \rangle})\}_{n \geq 3}$ of the genus 0 part of the the Deligne–Mumford–Knudsen moduli spaces of stable curves with marked points form a cyclic non-symmetric operad, with trivial summand in arities 1 and 2, where the structure maps are given by gluing curves at marked points, see [GK98, Section 6]. Similarly, the homology

$\{H_\bullet(\mathcal{M}_{0,\langle n \rangle}^\delta)\}_{n \geq 3}$ of the Brown's partial compactification [Bro09, DV17] also forms a cyclic non-symmetric operad.

REMARK 2.6. In order to avoid any confusion, we use the Roman font for properads (Section 1.2) and the calligraphic font for cyclic non-symmetric operads. For instance, the endomorphism properad of a dg vector space A is denoted by End_A and the endomorphism cyclic non-symmetric operad of a dg vector space V equipped with a scalar product is denoted by $\mathcal{E}nd_V$.

A morphism of cyclic non-symmetric operads is a map of cyclic modules which commutes with the respective partial composition maps. We denote the associated category by cyclic ns operads.

Definition 2.7 (Algebra over a cyclic non-symmetric operad). An *algebra structure over a cyclic non-symmetric operad* \mathcal{P} on a differential graded vector space $(V, d_V, \langle, \rangle)$ equipped with a symmetric bilinear form is given by the data of morphism of cyclic non-symmetric operads $\mathcal{P} \rightarrow \mathcal{E}nd_V$.

EXAMPLE 2.8. The category of algebras over the cyclic non-symmetric operad \mathcal{A} is the category of *cyclic associative algebras*, which are differential graded associative algebras (V, d_V, \cdot) equipped with a symmetric bilinear form \langle, \rangle satisfying $\langle a \cdot b, c \rangle = \langle a, b \cdot c \rangle$, for any $a, b, c \in V$.

REMARK 2.9. One can also consider the notion of a *unital* cyclic non-symmetric operad defined with the extra data of an arity 2 element which is a unit for the partial composition maps. The cyclic module $u\mathcal{A}(\langle n \rangle) := \mathbb{k}\mu_n$ with trivial \mathbb{C}_n action, for $n \geq 1$, equipped with the partial composition maps $\mu_n \circ_i \mu_{n'} := \mu_{n+n'-2}$ forms such a unital cyclic non-symmetric operad. The endomorphism cyclic non-symmetric operad $\mathcal{E}nd_V$ associated to a vector space (V, \langle, \rangle) equipped with a *non-degenerate* symmetric bilinear form, admits a unital structure given by $\sum_{i=1}^k v_i \otimes v_i^*$, where $\{v_1, \dots, v_k\}$ is a basis of V and where $\{v_1^*, \dots, v_k^*\}$ is the induced dual basis. A morphism of unital cyclic non-symmetric operads is required to preserve the respective units. In the present case, an algebra over the unital cyclic non-symmetric operad $u\mathcal{A}$ is a *unital* associative algebra satisfying $\langle a \cdot b, c \rangle = \langle a, b \cdot c \rangle$.

We will need the following variations of the notion of a cyclic non-symmetric operad. One can first consider the monad $\mathbb{P}\mathbb{T}_-$ of *signed planar trees*, which is given by the monad of planar trees equipped with an extra sign coming from the permutation of vertices.

Definition 2.10 (Anti-cyclic non-symmetric operad). An *anti-cyclic non-symmetric operad* is an algebra over the monad $\mathbb{P}\mathbb{T}_-$ of signed planar trees. Such a structure is equivalent to a cyclic module \mathcal{P} endowed with partial composition maps satisfying the parallel-sequential relation (11) and the equivariance property up to the following sign

$$(13) \quad (\mu \circ_2 \nu)^{\tau_{n+n'-2}} = -(-1)^{|\mu||\nu|} (\nu^{\tau_{n'}}) \circ_{n'} (\mu^{\tau_n}),$$

for any $\mu \in \mathcal{P}(\langle n \rangle), \nu \in \mathcal{P}(\langle n' \rangle)$, with $n \geq 2$ and $n' \geq 1$.

EXAMPLES 2.11.

- (1) The toy model of anti-cyclic non-symmetric operads is the endomorphism operad $\mathcal{E}nd_V$ of a vector space V equipped with a *skew-symmetric* bilinear form $\langle a, b \rangle = -(-1)^{|a||b|} \langle b, a \rangle$.
- (2) A conceptually important anti-cyclic non-symmetric operad is the *suspension operad* $s^2 \mathcal{E}nd_{\mathbb{k}S^{-1}}$.

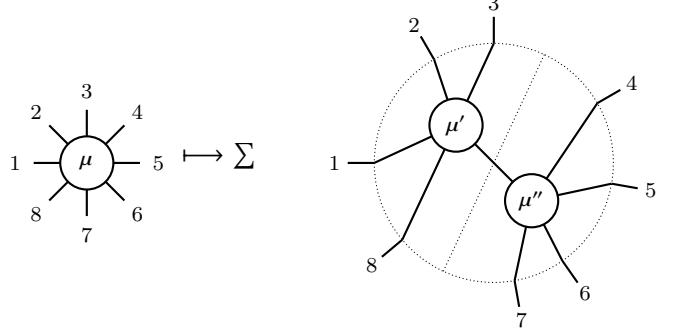
The endofunctor of signed planar trees can also be equipped with a comonad structure, denoted by $\mathbb{P}\mathbb{T}_-^c \rightarrow \mathbb{P}\mathbb{T}_-^c(\mathbb{P}\mathbb{T}_-^c)$, which amounts to sending a planar tree into the sum of all its partitions into planar sub-trees.

Definition 2.12 (Anti-cyclic non-symmetric cooperad). A *anti-cyclic non-symmetric cooperad* is a coalgebra over the comonad $\mathbb{P}\mathbb{T}_-^c$ of signed planar trees.

Similarly to the monad case, the comonad $\mathbb{P}\mathbb{T}_-^c$ is cogenerated by planar trees with two vertices.

Proposition 2.13. *An anti-cyclic non-symmetric cooperad structure on a cyclic module \mathcal{C} is equivalent to the data of partial decomposition maps*

$$\delta_i : \mathcal{C}(\langle n + n' - 2 \rangle) \rightarrow \mathcal{C}(\langle n \rangle) \otimes \mathcal{C}(\langle n' \rangle), \quad \text{for } n \geq 2, n' \geq 1, \text{ and } 2 \leq i \leq n,$$



satisfying the parallel-sequential relation

$$(14) \quad (\delta_i \otimes \text{id})\delta_j = \begin{cases} (23) \cdot ((\delta_j \otimes \text{id})\delta_{i+n'-2}), & \text{for } 2 \leq j \leq i-1, \\ (\text{id} \otimes \delta_{j-i+2})\delta_i, & \text{for } i \leq j \leq i+n'-2, \\ (23) \cdot ((\delta_{j-n'+2} \otimes \text{id})\delta_i), & \text{for } i+n'-1 \leq j \leq n+n'-2, \end{cases}$$

and the equivariance property

$$(15) \quad \begin{cases} \delta_i \tau_{n+n'-2}^{-1} = (\tau_n^{-1} \otimes \text{id}) \delta_i, & \text{for } i \geq 3, \\ \delta_2 \tau_{n+n'-2}^{-1} = -(\tau_{n'}^{-1} \otimes \tau_n^{-1})^{(23)} \delta_{n'}. \end{cases}$$

where $\tau_n^{-1}(\mu) := \mu^{\tau_n}$, for any $\mu \in \mathcal{C}(\langle n \rangle)$.

EXAMPLE 2.14. We consider the arity-wise linear dual of the suspension anti-cyclic non-symmetric operad:

$$\mathcal{B}^i(\langle n \rangle) := (\mathfrak{s}^2 \text{End}_{\mathbb{k}\mathfrak{s}^{-1}})^\vee(\langle n \rangle) \cong \mathbb{k}\mathfrak{s}^{n-2}, \quad \text{for } n \geq 3,$$

$\mathcal{B}^i(\langle 1 \rangle) = 0$, and $\mathcal{B}^i(\langle 2 \rangle) = 0$. Since $\text{End}_{\mathbb{k}\mathfrak{s}^{-1}}$ forms an arity-wise finite dimensional anti-cyclic non-symmetric operad, its linear dual is a well defined anti-cyclic non-symmetric cooperad. It is one-dimensional $\mathcal{B}^i(\langle n \rangle) := \mathbb{k}v_n$ in each arity $n \geq 3$, with the signature action of \mathbb{C}_n and the degree $|v_n| = n - 2$. Its partial decomposition map is equal to

$$\delta_i(v_{n+n'-2}) := (-1)^{in'} v_n \otimes v_{n'}.$$

EXAMPLE 2.15. In a similar way, we consider the following anti-cyclic non-symmetric cooperad

$$\mathcal{C}\mathcal{B}^i(\langle n \rangle) := (\mathfrak{s}^2 \text{End}_{\mathbb{k}\mathfrak{s}^{-1}})^\vee(\langle n \rangle) \cong \mathbb{k}\mathfrak{s}^{n-2}, \quad \text{for } n \geq 1.$$

REMARK 2.16. Like the case of modular operads [War19, DSVV20], the category of cyclic non-symmetric operads can be encoded by a groupoid-colored operad which is Koszul and whose Koszul dual (co)operad encodes anti-cyclic non-symmetric (co)operads.

Another range of generalisations can be obtained by shifting the underlying cyclic module.

Definition 2.17 (Shifted cyclic non-symmetric operad). A *shifted cyclic non-symmetric operad* structure on a cyclic module \mathcal{P} is a cyclic non-symmetric operad structure on the desuspended cyclic module $s^{-1}\mathcal{P}$. Such a data is equivalent to degree -1 partial composition maps satisfying the parallel-sequential relation up to the following sign

$$(16) \quad (\mu \circ_i \nu) \circ_j \omega = \begin{cases} -(-1)^{|\nu||\omega|} (\mu \circ_j \omega) \circ_{i+n'-2} \nu, & \text{for } 2 \leq j \leq i-1, \\ -\mu \circ_i (\nu \circ_{j-i+2} \omega), & \text{for } i \leq j \leq i+n'-2, \\ -(-1)^{|\nu||\omega|} (\mu \circ_{j-n'+2} \omega) \circ_i \nu, & \text{for } i+n'-1 \leq j \leq n+n'-2, \end{cases}$$

and the equivariance property up to the following sign

$$(17) \quad \begin{cases} (\mu \circ_i \nu)^{\tau_{n+n'-2}} = (\mu^{\tau_n}) \circ_{i-1} \nu, & \text{for } i \geq 3, \\ (\mu \circ_2 \nu)^{\tau_{n+n'-2}} = -(-1)^{|\mu||\nu|} (\nu^{\tau_{n'}}) \circ_{n'} (\mu^{\tau_n}), \end{cases}$$

for every $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, and $\omega \in \mathcal{P}(Z)$.

EXAMPLE 2.18. The paradigm of shifted cyclic non-symmetric operad is the endomorphism operad $\mathcal{E}nd_{sA \oplus A^*}$ associated to the dg vector space equipped with the degree -1 graded-symmetric paring $\langle f, sx \rangle := (-1)^{|f|} f(x)$ and $\langle sx, f \rangle := (-1)^{|f||x|} f(x)$.

In the same straightforward way, one can define notions of shifted (anti)-cyclic non-symmetric (co)operads. The details are left to the reader.

2.2. Deformation theory of morphisms of cyclic non-symmetric operads. In this section, we develop the deformation theory of morphisms of cyclic non-symmetric operads following a presentation similar to that of Section 1.3. Let us first recall the shifted version of the classical notion of a Lie algebra.

Definition 2.19 (Shifted Lie algebra). A *(differential graded) shifted Lie algebra* is a triple $(\mathfrak{g}, d, \{, \})$ made up of a differential graded vector spaces (\mathfrak{g}, d) and a degree -1 symmetric product $\{, \}: \mathfrak{g}^{\odot 2} \rightarrow \mathfrak{g}$ for which d is a derivation and satisfying the Jacobi relation

$$\{\{x, y\}, z\} + (-1)^{(|x|-1)(|y|+|z|)} \{\{y, z\}, x\} + (-1)^{(|z|-1)(|x|+|y|)} \{\{z, x\}, y\} = 0.$$

REMARK 2.20. A shifted Lie algebra structure on \mathfrak{g} is equivalent to a Lie algebra structure on the desuspension $s^{-1}\mathfrak{g}$.

Definition 2.21 (Maurer–Cartan equation). The *Maurer–Cartan equation* of a (shifted) Lie algebra is the equation

$$(18) \quad d\alpha + \frac{1}{2}\{\alpha, \alpha\} = 0.$$

We only consider solutions of degree $|\alpha| = -1$ (respectively degree $|\alpha| = 0$); their set is denoted by $MC(\mathfrak{g})$.

Let us now introduce our main example.

Definition 2.22 (Totalisation). The *totalisation* of a cyclic module \mathcal{P} is the graded vector space defined by

$$\widehat{\mathcal{P}} := \prod_{n \geq 1} \mathcal{P}(\langle n \rangle)^{C_n}.$$

REMARK 2.23. Since we are working here in characteristic 0, we could have equivalently considered coinvariants instead of invariants in the definition of the totalisation.

Lemma 2.24 ([KWZ15, Proposition 2.18]). *The following assignment defines a functor from (shifted) anti-cyclic non-symmetric operads to complete (shifted) Lie algebras*

(shifted) anti-cyclic ns operads \rightarrow complete (shifted) Lie algebras

$$(\mathcal{P}, d_{\mathcal{P}}, \circ_i) \mapsto (\widehat{\mathcal{P}}, d, \{, \}, \mathcal{F}) ,$$

where the differential d is induced by the differential $d_{\mathcal{P}}$ and where the Lie bracket is given by

$$\{\mu, \nu\} := \sum_{i=2}^n \mu \circ_i \nu - (-1)^{|\mu||\nu|} \sum_{j=2}^{n'} \nu \circ_j \mu ,$$

(with the sign $+(-1)^{|\mu||\nu|}$ in front of the second term in the shifted case), for any $\mu \in \mathcal{P}(\langle n \rangle)^{C_n}$ and $\nu \in \mathcal{P}(\langle n' \rangle)^{C_{n'}}$, and the decreasing filtration

$$\widehat{\mathcal{P}} = \mathcal{F}_{-1} \supset \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_N \supset \mathcal{F}_{N+1} \supset \cdots ,$$

where \mathcal{F}_N is made up of series such that the terms of arity $n < N + 2$ vanish.

Proof. The proof amounts to straightforward computations from the axioms (11), (12), and (13). \square

REMARK 2.25. The above Lie bracket is made up of the skew-symmetrisation of the product which is the usual pre-Lie product $\sum_{i=2}^n \mu \circ_i \nu$ on the totalisation $\prod_{n \geq 1} \mathcal{P}(\langle n \rangle)$ of the underlying non-symmetric operad [KM01]. Notice that this pre-Lie product is not stable on the sub-space of invariants with respect to the cyclic groups actions and so the skew-symmetrisation is mandatory here.

Lemma 2.26. *The assignment*

$$\begin{aligned} &(\text{anti-cyclic ns cooperads})^{\text{op}} \times (\text{shifted) cyclic ns operads} \rightarrow (\text{shifted) anti-cyclic ns operads} \\ &((\mathcal{C}, d_{\mathcal{C}}, \delta_i), (\mathcal{P}, d_{\mathcal{P}}, \circ_i)) \mapsto \text{Hom}(\mathcal{C}, \mathcal{P}) := (\{\text{Hom}(\mathcal{C}(\langle n \rangle), \mathcal{P}(\langle n \rangle))\}_{n \in \mathbb{N}^*}, \partial, \circ_i) \end{aligned}$$

defines a functor, where the cyclic groups act by conjugaison, where the differential is given by

$$\partial(f) := d_{\mathcal{P}} \circ f - (-1)^{|f|} f \circ d_{\mathcal{C}} ,$$

and where the partial compositions maps are given by

$$\circ_i(f \otimes g) := \circ_i(f \otimes g) \delta_i .$$

Proof. This follows in a straightforward way from the defining relations. \square

Definition 2.27 (Convolution algebra). The composite of the above two functors produces functorially the *convolution dg Lie algebra*:

$$\widehat{\text{Hom}}(\mathcal{C}, \mathcal{P}) := \left(\prod_{n \geq 1} \text{Hom}_{C_n}(\mathcal{C}(\langle n \rangle), \mathcal{P}(\langle n \rangle)), \partial, \{-, -\} \right) .$$

Definition 2.28 (Twisting morphism). The solutions to the Maurer–Cartan equation (18) in the convolution algebra $\widehat{\text{Hom}}(\mathcal{C}, \mathcal{P})$ are called *twisting morphisms* from \mathcal{C} to \mathcal{P} ; their set is denoted by $\text{Tw}(\mathcal{C}, \mathcal{P})$.

As first example, we consider the convolution dg Lie algebra associated to the anti-cyclic non-symmetric cooperad \mathcal{A}^i of Example 2.14 and the cyclic non-symmetric endomorphism operad End_V of Example 2.5.

Definition 2.29 (Cyclic A_{∞} -algebra). The algebraic structure defined by a twisting morphism in $\text{Tw}(\mathcal{A}^i, \text{End}_V)$ is called a *cyclic A_{∞} -algebra* structure on V .

Proposition 2.30. *A cyclic A_∞ -algebra structure on a dg module V induces products $m_n: V^{\otimes n} \rightarrow V$ of degree $n - 2$, for $n \geq 2$, satisfying*

$$(19) \quad \partial(m_n) = \sum_{p+q+r=n} (-1)^{pq+r+1} m_{p+1+r} \circ (\text{id}^{\otimes p} \otimes m_q \otimes \text{id}^{\otimes r}),$$

for any $n \geq 2$, and equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ satisfying

$$(20) \quad \langle v_1, m_n(v_2, \dots, v_{n+1}) \rangle = (-1)^{|v_1|(|v_2|+\dots+|v_{n+1}|)+n} \langle v_{n+1}, m_n(v_1, \dots, v_n) \rangle,$$

for any $n \geq 2$. Both structures are equivalent when the pairing $\langle \cdot, \cdot \rangle$ is non-degenerate.

Proof. We consider the morphism of chain complexes $\Theta: V^{\otimes(n+1)} \rightarrow \text{Hom}(V^{\otimes n}, V)$ defined by

$$\Theta(v_1, \dots, v_{n+1}) := v_1 \langle v_2, - \rangle \cdots \langle v_{n+1}, - \rangle.$$

Let $\alpha \in \text{Tw}(\mathcal{A}^i, \text{End}_V)$ be a twisting morphism. Under the notation $m_n := \Theta(\alpha(v_{n+1}))$, the image of the Maurer–Cartan equation satisfied by α under Θ gives the abovementioned equation (19) of A_∞ -algebras. The equation (20) comes from the signature representation on $\mathcal{A}^i(\langle n \rangle)$ and the equivariance of the map α . The map Θ is an isomorphism if and only if the symmetric bilinear form $\langle \cdot, \cdot \rangle$ is non-degenerate. In this case, the two algebraic structures are equivalent. \square

More generally, one can consider the anti-cyclic non-symmetric cooperad $c\mathcal{A}^i$ of Example 2.15. The canonical surjection $c\mathcal{A}^i \rightarrow \mathcal{A}^i$ of anti-cyclic non-symmetric cooperads induces an embedding of dg Lie algebras

$$\widehat{\text{Hom}}(\mathcal{A}^i, \text{End}_V) \hookrightarrow \widehat{\text{Hom}}(c\mathcal{A}^i, \text{End}_V).$$

Definition 2.31 (Cyclic curved A_∞ -algebra). The algebraic structure defined by a twisting morphism in $\text{Tw}(c\mathcal{A}^i, \text{End}_V)$ is called a *cyclic curved A_∞ -algebra* structure on V .

Proposition 2.32. *A cyclic curved A_∞ -algebra structure on a graded module V is made up of products $m_n: V^n \rightarrow V$ of degree $n - 2$, for $n \geq 0$, satisfying*

$$(21) \quad \sum_{p+q+r=n} (-1)^{pq+r+1} m_{p+1+r} \circ (\text{id}^{\otimes p} \otimes m_q \otimes \text{id}^{\otimes r}) = 0,$$

for any $n \geq 0$, and equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ satisfying

$$(22) \quad \langle v_1, m_n(v_2, \dots, v_{n+1}) \rangle = (-1)^{|v_1|(|v_2|+\dots+|v_{n+1}|)+n} \langle v_{n+1}, m_n(v_1, \dots, v_n) \rangle,$$

for any $n \geq 2$. Both structures are equivalent when the pairing $\langle \cdot, \cdot \rangle$ is non-degenerate.

Proof. This proof is similar to the one given above and uses the same identifications. \square

Therefore a cyclic A_∞ -algebra is a cyclic curved A_∞ -algebra such that the curvature operation m_0 is trivial. (In this case, the operation m_1 squares to zero and is considered as the differential of V .) The above-mentioned embedding of dg Lie algebras shows that the deformation theory of the former is included inside the deformation theory of the latter.

2.3. Generalised necklace algebra. In this section, we consider the special case $V = sA \oplus A^*$ equipped with its canonical degree -1 skew-symmetric pairing: $\langle f, sx \rangle := (-1)^{|f|} f(x)$.

Proposition 2.33. *The convolution algebra $\widehat{\text{Hom}}(\mathcal{A}^i, \text{End}_{sA \oplus A^*})$ is a shifted Lie algebra with underlying graded vector space isomorphic to*

$$(23) \quad s^2 \prod_{N \geq 3} \left(\bigoplus_{1 \leq m < N} \left(\bigoplus_{\lambda_1 + \dots + \lambda_m = n} A \otimes ((sA)^*)^{\otimes \lambda_1} \otimes A \otimes ((sA)^*)^{\otimes \lambda_2} \cdots \otimes A \otimes ((sA)^*)^{\otimes \lambda_m} \right)^{C_m} \oplus \left(((sA)^*)^{\otimes N} \right)^{C_N} \right),$$

where the sum runs over ordered partitions $\lambda_1 + \dots + \lambda_m = n$ of $n = N - m$ into non-negative integers, and with the bracket given by

$$\begin{aligned} & \{s^2 a_1 \otimes \dots \otimes a_N, s^2 b_1 \otimes \dots \otimes b_{N'}\} = \\ & s^2 \sum_{i=2}^N (-1)^{|a_1| + \dots + |a_{i-1}| + |b|(|a_{i+1}| + \dots + |a_N|)} \langle a_i, b_1 \rangle a_1 \otimes \dots \otimes a_{i-1} \otimes b_2 \otimes \dots \otimes b_{N'} \otimes a_{i+1} \otimes \dots \otimes a_N \\ & + s^2 \sum_{j=2}^{N'} (-1)^{|b_1| + \dots + |b_{j-1}| + |a|(|b_{j+1}| + \dots + |b_{N'}|)} \langle b_j, a_1 \rangle b_1 \otimes \dots \otimes b_{j-1} \otimes a_2 \otimes \dots \otimes a_N \otimes b_{j+1} \otimes \dots \otimes b_{N'} , \end{aligned}$$

with $a_k, b_l \in A \oplus (sA)^*$, $a := a_1 \otimes \dots \otimes a_N$, and $b := b_1 \otimes \dots \otimes b_{N'}$.

Proof. Since the cooperad \mathcal{A}^i is anti-cyclic non-symmetric and since the operad $\mathcal{E}nd_{sA \oplus A^*}$ is shifted cyclic non-symmetric, the convolution algebra $\widehat{\text{Hom}}(\mathcal{A}^i, \mathcal{E}nd_{sA \oplus A^*})$ is a shifted Lie algebra by Lemmata 2.24 and 2.26. Using now capital N for the arity in the cyclic operad context, the cyclic module $\mathcal{A}^i(\langle N \rangle)$ is isomorphic to s^{N-2} , for any $N \geq 3$. So the underlying graded space of the convolution algebra is isomorphic to

$$\prod_{N \geq 3} \text{Hom}_{\mathbb{C}_N}(s^{N-2}, (sA \oplus A^*)^{\otimes N}) \cong s^2 \prod_{N \geq 3} (s^{-N} \otimes (sA \oplus A^*)^{\otimes N})^{\mathbb{C}_N} \cong s^2 \prod_{N \geq 3} ((A \oplus (sA)^*)^{\otimes N})^{\mathbb{C}_N} .$$

Spitting according to which part of the direct sum comes from either A or sA^* , one obtains the graded space displayed above in (23). The abovementioned isomorphism is given explicitly by

$$\Phi(s^{N-2} \mapsto sa_1 \otimes \dots \otimes sa_N) := (-1)^{\frac{N(N+1)}{2} + N|a_N| + \dots + |a_1|} s^2 a_1 \otimes \dots \otimes a_N ,$$

where $a_1, \dots, a_N \in A \oplus (sA)^*$. Transporting the shifted Lie bracket of the convolution algebra (Definition 2.27) under this isomorphism gives the shifted Lie bracket of the above statement. \square

Definition 2.34 (Generalised necklace Lie algebra). The *generalised necklace Lie algebra* associated to the dg vector space A is the desuspension of the above shifted Lie sub-algebra:

$$\text{neck}_A := \left(s \prod_{N \geq 3} \left(\bigoplus_{1 \leq m < N} \left(\bigoplus_{\lambda_1 + \dots + \lambda_m = n} A \otimes ((sA)^*)^{\otimes \lambda_1} \otimes \dots \otimes A \otimes ((sA)^*)^{\otimes \lambda_m} \right)^{\mathbb{C}_m} \right), d, \{-, -\} \right) .$$

There is a crucial point for us in the present case: when $V = sA \oplus A^*$, the Lie bracket on the generalised necklace Lie algebra splits into two, depending on whether one applies the linearity pairing to $f \otimes sx$ or to $sx \otimes f$, where $f \in A^*$ and $x \in A$. More precisely, we denote by $X * Y$ the summand of $\{X, Y\}$ made up of the terms where one applies the linear pairing $\langle f, sx \rangle$, where $f \in A^*$ comes from X and $x \in A$ comes from Y . So the Lie bracket is equal to the skew-symmetrisation of the product $*$:

$$\{X, Y\} = X * Y - (-1)^{|X||Y|} Y * X .$$

Lemma 2.35. *The binary product $*$ satisfies the relation of a Lie-admissible algebra:*

$$\sum_{\sigma \in \mathbb{S}_3} \text{sgn}(\sigma) \text{assoc}(-, -, -)^\sigma = 0 ,$$

where assoc stands for the associator: $\text{assoc}(x, y, z) := (x * y) * z - x * (y * z)$, for every x, y, z .

Proof. This is actually the definition of a Lie-admissible bracket: its skew-symmetrised bracket satisfies the Jacobi relation. \square

REMARK 2.36. On the opposite to what is claimed in [IKV19, Section 2], the operation $*$ does not satisfy any stronger relation, like the pre-Lie relation, in general.

Definition 2.37 (Pre-Calabi–Yau algebra). A structure of a *pre-Calabi–Yau algebra* on a dg vector space A is a Maurer–Cartan element in the generalised necklace Lie(-admissible) algebra nect_A , that is a degree -1 element α satisfying:

$$\partial\alpha + \frac{1}{2} \{\alpha, \alpha\} = \partial\alpha + \alpha * \alpha = 0 .$$

Working *mutatis mutandis* with the anti-cyclic non-symmetric cooperad $c\mathcal{SB}^i$ encoding cyclic curved A_∞ -algebras, one gets the following more general context.

Definition 2.38 (Curvature necklace Lie-admissible algebra). The *curvature necklace Lie-admissible algebra* associated to the dg vector space A is defined by

$$\text{cnect}_A := \left(s \prod_{N \geq 1} \left(\bigoplus_{1 \leq m < N} \left(\bigoplus_{\lambda_1 + \dots + \lambda_m = n} A \otimes ((sA)^*)^{\otimes \lambda_1} \otimes \dots \otimes A \otimes ((sA)^*)^{\otimes \lambda_m} \right)^{\text{C}_m} \right), d, * \right) .$$

The skew-symmetrisation of the Lie-admissible product $*$ produces a Lie bracket and thus the curvature necklace Lie algebra.

Definition 2.39 (Curved pre-Calabi–Yau algebra). A structure of a *curved pre-Calabi–Yau algebra* on a graded vector space A is a Maurer–Cartan element in the curved necklace Lie-admissible algebra.

There is a canonical embedding of dg Lie-admissible algebras

$$\text{nect}_A \hookrightarrow \text{cnect}_A .$$

A pre-Calabi–Yau algebra is a curved pre-Calabi–Yau algebra with trivial components on A , $A \otimes A$, and $(A \otimes (sA)^* \oplus (sA)^* \otimes A)^{\text{C}_2}$.

REMARK 2.40. A (curved) pre-Calabi–Yau structure on a dg vector space A corresponds to a shifted cyclic (curved) A_∞ -algebra structure on $sA \oplus A^*$, i.e. a cyclic (curved) A_∞ -algebra structure on $A \oplus (sA)^*$, with trivial terms on $((sA)^*)^{\otimes N}$. In particular, it carries an (curved) A_∞ -algebra structure on A , which is a (curved) A_∞ -sub-algebra of $A \oplus (sA)^*$ by this latter condition.

2.4. Higher Hochschild complex and the main theorem. Recall that there is a canonical inclusion

$$(24) \quad \begin{aligned} W \otimes V^* &\hookrightarrow \text{Hom}(V, W) \\ x \otimes f &\mapsto (v \mapsto xf(v)), \end{aligned}$$

which is an isomorphism if and only if V is finite dimensional in each degree. It leads to considering $\text{Hom}((sA)^{\otimes \lambda_1} \otimes \dots \otimes (sA)^{\otimes \lambda_m}, A^{\otimes m})$ instead of $A \otimes ((sA)^*)^{\otimes \lambda_m} \otimes \dots \otimes A \otimes ((sA)^*)^{\otimes \lambda_1}$. Given two maps

$$F \in \text{Hom}_{\mathbb{C}_m} \left(\bigoplus_{\lambda_1 + \dots + \lambda_m = n} \bigotimes_{j=1}^m (sA)^{\otimes \lambda_j}, A^{\otimes m} \right) \quad \text{and} \quad G \in \text{Hom}_{\mathbb{C}_{m'}} \left(\bigoplus_{\lambda'_1 + \dots + \lambda'_{m'} = n'} \bigotimes_{j=1}^{m'} (sA)^{\otimes \lambda'_j}, A^{\otimes m'} \right),$$

we identify the suspension sF with the induced map $\bigotimes_{j=1}^m (sA)^{\otimes \lambda_j} \rightarrow sA \otimes A^{\otimes m-1}$, and similarly for sG . For any $1 \leq i \leq n$, we denote by $sF \otimes_i^1 sG$:

$$(sA)^{\otimes \lambda_1} \otimes \dots \otimes (sA)^{\otimes \lambda_{k-1}} \otimes (sA)^{\otimes \lambda'_1 + \lambda_k - l} \otimes (sA)^{\otimes \lambda'_2} \otimes \dots \otimes (sA)^{\otimes k-1 + \lambda'_{m'}} \otimes (sA)^{\otimes \lambda_{k+1}} \otimes \dots \otimes (sA)^{\otimes \lambda_m} \\ \rightarrow sA^{\otimes m+m'-1},$$

with $i = \lambda_1 + \dots + \lambda_{k-1} + l$ and $1 \leq l \leq \lambda_k$, the partial composite of the two multilinear operations sF and sG including the permutations of inputs and outputs depicted in Figure 4, for it to leave in the space of cyclically invariant maps.

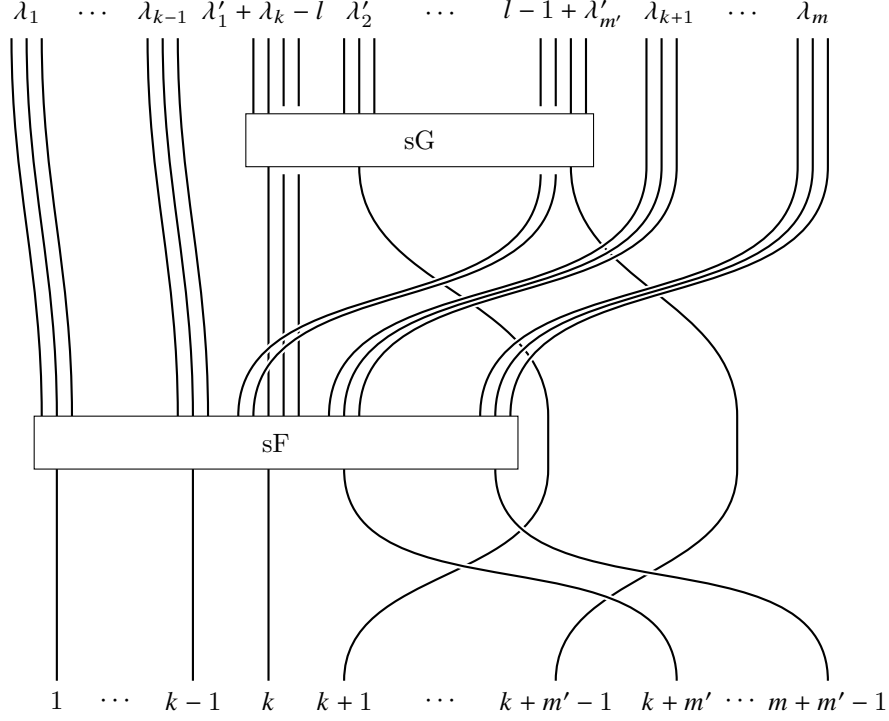


FIGURE 4. Illustration of the partial composite $\text{sF} \otimes_i^1 \text{sG}$.

Definition 2.41 (Higher Hochschild complex). The *higher Hochschild complex* associated to a dg vector space A is the Lie-admissible algebra

$$\mathfrak{hh}_A := \left(\text{s} \prod_{N \geq 1} \left(\bigoplus_{1 \leq m < N} \text{Hom}_{\mathbb{C}_m} \left(\bigoplus_{\lambda_1 + \dots + \lambda_m = n} \bigotimes_{j=1}^m (\text{sA})^{\otimes \lambda_j}, A^{\otimes m} \right) \right), \partial, \otimes \right),$$

where

$$\text{sF} \otimes \text{sG} := \sum_{i=1}^n \text{sF} \otimes_i^1 \text{sG}.$$

Proposition 2.42. *The inclusion (24) induces a monomorphism of dg Lie-admissible algebras*

$$\text{cnecf}_A \hookrightarrow \mathfrak{hh}_A,$$

which is an isomorphism if and only if A is finite dimensional in each degree.

Proof. It is straightforward to check that the product \otimes is well-defined, that is it lands in cyclically invariant maps, and that it is Lie-admissible. (This is also a direct corollary of the proof of Theorem 2.43 below.) The map $\Psi : \text{cnecf}_A \rightarrow \mathfrak{hh}_A$ is explicitly given by

$$\Psi(\text{s} a_1 \otimes f_1 \otimes \dots \otimes a_m \otimes f_m) := (-1)^\zeta \text{s} a_1 \otimes \dots \otimes a_m \otimes f_m \otimes \dots \otimes f_1,$$

with $a_j \in A$ and $f_j \in ((\text{sA})^*)^{\otimes \lambda_j}$, for $1 \leq j \leq m$, where

$$\zeta = |f_1|(|a_2| + |f_2| + \dots + |a_m| + |f_m|) + |f_2|(|a_3| + |f_3| + \dots + |a_m| + |f_m|) + \dots + |f_{m-1}|(|a_m| + |f_m|).$$

and where the right-hand term is interpreted as a map in $\text{sHom}((\text{sA})^{\otimes \lambda_1} \otimes \dots \otimes (\text{sA})^{\otimes \lambda_m}, A^{\otimes m})$ under the inclusion (24). Using the Koszul sign rule and the Koszul sign convention, it is straightforward to

check that

$$\begin{aligned} & \Psi(s a_1 \otimes f_1 \otimes \cdots \otimes a_m \otimes f_m) \otimes \Psi(s b_1 \otimes g_1 \otimes \cdots \otimes b_{m'} \otimes g_{m'}) \\ &= \Psi((s a_1 \otimes f_1 \otimes \cdots \otimes a_m \otimes f_m) * (s b_1 \otimes g_1 \otimes \cdots \otimes b_{m'} \otimes g_{m'})) . \end{aligned}$$

□

The main result of this paper is the following theorem.

Theorem 2.43. *For any dg vector space A , there is a canonical and functorial isomorphism of dg Lie-admissible algebras*

$$\mathfrak{cDPois}_A \cong \mathfrak{hh}_A .$$

Proof. Since the proof is a straightforward, long, and not really enlightening computation, we postpone it to Appendix B. □

In the end, we get the following monomorphism of dg Lie-admissible algebras

$$\mathfrak{cnect}_A \hookrightarrow \mathfrak{hh}_A \cong \mathfrak{cDPois}_A ,$$

which is an isomorphism if and only if A is degree-wise finite dimensional.

Corollary 2.44.

- (1) *Any curved pre-Calabi–Yau algebra carries a canonical curved homotopy double Poisson algebra structure.*
- (2) *When A is degree-wise finite dimensional, both notions are equivalent. In this case, any curved homotopy double Poisson algebra carries a canonical curved pre-Calabi–Yau algebra structure.*

Proof. It is a direct corollary of Theorem 2.43. □

3. INFINITY-MORPHISMS

The abovementioned theorem shows that the moduli spaces of curved homotopy double Poisson algebra structures and curved pre-Calabi–Yau algebra structures are equivalent. It also opens the doors to the properadic calculus developed in [HLV20, HLV22] and in [CV22]. As a direct application, we get a notion of an ∞ -morphism satisfying all the expected homotopical properties for these algebraic structures: homotopy transfer theorem, invertibility of ∞ -quasi-isomorphisms, formality, Koszul hierarchy, and twisting procedure.

3.1. Moduli spaces. Theorem 2.43 allows us to compare the moduli spaces of curved pre-Calabi–Yau algebras and the moduli spaces of curved homotopy double Poisson algebras. First, the moduli space of pre-Calabi–Yau algebras was introduced by W.-K. Yeung in [Yeu19, Section 3.1].

Definition 3.1 (Space of curved pre-Calabi–Yau structures). The *space of curved pre-Calabi–Yau structures* on a dg vector space A is the Kan complex $\mathfrak{cpCY}\{A\}$ defined by

$$\mathfrak{cpCY}\{A\} := \text{MC}(\mathfrak{cnect}_A \otimes \Omega_\bullet) ,$$

where Ω_\bullet is the Sullivan simplicial commutative algebra of polynomial differential forms of the standard simplices Δ^\bullet .

Similarly, the space of homotopy algebras, introduced by S. Yalin in [Yal16, Section 3.1], takes the following form here. We denote by \mathfrak{cDPois}_∞ the cobar construction of the partial coproperad \mathfrak{cDPois}^i , which is given by the quasi-free properad on the desuspension of \mathfrak{cDPois}^i . (We do not consider any coaugmentation ideal here as the partial coproperad \mathfrak{cDPois}^i fails to be coaugmented.) Since \mathfrak{cDPois}^i is non-negatively graded, the dg properad \mathfrak{cDPois}_∞ is cofibrant.

Definition 3.2 (Space of curved homotopy double Poisson gebra structures). The *space of curved homotopy double Poisson gebra structures* on a dg vector space A is the Kan complex $\text{cDPois}_\infty\{A\}$ defined by

$$\text{cDPois}_\infty\{A\} := \text{Hom}_{\text{dg properads}}(\text{cDPois}_\infty, \text{End}_A \otimes \Omega_\bullet),$$

where the properad $\text{End}_A \otimes \Omega_\bullet$ is defined by

$$(\text{End}_A \otimes \Omega_\bullet)(m, n) := \text{End}_A(m, n) \otimes \Omega_\bullet.$$

Proposition 3.3. *The embedding $\text{cnect}_A \hookrightarrow \text{cDPois}_A$ of dg Lie-admissible algebras induces an embedding of Kan complexes*

$$\text{cpCY}\{A\} \hookrightarrow \text{cDPois}_\infty\{A\},$$

which is an isomorphism when A is degree-wise finite dimensional.

Proof. This is a direct corollary of the isomorphism of Kan complexes

$$\text{cDPois}_\infty\{A\} \cong \text{MC}(\widehat{\text{Hom}}(\text{cDPois}^i, \text{End}_A) \otimes \Omega_\bullet),$$

see [Yal16, Theorem 3.12], and Theorem 2.43. □

3.2. Infinity-morphisms of homotopy double Poisson gbras. The advantage of homotopy double Poisson gbras over pre-Calabi–Yau algebras is that the properadic calculus recently developed in [HLV20] automatically equips them with a higher notion of ∞ -morphism which behaves well with respect the homotopical properties: homological invertibility of ∞ -quasi-isomorphisms, homotopy transfer theorem and Deligne groupoid, for instance.

Here are a few recollections in order to make the present text as self-contained as possible. Let us recall from [HLV20, Definition 3.12] that the *left* and *right infinitesimal composition products*

$$M \triangleleft_{(*)} N \quad \text{and} \quad M_{(*)} \triangleright N,$$

are the sub- \mathbb{S} -bimodules of $(I \oplus M) \boxtimes N$ and $M \boxtimes (I \oplus N)$ made up of the linear parts in M and N respectively. For $n \geq 1$, we consider their following summands

$$M \triangleleft_{(n)} N \quad \text{and} \quad M_{(n)} \triangleright N,$$

made up of one element of M on the bottom (resp. of N at the top) and n elements of N on the top (resp. of M at the bottom). Notice that the top (resp. bottom) level is saturated by elements of N (resp. M) and that the bottom (resp. top) level contains one element of M (resp. N) and possibly many copies of the identity element from I , see Figure 5.

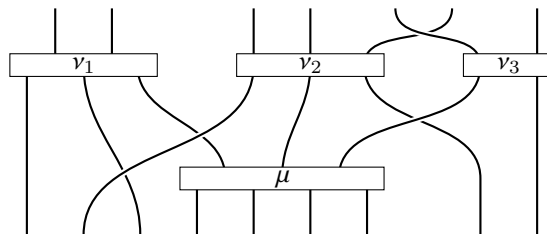


FIGURE 5. An element of $M \triangleleft_{(3)} N$.

Applying [HLV20, Definition 3.13] to the Koszul dual coproperad DPois^i , we get the *left* and *right infinitesimal decomposition maps*, defined respectively by

$$\begin{aligned}\Delta_{(*)} : \overline{\mathrm{DPois}}^i &\xrightarrow{\Delta} \mathrm{DPois}^i \boxtimes \mathrm{DPois}^i \xrightarrow{(\varepsilon; \mathrm{id}) \boxtimes \mathrm{id}} \overline{\mathrm{DPois}}^i \triangleleft_{(*)} \mathrm{DPois}^i, \\ {}_{(*)}\Delta : \overline{\mathrm{DPois}}^i &\xrightarrow{\Delta} \mathrm{DPois}^i \boxtimes \mathrm{DPois}^i \xrightarrow{\mathrm{id} \boxtimes (\varepsilon; \mathrm{id})} \mathrm{DPois}^i \triangleright_{(*)} \overline{\mathrm{DPois}}^i,\end{aligned}$$

which can be extended to DPois^i by setting the image of I to be trivial. Given $f \in \mathrm{Hom}_{\mathbb{S}}(\mathrm{DPois}^i, \mathrm{End}_B^A)$, $\alpha \in \mathrm{Hom}_{\mathbb{S}}(\mathrm{DPois}^i, \mathrm{End}_A)$, and $\beta \in \mathrm{Hom}_{\mathbb{S}}(\mathrm{DPois}^i, \mathrm{End}_B)$, the *left action* of β on f and the *right action* of α on f are defined respectively by

$$\begin{aligned}\beta \triangleleft f : \quad &\overline{\mathrm{DPois}}^i \xrightarrow{\Delta_{(*)}} \overline{\mathrm{DPois}}^i \triangleleft_{(*)} \mathrm{DPois}^i \xrightarrow{\beta \triangleleft_{(*)} f} \mathrm{End}_B \triangleleft_{(*)} \mathrm{End}_B^A \longrightarrow \mathrm{End}_B^A, \\ &I \xrightarrow{\cong} I \boxtimes I \xrightarrow{\beta \boxtimes f} \mathrm{End}_B \boxtimes \mathrm{End}_B^A \longrightarrow \mathrm{End}_B^A, \\ f \triangleright \alpha : \quad &\overline{\mathrm{DPois}}^i \xrightarrow{{}_{(*)}\Delta} \mathrm{DPois}^i \triangleright_{(*)} \overline{\mathrm{DPois}}^i \xrightarrow{f \triangleright_{(*)} \alpha} \mathrm{End}_B^A \triangleright_{(*)} \mathrm{End}_A \longrightarrow \mathrm{End}_B^A, \\ &I \xrightarrow{\cong} I \boxtimes I \xrightarrow{f \boxtimes \alpha} \mathrm{End}_B^A \boxtimes \mathrm{End}_A \longrightarrow \mathrm{End}_B^A,\end{aligned}$$

where the rightmost arrows are given by the usual composition of functions.

Definition 3.4 (∞ -morphism of homotopy double Poisson gebras). Let A, B be dg vector spaces equipped respectively with two homotopy double Poisson gebra structures viewed as two Maurer–Cartan elements $\alpha \in \widehat{\mathrm{Hom}}(\mathrm{DPois}^i, \mathrm{End}_A)$ and $\beta \in \widehat{\mathrm{Hom}}(\mathrm{DPois}^i, \mathrm{End}_B)$ in the respective convolution Lie-admissible algebras. An ∞ -morphism $f : (A, \alpha) \rightsquigarrow (B, \beta)$ is a degree 0 map of \mathbb{S} -bimodules $f : \mathrm{DPois}^i \rightarrow \mathrm{End}_B^A$ satisfying the equation

$$(25) \quad \partial(f) = f \triangleright \alpha - \beta \triangleleft f.$$

The *composite* of ∞ -morphisms is defined by

$$g \circ f : \mathrm{DPois}^i \xrightarrow{\Delta} \mathrm{DPois}^i \boxtimes \mathrm{DPois}^i \xrightarrow{g \boxtimes f} \mathrm{End}_C^B \boxtimes \mathrm{End}_B^A \longrightarrow \mathrm{End}_C^A.$$

Proposition 3.5. *Homotopy double Poisson gebras equipped with their ∞ -morphisms and the composite \circ form a category.*

Proof. This is a direct corollary of the general theory settled in [HLV20, Section 3]. \square

By definition, an ∞ -morphism of homotopy double Poisson gebras is a collection of maps

$$f_{\lambda_1, \dots, \lambda_m} : A^{\otimes n} \longrightarrow B^{\otimes m}$$

of degree $n - 1$, for any ordered partition $\lambda_1 + \dots + \lambda_m = n$ of $n \geq 1$, satisfying the relations given by (25). In order to make these relations and the composite of ∞ -morphisms fully explicit, one needs a neat description of the decomposition map of the coproperad DPois^i . We refer the reader to Appendix A for all the details.

Definition 3.6 (∞ -isotopy, ∞ -isomorphism, and ∞ -quasi-isomorphism). An ∞ -isotopy (resp. ∞ -isomorphism, ∞ -quasi-isomorphism) is an ∞ -morphism whose first component f_1 is the identity (resp. isomorphism, quasi-isomorphism).

Proposition 3.7. *The invertible ∞ -morphisms are the ∞ -isomorphisms.*

Proof. This is a special case of [HLV20, Theorem 3.22] applied to the Koszul dual coproperad DPois^i . \square

Recall that the *Deligne groupoid* $\text{Del}(\mathfrak{g})$ of a complete Lie algebra \mathfrak{g} is made up of the Maurer–Cartan elements for objects and the gauges for isomorphisms. We refer the reader to [DSV22, Chapter 1] for more details.

Proposition 3.8. *The Deligne groupoid associated to the convolution algebra of homotopy double Poisson gebras on A is isomorphic to the groupoid of homotopy double Poisson gebra structures on A with their ∞ -isotopies:*

$$\text{Del}\left(\widehat{\text{Hom}}\left(\text{DPois}^i, \text{End}_A\right)\right) \cong (\text{DPois}_\infty\text{-gebras}, \infty\text{-isotopies}) .$$

Proof. This is a special case of [CV22, Theorem 2.22] applied to the Koszul dual coproperad DPois^i . \square

Theorem 3.9 (Homological invertibility of ∞ -quasi-isomorphisms). *For any ∞ -quasi-isomorphism $f : A \rightsquigarrow B$ of homotopy double Poisson gebras, there exists an ∞ -quasi-isomorphism $g : B \rightsquigarrow A$ whose first component induces the homology inverse of the first component of f .*

Proof. This is a special case of [HLV20, Theorem 4.18] applied to the Koszul dual coproperad DPois^i . \square

Recall that a *contraction* of a dg vector space (A, d_A) is another dg vector space (H, d_H) equipped with chain maps i and p and a homotopy h of degree 1

$$h \circlearrowleft (A, d_A) \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} (H, d_H) ,$$

satisfying

$$pi = \text{id}_H , \quad ip - \text{id}_A = d_A h + h d_A , \quad hi = 0 , \quad ph = 0 , \quad \text{and} \quad h^2 = 0 .$$

Theorem 3.10 (Homotopy Transfer Theorem for homotopy double Poisson gebras). *For any contraction of dg vector space A and any homotopy double Poisson gebra structure on A , there exists a homotopy double Poisson gebra structure on H and extensions of the chain maps i and p into ∞ -quasi-isomorphisms.*

Proof. This is a special case of [HLV20, Theorem 4.14] applied to the Koszul dual coproperad DPois^i . \square

Notice that *loc. cit.* provides us with explicit formulas for the transferred structure and the extensions of the two maps. For instance, the homotopy transferred structure is given by

$$\overline{\text{DPois}}^i \xrightarrow{\tilde{\Delta}} \mathcal{G}^c\left(\overline{\text{DPois}}^i\right) \xrightarrow{\mathcal{G}^c(s\alpha)} \text{BEnd}_A \xrightarrow{\varphi} \text{End}_H ,$$

where $\tilde{\Delta}$ is the comonadic decomposition map of the coproperad DPois^i made up of all the ways to decompose its elements, where α is the initial homotopy double Poisson algebra structure on A , and where φ is the Van der Laan twisting morphism which amounts to labelling the edges of the graphs by the maps i , p , and h of the contraction in a levelled way, see [HLV20, Theorem 4.7] for more details.

In a forthcoming paper [HLV22], we go further and establish even more structural properties for ∞ -morphisms associated to Koszul properads like the following formality properties that automatically apply to the case of homotopy double Poisson gebras.

Proposition 3.11. *Two homotopy double Poisson gebra structures α and α' on the same underlying chain complex A are ∞ -isotopic if and only if they are related by a zig-zag of (strict) quasi-isomorphisms of homotopy double Poisson gebras:*

$$\begin{array}{ccc} \exists \infty\text{-isotopy} & & \exists \text{zig-zag of quasi-isomorphisms} \\ (A, \alpha) \rightsquigarrow (A, \alpha') & \iff & (A, \alpha) \rightsquigarrow \cdot \xleftarrow{\sim} \cdot \cdots \cdot \xleftarrow{\sim} \cdot \rightsquigarrow (A, \alpha') . \end{array}$$

Proof. This is a special case of [HLV22, Proposition 1.10] applied to the Koszul dual coproperad DPois^i . \square

Theorem 3.12. *Two homotopy double Poisson gebra structures (A, α) and (B, β) are ∞ -quasi-isomorphic if and only if they are related by a zig-zag of (strict) quasi-isomorphisms of homotopy double Poisson gebras:*

$$\begin{array}{ccc} \exists \infty\text{-quasi-isomorphism} & & \exists \text{zig-zag of quasi-isomorphisms} \\ (A, \alpha) \rightsquigarrow (B, \beta) & \iff & (A, \alpha) \xrightarrow{\sim} \cdot \xleftarrow{\sim} \cdot \cdots \cdot \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B, \beta) . \end{array}$$

Proof. This is a special case of [HLV22, Theorem 1.11] applied to the Koszul dual coproperad DPois^i . \square

The results of [HLV22] will also provide us with a simplicial enrichment of the category of homotopy double Poisson gebras and ∞ -morphisms such that the associated homotopy category coincides with the localisation with respect to ∞ -quasi-isomorphisms.

We finish this section with an application of the general theory of Koszul hierarchy given in [CV22, Section 3.3] which provides us here with a way to construct homotopy double Poisson gebras from shifted $\text{DPois}^!$ -gebra structures via universal formulas. The notion of a *shifted $\text{DPois}^!$ -gebra* is defined as a $\text{DPois}^!$ -gebra (Section 1.4) except that we require the generating binary product and “dual” double bracket to be of homological degree +1 instead of -1 . Recall that the original construction, given by J.-L. Koszul in [Kos85], produces a (shifted) homotopy Lie algebra from a commutative product on a dg vector space.

Theorem 3.13 (Koszul hierarchy). *Given a shifted $\text{DPois}^!$ -algebra structure on graded vector space A and a differential d on A , the action of the associated canonical element of the gauge group on the trivial double Poisson gebra structure on (A, d) produces a homotopy double Poisson gebra.*

Proof. This is a special case of [CV22, Section 3.3] applied to the Koszul dual coproperad DPois^i , but let us give it more details. Under the notation $\text{sDPois}^!$ for the properad encoding shifted $\text{DPois}^!$ -gebras, a structure of a gebra over it amounts to a morphism of properads $\Theta : \text{sDPois}^! \rightarrow \text{End}_A$. Identifying the dual treewise stairways basis given in Section 1.4, we get a canonical isomorphism of graded \mathbb{S} -bimodules $\overline{\text{DPois}}^i \cong \overline{\text{sDPois}}^!$ and thus a canonical element $\mathbb{1} + \theta$ of the gauge group integrating the dg Lie algebra $\mathfrak{D}\mathfrak{Pois}_A$, where

$$\theta : \overline{\text{DPois}}^i \cong \overline{\text{sDPois}}^! \xrightarrow{\Theta} \text{End}_A .$$

This gauge action, producing a homotopy double Poisson gebra structure by definition, is given by the explicit formula

$$(\mathbb{1} + \theta) \overset{d}{\bowtie} (\mathbb{1} + \theta)^{-1} ,$$

see *loc. cit.* for more details. \square

Notice that we do *not* require the shifted $\text{DPois}^!$ -gebra structure to be compatible with the differential. This construction applied to a shifted associative algebra viewed as a shifted $\text{DPois}^!$, with trivial dual double bracket, provides us with the A_∞ -algebra of K . Borjeson [Bor15].

3.3. Infinity-morphisms of curved homotopy double Poisson gebras. In this section, we extend the operadic calculus of [HLV20, CV22] along the lines of [DSV22] to handle the case of curved gebras. The first mandatory move is to work over the category of *complete* graded vector spaces in order to handle the infinite sums that will appear in this case.

REMARK 3.14. We write this section on the level of the convolution algebra $c\mathcal{D}\text{Pois}_A \cong \mathfrak{h}h_A$, but everything holds *mutatis mutandis* on the level of the curvature necklace Lie-admissible algebra $c\text{nect}_A$, when A is not necessarily degree-wise finite dimensional. As a consequence, all the results given here hold on for curved pre-Calabi–Yau algebras.

Let us recall that a graded vector space A is *complete* when it is equipped with a degree-wise decreasing filtration

$$A_n = F_0 A_n \supseteq F_1 A_n \supseteq F_2 A_n \supseteq \cdots \supseteq F_k A_n \supseteq F_{k+1} A_n \supseteq \cdots$$

of sub-vector spaces such that the associated topology is complete, i.e. $A_n \cong \lim_{k \rightarrow \infty} A_n / F_k A_n$. The category of complete graded vector space is made up of maps which preserve the respective filtrations. Equipped with the complete tensor product $\widehat{\otimes}$, it forms a symmetric monoidal category, where one can perform properadic calculus, see [DSV22, Chapter 2] for more details. For instance, its complete endomorphism properad is denoted by end_A .

On the opposite to the Koszul dual coproperad DPois^i which encodes homotopy double Poisson gebras, the partial coproperad $c\text{DPois}^i$ which encodes curved homotopy double Poisson gebras fails to form a coproperad: one can iterate its infinitesimal decomposition map in order to form two-levels graphs, but the upshot will be made up of infinite series and not finite sums. As a consequence the operations Δ and $\Delta_{(*)}$ are not well-defined on the level of $c\text{DPois}^i$. However, the global operations \odot , \triangleleft and \triangleright , defined by the same overall formulas, make sense as soon as one works with complete graded modules A, B , and with elements f satisfying $f_0(1) \in F_1 A$.

Definition 3.15 (∞ -morphism of complete curved homotopy double Poisson gebras). Let A, B be complete graded vector spaces equipped respectively with two curved homotopy double Poisson gebra structures $\alpha \in \text{MC}(\widehat{\text{Hom}}(c\text{DPois}^i, \text{end}_A))$ and $\beta \in \text{MC}(\widehat{\text{Hom}}(c\text{DPois}^i, \text{end}_B))$. An ∞ -morphism $f: (A, \alpha) \rightsquigarrow (B, \beta)$ is a degree 0 map of complete \mathbb{S} -bimodules $f: c\text{DPois}^i \rightarrow \text{end}_B^A$ such that $f_0(1) \in F_1 B$ and satisfying the equation

$$(26) \quad \partial(f) = f \triangleright \alpha - \beta \triangleleft f .$$

The *composite* of ∞ -morphisms is defined by $g \odot f$.

Proposition 3.16. *Complete curved homotopy double Poisson gebras equipped with their ∞ -morphisms and the composite \odot form a category.*

Proof. This proof is similar to that of Proposition 3.5 once one checks that all the terms make sense thanks to the completion assumption. \square

By definition, an ∞ -morphism of complete curved homotopy double Poisson gebras is a collection of filtration preserving maps

$$f_{\lambda_1, \dots, \lambda_m} : A^{\widehat{\otimes} n} \longrightarrow B^{\widehat{\otimes} m}$$

of degree $n - 1$, for any ordered partition $\lambda_1 + \cdots + \lambda_m = n$ of any $n \geq 0$ into non-negative integers, satisfying the relations given by (26). The combinatorics given in Appendix A, considered under the weaker assumption $\lambda_1, \dots, \lambda_m \geq 0$, provide us with explicit descriptions of these relations and the composite of ∞ -morphisms.

We conclude the present study with the twisting procedure, which is a universal way to produce new curved homotopy double Poisson gebras from the data of one element.

Theorem 3.17 (Twisting procedure). *Given any complete curved homotopy double Poisson algebra structure $\{m_{\lambda_1, \dots, \lambda_m}\}_{\lambda_1, \dots, \lambda_m \geq 0}$ on A and any element $a \in F_1 A_{-1}$, the sum of all the insertions of a at any possible*

inputs forms a complete curved homotopy double Poisson algebra:

$$m_{\lambda_1, \dots, \lambda_m}^a := \sum_{\substack{r_0^1, \dots, r_{\lambda_1}^1 \geq 0 \\ \dots \\ r_0^m, \dots, r_{\lambda_m}^m \geq 0}} (-1)^\eta m_{\lambda_1+r^1, \dots, \lambda_m+r^m} \left(a^{r_0^1}, -, a^{r_1^1}, -, \dots, -, a^{r_{\lambda_1}^1}, \dots, a^{r_0^m}, -, a^{r_1^m}, -, \dots, -, a^{r_{\lambda_m}^m} \right),$$

where $r^j := r_0^j + \dots + r_{\lambda_j}^j$, for $1 \leq j \leq m$, and where

$$\eta = rn + \frac{r(r-1)}{2} + \sum_{j=1}^{m-1} r^j \left(1 + \sum_{k=j+1}^m \lambda_k + r^k \right) + \sum_{j=1}^m \left(\sum_{i=0}^{\lambda_j} (r_i^j + 1) (r_0^j + \dots + r_{i-1}^j) + \frac{r_i^j (r_i^j - 1)}{2} \right),$$

with $r := r^1 + \dots + r^m$.

Proof. This is a direct corollary of [CV22, Section 3.2]. Let us denote by $\alpha \in \mathfrak{cD}\mathfrak{Pois}_A$ the Maurer–Cartan element corresponding to the original complete curved homotopy double Poisson algebra. Its image under the action of the element $\mathbb{1} - a$ of the gauge group integrating the convolution dg Lie algebra $\mathfrak{cD}\mathfrak{Pois}_A$ is given by

$$(\mathbb{1} - a) \cdot \alpha = (\mathbb{1} - a) \bowtie^\alpha (\mathbb{1} + a) = \alpha \odot (\mathbb{1} + a),$$

with the notations of *loc. cit.*. The right-hand side produces the formula for the twisted structure claimed in the statement. The sign is computed as in the proof of Proposition 1.34. \square

Notice that the original and the twisted complete curved homotopy double Poisson algebras are gauge equivalent by (conceptual) definition.

APPENDIX A. COMBINATORICS OF THE DECOMPOSITION MAPS

In order to make explicit the definition and the composition of ∞ -morphisms of (curved) homotopy double Poisson algebras (Section 3), we need to describe the decomposition map $\Delta: \text{DPois}^i \rightarrow \text{DPois}^i \boxtimes \text{DPois}^i$ of the Koszul dual coproperad. Since it is given by a distributive law (Proposition 1.28), let us start with the decomposition map of the coproperad DLie^i . The cyclic symmetry of its basis elements (Lemma 1.31) suggests to consider *bangles with n beads*, for any $n \leq 2$.

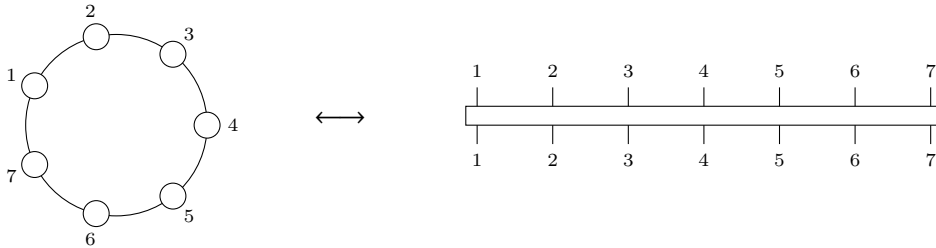


FIGURE 6. Bangle with 7 beads and the corresponding basis element of DLie^1 .

Definition A.1 (Elementary coloured cutting). An *elementary coloured cutting* of a bangle is defined by the following two-steps construction.

CUTTING: choose a bead of the bangle and cut the bangle into two parts, that is draw a line starting from the bead, splitting it into two, to an edge between two beads, such that each half-bangle contains at least one bead.

COLOURING: colour the beads on the clockwise side of the half-bean in white and the other ones, as well as the entire sector, in black, see Figure 7.

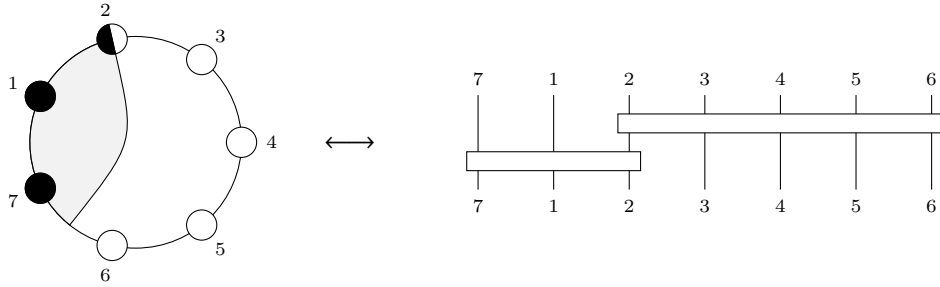


FIGURE 7. Elementary coloured cutting and the infinitesimal decomposition of $DLie^i$.

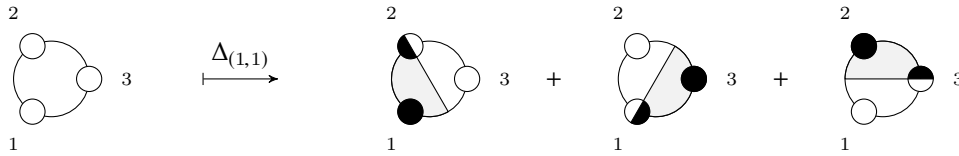
Lemma A.2. *The terms appearing in the infinitesimal decomposition map of the coproperad $DLie^i$ are in one-to-one correspondance with the elementary coloured cuttings of bangles.*

Proof. The images of the basis elements $v_{1,\dots,1}$ under the infinitesimal decomposition map

$$\Delta_{(1,1)}: DLie^i \rightarrow DLie^i \boxtimes_{(1,1)} DLie^i$$

are given by 2-steps cyclic stairways, see Proposition 1.34. Given an elementary coloured cutting of a bangle, we read it clockwise from the first full black bead: the black beads are inputs of a bottom basis element, the white beads are the inputs of the a top basis element, and the black-and-white bead corresponds to the attaching edge, see Figure 7. \square

EXAMPLE A.3 (3-BANGLE). The example of the 3-bangle gives the double Jacobi relation.



After an elementary coloured cutting, one can interpret the white part and the black part has proper bangles. This way, one can iterate elementary coloured cuttings such that half-beads cannot be further cut and such that the end of the new cut does not land on the same arc as a previous one.

Definition A.4 (Partitioned bangle). A bangle is *partitioned* when it is equipped with a non-negative number of iterations of elementary coloured cuttings. When there is no cutting, one can either paint all the vertices in white or in black. In other words, such a data amounts to a bangle partitioned into alternating black and white sections delimited by lines which start from a bead and lead to an arc such that each arc can receive at most one lines and such that the clockwise side of each half-bean is coloured in white, see Figure 8.

Proposition A.5. *The various terms appearing in the decomposition map of the coproperad $DLie^i$ are in one-to-one correspondance with partitioned bangles.*

Proof. Let us pursue the identification initiated in the above proof of Lemma A.2. To any partitioned bangle, we associate a 2-level connected graph as follows. Any black (respectively white) sector gives a cyclic bottom (respectively top) box with inputs and outputs labeled from left to right according to the

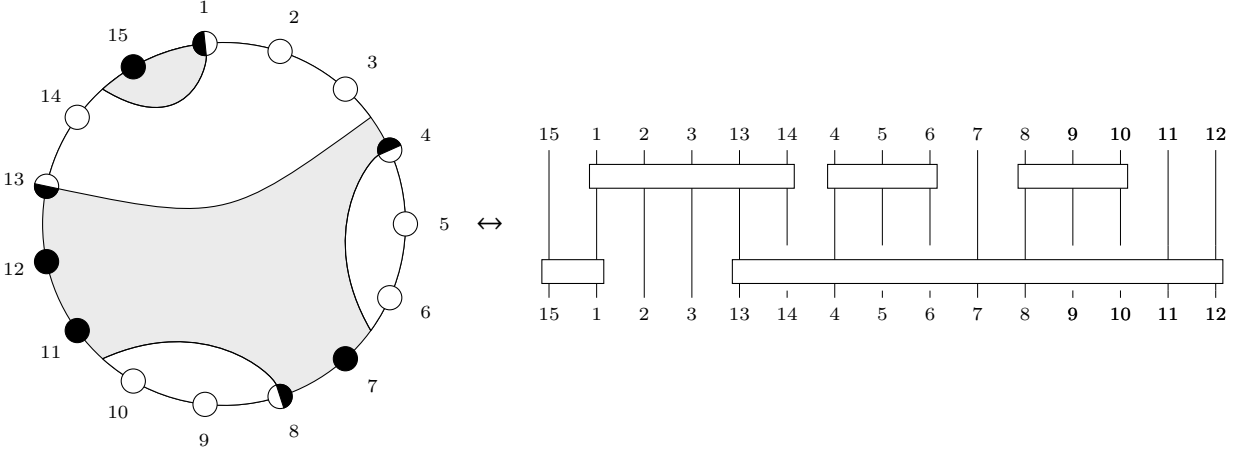


FIGURE 8. Partitioned bangle and the decomposition of DLie^1 .

clockwise order; we connect bottom and top boxes along the edges corresponding to black-and-white beads, see Figure 8 for an example. The bangle trivially partitioned with all of the beads coloured in black (respectively in white) gives the biggest bottom (respectively top) box with no top (respectively bottom) boxes. Under this map, the number of elementary coloured cuttings corresponds to internal edges.

Let us recall that the images $\Delta(v_{1,\dots,1})$ of the basis elements of DLie^1 under the decomposition map are made up of the 2-level connected graphs whose (dual) composite in the properad DLie^1 gives $v_{1,\dots,1}^*$ up to a sign, that is stairways with cyclically ordered labels. By a straightforward induction on the number $n \geq 0$ of elementary coloured cuttings, one can see that they are all obtained as images of partitioned bangles under the above-mentioned assignment. This is trivial for $n = 0$ and obvious for $n = 1$, by Lemma A.2. Suppose that this property holds true up to n and consider a bangle partitioned with $n + 1$ elementary coloured cuttings, that is into $n + 2$ sectors. We choose an external sector, that is a sector with consecutive beads, and we remove it from the bangle. By the induction hypothesis, the image of the remaining partitioned bangle under the above assignment produces a 2-level connected graph whose composite in the Koszul dual properad DLie^1 gives a stairway with leaves labeled cyclically in the same way. Then, Lemma A.2 shows that the image of the full bangle gives an element appearing in the image of the decomposition map. In the other way round, given a 2-level connected graph with $n + 1$ internal edge present in the image of the decomposition map, we consider similarly an extremal bottom or top box, that is one attached to the rest of the graph with just one edge. The 2-level connected graph obtained by removing it can be produced by a partitioned bangle labeled cyclically with the remaining indices of the leaves, thanks to the induction hypothesis. Finally Lemma A.2 shows that pruning the extremal box on the 2-level graph amounts to adding an external sector to this bangle. This concludes the proof. \square

Definition A.6 (Hairy partitioned bangle). A *hairy partitioned bangle* is a partitioned bangle equipped with the following treewise decorations.

AT BLACK BEADS: at least one corolla situated outside the the bangle.

AT WHITE BEADS: corollas outside and at least one leaf inside the bangle.

AT BLACK-AND-WHITE BEADS: corollas outside the bangle, corollas inside the black sector, and at least one leaf inside the white sector.

Black sectors are now allowed to have no full black bead provided that the number of their leaves (from the black-and-white bead) plus the number of outside leaves attached to the bead before in the cyclic order is greater or equal to one. The leaves receive indices which are altogether clockwise cyclically ordered: we read the leaves of white and black-and-white beads starting from the outer ones and pursuing with the inner ones, see Figure 9 for an example.

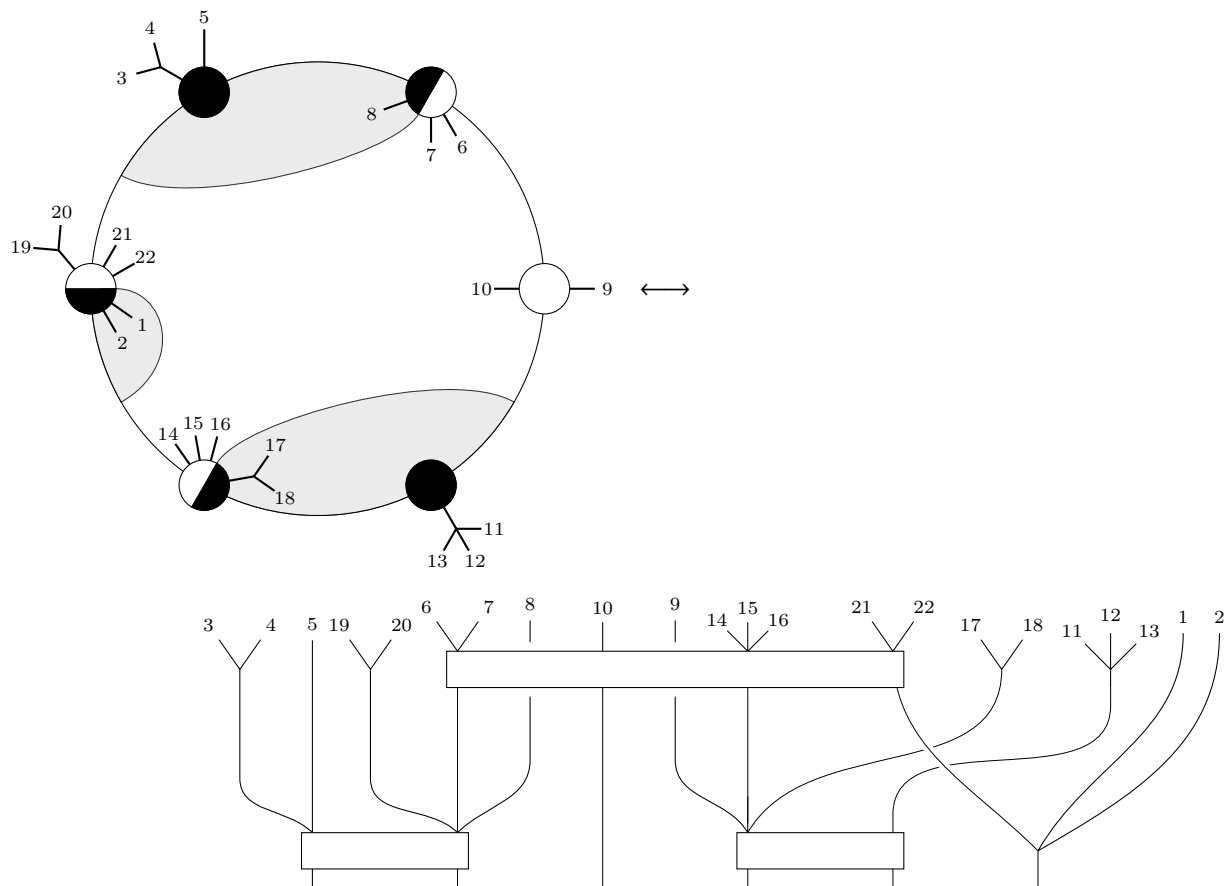


FIGURE 9. Hairy partitioned bangle and 2-level decomposition of DPois^i .

Proposition A.7. *The various terms appearing in the decomposition map of the coproperad DPois^i are in one-to-one correspondance with hairy partitioned bangles.*

Proof. We extend the above assignment to hairy partitioned bangles. The underlying partitioned bangle produces a 2-level connected graph. The corollas of the black beads are put on the top level. The leaves of the white sectors are the inputs of the corresponding top boxes. The corollas of the black-and-white beads situated in the black sector form the inputs of the bottom boxes situated at the right-hand side of the corresponding vertical edge but they are put on the top level. The outer leaves of a bead form the bottom box inputs situated on the left-hand side of the vertical edge corresponding to the next bead of this white sector; again they extend to the top level. The rest of the proof is a straightforward corollary of Proposition A.5 and the distributive law Proposition 1.28. \square

The composite $g \circledast f$ of ∞ -morphisms is given by the sum over all the hairy partitioned bangles of the composite of the associated 2-level connected directed graphs with upper maps labeled by f

and the lower maps labeled by g . The operator \triangleleft appearing in the relation (25) of an ∞ -morphism is given by hairy partitioned bangles with one black sector. The operator \triangleright is given by hairy partitioned bangles with either one white sector and no corolla of arity greater or equal to two on black beads or no white sector and a total number of one corolla of arity greater or equal to two on black beads.

APPENDIX B. PROOF OF THE MAIN THEOREM

This appendix contains the proof of the main result (Theorem 2.43) of the present paper, that is the existence of a canonical and functorial isomorphism of dg Lie-admissible algebras

$$c\mathcal{D}\text{Pois}_A \cong \mathfrak{h}\mathfrak{h}_A .$$

Proof of Theorem 2.43. The isomorphism $\Upsilon : c\mathcal{D}\text{Pois}_A \rightarrow \mathfrak{h}\mathfrak{h}_A$ from left to right is given by the following assignment:

$$\Upsilon(\bar{F})_{\lambda_1, \dots, \lambda_m} : sa_1 \otimes \dots \otimes sa_n \mapsto (-1)^{|\bar{F}| + \nabla(a_1, \dots, a_n) + \lambda_m + \frac{n(n+1)}{2}} s\bar{F}(v_{\lambda_1, \dots, \lambda_m})(a_1 \otimes \dots \otimes a_n) ,$$

for $\bar{F} : c\mathcal{D}\text{Pois}^i \rightarrow \text{End}_A$, where $v_{\lambda_1, \dots, \lambda_m}$ denotes the basis elements of the coproperad $c\mathcal{D}\text{Pois}^i$, and where

$$\nabla(a_1, \dots, a_n) := (n-1)|a_1| + (n-2)|a_2| + \dots + 2|a_{n-2}| + |a_{n-1}| .$$

First, we check that this map is equivariant with respect to the actions of the cyclic groups; so it will induce a well-defined map on the level of invariant elements. Under the notation,

$$\bar{F}(v_{\lambda_1, \dots, \lambda_m})(a_1 \otimes \dots \otimes a_n) = \sum b_1 \otimes \dots \otimes b_m ,$$

we get

$$\left(\tau_m^{-1} \cdot \Upsilon(\bar{F})_{\lambda_1, \dots, \lambda_m} \cdot \tau_{\lambda_1, \dots, \lambda_m} \right) (sa_{\lambda_1+1} \otimes \dots \otimes sa_n \otimes sa_1 \otimes \dots \otimes sa_{\lambda_1}) = \sum (-1)^A s b_2 \otimes \dots \otimes b_m \otimes b_1 ,$$

with

$$A = (|a_1| + \dots + |a_{\lambda_1}| + \lambda_1)(|a_{\lambda_1+1}| + \dots + |a_n| + n - \lambda_1) + |\bar{F}| + \nabla(a_1, \dots, a_n) + \lambda_m + \frac{n(n+1)}{2} + |b_1|(|b_2| + \dots + |b_m|) .$$

On the other hand, since the action of the cyclic groups on $\text{Hom}(c\mathcal{D}\text{Pois}^i, \text{End}_A)$ is given by conjugation, we have

$$\Upsilon(\bar{F} \cdot \tau_m)_{\lambda_2, \dots, \lambda_m, \lambda_1} (sa_{\lambda_1+1} \otimes \dots \otimes sa_n \otimes sa_1 \otimes \dots \otimes sa_{\lambda_1}) = \sum (-1)^B s b_2 \otimes \dots \otimes b_m \otimes b_1 ,$$

with

$$B = |\bar{F}| + \nabla(a_{\lambda_1+1}, \dots, a_n, a_1, \dots, a_{\lambda_1}) + \lambda_1 + \frac{n(n+1)}{2} + n\lambda_1 + \lambda_m + (|a_1| + \dots + |a_{\lambda_1}|)(|a_{\lambda_1+1}| + \dots + |a_n|) + |b_1|(|b_2| + \dots + |b_m|) ,$$

after the symmetry relation (7). Since $A \equiv B \pmod{2}$, this concludes that part of the proof.

Lemmata 1.33 and 1.39 show that

$$\Upsilon : \widehat{\text{Hom}}(c\mathcal{D}\text{Pois}^i, \text{End}_A) \xrightarrow{\cong} \prod_{N \geq 1} \left(\bigoplus_{1 \leq m < N} \text{Hom}_{\mathbb{C}_m} \left(\bigoplus_{\lambda_1 + \dots + \lambda_m = n} \bigotimes_{j=1}^m (sA)^{\otimes \lambda_j}, A^{\otimes m} \right) \right)$$

is indeed an isomorphism. It remains to check that it preserves the respective Lie-admissible products, that is

$$\Upsilon(\bar{F} \star \bar{G}) = \Upsilon(\bar{F}) \otimes \Upsilon(\bar{G}) .$$

Comparing the formulas given in Proposition 1.34 and in Definition 2.41, one can see that both $\Upsilon(\bar{F} \star \bar{G})$ and $\Upsilon(\bar{F}) \otimes \Upsilon(\bar{G})$ will be made up of the same terms, involving the same composites, possibly up to a sign.

For one of these terms, we will compute the signs appearing in the two formulas. Let $\lambda_1 + \dots + \lambda_m = n$ and $\lambda'_1 + \dots + \lambda'_{m'} = n'$ be two ordered partitions and let $p + 1 + q = \lambda_m$, with $p, q \geq 0$. For any elements $a_1, \dots, a_{\lambda_m+p}^{\zeta}, a_{\lambda_m+p+2}^{\zeta}, \dots, a_n, a'_1, \dots, a'_{n'} \in A$, we consider the following notations

$$\begin{aligned} \bar{G}(v_{\lambda_1, \dots, \lambda'_{m'}})(a'_1 \otimes \dots \otimes a'_{n'}) &= \sum b'_1 \otimes \dots \otimes b'_{m'} \quad \text{and} \\ \bar{F}(v_{\lambda_1, \dots, \lambda_m})(a_1 \otimes \dots \otimes a_{\lambda_m+p}^{\zeta} \otimes b'_1 \otimes a_{\lambda_m+p+2}^{\zeta} \otimes \dots \otimes a_n) &= \sum b_1 \otimes \dots \otimes b_m. \end{aligned}$$

On the one hand, we have

$$\begin{aligned} \left(\Upsilon(\bar{F})_{\lambda_1, \dots, \lambda_{m-1}} \otimes_i^1 \Upsilon(\bar{G})_{\lambda'_1, \dots, \lambda'_{m'}} \right) &\left(sa_1 \otimes \dots \otimes sa_{\lambda_m}^{\zeta} \otimes sa'_1 \otimes \dots \otimes sa'_{\lambda'_1} \otimes sa_{\lambda_m+p+2}^{\zeta} \otimes \dots \otimes sa_n \otimes \right. \\ &\left. sa'_{\lambda'_1+1} \otimes \dots \otimes a'_{\lambda'_{m'}} \otimes sa_{\lambda_{m+1}}^{\zeta} \otimes \dots \otimes sa_{\lambda_{m+p}}^{\zeta} \otimes sa'_{\lambda'_{m'}+1} \otimes \dots \otimes sa'_{n'} \right) = \\ &\sum (-1)^C s b_1 \otimes \dots \otimes b_m \otimes b'_2 \otimes \dots \otimes b'_{m'}, \end{aligned}$$

where $i = \tilde{\lambda}_m + p + 1$ and where

$$\begin{aligned} C &= \left(|a_{\lambda_m+p+2}^{\zeta}| + \dots + |a_n| + q \right) \left(|a'_1| + \dots + |a'_{\lambda'_1}| + \lambda'_1 \right) \\ &+ \left(|a_{\lambda_{m+1}}^{\zeta}| + \dots + |a_{\lambda_{m+p}}^{\zeta}| + p \right) \left(|a_{\lambda_m+p+2}^{\zeta}| + \dots + |a_n| + |a'_1| + \dots + |a'_{\lambda'_{m'}}| + q + \tilde{\lambda}'_{m'} \right) \\ &+ \bar{G}(|a| + n - 1) + |\bar{G}| + \nabla(a'_1, \dots, a'_{n'}) + \lambda'_m + \frac{n'(n'+1)}{2} \\ &+ (|b'_1| + 1) \left(|a_{\lambda_m+p+2}^{\zeta}| + \dots + |a_n| + q \right) \\ &+ |\bar{F}| + \nabla(a_1, \dots, a_{\lambda_m+p}^{\zeta}, b'_1, a_{\lambda_m+p+2}^{\zeta}, \dots, a_n) + \lambda_m + \frac{n(n+1)}{2}, \end{aligned}$$

under the convention $a := a_1 \otimes \dots \otimes a_{\lambda_m+p}^{\zeta} \otimes a_{\lambda_m+p+2}^{\zeta} \otimes \dots \otimes a_n$. In plain words, one starts by permuting the elements sa_j and sa'_k in order to have the first ones on the left-hand side of the second ones (first two lines of C). Then, one permutes $\Upsilon(\bar{G})_{\lambda'_1, \dots, \lambda'_{m'}}$ with the elements sa_j and one applies it to the elements sa'_k (third line of C). Finally, one permutes the output element sb'_1 with the last q elements sa_j (forth line of C) and one applies $\Upsilon(\bar{F})_{\lambda_1, \dots, \lambda_{m-1}}$ to the elements sa_j and b'_1 (fifth line of C).

On the other hand, we have

$$\begin{aligned} \Upsilon(\bar{F} \star \bar{G})_{\lambda_1, \dots, \lambda_{m-1}, \lambda'_1+q, \lambda'_2, \dots, \lambda'_{m'-1}, p+\lambda'_{m'}} &\left(sa_1 \otimes \dots \otimes sa_{\lambda_m}^{\zeta} \otimes sa'_1 \otimes \dots \otimes sa'_{\lambda'_1} \otimes \right. \\ &\left. sa_{\lambda_m+p+2}^{\zeta} \otimes \dots \otimes sa_n \otimes sa'_{\lambda'_1+1} \otimes \dots \otimes a'_{\lambda'_{m'}} \otimes sa_{\lambda_{m+1}}^{\zeta} \otimes \dots \otimes sa_{\lambda_{m+p}}^{\zeta} \otimes sa'_{\lambda'_{m'}+1} \otimes \dots \otimes sa'_{n'} \right) \end{aligned}$$

is equal to

$$\sum (-1)^D s b_1 \otimes \dots \otimes b_m \otimes b'_2 \otimes \dots \otimes b'_{m'},$$

with

$$\begin{aligned}
D = & |\bar{F}| + |\bar{G}| + \\
& \nabla \left(a_1, \dots, a_{\lambda_m}^{\zeta}, a'_1, \dots, a'_{\lambda'_1}, a_{\lambda_{m+p+2}}^{\zeta}, \dots, a_n, a'_{\lambda'_{1+1}}, \dots, a'_{\lambda'_{m'}}, a_{\lambda_{m+1}}^{\zeta}, \dots, a_{\lambda_{m+p}}^{\zeta}, a'_{\lambda'_{m'+1}}, \dots, a'_{n'} \right) + \\
& p + \lambda'_m + \frac{(n+n'-1)(n+n')}{2} + \\
& \left(|a_{\lambda_{m+p+2}}^{\zeta}| + \dots + |a_n| \right) \left(|a'_1| + \dots + |a'_{\lambda'_1}| \right) + \\
& \left(|a_{\lambda_{m+1}}^{\zeta}| + \dots + |a_{\lambda_{m+p}}^{\zeta}| \right) \left(|a_{\lambda_{m+p+2}}^{\zeta}| + \dots + |a_n| + |a'_1| + \dots + |a'_{\lambda'_{m'}}| \right) + \\
& q\lambda'_1 + p \left(q + \tilde{\lambda}'_{m'} \right) + (n-1)(n'-1) + |\bar{G}|(n-1) + \\
& (\bar{G} + n' - 1) |a| + |b'_1| \left(|a_{\lambda_{m+p+2}}^{\zeta}| + \dots + |a_n| \right) .
\end{aligned}$$

In plain words, the first signs (first three lines of D) come from the application of

$$\Upsilon(\bar{F} \star \bar{G})_{\lambda_1, \dots, \lambda_{m-1}, \lambda'_1+q, \lambda'_2, \dots, \lambda'_{m'-1}, p+\lambda'_{m'}} ,$$

which implies to unshifting all the elements sa_j and sa'_k for instance. As above, one permutes the elements a_j and a'_k in order to have the first ones on the left-hand side of the second ones (forth and fifth lines of D). Then, one computes the corresponding term in $\bar{F} \star \bar{G}$: first one decomposes the element $v_{\lambda_1, \dots, \lambda_{m-1}, \lambda'_1+q, \lambda'_2, \dots, \lambda'_{m'-1}, p+\lambda'_{m'}}$ in the partial coproperad $\text{cDPois}^!$ according to Proposition 1.34 (first three terms of the sixth line of D) and then one applies \bar{G} to the above element $v_{\lambda'_1, \dots, \lambda'_{m'}}$, thereby permuting \bar{G} with the bottom element $v_{\lambda_1, \dots, \lambda_m}$ (last term of the sixth line of D). It remains to permute the map $\bar{G}(v_{\lambda'_1, \dots, \lambda'_{m'}})$ with the elements a_j and to apply it to the elements a'_k (first term of the seventh line of D). Finally, one permutes the output element b'_1 with the last q elements a_j (last term of the seventh line of D) and one applies $\bar{F}(v_{\lambda_1, \dots, \lambda_m})$ to the elements a_j and b'_1 , which produces no extra sign.

Notice first that

$$(27) \quad \frac{n(n+1)}{2} + \frac{n'(n'+1)}{2} \equiv \frac{(n+n'-1)(n+n')}{2} + (n-1)(n'-1) + 1 \pmod{2} .$$

Then we compute the difference between the ∇ terms appearing in C and in D :

$$\begin{aligned}
& \nabla \left(a_1, \dots, a_{\lambda_m}^{\zeta}, a'_1, \dots, a'_{\lambda'_1}, a_{\lambda_{m+p+2}}^{\zeta}, \dots, a_n, a'_{\lambda'_{1+1}}, \dots, a'_{\lambda'_{m'}}, a_{\lambda_{m+1}}^{\zeta}, \dots, a_{\lambda_{m+p}}^{\zeta}, a'_{\lambda'_{m'+1}}, \dots, a'_{n'} \right) - \\
& \nabla \left(a_1, \dots, a_{\lambda_{m+p}}^{\zeta}, b'_1, a_{\lambda_{m+p+2}}^{\zeta}, \dots, a_n \right) - \nabla \left(a'_1, \dots, a'_{n'} \right) \\
= & (n'-1) \left(|a_1| + \dots + |a_{\lambda_m}^{\zeta}| \right) + (\lambda'_{m'} - q - 1) \left(|a_{\lambda_{m+1}}^{\zeta}| + \dots + |a_{\lambda_{m+p}}^{\zeta}| \right) \\
& + (n' - \lambda'_1 + p) \left(|a_{\lambda_{m+p+2}}^{\zeta}| + \dots + |a_n| \right) + q \left(|a'_1| + \dots + |a'_{\lambda'_1}| \right) + p \left(|a'_1| + \dots + |a'_{\lambda'_{m'}}| \right) - q |b'_1| .
\end{aligned}$$

So the difference, computed modulo 2, of all the terms in C and D of the form $|a_j|$ is equal to $(n'-1)|a|$, which matches with the one present in D . It remains to check the combinatorial terms which do not depend on the degrees of the elements a_j and a'_k : after removing the same terms which obviously appear both in C and in D , we get

$$q\lambda'_1 + p \left(q + \tilde{\lambda}'_{m'} \right) + q + \lambda_m$$

in C and

$$p + q\lambda'_1 + p(q + \tilde{\lambda}'_{m'}) + 1$$

in D , where the final term 1 comes from (27). Since $\lambda_m = p + q + 1$, we get $C \equiv D \pmod{2}$, which concludes the proof. \square

REFERENCES

- [BKR13] Yuri Berest, George Khachatryan, and Ajay Ramadoss. Derived representation schemes and cyclic homology. *Adv. Math.*, 245:625–689, 2013.
- [Bor15] Kaj Borjeson. A_∞ -algebras derived from associative algebras with a non-derivation differential. *J. Gen. Lie Theory Appl.*, 9(1):5 p., 2015.
- [Bro09] Francis Brown. Multiple zeta values and periods of moduli spaces $\overline{\mathfrak{M}}_{0,n}$. In *Annales scientifiques de l'École normale supérieure*, volume 42, pages 371–489, 2009.
- [CPT⁺17] Damien Calaque, Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted Poisson structures and deformation quantization. *J. Topol.*, 10(2):483–584, 2017.
- [CV22] Ricardo Campos and Bruno Vallette. Graph exponential map. *to appear*, 2022.
- [DSV22] Vladimir Dotsenko, Sergey Shadrin, and Bruno Vallette. *Maurer–Cartan methods in deformation theory: the twisting procedure*. London Mathematical Society Lecture Note Series (Cambridge University Press), 2022.
- [DSVV20] Vladimir Dotsenko, Sergey Shadrin, Arkady Vaintrob, and Bruno Vallette. Deformation theory of cohomological field theories. *arXiv preprint arXiv:2006.01649*, 2020.
- [DV17] Clément Dupont and Bruno Vallette. Brown’s moduli spaces of curves and the gravity operad. *Geometry & Topology*, 21(5):2811–2850, 2017.
- [Fai22] Maxime Fairon. Double brackets in mathematics. <https://www.maths.gla.ac.uk/~mfairon/DoubleBrackets.html>, 2022.
- [FH20] David Fernández and Estanislao Herscovich. Double quasi-Poisson algebras are pre-Calabi-Yau. *arXiv preprint arXiv:2002.10495*, 2020.
- [FH21] David Fernández and Estanislao Herscovich. Cyclic A_∞ -algebras and double Poisson algebras. *J. Noncommut. Geom.*, 15(1):241–278, 2021.
- [FOOO09] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2009.
- [Gan03] Wee Liang Gan. Koszul duality for dioperads. *Math. Res. Lett.*, 10(1):109–124, 2003.
- [GK95] Ezra Getzler and Michail M. Kapranov. Cyclic operads and cyclic homology. In *Geometry, topology and physics for Raoul Bott. Lectures of a conference in honor of Raoul Bott’s 70th birthday, Harvard University, Cambridge, MA, USA 1993*, pages 167–201. Cambridge, MA: International Press, 1995.
- [GK98] Ezra Getzler and Michail M. Kapranov. Modular operads. *Compositio Math.*, 110(1):65–126, 1998.
- [HLV20] Eric Hoffbeck, Johan Leray, and Bruno Vallette. Properadic Homotopical Calculus. *International Mathematics Research Notices*, 05 2020. rnaa091.
- [HLV22] Eric Hoffbeck, Johan Leray, and Bruno Vallette. Simplicial properadic homotopy. *to appear*, 2022.
- [IK18] Natalia Iyudu and Maxim Kontsevich. Pre-Calabi-Yau algebras as noncommutative Poisson structures. *IHES Prepublication*, 2018.
- [IK19] Natalia Iyudu and Maxim Kontsevich. Pre-Calabi-Yau algebras and $\xi\partial$ -calculus on higher cyclic Hochschild cohomology. *IHES Prepublication*, 2019.
- [IKV19] Natalia Iyudu, Maxim Kontsevich, and Yannis Vlassopoulos. Pre-Calabi-Yau algebras and double Poisson brackets. *arXiv preprint arXiv:1906.07134*, 2019.
- [IKV21] Natalia Iyudu, Maxim Kontsevich, and Yannis Vlassopoulos. Pre-Calabi-Yau algebras as noncommutative Poisson structures. *J. Algebra*, 567:63–90, 2021.
- [KM01] Michail M. Kapranov and Yuri I. Manin. Modules and Morita theorem for operads. *Amer. J. Math.*, 123(5):811–838, 2001.
- [Kos85] Jean-Louis Koszul. Crochet de Schouten–Nijenhuis et cohomologie. *Astérisque*, (Numero Hors Serie):257–271, 1985. The mathematical heritage of Élie Cartan (Lyon, 1984).
- [KR00] Maxim Kontsevich and Alexander L. Rosenberg. Noncommutative smooth spaces. In *The Gelfand Mathematical Seminars, 1996–1999. Dedicated to the memory of Chih-Han Sah*, pages 85–108. Boston, MA: Birkhäuser, 2000.
- [KTV21] Maxim Kontsevich, Alex Takeda, and Yiannis Vlassopoulos. Pre-Calabi-Yau algebras and topological quantum field theories. *arXiv preprint arXiv:2112.14667*, 2021.
- [KWZ15] Ralph M Kaufmann, Benjamin C Ward, and J Javier Zúñiga. The odd origin of gerstenhaber brackets, batalin-vilkovisky operators, and master equations. *Journal of Mathematical Physics*, 56(10):103504, 2015.

- [Ler19] Johan Leray. Protoperads I: Combinatorics and definitions. *arXiv preprint arXiv:1901.05653*, 2019.
- [Ler20] Johan Leray. Protoperads II: Koszul duality. *Journal de l'École polytechnique — Mathématiques*, 7:897–941, 2020.
- [LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2012.
- [Mar99] Martin Markl. Cyclic operads and homology of graph complexes. In *Proceedings of the 18th winter school “Geometry and physics”, Srní, Czech Republic, January 10–17, 1998*, pages 161–170. 1999.
- [Mel16] Valerio Melani. Poisson bivectors and Poisson brackets on affine derived stacks. *Adv. Math.*, 288:1097–1120, 2016.
- [MSS02] Martin Markl, Steve Shnider, and Jim Stasheff. *Operads in algebra, topology and physics*, volume 96. 2002.
- [MV09a] Sergei Merkulov and Bruno Vallette. Deformation theory of representations of prop(erad)s. I. *J. Reine Angew. Math.*, 634:51–106, 2009.
- [MV09b] Sergei Merkulov and Bruno Vallette. Deformation theory of representations of prop(erad)s. II. *J. Reine Angew. Math.*, 636:123–174, 2009.
- [ORS13] Alexander Odesskii, Vladimir Rubtsov, and Vladimir Sokolov. Double Poisson brackets on free associative algebras. In *Noncommutative birational geometry, representations and combinatorics*, volume 592 of *Contemp. Math.*, pages 225–239. Amer. Math. Soc., Providence, RI, 2013.
- [Pri20] Jonathan P. Pridham. Shifted bisymplectic and double Poisson structures on non-commutative derived prestacks. *arXiv preprint arXiv:2008.11698*, 2020.
- [PTVV13] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted symplectic structures. *Publ. Math., Inst. Hautes Étud. Sci.*, 117:271–328, 2013.
- [RiL22] Victor Roca i Lucio. Curved properadic calculus. *arXiv preprint arXiv:2201.07155*, 2022.
- [Sch09] Travis Schedler. Poisson algebras and Yang-Baxter equations. *Advances in Quantum Computation*, 482:91, 2009.
- [Sei12] Paul Seidel. Fukaya A_∞ -structures associated to Lefschetz fibrations. I. *J. Symplectic Geom.*, 10(3):325–388, 2012.
- [TZ07] Thomas Tradler and Mahmoud Zeinalian. Infinity structure of Poincaré duality spaces. With an appendix by Dennis Sullivan. *Algebr. Geom. Topol.*, 7:233–260, 2007.
- [Val07] Bruno Vallette. A Koszul duality for props. *Trans. Amer. Math. Soc.*, 359(10):4865–4943, 2007.
- [VdB08] Michel Van den Bergh. Double Poisson algebras. *Trans. Amer. Math. Soc.*, 360(11):5711–5769, 2008.
- [War19] Benjamin C. Ward. Massey products for graph homology, 2019.
- [Yal16] Sinan Yalin. Moduli stacks of algebraic structures and deformation theory. *Journal of Noncommutative Geometry*, 10(2):579–661, 2016.
- [Yeu19] Wai-Kit Yeung. Weak Calabi-Yau structures and moduli of representations. *arXiv preprint arXiv:1802.05398, version 3*, 2019.
- [Yeu22] Wai-Kit Yeung. Ribbon dioperads and modular ribbon properads. *arXiv preprint arXiv:2202.13269*, 2022.

UNIVERSITÉ DE NANTES, LABORATOIRE DE MATHÉMATIQUES JEAN LERAY (LMJL), CNRS, UMR 6629, 2 CHEMIN DE LA HOUSSINIÈRE, BP 92208, 44322 NANTES CEDEX 3, FRANCE
E-mail address: Johan.Leray@univ-nantes.fr

UNIVERSITÉ SORBONNE PARIS NORD, LABORATOIRE DE GÉOMÉTRIE, ANALYSE ET APPLICATIONS, LAGA, CNRS, UMR 7539, 93430, VILLETANEUSE, FRANCE
E-mail address: vallette@math.univ-paris13.fr