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► To cite this version:

A. Du Barry, V. Konané, D. Gadiaga. Contribution to the estimation by projection of the operator of a moving average with values in a Hilbert space. *Annales de l'ISUP*, 2018, 62 (1-2), pp.17-32. <hal-03605665>

HAL Id: hal-03605665

<https://hal.science/hal-03605665v1>

Submitted on 11 Mar 2022

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Contribution to the estimation by projection of the operator of a moving average with values in a Hilbert space

A. Du Barry, V. Konané, D. Gadiaga

July 19, 2017

Abstract

This paper is a contribution to the method of estimation by projection. We revisited the work of Bosq and Turbillon [3]. First, we propose extensions and simplified proofs of some of the results of the article [3]. Secondly, we proposed and demonstrated results of convergence. Finally, we apply the model on real data of the turnover of an insurance company.

1 Introduction

In the framework of the use of linear processes with values in a Hilbert space for prediction, Bosq and Turbillon [3] studied the moving average of order one. The main focus was the estimation of the moving average operator. Likewise, some asymptotics results have been derived. The technique used consist in projecting the MAH(1) covariance operator in an appropriate subspace and to study the eigenvalues.

In the process of doing that, Bosq and Turbillon established very important results which it's possible to extend.

the main contributions of this paper are twofold. The first one is to give extensions of lemmas 4.3, 4.4 and 4.5 of [3]. Secondly, we give a new rate of convergence of the estimator of the operator of the moving average MAH(1). The remainder of this paper is organized as follows. In section 2 we recall some notations and assumptions as given in [3] & [1]. In section 3, we give a result on convergence of the modified empirical eigenvalues of the covariance operator. In that section, we give a simplified version of the proof of lemma 4.4 of [3]. Section 4 is about result on the convergence of the estimators of eigenvalues of the linear operator of a MAH(1).

2 Notations, assumptions and the Problem

Unless otherwise stated, the notations and assumptions are those in [3]. We summarize some of them as follows.

We consider in the following the MAH(1) process defined by

$$X_n = \varepsilon_n + L(\varepsilon_{n-1}), \quad n \in \mathbb{Z}, \quad L \in \mathcal{L}(H),$$

where H is a separable real Hilbert space equipped with its norm $\|\cdot\|$ and its scalar product $\langle \cdot, \cdot \rangle$, $L \in \mathcal{L}(H)$, where $\mathcal{L}(H)$ denote the space of bounded linear operators from H to H and $(\varepsilon_n)_{n \in \mathbb{Z}}$ is a strong H -valued white-noise, with $\mathbb{E}(\|L(\varepsilon_{n-1})\|) > 0$.

Let (Ω, \mathcal{A}, P) be a probability space and $L^2_H(P) = L^2(\Omega, \mathcal{A}, P)$ the Hilbert space of (classes of) random variables X , H -valued and such that $\mathbb{E}\|X\|^2 < \infty$.

With n observations X_1, \dots, X_n of the process X , our purpose is to estimate the MAH(1)'s operator L . If X and Y are in $L^2_H(P)$ and are zero-mean, the cross-covariance operators of X and Y are defined as

$$C_{X,Y}(x) = \mathbb{E}(\langle X, x \rangle Y), \quad x \in H$$

and $C_{Y,X} = C_{X,Y}^*$, where $*$ denotes "adjoint". The covariance operator of X is defined as $C_X = C_{X,X}$. Let C and D denote respectively the autocovariance operator and the cross-covariance operator of the process X . It has been proved in [3] that

$$L^2 D^* - LC + D = 0. \quad (1)$$

Moreover, λ_k, c_k, d_k denote respectively the eigenvalue of L, C, D and v_k denote the eigenvector associated to the eigenvalue λ_k .

We make this following assumptions:

$$\mathbf{H}_1 : \exists j_0 \geq 1 : \|L^{j_0}\|_{\mathcal{L}} < 1,$$

$$\mathbf{H}_2 : \mathbb{E}(\|X_0\|^4) < \infty,$$

$$\mathbf{H}_3 : L \text{ is symmetric, compact, } L \text{ and } C_\varepsilon \text{ have the same orthonormal set of eigenvectors } (v_k, k \geq 1),$$

$$\mathbf{H}_4 : P(\langle X_0, v_k \rangle = 0) = 0, \text{ for } k \geq 1,$$

$$\mathbf{H}_5 : \sup_{\|y\| \leq 1, y \in C(H)} \|\rho(y)\| < \frac{1}{2}, \text{ where } \rho \text{ is defined by } \rho(y) = DC^{-1}(y),$$

$$\mathbf{H}_6 : d_k > 0, \quad k \geq 1.$$

Under assumption \mathbf{H}_3 , Equation (1) is reduced by diagonalization to solve real quadratic equations involving only the eigenvalues of the various operators. For estimates, we restrict ourselves to the subspace spanned by the k_n

first eigenvectors of C corresponding to the highest eigenvalues. The choice of k_n is validated, later on, with regard to the quality of the fit. The empirical eigenvalues of C and D are respectively given by:

$$\hat{c}_{kn} = \frac{1}{n} \sum_{i=0}^n \langle X_i | v_k \rangle^2$$

and

$$\hat{d}_{kn} = \frac{1}{n-1} \sum_{i=0}^n \langle X_i | v_k \rangle \langle X_{i+1} | v_k \rangle.$$

To have real solutions to equation (1), the eigenvalues are modified as follow:

$$\hat{d}'_{kn} = \max\left(\alpha_n, \hat{d}_{kn}\right), \text{ with } 0 < \alpha_n < 1 \text{ and } \alpha_n \rightarrow 0$$

$$\hat{c}'_{kn} = \max\left(\hat{c}_{kn}, 2\hat{d}'_{kn} + \frac{1}{n}\right).$$

The eigenvalues λ_k , $k \geq 0$ of L are estimated by

$$\hat{\lambda}_{kn} = \frac{\hat{c}'_{kn} - \sqrt{\hat{c}'_{kn}{}^2 - 4\hat{d}'_{kn}{}^2}}{2\hat{d}'_{kn}},$$

and the linear operator is estimated by

$$\hat{L}_{n,k_n} = \sum_{k=1}^{k_n} \hat{\lambda}_{kn} v_k \otimes v_k.$$

Remark 2.1. k_n is the number of eigenvalues on which we restrict ourselves for the estimations while kn denotes the index of the estimator of the k -th eigenvalue with n observations.

3 Convergence of the modified empirical eigenvalues of the covariance and cross-covariance operators

In this section, we bring about extensions of some of the results of [3].

This following lemma yields result on the convergence of the modified empirical eigenvalues of the cross-covariance operator. It is an extension of lemma 4.3 of [3].

Lemma 3.1.

$$\mathbb{E} \left(\sup_{k \geq 1} \left| \hat{d}'_{kn} - d_k \right|^2 \right) = O \left(\frac{1}{n} \right)$$

and

$$\sup_{k \geq 1} \left| \hat{d}'_{kn} - d_k \right| \longrightarrow 0, \text{ a.s.}$$

Proof. Let prove that

$$\mathbb{E} \left(\sup_{k \geq 1} \left| \hat{d}'_{kn} - d_k \right|^2 \right) = O \left(\frac{1}{n} \right).$$

From the definition of \hat{d}'_{kn} , we have:

$$\left| \hat{d}'_{kn} - d_k \right| = |\alpha_n - d_k| \mathbb{1}_{\{\hat{d}_{kn} \leq \alpha_n\}} + \left| \hat{d}_{kn} - d_k \right| \mathbb{1}_{\{\hat{d}_{kn} > \alpha_n\}}.$$

Thus, $\left(\hat{d}'_{kn} - d_k \right)^2 = (\alpha_n - d_k)^2 \mathbb{1}_{\{\hat{d}_{kn} \leq \alpha_n\}} + \left(\hat{d}_{kn} - d_k \right)^2 \mathbb{1}_{\{\hat{d}_{kn} > \alpha_n\}}.$

As, by assumption, $\alpha_n \longrightarrow 0$, there exists n so that $\alpha_n \leq \frac{d_k}{2}$.

For $\alpha_n \leq \frac{d_k}{2}$, we have

$$\left(\hat{d}'_{kn} - d_k \right)^2 \leq d_k^2 \mathbb{1}_{\{\hat{d}_{kn} \leq \alpha_n\}} + \left(\hat{d}_{kn} - d_k \right)^2. \quad (2)$$

Given $\ell \geq 1$, we have successively

$$\hat{d}_{\ell n} \leq \alpha_n \Rightarrow \left| \hat{d}_{\ell n} - d_\ell \right| \geq \frac{d_\ell}{2} \Rightarrow \sup_{k \geq 1} \left| \hat{d}_{kn} - d_k \right| \geq \frac{d_\ell}{2}.$$

Hence,

$$\hat{d}_{\ell n} \leq \alpha_n \Rightarrow \sup_{k \geq 1} \left| \hat{d}_{kn} - d_k \right| \geq \sup_{\ell \geq 1} \frac{d_\ell}{2} = \frac{\|D\|_{\mathcal{L}}}{2}.$$

So then,

$$\left\{ \hat{d}_{\ell n} \leq \alpha_n \right\} \subseteq \left\{ \sup_{k \geq 1} \left| \hat{d}_{kn} - d_k \right| \geq \frac{\|D\|_{\mathcal{L}}}{2} \right\}.$$

We get

$$\mathbb{1}_{\{\hat{d}_{\ell n} \leq \alpha_n\}} \leq \mathbb{1}_{\left\{ \sup_{k \geq 1} \left| \hat{d}_{kn} - d_k \right| \geq \frac{\|D\|_{\mathcal{L}}}{2} \right\}}.$$

Finally, we obtain

$$\sup_{k \geq 1} \mathbb{1}_{\{\hat{d}_{kn} \leq \alpha_n\}} \leq \mathbb{1}_{\{\sup_{k \geq 1} |\hat{d}_{kn} - d_k| \geq \frac{\|D\|_{\mathcal{L}}}{2}\}}. \quad (3)$$

Returning to inequality (2), we get :

$$\sup_{k \geq 1} \left| \hat{d}'_{kn} - d_k \right|^2 \leq \sup_{k \geq 1} d_k^2 \mathbb{1}_{\{\hat{d}_{kn} \leq \alpha_n\}} + \sup_{k \geq 1} \left| \hat{d}_{kn} - d_k \right|^2,$$

and

$$\sup_{k \geq 1} \left| \hat{d}'_{kn} - d_k \right|^2 \leq \|D\|_{\mathcal{L}}^2 \sup_{k \geq 1} \mathbb{1}_{\{\hat{d}_{kn} \leq \alpha_n\}} + \sup_{k \geq 1} \left| \hat{d}_{kn} - d_k \right|^2,$$

with (3) we conclude that

$$\sup_{k \geq 1} \left| \hat{d}'_{kn} - d_k \right|^2 \leq \|D\|_{\mathcal{L}}^2 \mathbb{1}_{\{\sup_{k \geq 1} |\hat{d}_{kn} - d_k| \geq \frac{\|D\|_{\mathcal{L}}}{2}\}} + \sup_{k \geq 1} \left| \hat{d}_{kn} - d_k \right|^2. \quad (4)$$

It follows that

$$\mathbb{E} \left(\sup_{k \geq 1} \left| \hat{d}'_{kn} - d_k \right|^2 \right) \leq \|D\|_{\mathcal{L}}^2 \mathbb{E} \left(\mathbb{1}_{\{\sup_{k \geq 1} |\hat{d}_{kn} - d_k| \geq \frac{\|D\|_{\mathcal{L}}}{2}\}} \right) + \mathbb{E} \left(\sup_{k \geq 1} \left| \hat{d}_{kn} - d_k \right|^2 \right).$$

Likewise,

$$\mathbb{E} \left(\sup_{k \geq 1} \left| \hat{d}'_{kn} - d_k \right|^2 \right) \leq \|D\|_{\mathcal{L}}^2 \mathbb{P} \left(\left\{ \sup_{k \geq 1} \left| \hat{d}_{kn} - d_k \right| \geq \frac{\|D\|_{\mathcal{L}}}{2} \right\} \right) + \mathbb{E} \left(\sup_{k \geq 1} \left| \hat{d}_{kn} - d_k \right|^2 \right).$$

By Markov's inequality,

$$\mathbb{E} \left(\sup_{k \geq 1} \left| \hat{d}'_{kn} - d_k \right|^2 \right) \leq 5 \mathbb{E} \left(\sup_{k \geq 1} \left| \hat{d}_{kn} - d_k \right|^2 \right).$$

We conclude with lemma 4.2 of [3].

Let us now give the proof of

$$\sup_{k \geq 1} \left| \hat{d}'_{kn} - d_k \right| \longrightarrow 0, \text{ a.s.}$$

Let $(Y_k)_{k \geq 1}$ be a sequence of real random variables such that $Y_k \longrightarrow 0$ almost surely. then $\forall \epsilon > 0, \exists n_0 \geq 0$ tel que $\forall n \geq 0, n \geq n_0 \Rightarrow |Y_n| < \epsilon$ almost surely. In particular $\exists n_0 \geq 0$ tel que $\forall n \geq 0, n > n_0 \Rightarrow \mathbb{1}_{|Y_n| \geq \epsilon} = 0$. That means $\mathbb{1}_{|Y_n| \geq \epsilon} \longrightarrow 0$ almost surely.

From expression (4) and from lemma 4.2 of [3] we have the expected result. \square

In the following lemma, we give a result on the convergence of the modified empirical eigenvalues of the covariance operator. This lemma is an extension of lemma 4.4 of [3].

Lemma 3.2.

$$\mathbb{E} \left(\sup_{k \geq 1} \left| \hat{c}'_{kn} - c_k \right|^2 \right) = O \left(\frac{1}{n} \right)$$

and

$$\sup_{k \geq 1} \left| \hat{c}'_{kn} - c_k \right| \rightarrow 0, \text{ a.s.}$$

Proof. Let prove that

$$\mathbb{E} \left(\sup_{k \geq 1} \left| \hat{c}'_{kn} - c_k \right|^2 \right) = O \left(\frac{1}{n} \right).$$

By the definition of \hat{c}'_{kn} we have

$$\hat{c}'_{kn} - c_k = \left(2\hat{d}'_{kn} + \frac{1}{n} - c_k \right) \mathbb{1}_{\{\hat{c}_{kn} \leq 2\hat{d}'_{kn} + \frac{1}{n}\}} + (\hat{c}_{kn} - c_k) \mathbb{1}_{\{\hat{c}_{kn} > 2\hat{d}'_{kn} + \frac{1}{n}\}},$$

and

$$\sup_{k \geq 1} \left| \hat{c}'_{kn} - c_k \right|^2 \leq \sup_{k \geq 1} \left| 2\hat{d}'_{kn} + \frac{1}{n} - c_k \right|^2 \mathbb{1}_{\{\hat{c}_{kn} \leq 2\hat{d}'_{kn} + \frac{1}{n}\}} + \sup_{k \geq 1} |\hat{c}_{kn} - c_k|^2. \quad (5)$$

Now, let's look at the term

$$\left| 2\hat{d}'_{kn} + \frac{1}{n} - c_k \right|^2 \mathbb{1}_{\{\hat{c}_{kn} \leq 2\hat{d}'_{kn} + \frac{1}{n}\}}.$$

It's null out of the set

$$\left\{ \hat{c}_{kn} \leq 2\hat{d}'_{kn} + \frac{1}{n} \right\}.$$

By restricting to the set

$$\left\{ \hat{c}_{kn} \leq 2\hat{d}'_{kn} + \frac{1}{n} \right\},$$

we have, sequentially,

$$c_k > 2d_k \Rightarrow -c_k < -2d_k \Rightarrow 2\hat{d}'_{kn} - c_k < 2 \left(\hat{d}'_{kn} - d_k \right) \Rightarrow$$

$$\hat{c}_{kn} - c_k < 2\hat{d}'_{kn} - c_k + \frac{1}{n} < 2\left(\hat{d}'_{kn} - d_k\right) + \frac{1}{n}.$$

It follows

$$\left|2\hat{d}'_{kn} - c_k + \frac{1}{n}\right|^2 \leq \max \left\{ |\hat{c}_{kn} - c_k|^2, \left|2\left(\hat{d}'_{kn} - d_k\right) + \frac{1}{n}\right|^2 \right\}.$$

Hence,

$$\sup_{k \geq 1} \left|2\hat{d}'_{kn} - c_k + \frac{1}{n}\right|^2 \leq \sup_{k \geq 1} |\hat{c}_{kn} - c_k|^2 + \sup_{k \geq 1} \left|2\left(\hat{d}'_{kn} - d_k\right) + \frac{1}{n}\right|^2. \quad (6)$$

Finally,

$$\left(\sup_{k \geq 1} \left|2\hat{d}'_{kn} - c_k + \frac{1}{n}\right|^2 \mathbb{1}_{\{\hat{c}_{kn} \leq 2\hat{d}'_{kn} + \frac{1}{n}\}}\right) \leq \mathbb{E}\left(\sup_{k \geq 1} |\hat{c}_{kn} - c_k|^2\right) + 8\mathbb{E}\left(\sup_{k \geq 1} |\hat{d}'_{kn} - d_k|^2\right) + \frac{2}{n^2}.$$

Coming back to inequality (5), we have

$$\mathbb{E}\left(\sup_{k \geq 1} |\hat{c}'_{kn} - c_k|^2\right) \leq 2\mathbb{E}\left(\sup_{k \geq 1} |\hat{c}_{kn} - c_k|^2\right) + 8\mathbb{E}\left(\sup_{k \geq 1} |\hat{d}'_{kn} - d_k|^2\right) + \frac{2}{n^2}.$$

We conclude with lemma 4.3 of [3] and lemma 3.1.

Let prove the second part of the lemma, i.e.,

$$\sup_{k \geq 1} |\hat{c}'_{kn} - c_k| \longrightarrow 0 \text{ a.s.}$$

We deduce from (5) and (6) that

$$\sup_{k \geq 1} |\hat{c}'_{kn} - c_k|^2 \leq 2 \sup_{k \geq 1} |\hat{c}_{kn} - c_k|^2 + 8 \sup_{k \geq 1} |\hat{d}'_{kn} - d_k|^2 + \frac{2}{n^2}.$$

Conclusion come from lemma 3.1 and from lemma 4.1 of [3]. □

4 Convergence of the estimators of eigenvalues of the linear operator of a MAH(1)

We give, in this section, an extension of lemma 4.5 of [3]. It's a result on the convergence of the modified empirical eigenvalues of the linear operator L .

Lemma 4.1.

$$\mathbb{E} \left(\sup_{k \geq 1} \left| \hat{\lambda}_{kn} - \lambda_k \right|^2 \right) = O \left(\frac{(\ln n)^\gamma}{n} \right) \quad \forall \gamma, 0 < \gamma < 1.$$

Proof. Let $k \geq 1$. From the expression of λ_k with regards to c_k, d_k and that of $\hat{\lambda}_{kn}$ with regards of \hat{c}'_{kn} and of \hat{d}'_{kn} [3] established that

$$\left| \frac{\hat{c}'_{kn}}{\hat{d}'_{kn}} - \frac{c_k}{d_k} \right| = \left| \hat{\lambda}_{kn} + \frac{1}{\hat{\lambda}_{kn}} - \lambda_k - \frac{1}{\lambda_k} \right|.$$

As, $\lambda_k, \hat{\lambda}_{kn} \in]0, 1[$, we have

$$\frac{1}{\lambda_k \hat{\lambda}_{kn}} > 1,$$

and

$$\frac{1}{\lambda_k \hat{\lambda}_{kn}} > \frac{1}{\lambda_k}.$$

It follows that

$$\left| \frac{\hat{c}'_{kn}}{\hat{d}'_{kn}} - \frac{c_k}{d_k} \right| > \left| \hat{\lambda}_{kn} - \lambda_k \right| \left(\frac{1}{\lambda_k} - 1 \right),$$

hence

$$\left| \hat{\lambda}_{kn} - \lambda_k \right| < \frac{\lambda_k}{1 - \lambda_k} \left| \frac{\hat{c}'_{kn}}{\hat{d}'_{kn}} - \frac{c_k}{d_k} \right|,$$

which leads to

$$\left| \hat{\lambda}_{kn} - \lambda_k \right|^2 < \left(\frac{\lambda_k}{1 - \lambda_k} \right)^2 \left| \frac{\hat{c}'_{kn}}{\hat{d}'_{kn}} - \frac{c_k}{d_k} \right|^2. \quad (7)$$

We have, subsequently

$$\left| \frac{\hat{c}'_{kn}}{\hat{d}'_{kn}} - \frac{c_k}{d_k} \right| = \frac{1}{\hat{d}'_{kn}} \left| \frac{c_k}{d_k} (d_k - \hat{d}'_{kn}) + (\hat{c}'_{kn} - c_k) \right|$$

and

$$\hat{d}'_{kn} \left| \frac{\hat{c}'_{kn}}{\hat{d}'_{kn}} - \frac{c_k}{d_k} \right| = \left| \frac{c_k}{d_k} (d_k - \hat{d}'_{kn}) + (\hat{c}'_{kn} - c_k) \right|$$

then

$$\hat{d}'_{kn} \left| \hat{\lambda}_{kn} - \lambda_k \right|^2 < 2 \left(\frac{\lambda_k}{1 - \lambda_k} \right)^2 \left[\frac{c_k^2}{d_k^2} \left(\hat{d}'_{kn} - d_k \right)^2 + \left(\hat{c}'_{kn} - c_k \right)^2 \right] \quad (8)$$

Let find an upper bound for $\frac{c_k^2}{d_k^2}$.

the following double inequality, established in [3],

$$\frac{d_k}{c_k} < \lambda_k < 2 \frac{d_k}{c_k}.$$

entails that

$$\frac{c_k^2}{d_k^2} < \frac{4}{\lambda_k^2}.$$

Coming back to inequality (8) we get :

$$\hat{d}'_{kn} \left| \hat{\lambda}_{kn} - \lambda_k \right|^2 < 2 \left(\frac{\lambda_k}{1 - \lambda_k} \right)^2 \left[\frac{4}{\lambda_k^2} \left(\hat{d}'_{kn} - d_k \right)^2 + \left(\hat{c}'_{kn} - c_k \right)^2 \right]$$

and

$$\hat{d}'_{kn} \left| \hat{\lambda}_{kn} - \lambda_k \right|^2 < \frac{8}{(1 - \lambda_k)^2} \left(\hat{d}'_{kn} - d_k \right)^2 + 2 \left(\frac{\lambda_k}{1 - \lambda_k} \right)^2 \left(\hat{c}'_{kn} - c_k \right)^2.$$

It follows that

$$\sup_{k \geq 1} \hat{d}'_{kn} \left| \hat{\lambda}_{kn} - \lambda_k \right|^2 \leq \frac{8}{(1 - \|L\|_{\mathcal{L}})^2} \sup_{k \geq 1} \left| \hat{d}'_{kn} - d_k \right|^2 + 2 \left(\frac{\|L\|_{\mathcal{L}}}{1 - \|L\|_{\mathcal{L}}} \right)^2 \sup_{k \geq 1} \left| \hat{c}'_{kn} - c_k \right|^2. \quad (9)$$

In particular, as $\hat{d}'_{kn} \geq \alpha_n$, we obtain :

$$\alpha_n^2 \mathbb{E} \left(\sup_{k \geq 1} \left| \hat{\lambda}_{kn} - \lambda_k \right|^2 \right) \leq \frac{8}{(1 - \|L\|_{\mathcal{L}})^2} \mathbb{E} \left(\sup_{k \geq 1} \left| \hat{d}'_{kn} - d_k \right|^2 \right) + 2 \left(\frac{\|L\|_{\mathcal{L}}}{1 - \|L\|_{\mathcal{L}}} \right)^2 \mathbb{E} \left(\sup_{k \geq 1} \left| \hat{c}'_{kn} - c_k \right|^2 \right).$$

From lemmas 3.1 and 3.2, we deduce that

$$\alpha_n^2 \mathbb{E} \left(\sup_{k \geq 1} \left| \hat{\lambda}_{kn} - \lambda_k \right|^2 \right) = O \left(\frac{1}{n} \right).$$

Set $\forall n > 1$,

$$\alpha_n = \frac{1}{(\ln n)^{\frac{\gamma}{2}}} \text{ with } 0 < \gamma < 1.$$

Then

$$\mathbb{E} \left(\sup_{k \geq 1} \left| \hat{\lambda}_{kn} - \lambda_k \right|^2 \right) = O \left(\frac{(\ln n)^\gamma}{n} \right) \quad \forall \gamma, 0 < \gamma < 1. \quad \square$$

Proposition 4.1. *Under assumptions \mathbf{H}_1 to \mathbf{H}_6 and by assuming that we have knowledge of the eigenvectors v_k of L , we have :*

$$\mathbb{E} \left(\left\| \hat{L}_{n,k_n} - L \right\|_{HS}^2 \right) = O \left(\frac{k_n (\ln n)^\gamma}{n} + \sum_{k > k_n} \frac{d_k^2}{c_k^2} \right) \quad \forall \gamma, 0 < \gamma < 1$$

Proof. We can write, from the definition of the norm of Hilbert-Schmidt, that

$$\mathbb{E} \left(\left\| \hat{L}_{n,k_n} - L \right\|_{HS}^2 \right) = \sum_{k=1}^{k_n} \mathbb{E} \left(\left(\hat{\lambda}_{kn} - \lambda_k \right)^2 \right) + \sum_{k > k_n} \lambda_k^2.$$

From lemma 4.1, we have

$$\sum_{k=1}^{k_n} \mathbb{E} \left(\left(\hat{\lambda}_{kn} - \lambda_k \right)^2 \right) = O \left(\frac{k_n (\ln n)^\gamma}{n} \right) \quad \forall \gamma, 0 < \gamma < 1.$$

Moreover, since

$$\frac{d_k}{c_k} < \lambda_k < 2 \frac{d_k}{c_k},$$

we get

$$\sum_{k > k_n} \lambda_k^2 = O \left(\sum_{k > k_n} \frac{d_k^2}{c_k^2} \right)$$

which end up the proof. \square

The previous result yields the rate of convergence of the estimator of the operator L .

What follows now is another new result that we obtain. We set it as follow:

Theorem 4.1. *Under assumptions \mathbf{H}_1 to \mathbf{H}_6 and if $c_k = k^{-\alpha}$, $d_k = k^{-\beta}$, where $\beta - \alpha > 1$, then $\forall \gamma, 0 < \gamma < 1$ and for any $k_n = \left(\frac{(\ln n)^\gamma}{n} \right)^{-\frac{1}{2}}$, we have*

$$\mathbb{E} \left(\left\| \hat{L}_{n,k_n} - L \right\|_{HS}^2 \right) = O \left(\left(\frac{(\ln n)^\gamma}{n} \right)^{\frac{1}{2}} \right).$$

Proof. Proposition 4.1, states that

$$\mathbb{E}\left(\left\|\hat{L}_{n,k_n} - L\right\|_{HS}^2\right) = O\left(\frac{k_n(\ln n)^\gamma}{n} + \sum_{k>k_n} \frac{1}{k^{2(\beta-\alpha)}}\right) \forall \gamma, 0 < \gamma < 1 \quad (10)$$

Let us look at the term $\sum_{k>k_n} \frac{1}{k^{2(\beta-\alpha)}}$. We have :

$$\sum_{k>k_n} \frac{1}{k^{2(\beta-\alpha)}} = \sum_{k>k_n} \frac{1}{k^{2(\beta-\alpha)-1}k} < \frac{1}{k_n} \sum_{k>k_n} \frac{1}{k^{2(\beta-\alpha)-1}}$$

Since, by assumption $\beta - \alpha > 1$, we have likewise $2(\beta - \alpha) - 1 > 1$, which allow us to conclude that the series $\sum_{k>k_n} \frac{1}{k^{2(\beta-\alpha)-1}}$ is convergent. Hence

$$\sum_{k>k_n} \frac{1}{k^{2(\beta-\alpha)}} = O\left(\frac{1}{k_n}\right)$$

Let us come back to equation (10). We have :

$$\mathbb{E}\left(\left\|\hat{L}_{n,k_n} - L\right\|_{HS}^2\right) = O\left(\frac{k_n(\ln n)^\gamma}{n} + \frac{1}{k_n}\right) \forall \gamma, 0 < \gamma < 1.$$

$k_n = \left(\frac{(\ln n)^\gamma}{n}\right)^{-\frac{1}{2}}$ is the minimum of the term $k_n \frac{(\ln n)^\gamma}{n} + \frac{1}{k_n}$. □

5 Application

We want to forecast for budgetary purposes, the evolution of the turnover of an insurance company. we purpose to model the turnover according to a MAH(1). Estimates were made with the Eviews software.

5.1 The Data

We have monthly data from 2008 to 2014. In Figure 1, a monthly series is created for each year. Each series can be likened to a function defined over the time interval [1-12].

5.2 Approximation in a basis of cubic spline

Wanting to make a functional forecast, we choose to project the different series in a basis of cubic splines defined on the interval [1-12]. Each spline

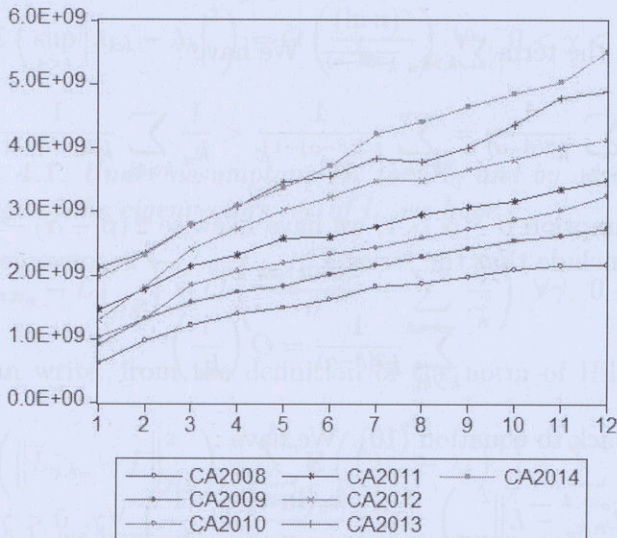


Figure 1: Annual turnover trends

function of the basis is approximated by a vector consisting of the values taken at the points $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

The projection consists of a linear regression of each series centered on the vectors of the spline functions. The simulations carried out on basis of dimension ranging from 4 to 10 (k_n in the method presented) led to roughly the same results. Figure 2 is the basis of splines of dimension 6.

Figure 3 gives an overview of the quality of adjustment of the 2012 and 2013 sales series with the spline basis. The top 2 curves of each graph are the initial and predicted series. The lower curves return the residual of the estimate.

5.3 Results and comments

The estimates have led to an approximation of the linear operator by the following matrix in the basis of cubic splines:

References

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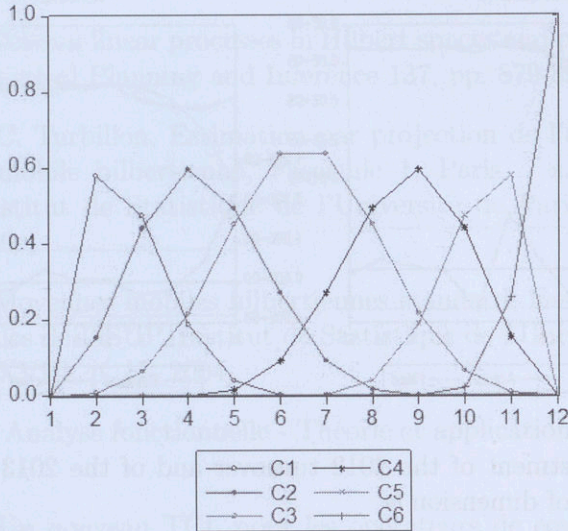


Figure 2: Basis of cubic splines of dimension 6 on the interval [1-12]

0.1025	0.0898	0.1454	0.0943	0.1641	0.1636
0.0898	0.0803	0.1174	0.0927	0.1598	0.1322
0.1454	0.1174	0.2723	0.0674	0.1274	0.3056
0.0943	0.0927	0.0674	0.1533	0.2566	0.0766
0.1641	0.1598	0.1274	0.2566	0.4304	0.1446
0.1636	0.1322	0.3056	0.0766	0.1446	0.3429

The hypothesis H_3 imposing the condition of the same orthonormal basis for the three operators ($C; D; L$) is not fulfilled. This is a fairly strong hypothesis that we could not get with any of the available series.

The forecast for 2014 is compared to the real data in figure 4.

The mean square error is of the order of $3.25E + 08$, for a coefficient of variation of about 8.5%, which can be considered very average.

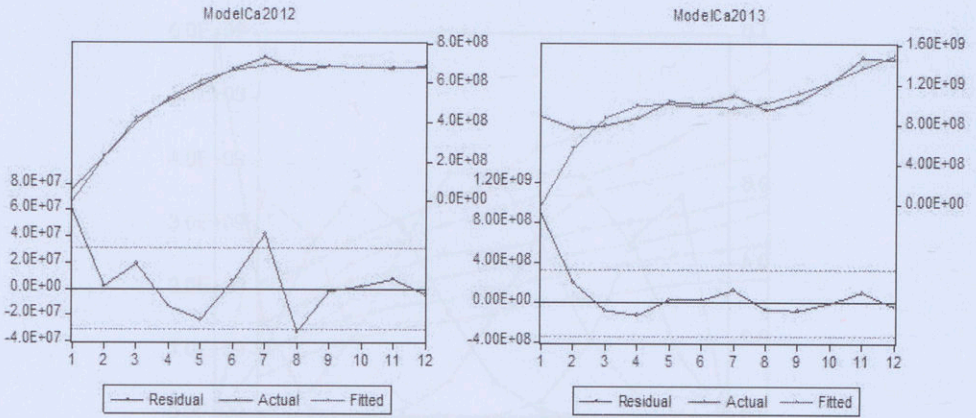


Figure 3: Adjustment of the 2012 turnover and of the 2013 turnover on a basis of splines of dimension 6

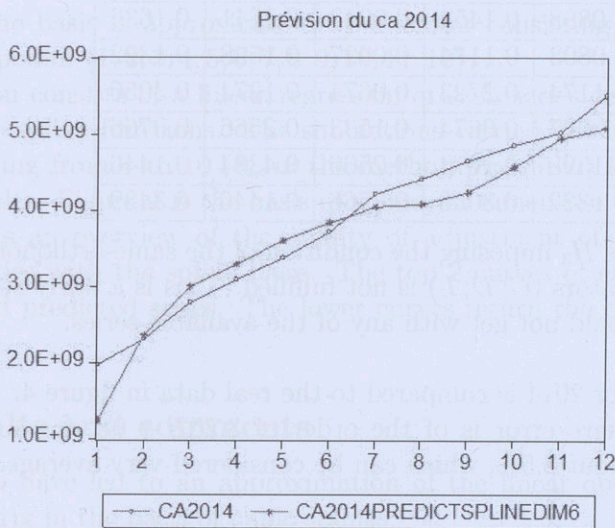


Figure 4: Real turnover vs predicted turnover

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