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# AN UNCOUNTABLE FAMILY OF NESTED SUBSHIFTS

MARTIN LUSTIG

ABSTRACT. A quick elementary construction of an uncountable family of nested subshifts over a finite alphabet  $\mathcal{A}$  is presented. Recall that subshifts over  $\mathcal{A}$  are special cases of algebraic laminations in the free group  $F(\mathcal{A})$ , which appear as support of currents on  $F(\mathcal{A})$  or as “dual support” of  $\mathbb{R}$ -trees with an isometric action of  $F(\mathcal{A})$ .

In this note we present the construction of an uncountable family of nested subshifts: For any sequence  $\mathbf{d} \in \{0, 1\}^{\mathbb{N}}$  (equipped with its natural lexicographic order) a subshift  $X(\mathbf{d}) \in \{a, b\}^{\mathbb{Z}}$  is constructed such that:

$$\mathbf{d} < \mathbf{d}' \quad \implies \quad X(\mathbf{d}) \subsetneq X(\mathbf{d}')$$

The author doesn't believe that the given uncountable nested subshift family is the first such known, although he cannot point to any specific reference. The main reason for this note to be written is the simplicity of the construction; also, the author freely admits that he actually started out by trying to show that nested families of pairwise distinct subshifts must always be countable. It thus appears fair to share with interested colleagues the author's surprise about both, the untruth of his intuitive conjecture and the simplicity of the exhibited counterexample.

Before starting the construction which fulfills the promise in the title of this note we need to set up the following:

For any alphabet  $\mathcal{A} = \{a, b\}$  and any integer  $\ell \geq 0$  consider an alphabet  $\mathcal{A}' = \{a', b'\}$  as well as the following morphism:

$$(1) \quad \sigma_{\mathcal{A}, \ell} : \mathcal{A}'^* \rightarrow \mathcal{A}^*, \quad a' \mapsto a^{\ell+1}b, \quad b' \mapsto a^{\ell+2}b$$

It is easy to see that  $\sigma_{\mathcal{A}, \ell}$  is recognizable in the full shift  $\mathcal{A}'^{\mathbb{Z}}$ , so that in particular the induced map  $X \mapsto \sigma_{\mathcal{A}, \ell}(X) := X(\sigma_{\mathcal{A}, \ell}(\mathcal{L}(X)))$  on the set of subshifts  $X \subset \mathcal{A}'^{\mathbb{Z}}$  is injective. (Here  $\mathcal{L}(X)$  denotes the language of the subshift  $X$  and  $X(\mathcal{L})$  the subshift generated by an infinite language  $\mathcal{L}$ .)

We now define iteratively for any infinite sequence  $\mathbf{d} = d_1 d_2 \dots \in \{0, 1\}^{\mathbb{N}}$  a subshift  $X(\mathbf{d}) \subset \mathcal{A}^{\mathbb{Z}}$  as follows, for  $\mathcal{A} = \{a, b\}$ :

Assume (by induction) that for a prefix  $\mathbf{d}_n := d_1 d_2 \dots d_n$  of  $\mathbf{d}$  a subshift  $X_n := X(\mathbf{d}_n) \subset \mathcal{A}^{\mathbb{Z}}$  is defined as union of some subshift  $X'_n \subset \mathcal{A}^{\mathbb{Z}}$  with the image  $\sigma_n(X')$ , for some alphabet  $\mathcal{A}_n = \{a_n, b_n\}$ , some morphism  $\sigma_n : \mathcal{A}_n^* \rightarrow \mathcal{A}^*$  and some subshift  $X' \subset \mathcal{A}_n^{\mathbb{Z}}$ .

Assume furthermore that  $X'$  satisfies the following condition: Let  $d_{m+1} d_{m+2} \dots d_n$  be the largest suffix of  $\mathbf{d}_n$  with all letters  $d_i$  equal to 1 (this includes of course the possibility  $m = n$ ). Then we assume that  $X'$  is the subshift which consists of all sequences

$$(2) \quad \dots b_n a_n^{n_j} b_n a_n^{n_{j+1}} b_n a_n^{n_{j+2}} \dots$$

subject to the condition

$$(3) \quad 1 \leq n_j \leq 2(n - m) + 2 \quad \text{for all} \quad n_j \in \mathbb{Z}.$$

We now define the subshift  $X_{n+1} = X(\mathbf{d}_{n+1}) \subset \mathcal{A}^{\mathbb{Z}}$  as follows:

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If  $d_{n+1} = 1$  we set  $\sigma_{n+1} := \sigma_n$  and

$$X_{n+1} := X'_n \cup \sigma_{n+1}(X''),$$

where the subshift  $X'' \subset \mathcal{A}_n^{\mathbb{Z}}$  is defined just as  $X'$  through sequences as specified in (2), but with the restriction (3) replaced by

$$(4) \quad 1 \leq n_j \leq 2(n-m) + 4 \quad \text{for all} \quad n_j \in \mathbb{Z}.$$

If  $d_{n+1} = 0$  we consider the morphism  $\sigma_{\mathcal{A}_n, \ell}$  as in (1) with  $\ell := 2(n-m) + 2$ , where we rename  $a' := a_{n+1}, b' := b_{n+1}$  and  $\mathcal{A}' := \mathcal{A}_{n+1}$ .

We then define  $X'' \subset \mathcal{A}_{n+1}^{\mathbb{Z}}$  through sequences as specified in (2) with  $a_n$  replaced by  $a_{n+1}$  and  $b_n$  by  $b_{n+1}$ , and with  $m = n$  in condition (3). We set  $\sigma_{n+1} := \sigma_n \circ \sigma_{\mathcal{A}_n, \ell}$  and define:

$$X_{n+1} := X_n \cup \sigma_{n+1}(X'')$$

From these definitions we see directly that in both cases one has  $X_n \subset X_{n+1}$  and  $X_n \neq X_{n+1}$ . Furthermore,  $X_{n+1}$  satisfies the inductive hypotheses listed above for  $X_n$ . Hence the iterative construction of the  $X_n = X(\mathbf{d}_n)$  is well defined once we set for the empty sequence  $\mathbf{d}_0$  the starting data to be  $X'_0 := \emptyset, \mathcal{A}_0 := \mathcal{A}, a_0 := a, b_0 := b$  and  $\sigma_0 = id_{\mathcal{A}^*}$ . The starting subshift  $X' \subset \mathcal{A}_0^{\mathbb{Z}}$  is then given by (2) and by (3) with  $n = m = 0$ .

We now define  $X(\mathbf{d})$  as closure of the increasing union of all  $X(\mathbf{d}_n)$  and observe:

**Lemma 1.** *For any sequence  $\mathbf{d} = d_1 d_2 \dots \in \{0, 1\}^{\mathbb{N}}$  and any prefix  $\mathbf{d}_n = d_1 \dots d_n$  of  $\mathbf{d}$  with  $d_{n+1} = 0$  one obtains for the finite sequence  $\mathbf{d}_n 1$  of length  $n + 1$  the following proper inclusion:*

$$X(\mathbf{d}) \subsetneq X(\mathbf{d}_n 1)$$

*Proof.* This is a direct consequence of the fact that from the assumption  $d_{n+1} = 0$  and the definition of our iterative construction it follows that for any  $n' \geq n$  the subshift  $X(\mathbf{d}_{n'})$  is contained in the union  $X_n \cup \sigma_{n+1}(\mathcal{A}_{n+1}^{\mathbb{Z}})$ , which in turn is easily seen to be contained in but not equal to the subshift  $X(\mathbf{d}_n 1)$ .  $\square$

We thus derive directly the desired result:

**Theorem 2.** *For any two sequences  $\mathbf{d} < \mathbf{d}'$  in  $\{0, 1\}^{\mathbb{N}}$  the associated subshifts in  $\{a, b\}^{\mathbb{Z}}$  satisfy:*

$$X(\mathbf{d}) \subsetneq X(\mathbf{d}')$$

*Proof.* The proof is a direct consequence of the statement of Lemma 1, together with the previously observed inclusion  $X(d_1 \dots d_n) \subset X(d_1 \dots d_n d_{n+1})$  for any  $\mathbf{d}' = d_1 d_2 \dots \in \{0, 1\}^{\mathbb{N}}$ .  $\square$

*Final remark and acknowledgements:* It is easy to exhibit uncountable families of pairwise distinct subshifts in  $\mathcal{A}^{\mathbb{Z}}$  for any finite alphabet  $\mathcal{A}$ . It is also not hard to find countable families of subshifts  $X_1 \subsetneq X_2 \subsetneq X_3 \subsetneq \dots$  or  $\dots \subsetneq X_3 \subsetneq X_2 \subsetneq X_1$ . Indeed, any presentation of a subshift  $X$  by an everywhere growing directive sequence gives directly an example of the second kind. Examples of the first kind are a bit harder to find “in nature”: the original motivation for this note came indeed from observing the existence of such families within the Pascal-adic subshift, see [1], §4. We would like to thank Pascal Hubert for having pointed our attention to this subshift, and Nicolas Bédaride for having unearthed the above reference.

## REFERENCES

- [1] X. Méla and K. Petersen. *Dynamical properties of the Pascal adic transformation*. Ergodic Theory Dynam. Systems **25** (2005), 227–256

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