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A Laplace type problem for a lattice of parallelepipeds with exponential distribution

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Dedicated to Marius Stoka

Abstract

In this paper we consider a lattice with fundamental cell a parallelepiped in the Euclidean space E_3 , and we determine the probability that a random segment with exponential distribution of constant length intersects a side of the lattice.

Keyword: Stochastic Geometry, Probability, Integral Geometry, Exponential distribution.

AMS Classification 2010: 60D05, 53C65, 52A22.

1 Introduction

Let $\mathfrak{R}(a_1, a_2, a_3)$ be a lattice in Euclidean space E_3 with fundamental cell P_0 a parallelepiped of sides a_1, a_2, a_3 . We have

$$\text{vol}P_0 = a_1 a_2 a_3. \quad (1)$$

We are interested to compute the probability that a random segment s with exponential distribution of constant length $l < \frac{1}{2} \inf(a_1, a_2, a_3)$ intersects a side of lattice \mathfrak{R} , i.e., the probability P_{int} that the segment s intersects a side of fundamental cell P_0 .

The position of the segment s is determined by the center and by the directors cosines of the line support of s :

$$\alpha_1 = \cos \theta, \quad \alpha_2 = \sin \theta \cos \varphi, \quad \alpha_3 = \sin \theta \sin \varphi.$$

To compute the probability P_{int} we consider the parallelepiped P_0^* with the vertex $A(\frac{l}{2} \cos \theta, \frac{l}{2} \sin \theta \cos \varphi, \frac{l}{2} \sin \theta \sin \varphi)$, with parallel sides of P_0 and with lengths

$$a_1 - l \cos \theta, \quad a_2 - l \sin \theta \cos \varphi, \quad a_3 - l \sin \theta \sin \varphi.$$

Then

$$\text{vol}P_o^* = (a_1 - l \cos \theta) (a_2 - l \sin \theta \cos \varphi) (a_3 - l \sin \theta \sin \varphi). \quad (2)$$

We denote by M , the set of all segments s that they have center in the cell P_0 , and by N the set of all segments s entirely contained in the cell P_o , we have ([3])

$$P_{int} = 1 - \frac{\mu(N)}{\mu(M)}, \quad (3)$$

where μ is the Lebesgue measure in the Euclidean plane E_3 .

To compute the measure $\mu(M)$ and $\mu(N)$ we use the kinematic measure of Blaschke in the space ([1])

$$dk = d\psi \wedge d\varphi \wedge \sin \theta d\theta \wedge dx \wedge dy \wedge dz,$$

where x, y, z are the coordinate of center of s , φ and θ is the fixed angle and ψ is an angle of rotation. We observe that

$$0 \leq \varphi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \psi \leq 2\pi.$$

2 Main Results

We suppose that φ and θ are independent random variables with the same distribution of probability $f(\varphi) = e^{-\varphi}$, $f(\theta) = e^{-\theta}$, we have

$$\begin{aligned} \mu(M) &= \int_0^{2\pi} d\psi \int_0^{\frac{\pi}{2}} e^{-\varphi} d\varphi \int_0^{\frac{\pi}{2}} e^{-\theta} \sin \theta d\theta \int \int \int_{\{(x,y,z) \in P_0\}} dx dy dz = \\ &2\pi \text{vol}P_0 \int_0^{\frac{\pi}{2}} e^{-\varphi} d\varphi \int_0^{\frac{\pi}{2}} e^{-\theta} \sin \theta d\theta = \\ &2\pi a_1 a_2 a_3 \int_0^{\frac{\pi}{2}} e^{-\varphi} d\varphi \int_0^{\frac{\pi}{2}} e^{-\theta} \sin \theta d\theta. \end{aligned} \quad (4)$$

By (2), we can write

$$\begin{aligned} \mu(N) &= \int_0^{2\pi} d\psi \int_0^{\frac{\pi}{2}} e^{-\theta} \sin \theta d\theta \int_0^{\frac{\pi}{2}} e^{-\varphi} d\varphi \int \int \int_{\{(x,y,z) \in P_o^*\}} dx dy dz = \\ &2\pi \int_0^{\frac{\pi}{2}} e^{-\theta} \sin \theta d\theta \int_0^{\frac{\pi}{2}} e^{-\varphi} (\text{vol}P_o^*) d\varphi = \\ &2\pi \int_0^{\frac{\pi}{2}} e^{-\theta} \sin \theta d\theta \int_0^{\frac{\pi}{2}} [a_1 a_2 a_3 - l(a_1 a_2 \sin \theta \sin \varphi + a_1 a_3 \sin \theta \cos \varphi + a_2 a_3 \cos \theta) + \\ &+ l^2(a_1 \sin^2 \theta \sin \varphi \cos \varphi + a_2 \sin \theta \cos \theta \sin \varphi + a_3 \sin \theta \cos \theta \cos \varphi) - \\ &- l^3 \sin^2 \theta \cos \theta \sin \varphi \cos \varphi] e^{-\varphi} d\varphi. \end{aligned} \quad (5)$$

Replacing in (3), the formulas (4) and (5) we deduce

$$P_{int} = \frac{l}{a_1 a_2 a_3 \int_0^{\frac{\pi}{2}} e^{-\varphi} d\varphi \int_0^{\frac{\pi}{2}} e^{-\theta} \sin \theta d\theta} \int_0^{\frac{\pi}{2}} e^{-\theta} \sin \theta d\theta \int_0^{\frac{\pi}{2}} [a_1 a_2 \sin \theta \sin \varphi + a_1 a_3 \sin \theta \cos \varphi + a_2 a_3 \cos \theta - l (a_1 \sin^2 \theta \sin \varphi \cos \varphi + a_2 \sin \theta \cos \theta \sin \varphi + a_3 \sin \theta \cos \theta \cos \varphi) + l^2 \sin^2 \theta \cos \theta \sin \varphi \cos \varphi] e^{-\varphi} d\varphi. \quad (6)$$

First, we compute the integral

$$I_1(\theta) = \int_0^{\frac{\pi}{2}} [a_1 a_2 \sin \theta \sin \varphi + a_1 a_3 \sin \theta \cos \varphi + a_2 a_3 \cos \theta - l \left(\frac{a_1}{2} \sin^2 \theta \sin 2\varphi + a_2 \sin \theta \cos \theta \sin \varphi + a_3 \sin \theta \cos \theta \cos \varphi \right) + \frac{l^2}{2} \sin^2 \theta \cos \theta \sin 2\varphi] e^{-\varphi} d\varphi.$$

As shown in [2]:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} e^{-\varphi} d\varphi &= 1 - e^{-\frac{\pi}{2}}; \\ \int_0^{\frac{\pi}{2}} e^{-\varphi} \sin \varphi d\varphi &= \frac{1}{2} (1 - e^{-\frac{\pi}{2}}), \\ \int_0^{\frac{\pi}{2}} e^{-\varphi} \cos \varphi d\varphi &= \frac{1}{2} (1 + e^{-\frac{\pi}{2}}); \\ \int_0^{\frac{\pi}{2}} e^{-\varphi} \sin 2\varphi d\varphi &= \frac{2}{5} (1 + e^{-\frac{\pi}{2}}), \\ \int_0^{\frac{\pi}{2}} e^{-\varphi} \cos 2\varphi d\varphi &= \frac{1}{5} (1 + e^{-\frac{\pi}{2}}). \end{aligned} \quad (7)$$

With these values we see that

$$\begin{aligned} I_1(\theta) &= \frac{1}{2} a_1 [a_2 (1 - e^{-\frac{\pi}{2}}) + a_3 (1 + e^{-\frac{\pi}{2}})] \sin \theta + a_2 a_3 (1 - e^{-\frac{\pi}{2}}) \cos \theta - \\ &- \frac{l}{2} \left\{ \frac{2}{5} a_1 (1 + e^{-\frac{\pi}{2}}) \sin^2 \theta + [a_2 (1 - e^{-\frac{\pi}{2}}) + a_3 (1 + e^{-\frac{\pi}{2}})] \sin \theta \cos \theta \right\} + \\ &+ \frac{l^2}{5} (1 + e^{-\frac{\pi}{2}}) \sin^2 \theta \cos \theta. \end{aligned}$$

Therefore, we observe

$$I_2 = \int_0^{\frac{\pi}{2}} e^{-\theta} I_1(\theta) \sin \theta d\theta = \int_0^{\frac{\pi}{2}} e^{-\theta} \left(\frac{1}{2} a_1 [a_2 (1 - e^{-\frac{\pi}{2}}) + a_3 (1 + e^{-\frac{\pi}{2}})] \right)$$

$$\begin{aligned} & \sin^2 \theta + \frac{1}{2} a_2 a_3 (1 - e^{-\frac{\pi}{2}}) \sin 2\theta - \frac{l}{2} \\ & \left\{ \frac{2}{5} a_1 (1 + e^{-\frac{\pi}{2}}) \sin^3 \theta + [a_2 (1 - e^{-\frac{\pi}{2}}) + a_3 (1 + e^{-\frac{\pi}{2}})] \sin^2 \theta \cos \theta \right\} + \\ & + \frac{l^2}{5} (1 + e^{-\frac{\pi}{2}}) \sin^3 \theta \cos \theta d\theta. \end{aligned}$$

Substituting θ with φ in (7) we obtain

$$\begin{aligned} \int_0^{\frac{\pi}{2}} e^{-\theta} \sin^2 \theta d\theta &= \frac{1}{2} \int_0^{\frac{\pi}{2}} e^{-\theta} (1 - \cos 2\theta) d\theta = \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} e^{-\theta} d\theta - \frac{1}{2} \int_0^{\frac{\pi}{2}} e^{-\theta} \cos 2\theta d\theta = \\ &= \frac{1}{2} (1 - e^{-\frac{\pi}{2}}) - \frac{1}{10} (1 + e^{-\frac{\pi}{2}}) = \frac{2}{5} - \frac{3}{5} e^{-\frac{\pi}{2}}. \end{aligned} \quad (8)$$

By the same arguments, we see that

$$\int_0^{\frac{\pi}{2}} e^{-\theta} 2 \sin \theta d\theta = \frac{2}{5} (1 + e^{-\frac{\pi}{2}}). \quad (9)$$

To compute the integral

$$J = \int_0^{\frac{\pi}{2}} e^{-\theta} \sin^3 \theta d\theta,$$

by denoting

$$J_1 = \int_0^{\frac{\pi}{2}} e^{-\theta} \cos 2\theta \sin \theta d\theta,$$

$$J_2 = \int_0^{\frac{\pi}{2}} e^{-\theta} \cos 2\theta \cos \theta d\theta,$$

$$J_3 = \int_0^{\frac{\pi}{2}} e^{-\theta} \sin 2\theta \sin \theta d\theta,$$

$$J_4 = \int_0^{\frac{\pi}{2}} e^{-\theta} \sin 2\theta \cos \theta d\theta$$

we conclude

$$\begin{aligned} J_1 &= -\frac{1}{10} + \frac{3}{10} e^{-\frac{\pi}{2}}, \\ J_2 &= \frac{3}{10} + \frac{1}{10} e^{-\frac{\pi}{2}}, \\ J_3 &= \frac{1}{5} + \frac{2}{5} e^{-\frac{\pi}{2}}, \\ J_4 &= \frac{2}{5} - \frac{1}{5} e^{-\frac{\pi}{2}}. \end{aligned} \quad (10)$$

With these values we get

$$\begin{aligned}
 J &= \frac{1}{2} \int_0^{\frac{\pi}{2}} e^{-\theta} (1 - \cos 2\theta) \sin \theta d\theta = \\
 &= \frac{1}{2} \left[\int_0^{\frac{\pi}{2}} e^{-\theta} \sin \theta d\theta - \int_0^{\frac{\pi}{2}} e^{-\theta} \cos 2\theta \sin \theta d\theta \right] = \\
 &= \frac{1}{2} \left[\frac{1}{2} (1 - e^{-\frac{\pi}{2}}) - J_1 \right] = \frac{3}{10} - \frac{2}{5} e^{-\frac{\pi}{2}}. \tag{11}
 \end{aligned}$$

Now, we compute the integral

$$L = \int_0^{\frac{\pi}{2}} e^{-\theta} \sin^2 \theta \cos \theta d\theta.$$

In view of (7) and (10), we observe

$$\begin{aligned}
 L &= \frac{1}{2} \int_0^{\frac{\pi}{2}} e^{-\theta} (1 - \cos 2\theta) \cos \theta d\theta = \\
 &= \frac{1}{2} \left[\int_0^{\frac{\pi}{2}} e^{-\theta} \cos \theta d\theta - \int_0^{\frac{\pi}{2}} e^{-\theta} \cos 2\theta \cos \theta d\theta \right] = \\
 &= \frac{1}{2} \left[\frac{1}{2} (1 + e^{-\frac{\pi}{2}}) - \left(\frac{3}{10} + \frac{1}{10} e^{-\frac{\pi}{2}} \right) \right] = \frac{1}{10} + \frac{1}{5} e^{-\frac{\pi}{2}}. \tag{12}
 \end{aligned}$$

In the end, we compute the integral

$$M = \int_0^{\frac{\pi}{2}} e^{-\theta} \sin^3 \theta \sin \theta \cos \theta d\theta.$$

We have

$$\begin{aligned}
 M &= \int_0^{\frac{\pi}{2}} e^{-\theta} \sin^2 \theta \sin \theta \cos \theta d\theta = \frac{1}{4} \int_0^{\frac{\pi}{2}} e^{-\theta} (1 - \cos 2\theta) \sin 2\theta d\theta = \\
 &= \frac{1}{4} \left(\int_0^{\frac{\pi}{2}} e^{-\theta} \sin 2\theta d\theta - \frac{1}{2} \int_0^{\frac{\pi}{2}} e^{-\theta} \sin 4\theta d\theta \right).
 \end{aligned}$$

By two parts integration we obtain

$$\int_0^{\frac{\pi}{2}} e^{-\theta} \sin 4\theta d\theta = \frac{4}{15} (1 - e^{-\frac{\pi}{2}})$$

This formula in conjunction with the fourth formula (7) yields

$$M = \frac{1}{4} \left[\frac{2}{5} (1 + e^{-\frac{\pi}{2}}) - \frac{2}{15} (1 - e^{-\frac{\pi}{2}}) \right] = \frac{1}{15} + \frac{2}{15} e^{-\frac{\pi}{2}}. \tag{13}$$

Putting (7), (8), (11), (12) and (13) together implies

$$\begin{aligned}
 I_2 = & \frac{1}{10} a_1 a_2 (1 - e^{-\frac{\pi}{2}}) (1 + 3e^{-\frac{\pi}{2}}) + \frac{1}{10} a_1 a_3 (1 + e^{-\frac{\pi}{2}}) \\
 & (1 + 3e^{-\frac{\pi}{2}}) + \frac{1}{5} a_2 a_3 (1 - e^{-\frac{\pi}{2}}) - \\
 & \frac{l}{10} \left[\frac{a_1}{5} (1 + e^{-\frac{\pi}{2}}) (3 - 4e^{-\frac{\pi}{2}}) + \frac{a_2}{2} (1 - e^{-\frac{\pi}{2}}) (1 + 2e^{-\frac{\pi}{2}}) + \right. \\
 & \left. \frac{a_3}{2} (1 + e^{-\frac{\pi}{2}}) (1 + 2e^{-\frac{\pi}{2}}) \right] + \frac{l^2}{75} (1 + e^{-\frac{\pi}{2}}) (1 + 2e^{-\frac{\pi}{2}}). \quad (14)
 \end{aligned}$$

By (6), (7) and (14) we obtain

$$\begin{aligned}
 P_{int} = & \frac{2l}{a_1 a_2 a_3 (1 - e^{-\frac{\pi}{2}})^2} \left\{ \frac{1}{10} a_1 a_2 (1 - e^{-\frac{\pi}{2}}) (1 + 3e^{-\frac{\pi}{2}}) + \right. \\
 & \frac{1}{10} a_1 a_3 (1 + e^{-\frac{\pi}{2}}) (1 + 3e^{-\frac{\pi}{2}}) + \frac{1}{5} a_2 a_3 (1 - e^{-\frac{\pi}{2}}) - \\
 & \frac{l}{10} \left[\frac{a_1}{5} (1 + e^{-\frac{\pi}{2}}) (3 - 4e^{-\frac{\pi}{2}}) + \frac{a_2}{2} (1 - e^{-\frac{\pi}{2}}) (1 + 2e^{-\frac{\pi}{2}}) + \right. \\
 & \left. \left. + \frac{a_3}{2} (1 + e^{-\frac{\pi}{2}}) (1 + 2e^{-\frac{\pi}{2}}) + \frac{l^2}{75} (1 + e^{-\frac{\pi}{2}}) (1 + 2e^{-\frac{\pi}{2}}) \right] \right\}.
 \end{aligned}$$

If $a_2 \rightarrow +\infty$, $a_3 \rightarrow +\infty$, the lattice R becomes a pattern of parallel planes at a distance a_1 and probability P_{int} can be written to the form

$$P_{int} = \frac{2l}{5a_1} \frac{1 + e^{-\frac{\pi}{2}}}{1 - e^{-\frac{\pi}{2}}}.$$

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