Predicting linear autoregressive processes with fixed jumps in derivative and exogenous random variables

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Pub. Inst. Stat. Univ. Paris
62, fasc.3, 2018, 75-94

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December 2, 2017

Abstract

In the context of autoregressive linear process, with exogenous random variables, say ARHX (p, q, k), We envisage derivative of order p, exogenous random variables of order q and k fixed jumps. Prediction and exponential rates are obtained in the context of exogenous random variables. Also, high frequency data holds with derivative, and one obtains jump in derivative of order $(p+1)$.

Keywords: Autoregressive functional data; Exogeneity; Intensity of derivative; Detecting instant of jumps; Prediction; High frequency data; Exponential rate of convergence; Hilbertian operators

MSC: 60G25, 60G86

1 Introduction

First, we consider a predictor of the form ARHX(q), where

$$X_n = \rho(X_{n-1}) + \sum_{h=1}^{q} a_h(Z_{n,h}) + \varepsilon_n,\ n \geq 1 \tag{1.1}$$

and

$$Z_{n,h} = u_h(Z_{n-1,h}) + \eta_{n,h}\ , 1 \leq h \leq q \tag{1.2}$$

cf Damon and Guillas [2002] p.763. Also, a simulation study appears in the context of ozone [cf Damon and Guillas, 2005].

Our objective is to consider the case of p derivatives with random jumps, say $X_n^{(p)}$, $Z_{n,h}^{(p)}$, $\varepsilon_n^{(p)}, n \geq 1$, and a fixed jump at $0 < t_1 < ... < t_k < 1$. Note that the case with $p = 0$ and without exogenous variables appear in Blanke and Bosq [2014], Bosq [2015], Blanke and Bosq [2016].

Thus, we envisage the model $ARHX(p, q, k)$ with

$$X_n^{(p)}(t) = \rho(X_{n-1}^{(p)})(t) + \sum_{h=1}^{q} a_{h,}(Z_{n,h}^{(p)})(t) + \varepsilon_n^{(p)}(t),\ n \geq 1 \tag{1.3}$$

and

$$Z_{n,h}^{(p)}(t) = u_h(Z_{n-1,h}^{(p)})(t) + \eta_{n,h}^{(p)}(t),\ 1 \leq h \leq q,\ n \geq 1,\ p \in \mathbb{N}, \tag{1.4}$$

where $0 \leq t \leq 1$.

Note that $\varepsilon_n^{(p)}$ and $\eta_{n,h}^{(p)}$ are a white noise, but there is a jump at t_j.

In this context we put $\sum_{n\geq 0} \left\|\rho_j^n\right\|_{\mathcal{L}} < \infty$, $\sum_{n\geq 0} \left\|a_{h,j}^n\right\|_{\mathcal{L}} < \infty$, $\sum_{n\geq 0} \left\|u_{h,j}^n\right\|_{\mathcal{L}} < \infty, 1 \leq h \leq q, 1 \leq j \leq k$, where $\|.\|_{\mathcal{L}}$ is the norm of bounded linear operators.

We now turn to estimation of jumps in derivative. Note that, big continuous variations may be considered as jump (cf Horváth and Kokoszka [2012] p.13 with credit card transaction). Here, applications appear in physic, finance, economy, seismology, shocks, avalanches, electricity consumption, waves propagation, climatic variation etc.... In Dmowska and Kostrov [1973], Takahashi [2004], Tanushev [2008], Çetin and Sheynzon [2014], crashes and beams are considered. In the context of derivative, we envisage Scherzer [1998], Mas and Pumo [2009], Blanke and Vial [2011, 2014]. We also consider Cates and Gelb [2007] with derivative at discontinuity and Joo and Qiu [2009] with regression curve in derivative. Jumps in derivative appear in Warsi [1964], Rauch [1988], Cai and Mehlum [1998] where shocks and waves hold. Also, electricity consumption appears in the context of El Hajj [2013].

In Section 2, we consider fixed jump in derivative with an exponential rate of convergence and convergence in distribution. In this context $(X_i^{(p)}(t) -$

$X_i^{(p)}(t-)$, $1 \leq i \leq n$) are independent (cf Proposition 2.1). Also a jump in derivative appears at $T_p = \{t_1, ..., t_k\}$ where $0 < t_1 < ... < t_k < 1, t \in T_p$, with $t_j = t_j(+)$ in the context of $X^{(p)}$. We also consider exogenous and random jump at $a_1(Z_{n,1}^{(p)})$ with fixed jumps $S_p = \{s_1, ..., s_h\}$,$0 < s_1 < ... < s_h < 1$, $s \in S_p$, $s_j = s_j(+)$.

For example, exogeneity appears in volatility models, cf Thieu [2016] and Bosq and Shen [1998].

Also, one obtains CLT and LLN, see Guillas [2000a,b], with derivatives (cf Proposition 3.5).

In the context of Section 3, a jump at $(a_1(Z_{n,1}^{(p)})(s) - a_1(Z_{n,1}^{(p)})(s-))$ is envisaged (see Lemma 3.1) and an exponential rate holds (cf Proposition 3.2).

Now, prediction holds in Section 4, we use $X_i^{(p)}(t), Z_{i,h}^{(p)}(t), 1 \leq h \leq q)$ with $1 \leq i \leq n$. Then, for obtaining empirical linear operators, it is necessary to estimate $\hat{\rho}_n, \hat{a}_{h,n}, \hat{u}_{h,n}$, $1 \leq h \leq q$ and three exponential rates are considered in Proposition 5.1 (cf Section 5.1). In the context of Hilbertian operators we consider projection with an orthonormal basis (cf Section 5.2)

Finally, high frequency data are obtained and we consider $X^{(p)}$ and exogenous variable of order p (see Section 6). Finally, we envisage some extension (cf Section 7).

Note that, we obtain main results in the context of Bosq [2000] and some change appears in Bosq [2016].

2 Fixed jumps with derivative

Now, we suppose that a jump appears at $X_n^{(p)}(t)$ where $t \in T_p = \{t_1, ..., t_k\}$ with $t_0 = 0 < t_1 ... < t_k < 1 = t_{k+1}$.

For convenience we envisage $q = 1$ since $q > 1$ is rather intricate, see also Remark 3.4.

We need the following assumption:

$$A1 - \rho(X_{n-1}^{(p)}) \in C([0, 1], n \geq 1, p \in \mathbb{N},$$

$$A2 - a_1(Z_{n,1}^{(p)}) \in C[0, 1] \, n \geq 1 \, p \in \mathbb{N},$$

$$A3 - \varepsilon_n^{(p)}(t) - \varepsilon_n^{(p)}(t-) \neq 0, \, a.s., t \varepsilon T_p, \, n \geq 1, \, p \varepsilon \mathbb{N}$$

where $C([0,\,1]$ is the space of continuous function.
Then, it follows that:

Proposition 2.1. *Under A1, A2, A3 it holds*

$$X_n^{(p)}(t) - X_n^{(p)}(t-) = \varepsilon_n^{(p)}(t) - \varepsilon_n^{(p)}(t-) \neq 0,\ a.s.\ t \in T_p\ n \geq 1,\ p \in \mathbb{N}, \quad (2.1)$$

Proof. We get

$$X_n^{(p)}(t) = \rho(X_{n-1}^{(p)})(t) + a_1(Z_{n,1}^{(p)})(t) + \varepsilon_n^{(p)}(t)$$

and

$$X_n^{(p)}(t-) = \rho(X_{n-1}^{(p)})(t-) + a_1(Z_{n,1}^{(p)})(t-) + \varepsilon_n^{(p)}(t-)$$

thus

$$\begin{aligned} X_n^{(p)}(t) - X_n^{(p)}(t-) = {} & [\rho(X_{n-1}^{(p)})(t) - \rho(X_{n-1}^{(p)})(t-)] \\ & + [a_1(Z_{n,1}^{(p)})(t) - a_1(Z_{n,1}^{(p)})(t-)] + [\varepsilon_n^{(p)}(t) - \varepsilon_n^{(p)}(t-)] \end{aligned}$$

Since A1 and A2 hold, one obtains

$$X_n^{(p)}(t) - X_n^{(p)}(t-) = \varepsilon_n^{(p)}(t) - \varepsilon_n^{(p)}(t-)$$

and there is a jump at T_p.

□

Thus $X^{(p)}$ and $\varepsilon^{(p)}$ have the same jump at T_p.
In order to obtain an exponential rate we make the following assumption

$$A4 - \left\|\varepsilon_n^{(p)}\right\|_\infty \leq b,\ n \geq 1,\ p \in \mathbb{N}$$

Proposition 2.2. *Under A1,A2,A3,A4 one get*

$$P\left(\frac{1}{n}\left|\sum_{i=1}^{n}[(X_i^{(p)}(t) - X_i^{(p)}(t-)) - E(X^{(p)}(t) - X^{(p)}(t-))]\right| \geq \alpha\right) \leq 2\exp\left(-\frac{n\alpha^2}{2b^2}\right) \quad (2.2)$$

$n \geq 1,\ p \in \mathbb{N},\ t \in T_p, \alpha > 0,$ *and*

$$\frac{1}{n}\sum_{i=1}^{n}[(X_i^{(p)}(t_j) - X_i^{(p)}(t_j-)) \to E(X^{(p)}(t_j) - X^{(p)}(t_j-)) \quad (2.3)$$

almost completely, $p \in \mathbb{N},\ 1 \leq j \leq k.$

Proof. We put

$$\bar{D}_n = \frac{1}{n}\sum_{i=1}^{n}[(X_i^{(p)}(t) - X_i^{(p)}(t-)) - E(X^{(p)}(t) - X^{(p)}(t-))]$$

and

$$\bar{\Delta}_n = \frac{1}{n}\sum_{i=1}^{n}[(\varepsilon_i^{(p)}(t) - \varepsilon_i^{(p)}(t-)) - E(\varepsilon^{(p)}(t) - \varepsilon^{(p)}(t-))].$$

Then from Proposition 2.1 we get

$$\bar{D}_n = \bar{\Delta}_n \tag{2.4}$$

Now, by using Hoeffding inequality (cf Bosq and Blanke [2007] p.297), and since $(\varepsilon_i^{(p)}(t) - \varepsilon_i^{(p)}(t-),\ 1 \leq i \leq n)$ are independent and bounded from A4, one obtains an exponential rate and (2.2) holds.

Then since

$$\sum_{n\geq 1} 2\exp(-\frac{n\alpha^2}{2b^2}) < \infty$$

We get the Borel-Cantelli lemma and (2.3) follows. □

Remark 2.3. Boundedness in A4 may be replaced with existence of some exponential moment (See Bosq [2000] p.221).

Remark 2.4. Finally a random jump with derivative can be considered (cf Bosq [2015] p.12-14).

Thus, a jump at t_j in the context of p derivatives is available.

It is also possible to slightly modify $\bar{D}_{n,p}(t)$, in the i.i.d. case we put

$$\bar{D}_{n,p}(t) = \frac{1}{n}\sum_{i=1}^{n}\left[X_i^{(p)}(t+\beta_n) - X_i^{(p)}(t-\alpha_n)\right]$$

$0 < t - \alpha_n < t + \beta_n < 1$,

and we make the following assumption:

B- $X^{(p)}$get a derivative on the right (resp. on the left), with Taylor rest and $X^{(p+1)}$ is bounded by b almost surely.

and

C- $0 < E\left|X^{(p)}(t) - X^{(p)}(t-)\right| < \infty$, where $T_p = \{t_1, ..., t_k\}$ has a jump in derivative.

Proposition 2.5. *If B and C hold, we have*

$$\bar{D}_{n,p}(t) \to E(X^{(p)}(t) - X^{(p)}(t-)), a.s.\ t \in T_p. \tag{2.5}$$

and

$$\bar{D}_{n,p}(t) \to 0\ a.s.\ if\ (\alpha_n, \beta_n) \to (0,0)\ t \notin T_p \tag{2.6}$$

Proof. From B one obtains

$$X_i^{(p)}(t+\beta_n) = X_i^{(p)}(t) + \beta_n X_i^{(p+1)}(t+\theta_n\beta_n), \tag{2.7}$$

$0 < t < t+\beta_n < 1$, $0 < \theta_n < 1$, $1 \le i \le n$, and

$$X_i^{(p)}(t-\alpha_n) = X_i^{(p)}(t-) - \alpha_n X_i^{(p+1)}(t-\theta_n'\alpha_n), \tag{2.8}$$

$0 < t-\alpha_n < t < 1$, $0 < \theta_n' < 1$, $1 \le i \le n$, hence

$$\bar{D}_{n,p}(t) = \frac{1}{n}\sum_{i=1}^{n}\left[X_i^{(p)}(t) - X_i^{(p)}(t-)\right] + \bar{Y}_n^{(p)}$$

where

$$\bar{Y}_n^{(p)} = \frac{1}{n}\sum_{i=1}^{n}\left[\beta_n X_i^{(p+1)}(t+\theta_n\beta_n) - \alpha_n X_i^{(p+1)}((t-) - \theta_n'\alpha_n)\right]$$

and now, from B, we get

$$\left|\bar{D}_{n,p}(t) - \frac{1}{n}\sum_{i=1}^{n}\left[X_i^{(p)}(t) - X_i^{(p)}(t-)\right]\right| \le b\,(\alpha_n + \beta_n) \to 0,\ a.s.$$

and (2.5) holds since C gives

$$\frac{1}{n}\sum_{i=1}^{n}\left[X_i^{(p)}(t) - X_i^{(p)}(t-)\right] \to E[X^{(p)}(t) - X^{(p)}(t-)]\ a.s.$$

If t has no jump, we get

$$\left|\bar{D}_{n,p}(t)\right| \le b\,(\alpha_n + \beta_n) \to 0,\ a.s.$$

□

Then we obtain:

Proposition 2.6. *(Convergence in distribution)*
If A1,A2,A3, and B, C holds, we get $\frac{\alpha_n+\beta_n}{\sqrt{n}} \to 0$, *and one obtains*

$$\frac{1}{\sqrt{n}} \sum \left[X_i^{(p)}(t+\beta_n) - X_i^{(p)}(t-\alpha_n) - E(X_i^{(p)}(t) - X_i^{(p)}(t-)) \right]$$

$$\Rightarrow N \sim \mathcal{N}(0, \sigma^2(p)), \quad (2.9)$$

$0 < t-\alpha_n < t+\beta_n < 1$, $t \varepsilon T_p$; *where* $\sigma^2(p) = V(X^{(p)}(t) - X^{(p)}(t-))$, $p \in \mathbb{N}$; *and*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[X_i^{(p)}(t+\beta_n) - X_i^{(p)}(t-\alpha_n) \right] \to 0, \quad (2.10)$$

in probability, with $t \notin T_p$, *since* $\sigma^2(p) = 0$.

Proof. We get

$$R_n^{(p)}(t)$$
$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[X_i^{(p)}(t+\beta_n) - X_i^{(p)}(t-\alpha_n) - E(X_i^{(p)}(t+\beta_n) - X_i^{(p)}(t-\alpha_n)) \right].$$

We put

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\beta_n X_i^{(p+1)}(t+\theta_n \beta_n) - \alpha_n X_i^{(p+1)}((t-) - \theta_n' \alpha_n) \right] := Z_n$$

and by using (2.7) and (2.8) we get

$$R_n^{(p)}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [(X_i^{(p)}(t) - X_i^{(p)}(t-) - E(X_i^{(p)}(t) - X_i^{(p)}(t-)] + Z_n - EZ_n$$

Now, in the context of B, one obtains

$$|Z_n - EZ_n| \leq 2 \left\| X^{(p+1)} \right\|_\infty \frac{\alpha_n + \beta_n}{\sqrt{n}}$$

and since

$$R_n^{(p)}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [(X_i^{(p)}(t) - X_i^{(p)}(t-) - E(X_i^{(p)}(t) - X_i^{(p)}(t-)] + Z_n - EZ_n$$

then, since $\frac{\alpha_n+\beta_n}{\sqrt{n}} \to 0$, one obtains Theorem 25.4 p. 332 [cf Billingsley, 1995]. Finally, we get the desired result from Proposition 2.1.

(2.10) is clear since the gaussian distribution is vanishing. □

3 Exogeneity

Now, we consider a fixed jump in derivative with exogenous random variables. For convenience, we envisage $q = 1$ (see also Remark 3.4).

A jump in derivative appears at $S_p = \{s_1, ..., s_h\}$ where $s_0 = 0 < s_1 < ... < s_h < 1 = s_{h+1}, s \in S_p$, in the context of $a_1(Z_{n,1}^{(p)})(t)$ (cf Section 2).

Now we get:

$$A5 - X_n^{(p)}(s) = X_n^{(p)}(s-),\ s \in S_p$$

Then

Lemma 3.1. *From A1,A3,A5 , we get*

$$a_1(Z_{n,1}^{(p)})(s-) - a_1(Z_{n,1}^{(p)})(s) = \varepsilon_n^{(p)}(s) - \varepsilon_n^{(p)}(s-) \neq 0\, a.s.\,, s \in S_p,\ n \geq 1,\ p \in \mathbb{N} \tag{3.1}$$

Proof. We put

$$X_n^{(p)}(s) = \rho(X_{n-1}^{(p)})(s) + a_1(Z_n^{(p)})(s) + \varepsilon_n^{(p)}(s)$$

and

$$X_n^{(p)}(s-) = \rho(X_{n-1}^{(p)})(s-) + a_1(Z_n^{(p)})(s-) + \varepsilon_n^{(p)}(s-)$$

Thus

$$\begin{aligned} X_n^{(p)}(s) - X_n^{(p)}(s-) = {} & [\rho(X_{n-1}^{(p)})(s) - \rho(X_{n-1}^{(p)})(s-)]] \\ & + [a_{,1}(Z_{n,1}^{(p)})(s) - a_{,1}(Z_{n,,1}^{(p)})(s-) + [\varepsilon_n^{(p)}(s) - \varepsilon_n^{(p)}(s-)] \end{aligned}$$

Then, from A1, A5 it is easy to see that (3.1) holds. □

Proposition 3.2. *Under A1,A3,A4,A5 , it holds*

$$\begin{aligned} P(\frac{1}{n} | \sum_{i=1}^{n} \{ & (a_1(Z_{i,1}^{(p)})(s_j) - (a_{1,}(Z_{i,1}^{(p)})(s_j-) \\ & - E(a_{1,}(Z_{i,1}^{(p)})(s_j) - (a_{1,}(Z_{i,1}^{(p)})(s_j-)\}| \geq \alpha) \leq 2\exp(-\frac{n\alpha^2}{2b^2}) \end{aligned} \tag{3.2}$$

$1 \leq j \leq h,\ n \geq 1,\ p \in \mathbb{N}$ *and*

$$\frac{1}{n} \sum_{i=1}^{n} \{(a_{1,j}(Z_{i,1}^{(p)})(s_j) - (a_{1,j}(Z_{i,1}^{(p)})(s_j-) \to E(a_{1,j}(Z_{i,1}^{(p)})(s_j) - a_{1,j}(Z_{i,1}^{(p)})(s_j-)) \tag{3.3}$$

almost completely, $1 \leq j \leq h,\ n \geq 1,\ p \in \mathbb{N}$.

Proof. Lemma 3.1 entails

$$a_{1,}(Z_{n,1}^{(p)})(s_j-) - a_{1,}(Z_{n,1}^{(p)})(s_j) = \varepsilon^{(p)}(s_j) - \varepsilon^{(p)}(s_j-) \neq 0 \, a.s.$$

and, one obtains Hoeffding inequality (cf Bosq and Blanke [2007] p.297).

Finally, we get

$$\sum_{n=1}^{\infty} 2\exp(-\frac{n\alpha^2}{2b^2}) < \infty$$

and the convergence is almost complete by Borel-Cantelli lemma, cf (3.3). □

Remark 3.3. Then, a jump at s_j in a context of p derivatives holds.

Remark 3.4. Now, it is possible to envisage q jumps. We set $a_j(Z_{n,j}^{(p)})(s)$ where $s \in S_j^{(p)} = \left\{s_{j,1}, ..., s_{j,k_j}\right\}$, with $0 < s_{j,1} < ... < s_{j,k_j} < 1$, $1 \leq j \leq q$ where k_j is an integer. Then, it holds

$$R_p = T_p \cap (\cap_{j=1}^{q} S_j^{(p)}) = \emptyset$$

where $p \in \mathbb{N}$. Thus, Lemma 3.1 and Proposition 3.2 are clear.

Note also that the CLT and the LLN hold with exogeneity and derivative:

Proposition 3.5. *(Guillas)*
Let $X^{(p)}, Z^{(p)}, \eta^{(p)}, \varepsilon^{(p)}$ be a bounded ARHX(p), if $p \in \mathbb{N}$, we get

$$Z_n^{(p)} = u(Z_{n-1}^{(p)}) + \eta_n^{(p)}$$
$$X_n^{(p)} = \rho(X_{n-1}^{(p)}) + a(Z_n^{(p)}) + \varepsilon_n^{(p)}$$

with $\|u\| \leq \|\rho\| < 1$, then

$$\frac{X_1^{(p)} + ... + X_n^{(p)}}{n} \to 0$$

almost surely and in L_H^2, and

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i^{(p)} \Longrightarrow \mathcal{N}(0, C_p)$$

Proof. In the context of Guillas [2000a,b], Theorems 1 and 2, we may replace X, Y, η, ε with $X^{(p)}, Y^{(p)}, \eta^{(p)}, \varepsilon^{(p)}$ and the result is straightforward. □

4 Prediction

We now consider forecasting over $t \in [0, 1]$.

We make the following assumption:

$$A6 - (X_{n-1}^{(p)}, Z_{n,1}^{(p)}, \varepsilon_n^{(p)}, \eta_n^{(p)}) \text{ are globally independent } n \geq 1,\ p \in \mathbb{N}$$

Then, we put

$$U_{n,}^{(p)}(t) = (X_{n-1}^{(p)}(t), Z_{n,1}^{(p)}(t))$$

Consequently

$$E(X_n^{(p)}(t)|U_n^{(p)}(t)) = \rho(X_{n-1}^{(p)})(t) + a(Z_n^{(p)})(t) + E(\varepsilon_n^{(p)}(t)|U_n^{(p)}(t))$$

and, since A6 are independent, one obtains

$$E(\varepsilon_n^{(p)}(t)|U_{n,j}^{(p)}(t)) = E(\varepsilon_n^{(p)}(t)) = 0.$$

Then, for obtaining prediction, it is necessary to estimate ρ, a, u in the context of exogenous random variables (cf Section 5).

There is a lot of papers about prediction but Mas [2007] and Brunel et al. [2016] are interesting.

5 Estimating linear operators

5.1 Exponential rate

In the context of $t \in [0, 1]$ it holds

$$D = \rho C,$$

where D is a cross-covariance operator and C a covariance operator.

Also, from Mercer Lemma, we have

$$C = \sum_{r=1}^{\infty} \lambda_r\, v_r \otimes v_r,$$

with

$$C(v_r) = \lambda_r v_r,\ r \geq 1,.$$

To this aim, we envisage the empirical covariance in the context of derivative, given by

$$C_{n,}^{(p)} = \frac{1}{n}\sum_{i=1}^{n} X_i^{(p)} \otimes X_i^{(p)}$$

and the empirical cross-covariance operator get the form

$$D_{n,}^{(p)} = \frac{1}{n-1}\sum_{i=1}^{n-1} X_i^{(p)} \otimes X_{i+1}^{(p)},\ n \geq 2.$$

Since C is not invertible, we must obtain a truncation at k_n, that is

$$\hat{\rho}_{n,} = \Pi^{k_n} D_n^{(p)}\, C_n^{-1,(p)} \Pi^{k_n}$$

where $\Pi^{k_n}$is the projector over $v_1, ..., v_{k_n}$.

Similarly, we set

$$\Gamma_n^{(p)} = \frac{1}{n}\sum_{i=1}^{n} Z_i^{(p)} \otimes Z_i^{(p)}$$

and

$$\Delta_n^{(p)} = \frac{1}{n-1}\sum_{i=1}^{n-1} Z_i^{'p)} \otimes Z_{i+1}^{(p)},\ n \geq 2.$$

And, one obtains:

$$\hat{a_n}^{(p)} = \Pi^{k_n} \Delta_n^{(p)}\, \Gamma_n^{-1,(p)} \Pi^{k_n}$$

An analogous result holds concerning u.

Finally the predictor takes the form

$$\hat{X}_n^{(p)}(t) = \hat{\rho}_n(X_{n-1}^{(p)})(t) + \hat{a}_{1,n}(Z_{n,1}^{(p)})(t),$$

where

$$\hat{Z}_{n,h}^{(p)}(t) = \hat{u}_n(Z_{n-1}^{(p)})(t).$$

Now, we need to obtain an exponential rate with two assumptions:

$$A7 - \lambda_r = b\, r^{-\gamma},\ (b > 0, \gamma > 1),\ r > 1$$

and

$$A8 - \left\|X_n^{(p)}\right\|_\infty \textit{ is bounded},\ n \geq 1,\ p \in \mathbb{N}$$

Then we have

Proposition 5.1. *Under assumptions A1, A3, A5, A7, A8, one obtains*

$$P(\left\|\hat{\rho}_{n,}(X_n^{(p)}) - \rho(X_n^{(p)})\right\|_{\mathcal{L}} \geq \varepsilon) \leq c_{1,}(\varepsilon, p)\, \exp(-c_{2,}(\varepsilon,\, p)\, \frac{n}{(\log n)^{4(1+\gamma)}}), \tag{5.1}$$

$\eta > 0,$, $n \geq n_{1,\eta}$, $\gamma > 1$, $p \in \mathbb{N}$.

And

$$P(\left\|\hat{a}_{h,n,}(Z_n^{(p)}) - a_{h,}(Z_n^{(p)})\right\|_{\mathcal{L}} \geq \varepsilon) \leq d_{1,}(\varepsilon,\, p)\, \exp(-d_{2,}(\varepsilon,\, p)\, \frac{n}{(\log n)^{4(1+\gamma)}}) \tag{5.2}$$

$\eta > 0,$, $n \geq n_{2,\eta}$, $\gamma > 1$, $p \in \mathbb{N}$.

And, finally

$$P(\left\|\hat{u}_{h,n,}(Z_n^{(p)}) - u_{h,}(Z_n^{(p)})\right\|_{\mathcal{L}} \geq \varepsilon) \leq e_{1,}(\varepsilon,\, p)\, \exp(-e_{2,}(\varepsilon,\, p)\, \frac{n}{(\log n)^{4(1+\gamma)}}) \tag{5.3}$$

$\eta > 0,$, $n \geq n_{3,\eta}$, $\gamma > 1$, $p \in \mathbb{N}$.

Proof. From Theorem 8.8 and Example 8.6 p.221 of Bosq [2000], we get

$$P(\|\hat{\rho}_{n,} - \rho\|_{\mathcal{L}} \geq \eta) \leq c_{1,j}(\eta)\, \exp(-c_{2,j}(\eta)\, \frac{n}{(\log n)^{4(1+\gamma)}}) \tag{5.4}$$

$$\eta > 0,\, n \geq n_{\eta},\, c_{1,j}(\eta) > 0,\, c_{2,J}(\eta) > 0,\, ,\, \gamma > 1,\, p \in \mathbb{N};$$

cf p.232-234 in Bosq [2000] with some little modifications.

Now, since A8 is bounded, we have

$$P(\left\|\hat{\rho}_{n,}(X_n^{(p)}) - \rho(X_n^{(p)})\right\|_{\mathcal{L}} \geq \eta) \leq P(\|\hat{\rho}_{n,} - \rho\|_{\mathcal{L}} \geq \frac{\eta}{\left\|X_n^{(p)}\right\|_{\infty}})$$

and from (5.4), one obtains

$$P(\left\|\hat{\rho}_{n,}(X_n^{(p)}) - \rho(X_n^{(p)})\right\|_{\mathcal{L}} \geq \varepsilon) \leq c_{1,}(\varepsilon, p)\, \exp(-c_{2,}(\varepsilon, p)\, \frac{n}{(\log n)^{4(1+\gamma)}})$$

where $\frac{\eta}{\left\|X_n^{(p)}\right\|} = \varepsilon$, then (5.1) holds.

(5.2) and (5.3) are similar. □

Remark 5.2. Note that we get $k_n \cong \log n$.

Then, the next statement entails:

Corollary 5.3. *Assumptions A1, A3, A5, A7, A8 give*

$$P(\sup_{0\leq t\leq 1}|\hat{X}_n^{(p)}(t)-X_n^{(p)}(t)|\geq\eta)=\mathcal{O}(\exp(-\delta(\eta,q)\frac{n}{(\log n)^{4(1+\gamma)}})$$

$p\in\mathbb{N}$, $n\geq n(\eta,q)$.

Proof. Clear, from Proposition 5.1. □

Remark 5.4. It is also possible to modify ρ by using regression in derivative. We put

$$\hat{r}_{n,j}(X_n^{(p)})=\frac{\sum_{i=1}^{n-1}X_{i+1}^{(p)}K(\frac{\left\|X_i^{(p)}-X_n^{(p)}\right\|_H}{h_n})}{\sum_{i=1}^{n-1}K(\frac{\left\|X_i^{(p)}-X_n^{(p)}\right\|_H}{h_n})},$$

$0\leq j\leq k$, where K is gaussian and $h_n\to 0$ (See Damon and Guillas [2002] p.765, equation (5)).

5.2 Almost sure convergence

In the context of Hilbertian operators, we get ρ and a with

$$X_n=\rho(X_{n-1})+a(Z_n)+\varepsilon_n,\ n\geq 1$$

and we envisage projection with an orthonormal basis. Also, we suppose that ρ and a are symmetric.

We put

$$\rho(v_j)=\lambda_j v_j,\ j\geq 1, \tag{5.5}$$

and

$$a(v_j)=\mu_j v_j,\ j\geq 1, \tag{5.6}$$

where the $v_j's$ are orthonormal.

We consider the following assumption

$$A10-(\varepsilon_n)\,\mathrm{II}\,(Z_n)$$

cf Damon and Guillas [2005] p. 187.

We need the following hypothesis:

$$A11-E(\langle Z_n,v_j\rangle^2)<\infty\ and\ E(\langle X_n,v_j\rangle^2)<\infty$$

Proposition 5.5. *From A10 and A11, one obtains*

$$\hat{\lambda}_{jn} = \frac{\frac{1}{n-1}\sum_{i=1}^{n-1} \langle X_i, v_j \rangle \langle X_{i+1}, v_j \rangle}{\frac{1}{n}\sum_{i=1}^{n} \langle X_i, v_j \rangle^2}, \; j \geq 1, n \geq 2 \tag{5.7}$$

and

$$\hat{\mu}_{jn} = \frac{\frac{1}{n}\sum_{i=1}^{n} \langle X_i, v_j \rangle \langle Z_i, v_j \rangle}{\frac{1}{n}\sum_{i=1}^{n} \langle Z_i, v_j \rangle^2}, \; j \geq 1, n \geq 1 \tag{5.8}$$

Proof. By using v_j, it holds

$$\langle X_n, v_j \rangle = \langle \rho(X_{n-1}), v_j \rangle + \langle a(Z_n), v_j \rangle + \langle \varepsilon_n, v_j \rangle$$

and from (5.5) and (5.6) (7.1) and (7.2) we get

$$\langle X_n, v_j \rangle = \lambda_j \langle X_{n-1}, v_j \rangle + \mu_j \langle Z_n, v_j \rangle + \langle \varepsilon_n, v_j \rangle, \tag{5.9}$$

and

$$\langle X_n, v_j \rangle \langle X_{n-1}, v_j \rangle = \lambda_j \langle X_{n-1}, v_j \rangle^2 + \mu_j (\langle Z_n, v_j \rangle \langle X_{n-1}, v_j \rangle) + \langle \varepsilon_n, v_j \rangle \langle X_{n-1}, v_j \rangle$$

but since

$$X_{n-1} = \sum_{j=0}^{\infty} \rho^j(\varepsilon_{n-1-j})$$

(see Bosq and Blanke [2007] p.245), then from A10 , we obtain

$$\langle Z_n, v_j \rangle \amalg \langle X_{n-1}, v_j \rangle$$

and also

$$\langle \varepsilon_n, v_j \rangle \amalg \langle X_{n-1}, v_j \rangle$$

thus

$$E(\langle X_n, v_j \rangle \langle X_{n-1}, v_j \rangle) = \lambda_j E(\langle X_{n-1}, v_j \rangle^2)$$

and one obtains an empirical form in the context of (5.7).

Now, by using (5.9) (7.5), we get

$$\langle X_n, v_j \rangle = \lambda_j \langle X_{n-1}, v_j \rangle + \mu_j \langle Z_n, v_j \rangle + \langle \varepsilon_n, v_j \rangle$$

and inserting $\langle Z_n, v_j \rangle$ one obtains, by using A11

$$E(\langle X_n, v_j \rangle \langle Z_n, v_j \rangle) = \mu_j E(\langle Z_n, v_j \rangle^2)$$

and the empirical form appears in (5.8). □

Then

Corollary 5.6. *From A10 and A11, it holds*

$$\hat{\lambda}_{jn} \to \lambda_j \; a.s., \; j \geq 1, \tag{5.10}$$

and

$$\hat{\mu}_{jn} \to \mu_j, \; a.s., \; j \geq 1, \tag{5.11}$$

Proof. From A11, one obtains

$$E(|\langle X_n, v_j \rangle \langle X_{n-1}, v_j \rangle|) \leq [E(\langle X_n, v_j \rangle^2)]^{\frac{1}{2}} [E(\langle X_{n-1}, v_j \rangle^2)]^{\frac{1}{2}} < \infty$$

Then, from the strong law of large numbers (cf. Shiryaev [1996], p.325), we get

$$\frac{1}{n-1} \sum_{i=1}^{n-1} \langle X_i, v_j \rangle \langle X_{i+1}, v_j \rangle \to E(\langle X_1, v_j \rangle \langle X_2, v_j \rangle), \; a.s.$$

and

$$\frac{1}{n} \sum_{i=1}^{n} \langle X_i, v_j \rangle^2 \to E(\langle X, v_j \rangle^2)$$

then

$$\hat{\lambda}_{jn} \to \lambda_j \; a.s$$

and (5.10) holds.

On the other hand

$$E(|\langle X_i, v_j \rangle \langle Z_i, v_j \rangle|) \leq [E(\langle X_i, v_j \rangle^2)]^{\frac{1}{2}} [E(\langle Z_i, v_j \rangle^2)]^{\frac{1}{2}} < \infty$$

thus

$$\frac{1}{n} \sum_{i=1}^{n} \langle X_i, v_j \rangle \langle Z_i, v_j \rangle \to E(\langle X, v_j \rangle \langle Z, v_j \rangle), \; a.s.$$

and also

$$\frac{1}{n} \sum_{i=1}^{n} \langle Z_i, v_j \rangle^2 \to E(\langle Z, v_j \rangle^2), \; a.s.$$

then

$$\hat{\mu}_{jn} \to \mu_j, \; a.s$$

and (5.11) holds. □

Remark 5.7. An infinite dimensional vector get the form

$$\langle X_n, v_j \rangle = \hat{\lambda}_{jn} \langle X_{n-1}, v_j \rangle + \hat{\mu}_{jn} \langle Z_n, v_j \rangle + \langle \varepsilon_n, v_j \rangle, \ j \geq 1$$

Remark 5.8. An exponential rate may be envisaged, if $\|X_n\|$ is bounded, we get

$$P(\sup_{j \geq 1} |\hat{\lambda}_{jn} - \lambda_j| \geq \eta) \leq 4 \exp(-\frac{n\eta^2}{\alpha + \beta\eta})$$

n≥ 1,$\alpha > 0$,$\beta > 0$,$\eta > 0$, cf Bosq [2000] p. 99 and 104.

$\hat{\mu}_{jn}$ is similar.

Remark 5.9. A derivative of order p can also be considered.

6 High frequency data

It is also possible to observe high frequency data, namely $(X_i^{(p)}(\frac{j}{r_n}),\mathrm{r}_n \geq 1, 0 \leq j \leq r_n, 1 \leq i \leq n)$ with $p \in \mathbb{N}$ and to envisage exogenous random variables.

In order to simplify the exposition, we suppose that $X^{(p)}$ has a single jump at $t_0 \in [0,1[$.

Since data are discrete, we need a regularity assumption for the sample path, that is

$$A9 - |X^{(p)}(t) - X^{(p)}(s)| \leq c|t-s|^\alpha, \ (s,t) \in [0,t_0[^2 \cup [t_0,1]^2, \ c > 0, \ \alpha \in]0,1].$$

Now, there exists an integer $j_0(n)$ such that

$$\frac{j_0(n) - 1}{r_n} < t_0 \leq \frac{j_0(n)}{r_n} < 1$$

Then, we construct a detector of t_0 by putting

$$\hat{t}_{0,n} = \frac{1}{r_n} \arg \max_{1 \leq j \leq r_n} S_{j,n}^{(p)}$$

where

$$S_{j,n}^{(p)} = \frac{1}{n} \sum_{i=1}^{n} (X_i^{(p)}(\frac{j}{r_n}) - X_i^{(p)}(\frac{j-1}{r_n})),$$

$1 \leq j \leq r_n$,, $p \in \mathbb{N}$.

Then we get the following statement

Proposition 6.1. *From A9 we obtain*

$$\hat{t}_{0,n} \to t_0(+) \ a.s.$$

and

$$\hat{t}_{0,n} - \frac{1}{r_n} \to t_0(-) \ a.s.$$

with

$$\frac{1}{n}\sum_{i=1}^{n}(X_i^{(p)}(\hat{t}_{0,n}) - X_i^{(p)}(\hat{t}_{0,n} - \frac{1}{r_n})) \to E(X^{(p)}(t_0) - X^{(p)}(t_0-)) \ a.s.$$

where $r_n \to \infty$.

Proof. In the proof of Proposition 4 we envisage some little modifications but the proof is clear [see Bosq, 2015]. □

Remark 6.2. In the context of exogenous random variables one consider $a_h(Z_{n,h}^{(p)})$ and the proof is clean.

Remark 6.3. It is also possible to obtain a jump at $X^{(p+1)}$, namely:

$$\bar{D}_n^{(p)}(t_0) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{X_i^{(p)}(t_0+\alpha_n) - X_i^{(p)}(t_0+)}{\alpha_n} - \frac{X_i^{(p)}(t_0-\beta_n) - X_i^{(p)}(t_0-)}{-\beta_n}\right]$$

$0 < t_0 - \beta_n < t_0 < t_0 + \alpha_n < 1$ with $\alpha_n \downarrow 0(+)$ and $\beta_n \downarrow 0(+)$.

We consider one jump at t_0, a Taylor expansion and a bounded jump at $X^{(p)}$, then we have

$$\bar{D}_n^{(p)}(t_0) \to E(X^{(p+1)}(t_0+) - X^{(p+1)}(t_0-)) \ a.s.$$

cf Bosq [2016] Proposition 3.2 p.731-732.

7 Possible extension

Remark 7.1. The $ARMAD^{(p)}(k,r)$ process can also be considered (cf Bosq [2016] p.730) in this context:

$$X_n^{(p)} = \sum_{i=1}^{k} l_i(X_{n-i}^{(p)}) + Y_n^{(p)} - \sum_{i=1}^{r}\lambda_i(Y_{n-i}^{(p)}), \ \ p \in \mathbb{N}.$$

But this expression is rather complicate.

Remark 7.2. We may envisage a fixed jump in the context of a continuous function cf $C_j = C([t_j, t_{j+1}])$, $0 \leq j \leq k+1$.

Then, we consider a jump at $t_j \leq s$, $t \leq t_{j+1}$, where

$$C_{n,j}(s, t) = \frac{1}{n} \sum_{i=1}^{n} X_i(s)\, X_i(t) \,\|_{t_j \leq s \leq t_{j+1},\, t_j \leq t \leq t_{j+1},}$$

and

$$D_{n,j}(s, t) = \frac{1}{n-1} \sum_{i=1}^{n} X_i(s)\, X_{i+1}(t) \,\|_{t_j \leq s \leq t_{j+1},\, t_j \leq t \leq t_{j+1},}$$

with

$$\rho_{n,j} = \Pi^{k_n} D_{n,j}\, C_{n,j}^{-1} \Pi^{k_n}$$

where $\Pi^{k_n}$is the projector over $v_1, ..., v_{k_n}$.
$a_{n,j}$ and $u_{n,j}$ are similar (cf Section 5).

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