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Numéro spécial en l'honneur des 80 ans de Denis Bosq /

Special issue in honour of Denis Bosq's 80th birthday

Almost Sure convergence of the Resolvent Estimators for Hilbertian Autoregressive Processes

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Abstract: We consider the class of resolvent estimators of the correlation operator ruling the functional autoregressive processes introduced by Mas, A. ([10][11]). Under mild conditions on smoothing parameter, we establish exponential bounds and almost sure convergence of the resolvent estimators as well as convergence rates improving the existing results. As a consequence we derive asymptotic results on the resolvent predictors. Numerical studies illustrate the performance of the resolvent predictors giving a comparison with other existing prediction methods both on simulated and real functional data sets showing competitive results.

1. Introduction

In common many fields such as economy, finance, industry, biology, medicine..., data acquisition (measurements at several consecutive time points or during a continuous time interval) and processing techniques have lead to a continuous flow of data putting statisticians to consider them as "high dimensional" vectors. Such data are conveniently described as realizations of random curves and seen as a sample of a valued function space random variable.

Functional Autoregressive Processes has been largely investigated by many authors for modeling and predicting continuous time random processes. Applications in a variety of domains have been successfully performed. Examples range from prediction of electricity consumption, road traffic, El-Nino temperature to concentration levels in air pollution (see the monograph by Bosq [5]). An application to environmental data processing was given by Omatu [15] predicting the concentration levels of pollutants in air pollution analyzed through a Bucy-Kalman filter system. These generalizations lead to a growing demand for developing statistical methods which are carried out through the Functional Data Analysis (FDA). The latter is now fairly well understood and has yet reached a unified form and several functional models have been covered. For a fairly recent account on statistical inference with various generalizations to functional models with interesting nonparametric issues we mention but a few ([5][8][16]) and references therein).

In this paper, we are interested by the class of resolvent estimators studied by Mas [10][11] in the framework of Functional Autoregressive Processes. The estimation methods are related to the well known linear ill-posed inverse problem and on a very basic facts of perturbation theory (see [9] or [7]). Mas in [10] provides asymptotic distribution and deduces convergence in probability for this class of estimators. In this work we investigate the asymptotic results of this class of resolvent estimators at the light on techniques martingale giving exponential bounds and recent results on empirical covariance operator of this class of processes ([5] Chap. 2). This allows us to obtain exponential bounds and almost sure convergence of the class of resolvent estimators as well as convergence rate depending on smoothing parameter. Then we provide similar results on resolvent predictors. Numerical simulations and practical use of these results are handled to show the ability of these estimators and their performance on the base of two errors. Some perspectives could be investigated on the choice of optimal smoothing parameter.

This paper is organized as follows. Section 2 introduces notations and definitions. Section 3 deals with main results. Section 4 contains numerical simulations and examples . Proofs are

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postponed to Section 5.

2. Definitions and Notations

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space and (H, \mathcal{H}) a real separable Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle$, the associate norm $\|\cdot\|$ and endowed with its Borel σ -field \mathcal{H} . We denote by $\mathcal{L}(H)$ the separable space of bounded linear operators defined on H into H , and by $\|\cdot\|_{\mathcal{L}}$ the usual norm of bounded linear operators. A strong H -valued white noise $(\varepsilon_n, n \in \mathbb{Z})$ defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in H , is a sequence of H -valued i.i.d. rv's with zero mean and $E\|\varepsilon_n\|^2 := \sigma_\varepsilon^2 > 0$. Let ρ be a bounded linear operator in $\mathcal{L}(H)$ endowed with its Borel σ -field. We consider a H -valued sequence $(X_n, n \in \mathbb{Z})$ of random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ which verifies the equation

$$(2.1) \quad X_n - \mu = \rho(X_{n-1} - \mu) + \varepsilon_n, \quad n \in \mathbb{Z},$$

where $(\varepsilon_n, n \in \mathbb{Z})$ is a H -valued white noise, $\rho \in \mathcal{L}(H)$ and $\mu \in H$.

The sequence $(X_n, n \in \mathbb{Z})$ is then called Hilbert space valued Autoregressive Process (we abbreviate it by *ARH*). We study general model defined by (2.1) under a condition ensuring a unique strictly stationary solution of (2.1) ([5] Chap. 2) :

Condition C:

- (i) $\|\rho\|_{\mathcal{L}}^j < 1$ for $j \geq j_0$,
- (ii) $\mathbb{E}\|X_0\|^4 < \infty$.

Under **Condition C** the strictly stationary solution of (2.1) is such that $E(X_n) = \mu$ and is given by

$$(2.2) \quad X_n - \mu = \sum_{i=0}^{\infty} \rho^i \varepsilon_{n-i}, \quad n \in \mathbb{Z},$$

where the series converges a.s. and in L_H^2 .

We consider the space of Hilbert-Schmidt operators

$$\mathcal{S}(\mathcal{H}) = \left\{ s \in \mathcal{L}(H) : \sum_{p \in \mathbb{N}} \|s(e_p)\|^2 < \infty \right\}$$

where $(e_i)_{i \in \mathbb{N}}$ is any complete orthonormal basis in H . The space \mathcal{S} becomes a separable Hilbert space with the scalar product on \mathcal{S} defined by: (see [7])

$\langle T_1, T_2 \rangle_{\mathcal{S}} = \sum_{i,j=1}^{\infty} \langle T_1(f_i), f_j \rangle_H \langle T_2(f_i), f_j \rangle_H$ and norm $\|\cdot\|_{\mathcal{S}}$. The covariance and cross covariance operators are respectively defined by

$$C_{X_0}(x) = \mathbb{E}(\langle X_0, x \rangle X_0), \quad R_{X_0, X_1}(x) = \mathbb{E}(\langle X_0, x \rangle X_1).$$

C_{X_0} is a bounded symmetric, compact, positif and nuclear operator. Their eigenvalues $(\lambda_i)_{i \geq 0}$ are ranked in decreasing order associated to the eigenvectors $(h_i)_{i \geq 0}$. From a sample $(X_i)_{1 \leq i \leq n}$ of (1), empirical covariance and cross covariance operators associated to C_{X_0} and R_{X_0, X_1} respectively are defined by

$$C_n(\cdot) = \frac{1}{n} \sum_{i=1}^n \langle X_i, \cdot \rangle X_i, \quad R_n(\cdot) = \frac{1}{n} \sum_{i=1}^n \langle X_i, \cdot \rangle X_{i+1}.$$

Estimation problem of the operator ρ , parameter of interest in the model (1), deeply relies on inverse problem in the moment equation: $R_{X_0, X_1} = \rho C_{X_0}$ or equivalently

$$R_{X_0, X_1}^* = C_{X_0} \rho^*$$

Mas in [10][11] defined the class of resolvent estimators of the adjoint of the parameter ρ . It is well known that the inverse operator of C_{X_0} exists whenever $\ker C_{X_0} = 0$ but it is unbounded (or not continuous) ([5][7]). Hence in the sequel we will assume that all the eigenvalues are strictly positive and that $\lambda_1 \geq \lambda_2 \geq \dots > 0$. The estimate of the inverse of C_{X_0} is based on empirical covariance operator C_n (which have the rank less than n thus it is no invertible on H). As it discussed in many books ([7][9][5][11]...), classical techniques carry out in inverse problems theory lead to regularize via spectral decomposition to define such continuous inverse. Firstly a regularized inverse of C_n is chosen as follows :

$$(C_n + \alpha_n I)^{-(p+1)} C_n^p$$

where $p \in \mathbb{N}^*$ and α_n is a smoothing parameter that is a decreasing sequence and positive reals. Let the sequence of functions $f_{n,p} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, (p integer indexing the function) defined by:

$$f_{n,p}(x) = \frac{x^p}{(x + \alpha_n)^{p+1}}$$

here α_n is a deterministic decreasing sequence and strictly positive. Functional calculus for bounded operators allows us to write $f_{n,p}(C_n)$ (as well as $f_{n,p}(C_{X_0})$) which give

$$f_{n,p}(C_n) = (C_n + \alpha_n I)^{-(p+1)} C_n^p$$

Mas in [10][11] defined the resolvent estimators of the adjoint of the parameter ρ by setting

$$(2.3) \quad \rho_{n,p}^* = f_{n,p}(C_n) R_n^*$$

Their name comes from the fact that the application $R_C(\lambda)$ on \mathbb{R} (or \mathbb{C}) defined by $R_C(\lambda) = (C + \lambda I)^{-1}$ is the resolvent of the operator C .

In the case of symmetric operator ρ , the resolvent estimators are simply defined by $\rho_{n,p}$.

Remark 1.

- The supremum norm of the operator $f_{n,p}(C_n)$ is equal to α_n^{-1} (almost surely).
- Since the operators C_n and C_{X_0} are positive, this implies that operators $(C_n + \alpha_n I)^p$ and $(C + \alpha_n I)^p$ are invertible with continuous inverse for all p .
- For $p \geq 1$, $f_{n,p}(C_n)$ is a compact operator since $f_{n,p}(0) = 0$.

Mas, A. in [10][11] provides the limit law of the resolvent estimators defined in (3) and deduced the convergence in probability under **Condition C** and $\ker C_{X_0} = 0$:

Theorem 2.1. ([11] Th.3 Chap.3). If $\sqrt{n}\alpha_n^{p+1} \rightarrow \infty$ and $\mathbb{E}\|C_{X_0}^{-1}\varepsilon_0\|^2 < \infty$, then we have for the uniform norm

$$\sqrt{n}(\rho_{n,p}^* - f_{n,p}(C_{X_0})C_{X_0}\rho^*) \Rightarrow_{n \rightarrow \infty} G_\rho$$

where G_ρ is a zero mean Gaussian random operator.

Furthermore

$$\begin{aligned} \|\rho_{n,p}^* - f_{n,p}(C_{X_0})C_{X_0}\rho^*\|_{\mathcal{L}} &\xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \\ \|\rho_{n,p}^*(x) - \rho^*(x)\|_{\mathcal{L}} &\xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \end{aligned}$$

for all $x \in H$.

We provide exponential bounds and almost sure convergence for this class of estimators as well as convergence rate depending on smoothing parameter. Then we give similar results on resolvent predictors. Numerical simulations illustrate their performance on the base of two errors. Some perspectives could be investigated on the choice of optimal smoothing parameter value.

3. Main results

With the help of the notations of Section 2, our aim is to give additional conditions to obtain exponential bounds and the almost surely convergence of the resolvent estimators as well as rate of convergence.

Theorem 3.1. Assume *Condition C (i)* and $\|X_0\| \leq c$ for some $c > 0$. Then for all $\eta > 0$,

$$\mathbb{P}(\|\rho_{n,p}^* - f_{n,p}(C_{X_0})C_{X_0}\rho^*\|_{\mathcal{L}} > \eta) \leq 4 \exp\left(-\frac{n\alpha_n^{2(p+1)}\eta^2}{\mu_1 + \mu_2\alpha_n^{p+1}\eta}\right)$$

where μ_1, μ_2 are positive constants.

Moreover we have,

$$\|\rho_{n,p}^* - f_{n,p}(C_{X_0})C_{X_0}\rho^*\|_{\mathcal{L}} = \mathcal{O}\left(\left(\frac{\log n}{n\alpha_n^{2(p+1)}}\right)^{\frac{1}{2}}\right) \text{ a.s.}$$

If $\alpha_n^{p+1} \left(\frac{n}{\log n}\right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} \infty$, then

$$\|\rho_{n,p}^* - f_{n,p}(C_{X_0})C_{X_0}\rho^*\|_{\mathcal{L}} \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.}$$

For the resolvent predictors $\rho_{n,p}^*(x)$ we have the following almost sure convergence.

Proposition 3.2. Assume *Condition C (i)* and $\|X_0\| \leq c$ for some $c > 0$.

If $\alpha_n^{p+1} \left(\frac{n}{\log n}\right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} \infty$, then for all $x \in H$:

$$\|\rho_{n,p}^*(x) - f_{n,p}(C_{X_0})C_{X_0}\rho^*(x)\|_{\mathcal{L}} \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.}$$

Remark 2. Exponential bounds and almost sure convergence of resolvent estimators are obtained from condition $\alpha_n^{p+1} \left(\frac{n}{\log n}\right)^{\frac{1}{2}} \xrightarrow{n \rightarrow \infty} \infty$ which is stronger than the condition $\sqrt{n}\alpha_n^{p+1} \rightarrow \infty$ of Th.3 Chap.3 in [10] giving the weak convergence.

Also our results are under the bound condition $\|X_0\| \leq c$ for some $c > 0$ making the statistical proofs of estimators more tractable. Other weak conditions are possible, but this leads to making the proofs more technical and will be the subject of a forthcoming work.

4. Numerical Simulations and Example

4.1. Simulation of ARH processes

In this part we simulate trajectories of Hilbert space valued AR process given by (1) where $H = L^2[0, 1]$ and Resolvent Predictors using *R-package* far developed by J. Damon and S. Guillas [6]. The increments of Brownian Motion are given by Karhunen Loève decomposition and generate a strong white noise and ρ is a bounded linear operator.

Figure 1 shows a trajectory of ARH process for $n = 80$ observations over successive unit intervals calculated at $m = 30$ discretization points.

Resolvent Predictors.

To predict the trajectory X_{n+1} we observe a sample X_1, \dots, X_n of (1). Resolvent predictor \widehat{X}_{n+1} (in symmetric case) is calculated at discrete points $(t_j, j = 1, \dots, m)$ by :

$$\widehat{X}_{n+1}(t_j) = \rho_{n,p}(X_n(t_j))$$

where $\rho_{n,p}$ is defined by (3) and we take $\alpha_n = \frac{1}{100 \cdot \log n}$. To evaluate the performance of the predictors, we calculate Mean Square Error (*MSE*) and Relative Mean Absolute Error (*RMAE*) defined respectively by (for $m_1 = 20$ replications):

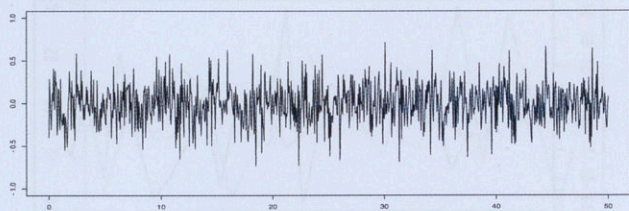


FIGURE 1.

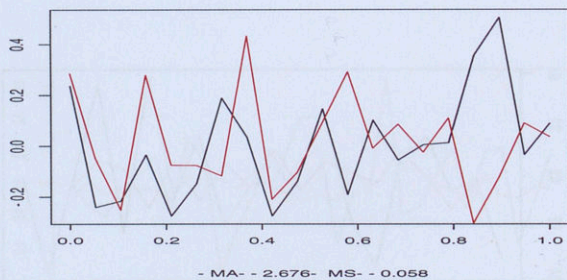
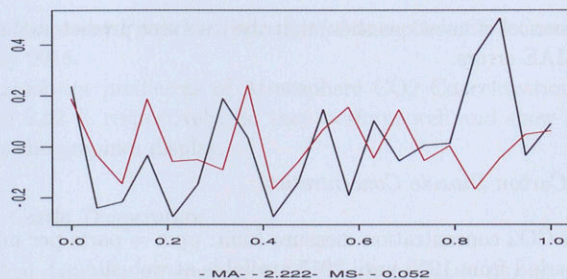
$$MSE = \frac{1}{m_1} \sum_{i=1}^{m_1} |\hat{X}_{n+1}(t_i) - X_{n+1}(t_i)|^2, \quad RMAE = \frac{1}{m_1} \sum_{i=1}^{m_1} \frac{(|\hat{X}_{n+1}(t_i) - X_{n+1}(t_i)|)}{|X_{n+1}(t_i)|}.$$

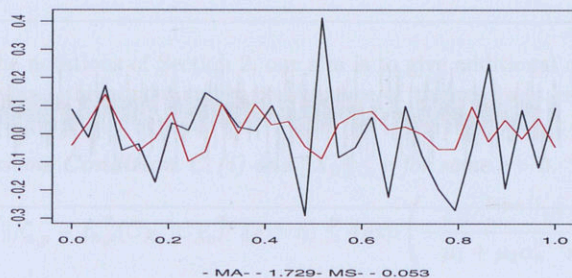
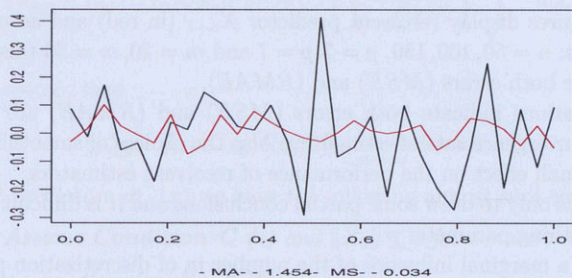
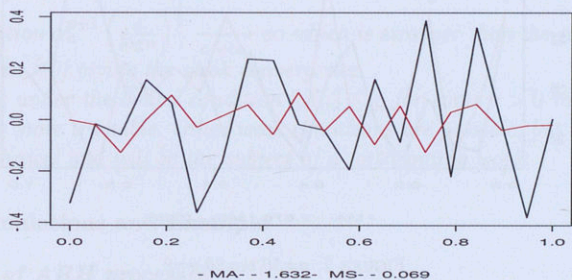
The following figures display resolvent predictor \hat{X}_{n+1} (in red) and actual trajectory X_{n+1} (in black) for values: $n = 50, 100, 150$, $p = 2, p = 7$ and $m = 20, m = 30$ (discretization points). In each case we give both errors (MSE) and ($RMAE$).

Numerical simulations indicate both errors (MSE) and ($RMAE$) are stabilized from on $n = 50$ as well as convergence rate of estimators. Also the choices of smoothing parameter $p = 2$ and $p = 7$ have a small effect on the performance of resolvent estimators.

However this leads only to draw some partial conclusions and it is difficult to indicate optimal values for the smoothing parameter p .

We may observe a marginal influence of the number m of discretization points.

FIGURE 2. $n=50, m=20, p=2$ FIGURE 3. $n=50, m=20, p=7$

FIGURE 4. $n=50, m=30, p=2$ FIGURE 5. $n=50, m=30, p=7$ FIGURE 6. $n=100, m=20, p=2$

In conclusion numerical simulations show that the resolvent predictors perform well and give stable MSE and RMAE errors.

4.2. Examples

4.2.1. Atmosphere Carbon Dioxide Concentration

We consider data of CO₂ concentration measure (unit: ppm = parts per million) in the atmosphere during the period from 1959 until 2015 available at web site : <http://cdiac.esd.ornl.gov/pub/maunaloa-co2/maunaloa.co2>. Figure 14 shows the growth of CO₂ emissions in the atmosphere showing a clear sign of global warming.

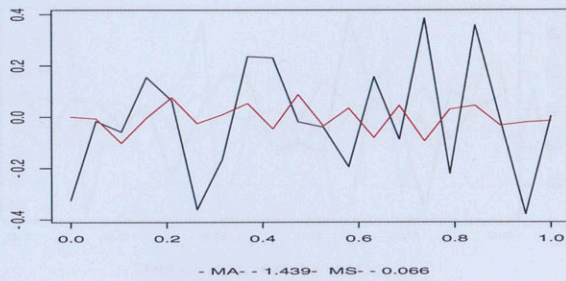
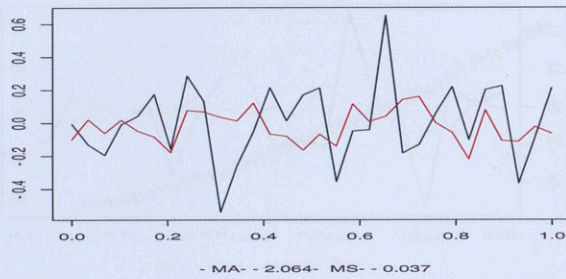
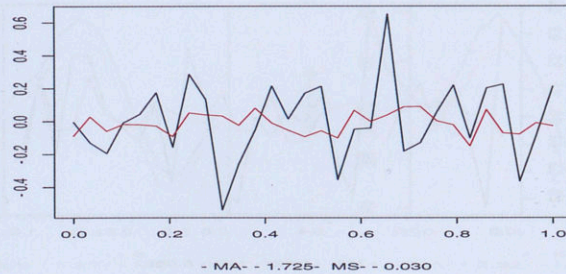
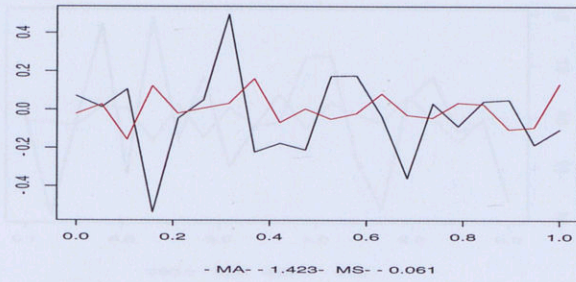
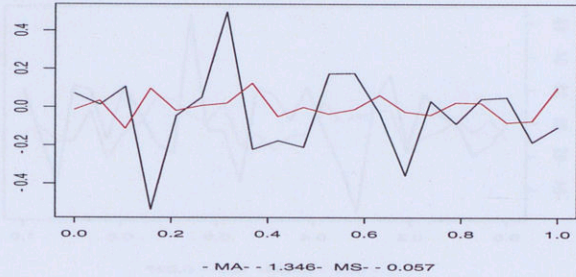
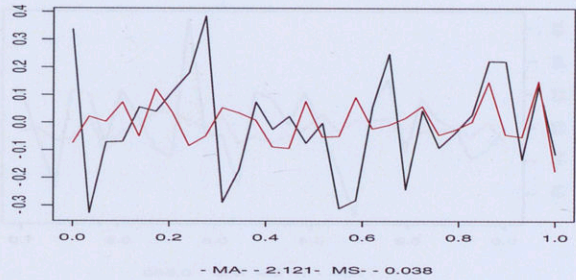
FIGURE 7. $n=100, m=20, p=7$ FIGURE 8. $n=100, m=30, p=2$ FIGURE 9. $n=100, m=30, p=7$

Figure 15 displays Resolvent Predictor (in red) of Atmosphere CO₂ Concentration (in black) for 1975 respectively 2015.

As we observe, resolvent predictors of Atmosphere CO₂ Concentration give small RMAE errors of 1.67% and 2.32 % respectively. So they perform well and show a good trend of the actual trajectory as the graphics display.

4.2.2. Nottingham Castle Temperature

We consider historical data of temperatures in Nottingham Castle during the period from January 1st, 1920 to December 31, 1939. This series has been analyzed by many authors. To compare the performance of resolvent predictors with others existing prediction methods ([2][1][3][6][13]),

FIGURE 10. $n=150, m=20, p=2$ FIGURE 11. $n=150, m=20, p=7$ FIGURE 12. $n=150, m=30, p=2$

we calculate RMAE errors of prediction of the reference year 1939.

Figure 16 displays temperature values in Nottingham Castle during the period: 1920 - 1939.

Figure 17 displays Resolvent Predictor (in red) of Nottingham Castle temperatures (in black) for 1939.

Table 1 displays RMAE errors for different available prediction methods.

We remark that in the case of Castle Nottingham temperatures we have only 19 observations which is very little.

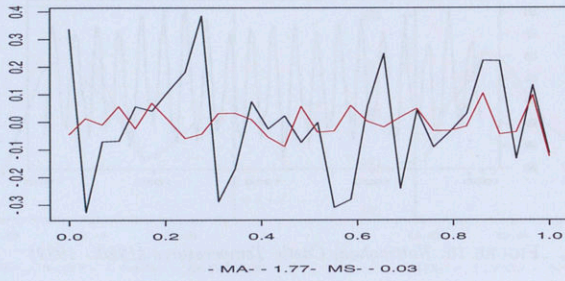


FIGURE 13. $n=150, m=30, p=7$

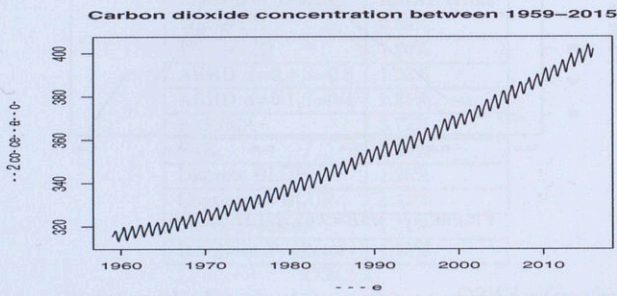


FIGURE 14.

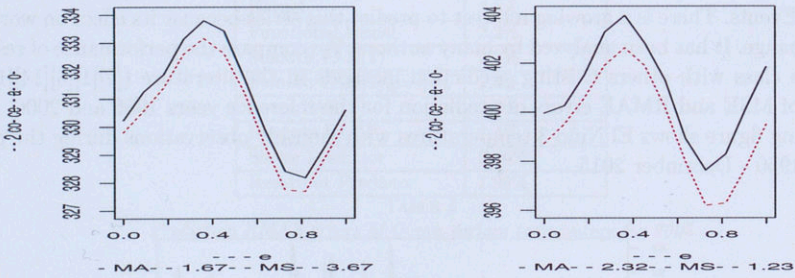


FIGURE 15.

Prediction methods	RMAE errors
Wavelet-Kernel	3%
Spline Smoothing	2.8%
SARIMA	3.1%
Sieves Predictor	2.95%
Continuous BLUP	2.96%
Resolvent Predictor	7.07%

TABLE 1

Prediction RMAE errors of Castle Nottingham temperatures for 1939

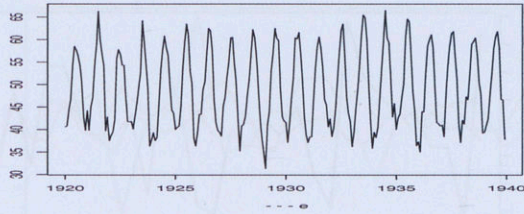
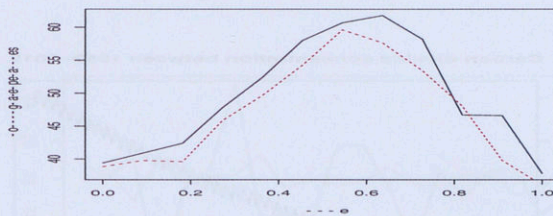


FIGURE 16. Nottingham Castle Temperatures (1920 - 1939)

FIGURE 17. $MSE=9.65, RMAE=0.0707$

4.2.3. Climatological series ENSO

We consider historical data of El Niño 3 (ENSO) Ocean surface temperatures during the period from January 1950 until December 2015 where monthly average values are recorded available at web site : Climate Prediction Center - Monitoring and Data: ENSO Impacts on the U.S. - Previous Events. There is a growing interest to predict this series because its effect on worldwide climate change. It has been analyzed by many authors. We compare the performance of resolvent predictors class with others existing prediction methods in the literature ([2][1][5][14][13]) on the base of MSE and RMAE errors of prediction for the reference years 1986 and 2006.

Following figure shows El Niño 3 temperatures with monthly observations during the period: January 1950 - December 2015.

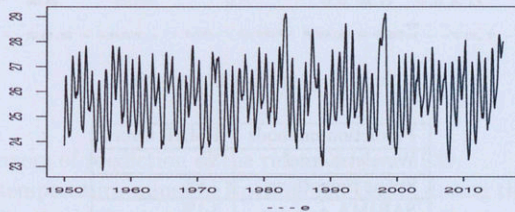


FIGURE 18.

Figure 19 shows resolvent predictor (in red) and Ocean surface temperature (in black) for 1986 respectively 2006.

Tables 2 and 3 display RMAE errors values of known prediction methods ([1][2][3][6][13]) for 1986 respectively 2006.

Conclusion. In all datasets analyzed in this part, we may observe that resolvent predictors perform well and exhibit good predictions. They give good competitive results on the base on

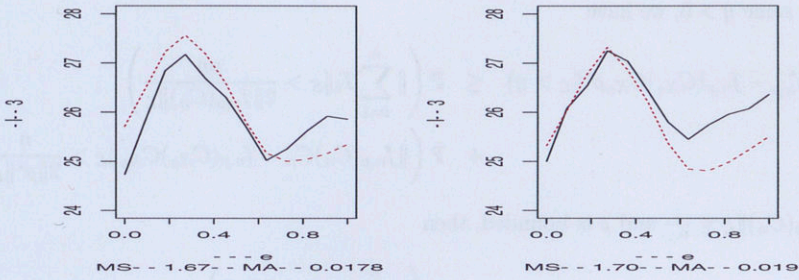


FIGURE 19.

Prediction methods	RMAE errors
Wavelet II	0.89%
Wavelet III	1.20%
ARHD $\alpha=0.4, \beta=0.8$	1.33%
ARHD $\alpha=0.1, \beta=0.4$	1.25%
SARIMA	3.72%
FAR	0.89%
Discrete BLUP	1.25%
Continuous BLUP	1.31%
Sieves Predictor	1.31%
Resolvent Predictor	1.78%

TABLE 2

Prediction RMAE errors of Ocean surface temperatures for 1986

Prediction methods	RMAE errors
Climatologie	2.5%
SARIMA	3.7%
Kernel	2.3%
Functional kernel	2.2%
Smooth FAR(1)	2.3%
Smooth FAR(1), $p=q=1$	2.4%
Local FAR(1)	2.2%
Discrete BLUP	1.4%
Sieves Predictor	2.46%
Resolvent Predictor	1.99%

TABLE 3

Prediction RMAE errors of Ocean surface temperature for 2006

both errors MSE and RMAE. Their performance give new insights on prediction perspectives with respect to other well-known parametric and nonparametric techniques existing in the literature.

5. Proofs

Proof of Theorem 3.1. Introduce the random operators $U_n = \frac{1}{n} \sum_{i=1}^n T_i$, where $T_i = \langle \varepsilon_{i+1}, \cdot \rangle X_i$ with $T_i \in \mathcal{S}$. We can write

$$\rho_{n,p}^* - f_{n,p}(C_{X_0})C_{X_0}\rho^* = (f_{n,p}(C_n)C_n - f_{n,p}(C_{X_0})C_{X_0})\rho^* + f_{n,p}(C_n)U_n$$

Then for some $\eta > 0$, we have

$$\begin{aligned} \mathbb{P}(\|\rho_{n,p}^* - f_{n,p}(C_{X_0})C_{X_0}\rho^*\|_{\mathcal{L}} > \eta) &\leq \mathbb{P}\left(\left\|\sum_{i=1}^n T_i\right\|_{\mathcal{S}} > \frac{n\eta}{2\|f_{n,p}(C_n)\|_{\mathcal{L}}}\right) \\ &+ \mathbb{P}\left(\|f_{n,p}(C_n)C_n - f_{n,p}(C_{X_0})C_{X_0}\|_{\mathcal{L}} > \frac{\eta}{2\|\rho^*\|_{\mathcal{L}}}\right) \end{aligned}$$

But $\|f_{n,p}(C_n)\|_{\mathcal{L}} \leq \frac{1}{\alpha_n}$ and ρ is bounded, then

$$\begin{aligned} (5.1) \mathbb{P}(\|\rho_{n,p}^* - f_{n,p}(C_{X_0})C_{X_0}\rho^*\|_{\mathcal{L}} > \eta) &\leq \mathbb{P}\left(\left\|\sum_{i=1}^n T_i\right\|_{\mathcal{S}} > \frac{n\alpha_n\eta}{2}\right) \\ &+ \mathbb{P}\left(\|f_{n,p}(C_n)C_n - f_{n,p}(C_{X_0})C_{X_0}\|_{\mathcal{L}} > \frac{\eta}{2\|\rho^*\|_{\mathcal{L}}}\right) \end{aligned}$$

The first term in (5.1):

The rv's T_i is a martingale difference with respect to the filtration $\mathcal{B}_i = \sigma(\varepsilon_j, j \leq i)$, moreover

$$\|T_i\|_{\mathcal{S}} \leq \|X_i\| \|\varepsilon_{i+1}\| \leq c(1 + \|\rho\|_{\mathcal{L}})$$

Hence $\mathbb{E}^{B_{i-1}}\|T_i^2\|_{\mathcal{S}} \leq c^2\sigma^2$. So by Theorem 2.14 in [5] we have

$$\mathbb{P}\left(\left\|\sum_{i=1}^n T_i\right\|_{\mathcal{S}} > \frac{n\alpha_n\eta}{2}\right) \leq 2 \exp\left(-\frac{n\alpha_n^2\eta^2}{8c^2\sigma^2 + \frac{4}{3}c(1 + \|\rho\|_{\mathcal{L}})\alpha_n\eta}\right)$$

The second term in (5.1):

We have

$$\begin{aligned} f_{n,p}(C_n)C_n - f_{n,p}(C_{X_0})C_{X_0} &= \underbrace{(C_n + \alpha_n I)^{-(p+1)}[C_n^{p+1} - C_{X_0}^{p+1}]}_{A_{n,p}} \\ &+ \underbrace{(C_n + \alpha_n I)^{-(p+1)}[(C_{X_0} + \alpha_n I)^{p+1} - (C_n + \alpha_n I)^{p+1}]C_{X_0}^{p+1}}_{B_{n,p}}. \end{aligned}$$

We can easily verify that

$$\|A_{n,p}\|_{\mathcal{L}} \leq \frac{1}{\alpha_n^{p+1}} K_{p,n} \|C_n - C_{X_0}\|_{\mathcal{S}}$$

and

$$\|B_{n,p}\|_{\mathcal{L}} \leq \frac{1}{\alpha_n^{p+1}} K'_{p,n} \|C_n - C_{X_0}\|_{\mathcal{S}}$$

where $K_{p,n} := \sum_{i=0}^p \lambda_{0,n}^{p-i} \lambda_0^i$ and $K'_{p,n} = \sum_{i=0}^p (\lambda_{0,n} + \alpha_n)^{p-i} (\lambda_0 + \alpha_n)^i$.

Then for all $\eta > 0$, we obtain

$$\mathbb{P}\left(\|f_{n,p}(C_n)C_n - f_{n,p}(C_{X_0})C_{X_0}\|_{\mathcal{L}} > \frac{\eta}{2\|\rho^*\|_{\mathcal{L}}}\right) \leq \mathbb{P}\left((K_{p,n} + K'_{p,n})\|C_n - C_{X_0}\|_{\mathcal{S}} > \frac{\eta\alpha_n^{p+1}}{2\|\rho^*\|_{\mathcal{L}}}\right)$$

Consider the event E_n defined by $E_n = \left\{\lambda_{0,n} > \frac{3\lambda_0}{2}\right\} = \left\{\lambda_{0,n} - \lambda_0 > \frac{\lambda_0}{2}\right\}$ where $\lambda_0 > 0$.

We have

$$\mathbb{P}(E_n) \leq \mathbb{P}\left(|\lambda_{0,n} - \lambda_0| > \frac{\lambda_0}{2}\right) \leq \mathbb{P}\left(\|C_n - C_{X_0}\|_S > \frac{\lambda_0}{2}\right)$$

Hence from Theorem 2.4 [5], we imply

$$\mathbb{P}(E_n) \leq 4 \exp\left(-\frac{n\lambda_0^2}{2(2\alpha_1 + \beta_1\lambda_0)}\right)$$

where α_1 et β_1 are constants depending only on ρ et $\mathbb{P}_{\varepsilon_0}$.

Note that on the event \bar{E}_n we have $K_{p,n} + K'_{p,n} < 2(p+1)\lambda_0^p$ a.s.
So

$$\begin{aligned} \mathbb{P}\left(\left[(K_{p,n} + K'_{p,n})\|C_n - C_{X_0}\|_S > \frac{\eta\alpha_n^{p+1}}{2\|\rho^*\|_{\mathcal{L}}}\right] \cap \bar{E}_n\right) \\ \leq \mathbb{P}\left(\|C_n - C_{X_0}\|_S > \frac{\eta\alpha_n^{p+1}}{4(p+1)\lambda_0^p\|\rho^*\|_{\mathcal{L}}}\right) \end{aligned}$$

We will deduce from Theorem 2.4 [5]

$$\begin{aligned} \mathbb{P}\left(\left[(K_{p,n} + K'_{p,n})\|C_n - C_{X_0}\|_S > \frac{\eta\alpha_n^{p+1}}{2\|\rho^*\|_{\mathcal{L}}}\right] \cap \bar{E}_n\right) \\ \leq 4 \exp\left(-\frac{n\alpha_n^{2(p+1)}\eta^2}{4(p+1)\lambda_0^p\|\rho^*\|_{\mathcal{L}}[4\alpha_1(p+1)\lambda_0^p\|\rho^*\|_{\mathcal{L}}\alpha_1 + \beta_1\alpha_n^{p+1}\eta]}\right). \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{P}\left(\|f_{n,p}(C_n)C_n - f_{n,p}(C_{X_0})C_{X_0}\|_{\mathcal{L}} > \frac{\eta}{2\|\rho^*\|_{\mathcal{L}}}\right) \\ \leq \mathbb{P}\left(\left[(K_{p,n} + K'_{p,n})\|C_n - C_{X_0}\|_S > \frac{\eta\alpha_n^{p+1}}{2\|\rho^*\|_{\mathcal{L}}}\right] \cap \bar{E}_n\right) + \mathbb{P}(E_n) \\ \leq 4 \exp\left(-\frac{n\alpha_n^{2(p+1)}\eta^2}{4(p+1)\lambda_0^p\|\rho^*\|_{\mathcal{L}}[4\alpha_1(p+1)\lambda_0^p\|\rho^*\|_{\mathcal{L}}\alpha_1 + \beta_1\alpha_n^{p+1}\eta]}\right) \\ + 4 \exp\left(-\frac{n\lambda_0^2}{2(2\alpha_1 + \beta_1\lambda_0)}\right) \\ \leq 4 \exp\left(-\frac{n\alpha_n^{2(p+1)}\eta^2}{\mu_1 + \mu_2\alpha_n^{p+1}\eta}\right) \end{aligned}$$

where $\mu_1 = \max(16(p+1)^2\lambda_0^2\|\rho^*\|_{\mathcal{L}}^2, 2(2\alpha_1 + \beta_1\lambda_0))$ et $\mu_2 = 4(p+1)\lambda_0^p\|\rho^*\|_{\mathcal{L}}\beta_1$.

Finally regrouping the latter results and from (5.1) we arrive at

$$\begin{aligned} \mathbb{P}\left(\|\rho_{n,p}^* - f_{n,p}(C_{X_0})C_{X_0}\rho^*\|_{\mathcal{L}} > \eta\right) \\ \leq 2 \exp\left(-\frac{n\alpha_n^2\eta^2}{24c^2\sigma^2 + \frac{4}{3}c(1 + \|\rho\|_{\mathcal{L}})\alpha_n\eta}\right) + 4 \exp\left(-\frac{n\alpha_n^{2(p+1)}\eta^2}{\mu_1 + \mu_2\alpha_n^{p+1}\eta}\right) \\ \leq 4 \exp\left(-\frac{n\alpha_n^{2(p+1)}\eta^2}{\mu_3 + \mu_4\alpha_n^p}\right) \end{aligned}$$

where μ_3, μ_4 are positive constants.

Choosing now $\eta = A \left(\frac{\log n}{n\alpha_n^{2(p+1)}} \right)^{1/4}$, $A > 0$, we obtain

$$\begin{aligned} \mathbb{P} \left(\left(\frac{n\alpha_n^{2(p+1)}}{\log n} \right)^{1/2} \|\rho_{n,p}^* - f_{n,p}(C_{X_0})C_{X_0}\rho^*\|_{\mathcal{L}} \geq A \right) &\leq \gamma \exp \left(-\frac{A^2 \log n}{\mu_3 + \mu_4 A} \right) \\ &\leq \gamma_1 n^{-\frac{A^2}{\mu_3 + \mu_4 A}}. \end{aligned}$$

By $A^2 > \mu_3 + \mu_4 A$, we may conclude by Borel-Cantelli Lemma. This ends the proof of the theorem.

Proof of Proposition 3.2.

Direct so omitted.

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