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# On natural approaches related to classical trigonometric inequalities

Abd Raouf Chouikha \*

## Abstract

In this paper, new refinements and improvements of Mitrinovic-Cusa and related inequalities are presented. We prove in particular for  $0 < x < \frac{\pi}{2}$  and any  $n \geq 5$

$$P_n(x) < (\sin x)^2 - x^3 \cot x < P_{n-1}(x) + \left[ \left(\frac{2}{\pi}\right)^{2n} - \sum_{k=3}^{n-1} a_k \left(\frac{2}{\pi}\right)^{2n-2k} \right] x^{2n}$$

where  $P_n(x) = \frac{x^6}{15} - \frac{x^8}{945} + \sum_{k=5}^n a_k x^{2k}$  is a  $n$ -polynomial, with positive coefficients  $a_k = \frac{2^{2k-2}}{(2^{k-2})!} \left( |B_{2k-2}| + \frac{(-1)^{k+1}}{(2^{k-1})^k} \right)$ ,  $B_{2k}$  are Bernoulli numbers. We also prove for any  $n \geq 5$

$$\left(\frac{1 + \cos x}{2}\right)^2 < \cos x + \frac{P_n(x)}{x^3} < \left(\frac{\sin(x)}{x}\right)^3.$$

The analysis results show that our bounds are tighter than previous methods.

*Key Words and phrases:* Trigonometric functions; Sinc function; Inequalities.<sup>1</sup>

## 1 Introduction

Inequalities involving trigonometric functions are used in many applications in various fields of mathematics. The method to compare functions to their corresponding Taylor polynomials has been successfully applied to prove and approximate a lot of trigonometric inequalities [8].

A method called the natural approach, introduced by Mortici in [13], uses the idea of comparing functions to their corresponding Taylor polynomials. This method has been successfully applied to prove and approximate a wide category of trigonometric inequalities.

Let us consider the double inequality

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$$(\cos x)^{\frac{1}{3}} < \frac{\sin x}{x} < \frac{2 + \cos x}{3}. \quad (1)$$

The left-hand side is known as Adamovic-Mitrinovic inequality (see [11], [12]), while the right-hand side is known as Cusa inequality. The latter one which was proved by Huygens was used in order to estimate the number  $\pi$ .

In this paper we provide another natural approach by comparing functions with their corresponding Taylor polynomials. This approach is analog to that given by [13]. As a corollary, that permits us to extend and sharpen results related to trigonometric inequalities and give generalizations and refinements.

In [10, Theorem 1 p.4] it was proved for  $0 < x < \pi/2$  and every  $n \in \mathbb{N}$

$$\sum_{k=2}^{2n} A(k)x^{2k} < \cos x - \left(\frac{\sin x}{x}\right)^3 < \sum_{k=2}^{2n+1} A(k)x^{2k},$$

where  $A(k) = \frac{3^{2k+3} - 32k^3 - 96k^2 - 88k - 27}{4(2k+3)!}$ . An error estimation is also given.

In particular, the following inequalities have been established

**Statement**

$$(i) \quad -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} < \cos x - \left(\frac{\sin x}{x}\right)^3 < -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600}.$$

Notice that these inequalities are finer than provided by Mortici [13] :

$$-\frac{x^4}{15} < \cos x - \left(\frac{\sin x}{x}\right)^3 < -\frac{x^4}{15} + \frac{23x^6}{1890}.$$

$$(ii) \quad \cos x - \left(\frac{\sin x}{x}\right)^3 < -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} - \frac{74677x^{12}}{27243216000}.$$

$$(iii) \quad -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} - \frac{74677x^{12}}{27243216000} < \cos x - \left(\frac{\sin x}{x}\right)^3 < -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} - \frac{74677x^{12}}{27243216000} + \frac{989x^{14}}{10897286400}.$$

In this paper, we aim to refine some above inequalities and to generalize trigonometric chains.

## 2 Main results

In [4] we proved for  $0 < x < \frac{\pi}{2}$

$$\cos x + x^3 \left(1 - \frac{x^2}{63}\right) \frac{\sin x}{15} < \left(\frac{\sin x}{x}\right)^3 < \cos x + \frac{x^3 \sin x}{15}. \quad (2)$$

We easily derive the following

$$\begin{aligned} \frac{x^6}{15} \left(1 - \frac{x^2}{63}\right) &< (\sin x)^2 - x^3 \cot x < \frac{x^6}{15}, \\ -\frac{x^8}{945} &< (\sin x)^2 - x^3 \cot x - \frac{x^6}{15} < 0. \end{aligned}$$

In the sequel we provide lower bound of (1) which is finer than (3) and appears to be sharper than many known bounds .

To that end, we will consider a function

$$f(x) = (\sin x)^2 - x^3 \cot x - \frac{x^6}{15} + \frac{x^8}{945}$$

defined for  $0 < x < \pi/2$  and we will provide better estimations.

### 2.1 Estimation for the function $f(x)$

At first, consider the following technical lemma which has been proved by [3, Lemma 3.1], but the proof we give here is slightly different.

**Lemma 2-1** For  $0 < x < \pi/2$  consider the function

$$f(x) = (\sin x)^2 - x^3 \cot x - \frac{x^6}{15} + \frac{x^8}{945}$$

then  $f(x)$  can be expanded as power series  $f(x) = \sum_{k \geq 5} a_k x^{2k}$ ,

$$a_k = \frac{2^{2k-2} |B_{2k-2}|}{(2k-2)!} + \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!} = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)k} \right),$$

where  $B_{2k}$  are the Bernoulli numbers. Moreover, the coefficients  $a_k, k \geq 5$  are all positive:

$$a_5 = \frac{1}{2735}, \quad a_6 = \frac{8}{467775}, \quad a_7 = \frac{206}{91216125}, \dots$$

*Proof of Lemma 2-1*

The following series expansions can be found in [6] or [7]

$$\cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1}, \quad x \in (0, \pi)$$

and

$$\sin^2 x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^{2k-1}}{(2k)!} x^{2k}, \quad x \in (0, \frac{\pi}{2})$$

$$\begin{aligned} (\sin x)^2 - x^3 \cot x = & \left( \frac{1}{15} x^6 - \frac{1}{945} x^8 + \frac{1}{2835} x^{10} + \frac{8}{467775} x^{12} + \frac{206}{91216125} x^{14} + \right. \\ & \left. \frac{139}{638512875} x^{16} + \frac{10861}{488462349375} x^{18} + \frac{438628}{194896477400625} x^{20} + O(x^{22}) \right). \end{aligned}$$

To prove the positivity of the coefficients  $a_k$  we will use the following inequality for Bernoulli numbers established by D'Aniello [5]:

$$\frac{2(2k)!}{\pi^{2k}(2^{2k}-1)} < |B_{2k}| < \frac{2(2k)!}{\pi^{2k}(2^{2k}-2)}.$$

For any odd value of  $k$ , we have

$$a_k = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)k} \right) = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| + \frac{1}{(2k-1)k} \right) > 0.$$

Consider the even case, then

$$a_k = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| + \frac{-1}{(2k-1)k} \right).$$

By definition of Bernoulli numbers

$$S_n(p) = \sum_{k=0}^{n-1} k^p = \frac{1}{p+1} \sum_{k=0}^p \binom{p+1}{k} B_k n^{p+1-k}.$$

Then for  $k$  even

$$a_k = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| - \frac{1}{(2k-1)k} \right) > \frac{2^{2k-2}}{(2k-2)!} \left( \frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-2)} - \frac{1}{(2k-1)k} \right)$$

and

$$a_k < \frac{2^{2k-2}}{(2k-2)!} \left( \frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-1)} - \frac{1}{(2k-1)k} \right).$$

Therefore

$$\frac{2^{2k-2} \frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-1)}}{(2k-2)!} + \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!} < a_k = \frac{2^{2k-2} |B_{2k-2}|}{(2k-2)!} + \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!}$$

$$< \frac{2^{2k-2} \frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-2)}}{(2k-2)!} + \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!}.$$

and

$$\sum_{k \geq 5} \left( \frac{2^{2k-1}}{2\pi^{2k-2}(2^{2k-2}-1)} + \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!} \right) x^{2k} < f(x).$$

Thus it implies the left hand of (4) since for any integer  $n \geq 5$  we have

$$\begin{aligned} \cos x &\leq \cos x + \sin x \left( \frac{x^3}{15} - \frac{x^5}{945} \right) + \sin x \sum_{5 \leq k \leq n} a_k x^{2k-3} \\ &\leq \cos x + \sin x \left( \frac{x^3}{15} - \frac{x^5}{945} \right) + \sin x \sum_{5 \leq k \leq \infty} a_k x^{2k-3} = \left( \frac{\sin x}{x} \right)^3. \end{aligned}$$

Notice that

$$\begin{aligned} (\sin x)^2 - x^3 \cot x &= (\sin x)^2 - x^3 \left( \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1} \right) = \\ &(\sin x)^2 - x^2 + \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k+2} < (\sin x)^2 - x^2 + \sum_{k=1}^{\infty} \frac{2^{2k+1} x^{2k+2}}{\pi^{2k} (2^{2k}-2)}. \end{aligned}$$

In the other hand we know that for  $k > 1$

$$(2k)! > \sqrt{4\pi k} \left( \frac{2k}{e} \right)^{2k} e^{\frac{1}{24k+1}}.$$

It implies

$$\begin{aligned} \left( \frac{2(2k-2)!}{\pi^{2k-2}(2^{2k-2}-1)} - \frac{1}{(2k-1)k} \right) &> \left( \frac{2\sqrt{4\pi(k-1)} \left( \frac{2k-2}{e} \right)^{2k-2} e^{\frac{1}{48k-23}}}{\pi^{2k-2}(2^{2k-2}-1)} - \frac{1}{(2k-1)k} \right) > \\ \left( \frac{2\sqrt{4\pi(k-1)} \left( \frac{2k-2}{\pi e} \right)^{2k-2}}{(2^{2k-2}-1)} - \frac{1}{(2k-1)k} \right) &> \left( \frac{2\sqrt{4\pi(k-1)} \left( \frac{2k-2}{\pi e} \right)^{2k-2}}{2^{2k-2}} - \frac{1}{(2k-1)k} \right) > \\ \left( 2\sqrt{4\pi(k-1)} \left( \frac{k-1}{\pi e} \right)^{2k-2} - \frac{1}{(2k-1)k} \right). & \end{aligned}$$

Thanks to *Maple* we may easily verify that the last expression is non negative as soon as  $k > 6$ . This means that  $a_k$  are non negative.

Our first result Theorem 2-2 motivated us to further refine the Adamovic-Mitrinovic inequality.

It permits us to deduce the lower bound of  $\left( \frac{\sin x}{x} \right)^3$  which appears to be finer than the corresponding bounds in Statement .

**Theorem 2-2** For  $0 < x < \frac{\pi}{2}$  for any  $n \geq 5$  the following inequalities hold

$$\begin{aligned} \sin x \sum_{k=5}^n a_k x^{2k-3} &\leq \left(\frac{\sin x}{x}\right)^3 - \cos x - \sin x \left(\frac{x^3}{15} - \frac{x^5}{945}\right) \leq \\ \sin x \sum_{k=5}^{n-1} a_k x^{2k-3} + \sin x \left[ \left(\frac{2}{\pi}\right)^{2n} - \sum_{k=5}^{n-1} a_k \left(\frac{2}{\pi}\right)^{2n-2k} \right] x^{2n-3} &\quad (3) \end{aligned}$$

where

$$a_k = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)k} \right), \quad k \geq 5$$

and  $B_{2k}$  are the Bernoulli numbers.

**Proof of Theorem 2-2** By Lemma 2-1 the function

$$f(x) = (\sin x)^2 - x^3 \cot x - \frac{x^6}{15} + \frac{x^8}{945}$$

is positive. We also need the following Lemma which gives a upper and lower bound for the preceding function.

**Lemma 2-3** consider the real analytic functions  $f$  defined on the interval  $[a, b]$  :

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

where  $a_k \in \mathbb{R}, a_k \geq 0$  for all  $k \in \mathbb{N}$  Then  $f(x)$  may be bounded by Taylor's approximation for any  $n \geq 1$

$$\sum_{k=0}^n \frac{f^{(k)}(a+)}{k!} (x-a)^k \leq f(x) \leq \sum_{k=0}^n \frac{f^{(k)}(a+)}{k!} (x-a)^k + \frac{1}{(b-a)^n} R_n(b) (x-a)^n,$$

where  $R_n(x) = f(x) - \sum_{k=0}^n \frac{f^{(k)}(a+)}{k!} (x-a)^k$ .

We may find an elegant proof of this Lemma in [11, Theorem 4] .

Moreover, if we suppose in addition  $f^{(n)}$  is increasing in  $[a, b]$  we easily deduce

$$\begin{aligned} \sum_{k=0}^n \frac{f^{(k)}(a+)}{k!} (x-a)^k + \frac{f^{(n)}(a+)}{n!} (x-a)^n &\leq f(x) \leq \\ \sum_{k=0}^n \frac{f^{(k)}(a+)}{k!} (x-a)^k + \frac{f^{(n)}(a+)}{n!} (x-a)^n &+ \frac{1}{(b-a)^n} R_n(b) (x-a)^n. \end{aligned}$$

The coefficients  $\frac{f^{(n)}(a+)}{n!}$  and  $\frac{1}{(b-a)^n}R_n(b)$  are the best possible constants. This result may also be deduced from [3 , Theorem 3.2].

Applying the preceding to the function  $f(x) = (\sin x)^2 - x^3 \cot x = \sum_{k \geq 5} a_k x^{2k}$  we then derive for  $0 < x < \pi/2$  and  $n \geq 5$  the following inequalities

$$\sum_{k \geq 5}^n a_k x^{2k} \leq f(x) \leq \sum_{k \geq 5}^{n-1} a_k x^{2k} + \left[ \left( \frac{2}{\pi} \right)^{2n} - \sum_{k \geq 5}^{n-1} a_k \left( \frac{2}{\pi} \right)^{2n-2k} \right] x^{2n}$$

hold. Therefore,

$$\begin{aligned} \sin x \sum_{k \geq 5}^n a_k x^{2k-3} &\leq f(x) \frac{\sin x}{x^3} \leq \sin x \sum_{k \geq 5}^{n-1} a_k x^{2k-3} + \\ &\sin x \left[ \left( \frac{2}{\pi} \right)^{2n} - \sum_{k \geq 5}^{n-1} a_k \left( \frac{2}{\pi} \right)^{2n-2k} \right] x^{2n-3}. \end{aligned}$$

This means

$$\begin{aligned} \sin x \sum_{k \geq 5}^n a_k x^{2k-3} &\leq \left( \frac{\sin x}{x} \right)^3 - \cos x - \frac{x^3 \sin x}{15} + \frac{x^5 \sin x}{945} \leq \\ \sin x \sum_{k \geq 5}^{n-1} a_k x^{2k-3} &+ \sin x \left[ \left( \frac{2}{\pi} \right)^{2n} - \sum_{k \geq 5}^{n-1} a_k \left( \frac{2}{\pi} \right)^{2n-2k} \right] x^{2n-3}. \end{aligned}$$

This proves the theorem.

Some particular cases of Theorem 2-2 are given below. Let us introduce some examples of the inequalities obtained for  $n = 5, 6, 7, 8, \dots$

- Putting  $n = 5$  , we obtain the following

$$\begin{aligned} \cos x + x^3 \left( 1 - \frac{x^2}{63} + \frac{x^4}{189} \right) \frac{\sin x}{15} &< \left( \frac{\sin x}{x} \right)^3 < \\ \cos x + x^3 \left( 1 - \frac{x^2}{63} \right) \frac{\sin x}{15} + \frac{1024}{\pi^{10}} x^7 \sin x &\quad (4) \end{aligned}$$

Moreover, for  $0 < x < \pi/2$  the following inequalities hold

$$\cos x + x^3 \left( 1 - \frac{x^2}{63} + \frac{x^4}{189} \right) \frac{\sin x}{15} < \left( \frac{\sin x}{x} \right)^3. \quad (5)$$



- Taking  $n = 6$ , one has for  $0 < x < \pi/2$  the following inequality

$$\cos x + x^3 \left( 1 - \frac{x^2}{63} + \frac{x^4}{189} + \frac{8}{31185}x^6 \right) \frac{\sin x}{15} < \left( \frac{\sin x}{x} \right)^3 < \quad (6)$$

$$\cos x + x^3 \left( 1 - \frac{x^2}{63} + \frac{x^4}{189} \right) \frac{\sin x}{15} + \left( \frac{4096}{\pi^{12}} - \frac{4}{2835\pi^2} \right) x^9 \sin x.$$

- Taking  $n = 7$ , one has for  $0 < x < \pi/2$  the following inequality

$$\cos x + \frac{x^3}{15} \left( 1 - \frac{x^2}{63} \left( 1 - \frac{x^2}{3} - \frac{8x^4}{495} - \frac{206x^6}{96525} \right) \right) \sin x < \frac{(\sin x)^3}{x^3} < \quad (7)$$

$$\cos x + \frac{x^3}{15} \left( 1 - \frac{x^2}{63} \left( 1 - \frac{x^2}{3} - \frac{8x^4}{495} \right) \right) \sin x + \left( \frac{16384}{\pi^{14}} - \frac{16}{2835\pi^4} - \frac{32}{5\pi^2} \right) x^{11} \sin x.$$

- Putting  $n = 8$ , one has for  $0 < x < \pi/2$  the following inequality

$$\cos x + \frac{x^3}{15} \left( 1 - \frac{x^2}{63} \left( 1 - \frac{x^2}{3} - \frac{8x^4}{495} - \frac{206x^6}{96525} - \frac{139x^8}{675675} \right) \right) \sin x < \frac{(\sin x)^3}{x^3} < \quad (8)$$

$$\begin{aligned} & \cos x + \frac{x^3}{15} \left( 1 - \frac{x^2}{63} \left( 1 - \frac{x^2}{3} - \frac{8x^4}{495} - \frac{206x^6}{96525} \right) \right) \sin x + \\ & \left( \frac{65536}{\pi^{16}} - \frac{64}{2825\pi^6} - \frac{128}{467775\pi^4} - \frac{824}{91216125\pi^2} \right) x^{13} \sin x. \end{aligned}$$

Etc...

**Remarks 2-4:** Notice that when the degree  $n$  of the polynomial increases, the function  $\sin x \sum_{5 \leq k \leq n} a_k x^{2k-3}$  approaches the upper bound  $\left(\frac{\sin x}{x}\right)^3 - \cos x - \frac{x^3}{15} + \frac{x^5}{945}$  since the coefficients  $a_k > 0$ . In other words, the precision increases with  $n$  and allows us to have a good estimate of the error.

Indeed, consider the difference

$$\cos x + \frac{x^3}{15} \left( 1 - \frac{x^2}{63} \left( 1 - \frac{x^2}{3} + bx^4 \right) \right) \sin x - \frac{(\sin x)^3}{x^3}$$

For  $b = -\frac{8}{495}$  we have

$$\begin{aligned} & \cos x + \frac{x^3}{15} \left( 1 - \frac{x^2}{63} \left( 1 - \frac{x^2}{3} - \frac{8x^4}{495} \right) \right) \sin x - \frac{(\sin x)^3}{x^3} \\ & < -\frac{206x^{12}}{91216125} + \frac{304x^{14}}{1915538625} - \frac{373x^{16}}{78153975900} < 0 \end{aligned}$$

For  $a = \frac{1}{3}$ ,  $b = -\frac{8}{495}$  and  $c = -\frac{206}{96525}$  we get

$$\begin{aligned} & \cos x + \frac{x^3}{15} \left( 1 - \frac{x^2}{63} \left( 1 - \frac{x^2}{3} + bx^4 + cx^6 \right) \right) \sin x - \frac{(\sin x)^3}{x^3} \\ & < -\frac{139x^{14}}{638512875} + \frac{13723x^{16}}{976924698750} - \frac{559483x^{18}}{1559171819205000} < 0 \end{aligned}$$

For  $a = \frac{1}{3}$ ,  $b = -\frac{8}{495}$ ,  $c = -\frac{206}{96525}$ ,  $d = -\frac{139}{675675}$  we get

$$\begin{aligned} & \cos x + \frac{x^3}{15} \left( 1 - \frac{x^2}{63} \left( 1 - \frac{x^2}{3} - \frac{8x^4}{495} - \frac{206x^6}{96525} - \frac{139x^8}{675675} + dx^{10} \right) \right) \sin x - \frac{(\sin x)^3}{x^3} \\ & < -\frac{10861x^{16}}{488462349375} + \frac{567257x^{18}}{389792954801250} - \frac{13763x^{20}}{359808881355000} < 0 \end{aligned}$$

## 2.2 Another estimation for $f(x)$

The following result provided new bounds for the function  $f(x) = (\sin x)^2 - x^3 \cot x$ . It was proved in [3, Theorem 3.3, p.11]

**Lemma 2-5** For  $0 < x < \pi/2$  and  $n \geq 0$  the following inequalities hold

$$\begin{aligned} (\sin x) \sum_{k=0}^n b_{k+2} x^{2k+5} < (\sin x)^2 - x^3 \cot x < (\sin x) \sum_{k=0}^{n-1} b_{k+2} x^{2k+5} + \\ & \left[ \left( \frac{2}{\pi} \right)^{2n+5} - \sum_{k=0}^{n-1} b_{k+2} \left( \frac{2}{\pi} \right)^{2n-2k} \right] x^{2n+3} \sin x \end{aligned}$$

where

$$b_k = \frac{2(2k-1)(2k+1)(2^{2k-1}-1) |B_{2k}| + (-1)^k}{(2k+1)!}.$$

Moreover, all the coefficients  $b_k$  are non negative for  $k \geq 2$ .

Let  $f(x) = (\sin x)^2 - x^3 \cot x$  then following [3] one has the expansion

$$f(x) = \left( \frac{1}{15} + \frac{19x^2}{1890} + \frac{167x^4}{113400} + \frac{479x^6}{2494800} + \dots \right) x^5 \sin x.$$

Here is a slight improvement of this Lemma which will be useful to refine the bounds of inequalities.

**Lemma 2-6** For  $0 < x < \pi/2$  and  $n \geq 0$  the following inequalities hold

$$\begin{aligned}
& (\sin x) \left[ \sum_{k=0}^n b_{k+2} x^{2k+5} - \left( \frac{x^5}{15} - \frac{x^7}{945} \right) \left( 1 + \sum_{k=1}^{\infty} 2 \frac{(2^{2k-1} - 1) |B_{2k}| x^{2k}}{(2k)!} \right) \right] < \\
(\sin x)^2 - x^3 \cot x - \frac{x^5}{15} + \frac{x^7}{945} & < (\sin x) \left[ \sum_{k=0}^{n-1} b_{k+2} x^{2k+1} - \left( \frac{x^6}{15} - \frac{x^8}{945} \right) \left( 1 + \sum_{k=1}^{\infty} 2 \frac{(2^{2k-1} - 1) |B_{2k}| x^{2k}}{(2k)!} \right) \right] + \\
& \left[ \left( \frac{2}{\pi} \right)^{2n+5} - \sum_{k=1}^{n-1} \left( \frac{2}{\pi} \right)^{2n-2k} \right] x^{2n+3} \sin x
\end{aligned}$$

where

$$b_k = \frac{2(2k-1)(2k+1)(2^{2k-1}-1) |B_{2k}| + (-1)^k}{(2k+1)!}.$$

Moreover, all the coefficients  $b_k$  are non negative for  $k \geq 2$ .

Indeed, since [9,p.145]

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{k=1}^{\infty} 2 \frac{(2^{2k-1} - 1) |B_{2k}| x^{2k-1}}{(2k)!}$$

this Lemma may be deduced from Lemma 1.5 in writing

$$\begin{aligned}
-\frac{x^6}{15} + \frac{x^8}{945} &= (\sin x) \left( -\frac{x^6}{15} + \frac{x^8}{945} \right) \left( \frac{1}{x} + \sum_{k=1}^{\infty} 2 \frac{(2^{2k-1} - 1) |B_{2k}| x^{2k-1}}{(2k)!} \right) = \\
& (\sin x) \left( -\frac{x^5}{15} + \frac{x^7}{945} \right) \left( 1 + \sum_{k=1}^{\infty} 2 \frac{(2^{2k-1} - 1) |B_{2k}| x^{2k}}{(2k)!} \right)
\end{aligned}$$

Now, we will proceed as above, we will use Taylor's approximation for this function to provide bounds for  $\left(\frac{\sin x}{x}\right)^3 - \cos x$ . By Lemma 1-5 we derive

**Theorem 2-7** For  $0 < x < \pi/2$  the following inequalities hold

$$\begin{aligned}
(\sin x)^2 \sum_{k=0}^n b_{k+2} x^{2k+2} &< \frac{(\sin x)^3}{x^3} - \cos x < (\sin x)^2 \sum_{k=0}^{n-1} b_{k+2} x^{2k+2} + \\
& \left[ \left( \frac{2}{\pi} \right)^{2n+5} - \sum_{k=0}^{n-1} b_{k+2} \left( \frac{2}{\pi} \right)^{2n-2k} \right] x^{2n+2} (\sin x)^2
\end{aligned}$$

where

$$b_k = \frac{2(2k-1)(2k+1)(2^{2k-1}-1) |B_{2k}| + (-1)^k}{(2k+1)!},$$

$B_{2k}$  are the Bernoulli numbers.

### 3 A trigonometric chain

By using inequalities involving several means, Neuman [15] presented the following inequality chain generalizing the Adamovic-Mitrinovic inequality :

$$\left(\frac{\sin x}{x}\right)^3 > \cos x. \quad (9)$$

For  $x \in (0, \frac{\pi}{2})$  one has

$$(\cos(x))^{1/3} < \left(\cos(x)\frac{\sin(x)}{x}\right)^{1/4} < \left(\frac{\sin(x)}{\arctan(\sin(x))}\right)^{1/2} < \quad (10)$$

$$\left[\frac{1}{2}(\cos(x) + \frac{\sin(x)}{2x})\right]^{1/2} < \left[\frac{1 + 2\cos(x)}{3}\right]^{1/2} < \left[\frac{1 + \cos(x)}{2}\right]^{2/3} < \frac{\sin(x)}{x}.$$

Thus the left inequality of (1) is improved.

Notice that Yang [17] proved that for  $0 < x < \frac{\pi}{2}$ ,

$$\frac{\sin(x)}{x} < \left(\frac{2}{3}\cos\left(\frac{x}{2}\right) + \frac{1}{3}\right)^2 < (\cos\left(\frac{x}{3}\right))^3 < \frac{2 + \cos(x)}{3} \quad (11)$$

which improves the right inequality of (1).

We are interested in last inequality of the chain (9) for  $x \in (0, \frac{\pi}{2})$

$$\left[\frac{1 + \cos(x)}{2}\right]^{2/3} < \frac{\sin(x)}{x} \Leftrightarrow \left[\frac{1 + \cos(x)}{2}\right]^2 < \left(\frac{\sin(x)}{x}\right)^3.$$

Theorem 1-2 asserts that for any  $n \geq 5$

$$\begin{aligned} \sin x \sum_{k=5}^n a_k x^{2k-3} &\leq \left(\frac{\sin x}{x}\right)^3 - \cos x - \sin x \left(\frac{x^3}{15} - \frac{x^5}{945}\right) \leq \\ \sin x \sum_{k=5}^{n-1} a_k x^{2k-3} + \sin x &\left[\left(\frac{2}{\pi}\right)^{2n} - \sum_{k=5}^{n-1} a_k \left(\frac{2}{\pi}\right)^{2n-2k}\right] x^{2n-3}, \end{aligned}$$

where

$$a_k = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)k} \right), \quad k \geq 5$$

and  $B_{2k}$  are the Bernoulli numbers. Then by Theorem 1-2

$$\sin x \sum_{k=5}^n a_k x^{2k-3} + \cos x + \sin x \left(\frac{x^3}{15} - \frac{x^5}{945}\right) \leq \left(\frac{\sin x}{x}\right)^3,$$

where

$$a_k = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)k} \right), \quad k \geq 5$$

and  $B_{2k}$  are the Bernoulli numbers.  
The function

$$u(x) = \left[ \frac{1 + \cos(x)}{2} \right]^2 - \cos x - \sin x \left( \frac{x^3}{15} - \frac{x^5}{945} \right)$$

has no constant sign. Indeed,

$$u\left(\frac{\pi}{2}\right) = \frac{1}{4} - \left(\frac{\pi^3}{120}\right) \left(1 - \left(\frac{\pi^2}{252}\right)\right) \approx 0.00174,$$

$$u\left(\frac{\pi}{4}\right) = \left(\frac{1}{2} + \left(\frac{\sqrt{2}}{4}\right)\right)^2 - \left(\frac{\sqrt{2}}{2}\right) - \left(\pi^3 \frac{\sqrt{2}}{1920}\right) \left(1 - \left(\frac{\pi^2}{1008}\right)\right) \approx -0.001164.$$

However, we obtains the following which is an improvement of the chain (9)

**Theorem 3-1** For  $x \in (0, \frac{\pi}{2})$  the following inequalities hold

$$\left[ \frac{1 + \cos x}{2} \right]^2 < \cos x + \sin x \left( \frac{x^3}{15} - \frac{x^5}{945} + \frac{x^7}{2835} \right) < \left( \frac{\sin x}{x} \right)^3,$$

**Proof of Theorem 3-1** The right hand of the inequalities are deduced from Theorem 2-2. Let us consider expansions trigonometric functions with power series, [6], [7]. We will use the Taylor expansions of  $\sin x, \cos x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} + (-1)^k \frac{\sin \theta x}{(2k+1)!} x^{2k+1}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^k \frac{x^{2k}}{(2k)!} + (-1)^{k+1} \frac{\cos \theta x}{(2k)!} x^{2k}$$

where  $0 < \theta < 1$ . It is easy to remark that

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} < \cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}.$$

for  $0 < x < \frac{\pi}{2}$ . The left hand of the theorem can be majored for  $x \in (0, \frac{\pi}{2})$

$$\begin{aligned} & \left[ \frac{1 + \cos x}{2} \right]^2 - \cos x + \sin x \left( \frac{x^3}{15} - \frac{x^5}{945} + \frac{x^7}{2835} \right) < \left( 1 - \frac{x^2}{4} + \frac{x^4}{48} - \frac{1}{1440} x^6 + \frac{1}{80640} x^8 \right)^2 - \\ & 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{1}{720} x^6 - \frac{x^3}{15} \left( x - \frac{x^3}{6} + \frac{1}{120} x^5 - \frac{1}{5040} x^7 \right) \left( 1 - \frac{1}{63} x^2 + \frac{1}{189} x^4 \right) = \end{aligned}$$

$$-\frac{x^4}{240} + \frac{53x^6}{30240} - \frac{337x^8}{1209600} + \frac{199x^{10}}{4354560} - \frac{437x^{12}}{203212800} + \frac{193x^{14}}{3657830400} + \frac{x^{16}}{6502809600} < 0.$$

**Corollary 3-2** For  $x \in (0, \frac{\pi}{2})$  and  $n \geq 5$  the following inequalities hold

$$\left[ \frac{1 + \cos x}{2} \right]^2 < \cos x + \sin x \left( \frac{x^3}{15} - \frac{x^5}{945} + \sum_{k=5}^n a_k x^{2k-3} \right) < \left( \frac{\sin x}{x} \right)^3.$$

where

$$a_k = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)k} \right), \quad k \geq 5$$

and  $B_{2k}$  are the Bernoulli numbers.

Indeed, by induction on  $n$  we derive

$$\sin x \left( \frac{x^3}{15} - \frac{x^5}{945} + \sum_{k=5}^p a_k x^{2k-3} \right) < \sin x \left( \frac{x^3}{15} - \frac{x^5}{945} + \sum_{k=5}^{p+1} a_k x^{2k-3} \right)$$

since the coefficient  $a_{p+1} > 0$  which implies the corollary.

## 4 Bounds of Adamovic-Mitrinovic inequalities

By Theorem 2-2 we are able to improve the left hand inequality (3). Indeed, one has the following

$$\cos x - \left( \frac{\sin x}{x} \right)^3 < -\sin x P_n(x) < -\frac{x^3}{15} \left( 1 - \frac{x^2}{63} \right) \sin(x) = -\sin x P_5(x)$$

where  $P_n(x), n \geq 5$  is the  $n$ -polynomial

$$P_n(x) = \sum_{5 \leq k \leq n} a_k x^{2k-2} = \sum_{5 \leq k \leq n} \left( \frac{2^{2k-2}}{(2k-2)!} \left( \frac{(-1)^{k+1}}{(k)(2k-1)} + |B_{2k}| \right) \right) x^{2k-2}.$$

Turn to the Statement. In [8, Theorem 1] the authors proved for  $0 < x < \pi/2$

$$-\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} < \cos x - \left( \frac{\sin x}{x} \right)^3 < -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600}$$

The authors used the following frame

$$\sum_{k=2}^{2n} (-1)^k A(k) x^{2k} < \cos x - \left( \frac{\sin x}{x} \right)^3 < \sum_{k=2}^{2n+1} (-1)^k A(k) x^{2k},$$

where  $A(k) = \frac{3^{2k+3} - 32k^3 - 96k^2 - 88k - 27}{4(2k+3)!}$ .

## 4.1 Examples

By increasing the degree  $n$  of  $P_n(x)$  in Theorem 2-2, one can improve the precision so that the bounds obtained are sometime better than the one provided by [8].

In this case, taking  $n = 7$  is enough to prove right hand of Statement (i)

$$P_7(x) = \frac{x^3}{15} \left( 1 - \frac{x^2}{63} + \frac{x^4}{189} + \frac{8x^6}{31185} \right) \sin x.$$

recall that we have for  $x \in (0, \pi/2)$

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880}.$$

Then we obtain thanks to *Maple*

$$\begin{aligned} & -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} + \frac{x^3}{15} \left( 1 - \frac{x^2}{63} + \frac{x^4}{189} + \frac{8x^6}{31185} \right) \sin x \\ > -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} + \frac{x^3}{15} \left( 1 - \frac{x^2}{63} + \frac{x^4}{189} + \frac{8x^6}{31185} \right) \left( x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \right) = \\ & \frac{47x^{12}}{157172400} + \frac{19x^{14}}{261954000} - \frac{x^{16}}{294698250} = -\frac{x^{12}(-705 - 171x^2 + 8x^4)}{2357586000}. \end{aligned}$$

It is easy to see that the last expression is non negative for  $0 < x < \pi/2$ . Then for  $x \in (0, \pi/2)$  the following inequalities hold

$$\begin{aligned} -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} &> -\frac{x^3}{15} \left( 1 - \frac{x^2}{63} + \frac{x^4}{189} + \frac{8x^6}{31185} \right) \sin x > \\ &\cos x - \left( \frac{\sin x}{x} \right)^3. \end{aligned}$$

Thus, for the left hand our estimate appears to be finer than that provided by Statement (i) [8]. However, for the right hand one has the following

$$\begin{aligned} & (\sin x) \left[ -\frac{x^3}{15} + \frac{x^5}{945} - \frac{x^7}{2835} - \frac{8x^9}{467775} - \frac{206x^{11}}{91216125} - \right. \\ & \left. \left( \frac{16384}{\pi^{14}} - \frac{16}{2835\pi^4} - \frac{32}{467775\pi^2} \right) x^{11} \right] + \frac{x^4}{15} - \frac{23x^6}{1890} + \frac{41x^8}{37800} - \frac{53x^{10}}{831600} + \frac{74677x^{12}}{27243216000} < 0, \end{aligned}$$

which means that the following holds for  $0 < x < \pi/2$

$$\cos x - \left( \frac{\sin x}{x} \right)^3 < (\sin x) \left[ -\frac{x^3}{15} + \frac{x^5}{945} - \frac{x^7}{2835} - \frac{8x^9}{467775} - \frac{206x^{11}}{91216125} - \right.$$

$$\left(\frac{16384}{\pi^{14}} - \frac{16}{2835\pi^4} - \frac{32}{467775\pi^2}\right)x^{11}] < -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} - \frac{74677x^{12}}{27243216000}$$

and for this case the right hand of our estimate appears to be finer than that provided by Statement **(ii)** [8].

By the same way one proves Statement **(iii)**

$$-\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} - \frac{74677x^{12}}{27243216000} + \frac{989x^{14}}{10897286400} <$$

$$(\sin x) \left[ -\frac{x^3}{15} + \frac{x^5}{945} - \frac{x^7}{2835} - \frac{8x^9}{467775} - \frac{206x^{11}}{91216125} - \frac{139x^{13}}{42567525} \right] < \cos x - \left( \frac{\sin x}{x} \right)^3.$$

## 4.2 Remarks

By the same way we may improve another bound of [8] : for  $0 < x < \pi/2$

$$\cos x - \left( \frac{\sin x}{x} \right)^3 < -\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} - \frac{74677x^{12}}{27243216000} + \frac{989x^{14}}{10897286400}.$$

We must then use in Theorem 1-2 the polynomial  $P_n(x)$  with a degree of order  $n = 9$  to get a better estimate. Using *Maple* consider

$$\frac{x^3}{15} \sin x \left( 1 - \frac{x^2}{63} \left( 1 - \frac{x^2}{3} - \frac{8x^4}{495} - \frac{206x^6}{96525} \right) \right) - \left( \frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} - \frac{74677x^{12}}{27243216000} + \frac{989x^{14}}{10897286400} \right) >$$

$$\frac{x^3}{15} \left( x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \right) \left( 1 - \frac{x^2}{63} \left( 1 - \frac{x^2}{3} - \frac{8x^4}{495} - \frac{206x^6}{96525} \right) \right) -$$

$$\left( \frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} - \frac{74677x^{12}}{27243216000} + \frac{989x^{14}}{10897286400} \right) =$$

$$-\frac{x^{12}}{5443200} - \frac{14929x^{14}}{70053984000} + \frac{197x^{16}}{12770257500} - \frac{103x^{18}}{229864635000} = -\frac{x^{12}(1351350 + 1567545x^2 - 113472x^4 + 3296x^6)}{7355668320000} > 0.$$

That means for  $0 < x < \pi/2$  we refine Statement **1 (iii)** and the following hold

$$\cos x - \left( \frac{\sin x}{x} \right)^3 < -\frac{x^3}{15} \sin x \left( 1 - \frac{x^2}{63} \left( 1 - \frac{x^2}{3} - \frac{8x^4}{495} - \frac{206x^6}{96525} \right) \right) <$$

$$-\frac{x^4}{15} + \frac{23x^6}{1890} - \frac{41x^8}{37800} + \frac{53x^{10}}{831600} - \frac{74677x^{12}}{27243216000} + \frac{989x^{14}}{10897286400}.$$

## Other refinements

We may prove the following frame which also improves the one of Mortici [13]

$$-\frac{x^6}{945} + \frac{x^8}{1890} - \frac{x^{10}}{19800} < \frac{(\sin x)^3}{x^3} - \cos x - \frac{x^3}{15} \sin x$$



$$< -\frac{x^6}{945} + \frac{x^8}{1890} - \frac{x^{10}}{19800} + \frac{2903x^{12}}{1135134000} < 0$$

The following are stronger

$$-\frac{x^8}{5670} + \frac{113x^{10}}{1247400} - \frac{5293x^{12}}{486486000} < \frac{(\sin x)^3}{x^3} - \cos x$$

$$-\frac{x^3}{15} \sin x \left(1 - \frac{x^2}{63} \cos x\right) < -\frac{x^8}{5670} + \frac{113x^{10}}{1247400} < 0$$

For  $a \leq 1$  we have

$$\frac{(\sin x)^3}{x^3} - \cos x - \frac{x^3 \sin x}{15} + \frac{x^5 \sin x}{945} - \frac{ax^7 \sin x}{2835}$$

$$> \left(\frac{1}{2835} - \frac{a}{2835}\right)x^8 + \left(-\frac{13}{311850} + \frac{a}{17010}\right)x^{10} + \left(\frac{571}{243243000} - \frac{a}{340200}\right)x^{12} > 0$$

For  $a = 1$  we have

$$\frac{(\sin x)^3}{x^3} - \cos x - \frac{1}{15}x^3 \sin x + \frac{1}{945}x^5 \sin x - \frac{1}{2835}x^7 \sin x$$

$$> \frac{8x^{10}}{467775} - \frac{2x^{12}}{3378375} - \frac{31x^{14}}{1915538625} + \frac{x^{16}}{724990500} - \frac{43789x^{18}}{1039447879470000} > 0$$

In particular

$$\frac{8x^{10}}{467775} - \frac{2x^{12}}{3378375} > 0.$$

### 4.3 Other examples

Theorem 2-7 appears to be better in order to find again the Statement .  
Notice at first by the Taylor approximation we get the bound [7,p.145]

$$(\sin x)^2 < x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \dots = \sum_1^{2n+1} (-1)^{k+1} \frac{2^{2k+1} x^{2k}}{(2k)!}.$$

For the right hand of the inequality, Theorem 2-7 implies

$$\frac{(\sin x)^3}{x^3} - \cos x < (\sin x)^2 \left[ \sum_{k=0}^{n-1} b_{k+2} x^{2k+2} + \left[ \left(\frac{2}{\pi}\right)^{2n+5} - \sum_{k=0}^{n-1} b_{k+2} \left(\frac{2}{\pi}\right)^{2n-2k} \right] x^{2n+2} \right] <$$

$$\left(x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \dots\right) \left[ \sum_{k=0}^{n-1} b_{k+2} x^{2k+2} + \left[ \left(\frac{2}{\pi}\right)^{2n+5} - \sum_{k=0}^{n-1} b_{k+2} \left(\frac{2}{\pi}\right)^{2n-2k} \right] x^{2n+2} \right].$$

Putting  $n = 2$  one gets thanks to *Maple*

$$\frac{(\sin x)^3}{x^3} - \cos x < (\sin(x))^2 \left( \frac{x^2}{15} + \frac{19}{1890}x^4 + \frac{167}{113400}x^6 \right) + \left( \frac{2048}{\pi^{11}} - \frac{64}{15\pi^6} - \frac{152}{945\pi^4} - \frac{167}{28350\pi^2} \right) x^8 (\sin(x))^2$$

For the left hand one Theorem 2-7 yields

$$\begin{aligned} \frac{(\sin x)^3}{x^3} - \cos x &> (\sin x)^2 \left( \frac{x^2}{15} + \frac{19}{1890}x^4 + \frac{167}{113400}x^6 \right) > \\ &\left( x - \frac{x^3}{6} + \frac{1}{120}x^5 - \frac{1}{5040}x^7 \right)^2 \left( \frac{x^2}{15} + \frac{19}{1890}x^4 + \frac{167}{113400}x^6 \right) > \\ &\frac{x^4}{15} - \frac{23}{1890}x^6 + \frac{41}{37800}x^8 - \frac{29}{113400}x^{10} + \frac{12167}{285768000}x^{12} - \frac{673}{190512000}x^{14} + \\ &\frac{5797}{34292160000}x^{16} - \frac{179}{40007520000}x^{18} + \frac{167}{2880541440000}x^{20} > \\ &\frac{x^4}{15} - \frac{23}{1890}x^6 + \frac{41}{37800}x^8 - \frac{53}{83160}x^{10}. \end{aligned}$$

That means our bounds are finer than Statement (i).

Putting  $n = 6$  one gets

$$\begin{aligned} \frac{(\sin x)^3}{x^3} - \cos x &> (\sin(x))^2 \left( \frac{x^2}{15} + \frac{19}{1890}x^4 + \frac{167}{113400}x^6 + \frac{479}{2494800}x^8 + \frac{1944919}{81729648000}x^{10} + \right. \\ &\left. \frac{465863}{163459296000}x^{12} + \frac{246913}{741015475200}x^{14} \right) \end{aligned}$$

and

$$\begin{aligned} &(\sin(x))^2 \left( \frac{x^2}{15} + \frac{19}{1890}x^4 + \frac{167}{113400}x^6 + \frac{479}{2494800}x^8 + \frac{1944919}{81729648000}x^{10} + \right. \\ &\left. \frac{465863}{163459296000}x^{12} + \frac{246913}{741015475200}x^{14} \right) - \\ &\frac{x^4}{15} + \frac{23}{1890}x^6 - \frac{41}{37800}x^8 + \frac{53}{831600}x^{10} - \frac{74677}{27243216000}x^{12} + \frac{989}{10897286400}x^{14} > \\ &\frac{79649}{33345696384000}x^{16} - \frac{80489729}{2100778872192000}x^{18} + \frac{148949261}{17685128260800000}x^{20} - \frac{73252620769}{98789126464828800000}x^{22} > 0. \end{aligned}$$

That means this bound is finer than Statement (iii).

#### 4.4 Concluding remarks

More generally, all the examples derived from Theorems 2-2 and 2-7, naturally suggest that we may expect that the following inequalities hold or at least for some values of  $n$

$$\cos x - \left( \frac{\sin x}{x} \right)^3 < -\frac{x^6 \sin x}{15} + \frac{x^8 \sin x}{945} - \sin x \sum_{k=5}^n a_k x^{2k-3} < \sum_{k=2}^{2n+1} (-1)^k A(k) x^{2k},$$

where

$$a_k = \frac{2^{2k-2}}{(2k-2)!} \left( |B_{2k-2}| + \frac{(-1)^{k+1}}{(2k-1)k} \right), \quad A(k) = \frac{3^{2k+3} - 32k^3 - 96k^2 - 88k - 27}{4(2k+3)!}.$$

We may also expect (following Theorem 2-7) that

$$\frac{(\sin x)^3}{x^3} - \cos x < (\sin x)^2 \sum_{k=1}^{n-1} b_{k+2} x^{2k-2} +$$

$$\left[ \left( \frac{2}{\pi} \right)^{2n+5} - \sum_{k=1}^{n-1} b_{k+2} \left( \frac{2}{\pi} \right)^{2n-2k} \right] x^{2n} (\sin x)^2 < - \sum_{k=2}^{2n} (-1)^k A(k) x^{2k}$$

hold.

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