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On the Measurement of Opportunity-dependent Inequality under Uncertainty

Xiangyu Qu *

Abstract

The actual level of income inequality and the scope of allocation options influence the individual as well as social perception of inequality under uncertainty. We show how incorporating opportunity concerns under uncertainty would affect inequality perception. An axiomatic system on preferences is suggested and demonstrated to characterize a measure of opportunity-dependent index under uncertainty, which would potentially provide a satisfactory answer to the puzzling observation that many individuals perceive higher level of income inequality although its actual level is low.

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1 INTRODUCTION

It is increasingly understood that inequality has impacted nearly every aspect of economics. Now, as then, most economists agree that inequality studies lie at the heart of economic analysis. The many inequality measurement studies, [Dalton \[1920\]](#), [Kolm \[1969\]](#), [Atkinson \[1983\]](#), [Sen and Foster \[1997\]](#) among many others, carried out over the past several decades have provided a precise snapshot of inequality, especially in situations of perfect certainty.

To alleviate inequality, effective policies are needed, which, accordingly, require a trustworthy inequality estimate to signal and predict the effectiveness of any given policy. Except for a few studies, most inequality analyses assume perfect knowledge of the economic present and foresight of the economic future. Since the effects of uncertainty on policies determining inequality are widely acknowledged, assuming away uncertainty largely limits our ability to understand the elements that determine inequality, as well as the reliability of selected policies. Therefore, little progress can be made in the analysis of economic policy toward inequality without a satisfactory measurement under uncertainty.

There are a variety of stylized answers to how to incorporate uncertainty into inequality measurements. Two very natural approaches are, as [Ben-Porath, Gilboa, and Schmeidler \[1997\]](#) pointed out, (1) first to calculate each individual allocation by its expected value and then to measure the inequality index based on individual expected values; and (2) first to measure the inequality index at each state and then to calculate the expected inequality index. For reasons of social principles or statistical tractability, it is reasonable to apply the different approaches to different types of problems. To avoid the inability to address some type of injustice, see, for example, [Ben-Porath et al. \[1997\]](#) and [Chew and Sagi \[2012\]](#), two approaches can be generalized in some way to reconcile different principles in distributional problems.

While the above generalized approach seems to be a very natural extension of inequality measurements under conditions of pure certainty to uncertainty, it nevertheless still possesses some troubling aspects. The present inequality measures are grounded on the allocation comparison among the *universal* set of allocations, which includes both *feasible* and *unfeasible* allocations. A serious measurement of inequality should only be based on the feasible allocations to which the allocation judgement is being applied: what are the attainable allocations given the implementable policies? In other words, the ranking of feasible allocations should not solely depend on the income or wealth distributions and social justice principles we choose. [Alesina and Angeletos \[2005\]](#) observed that the United States and the Europe, though are quite similar in tastes, technologies and political systems, have considerably various social policies, such as redistribution policy. One possible explanation is that two societies have different views about the feasible allocations. As

Bénabou [2000] claimed, *European voters choose to sacrifice more employment and growth to social insurance that their American counterparts, even though both populations have the same basic preferences*. The feasible allocations in the United States is relatively restricted compared to the Europe, which might generate the different perception of inequality. Therefore, incorporating the concern of feasible allocations into inequality measurement would help explain the redistribution puzzle observed by Alesina and Angeletos [2005].

We should also consider the possible allocations by introducing all practical modifications to the economy. Adopting an inequality measure that ignores the national ‘cakes’ makes it hard to understand why a less equal society, in terms of a lower index of equality, sometimes feels more equal than a society with a higher index of equality. However, the most ubiquitous inequality indices are not characterized based on the opportunity set of allocations, which may lead to a trap that an unequal society is preferred.

The principal aim of this paper is to tackle the inequality measurement problem within a context-dependent framework that takes both uncertainty and feasibility of allocations into account. The inequality index we suggest is in a specific quantitative sense, so it would be clear what can sensibly be used to measure inequality. Specifically, let M denote the opportunity set of contingent allocations. Then, the inequality degree of f given M is measured by

$$(1) \quad \mathcal{I}_{un}(f, M) = \min_{p \in P} \int_S \{ \phi(f^s) - \max_{h \in M} \phi(h^s) \} dp(s).$$

Function ϕ is a classical Gini index measurement of allocation whenever state s is realized. The set of probabilities, P , represents society’s probability estimation over states. The expression $\max_{h \in M} \phi(h^s)$ shows the most egalitarian allocation we can obtain from M at state s . Therefore, $\phi(f^s) - \max_{h \in M} \phi(h^s)$ measures exactly the *equality loss* at state s if allocation f is selected. Hence, the index we propose reflects the minimum expected equality loss. Moreover, this index can be regarded as deriving from the psychological notion of regret. That is, the reaction to receiving allocation f^s when there is a best alternative would have led to allocation $\max_{h \in M} \phi(h^s)$. It is worth emphasizing the distinction between uncertainty and risk, which is why our index takes a set of probabilities to represent that the events are not perfectly foreknown. In fact, our index coincides with the second approach discussed above whenever the set of probabilities is a singleton.

The main goal of this paper is to axiomatize the above inequality index. We shall provide principles on which comparisons of allocations can be justified. We shall also discuss the normative arguments for and against those principles. As a result, we shall show how our principles can be equivalently translated into our proposed index, which can be used in practice.

It is worth mentioning that our index is a natural extension of both [Ben Porath and Gilboa \[1994\]](#) and [Ben-Porath et al. \[1997\]](#). On the one hand, it is a generalization of the principles used in [Ben Porath and Gilboa \[1994\]](#) for evaluating inequality under certainty. On the other hand, [Ben-Porath et al. \[1997\]](#) is a special case of our index under uncertainty whenever a feasible set M is universal or the opportunity cost is *nil*. It is well-known that the inequality measured by [Ben Porath and Gilboa \[1994\]](#) can have either Choquet integral form or *maxmin* expected utility form. However, in the face of uncertainty, both ex-ante and ex post inequality matter. Therefore, to maintain the consistency, it would be plausible that the form is invariant if we apply the same form over states first and, then over individuals, or vice versa. As [Ben-Porath et al. \[1997\]](#) demonstrated, only MEU form has invariant property, which, alternatively, justify the rationale of our index to adopting the MEU form.

However, there are many situations, for example, where a society considers how climate changes affect allocations, in which this attaching of probabilities to various alternative possible events in advance cannot reasonably be performed because the outcome is strongly influenced by some elements about which we have little or no prior information. While the classical indices can readily be extended to cover cases in the face of uncertainty, they cannot easily be extended to do so in the face of ignorance. Therefore, we revise the above index using the *minmax* approach to achieve a measurement under ignorance.

$$(2) \quad \mathcal{I}_{ig}(f, M) = \min_{s \in S} \{ \phi(f^s) - \max_{h \in M} \phi(h^s) \}.$$

This is actually a special case of the \mathcal{I}_{un} in which the set of probabilities is the universal set of probabilities. We also axiomatize this inequality index and discuss the related principles.

Section 2 first presents the framework, followed by a discussion of the axioms in subsections. They include a set of axioms, which extends the assumptions from certainty to an uncertainty framework. The representation result is presented in Subsection 2.3. In subsection 2.4, we also discuss the index measurement under ignorance. Section 3 concludes with a discussion. All proofs are provided in the Appendix.

2 THE MODEL

2.1 Framework

Let $S = \{1, \dots, m\}$ be a set of *states* of world and let $I = \{1, \dots, n\}$ be a set of *individuals*. A *contingent* allocation is denoted by $f = (f_i^s)_{i \in I, s \in S} \in [-\ell, \ell]^{mn} \subset \mathbb{R}^{mn}$, where f_i^s describes the

income attained by individual i at state s . We allow the negative incomes to illustrate the situation of *debt*. Let F denote the space of all contingent allocations. We say an allocation f is *null*, write $f = \mathbf{0}$, if $f_i^s = 0$ for all s and i .

Every nonempty compact and convex subset M of F is a *menu*, representing society's *opportunity set* of feasible choice objects. A society that is supposed to make a choice from menu M is assumed to have a well-defined ranking on M . Let \mathbb{M} be the set of menus. For every $M \in \mathbb{M}$, we denote society's preference relation by $\succsim_M \subset M \times M$. As usual, \succ_M and \sim_M represent the asymmetric and symmetric parts of \succsim_M , respectively.

2.2 Axioms

In this subsection, we set forth the axioms of egalitarian relations and discuss the principles for constructing a measure of inequality in ambiguous environments.

A1. (*Weak order.*) For every $M \in \mathbb{M}$, \succsim_M is complete and transitive on M .

Completeness and transitivity are standard assumptions on preference relation \succsim_M . There are many intuitions that only support partial orderings claimed by [Sen and Foster \[1997\]](#). However, the inequality indices we consider completely order all the contingent allocations among every allocation menu M .

To state the next axioms, we need to define the mixture of two allocations. For every pair of allocations f, g and each number $\lambda \in [0, 1]$, the mixture allocation $h = \lambda f + (1 - \lambda)g$ is defined by $h_i^s = \lambda f_i^s + (1 - \lambda)g_i^s$ for each $s \in S$ and $i \in I$.

A2. (*Mixture Continuity.*) For all $f, g, h \in M \in \mathbb{M}$, if $f \succ_M g \succ_M h$, then there are $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1 - \alpha)h \succ_M g \succ_M \beta f + (1 - \beta)h$.

Mixture continuity is a standard technical assumption, which we do not elaborate further. With regard to intuitive appeal, a natural inequality measure under certainty is the generalized Gini index (GGI) proposed by [Weymark \[1981\]](#) and [Ben Porath and Gilboa \[1994\]](#). In this paper, we assume that society selects GGI to express its attitude toward inequality. We realize that there are many other very good measures of inequality under certainty.¹ The selection of GGI is not critical for our results. It is only a matter of preference. The next two axioms along with the previous two would characterize a GGI.

We need more notations to state the next axiom. A contingent allocation f is said to be *constant* if there is a $x = (x_i)_{i \in I} \in \mathbb{R}^n$ such that $f_i^s = x_i$ for all $s \in S$ and $i \in I$. Without fear of confusion,

¹See [Cowell \[2000\]](#) for an excellent survey about inequality measures under certainty.

we identify the set of constant allocations by $X = [-\ell, \ell]^n$. Clearly, the set of constant allocations is compact and convex and, hence, belongs to \mathbb{M} .

For a permutation $\pi : I \rightarrow I$ and $x \in X$, define $\pi \circ x \in X$ by $(\pi \circ x)_i = x_{\pi(i)}$. Obviously, $\pi \circ X = X$.

A3. (*Constant Anonymity.*) For every $x, y \in X$, if there is a permutation π such that $x = \pi \circ y$, then $x \sim_X y$.

This axiom states that any permutations of personal labels are regarded as inequality equivalent. This means that a society only cares about the information related to the income variable, not about some characteristics of individuals. It has been realized that this assumption is not always self-evident, especially under some circumstances; for instance, there is a link between past and present distributions. However, it is a promising assumption whenever sensitive information about income distributions is absent.

Two constant allocations $x, y \in X$ are *constantly commontonic* if $x_i \geq x_j \Leftrightarrow y_i \geq y_j$ for all $i, j \in I$.

A4. (*Comonotonic Translation Invariance.*) For every pairwise comonotonic $x, y, z \in X$, if $x \succsim_X y$, then $x + z \succsim_X y + z$.

A4 states that the ranking that one allocation x is more egalitarian than another comonotonic allocation y is invariant if both allocations receive the same ‘donation’ z , which also respects the same income ordering as x and y . In contrast to classical translation invariance, in which the rankings remain invariant under uniform addition or subtraction, this assumption is in a strong form that requires the rankings to hold when the rank order of allocations is identical to the addition or subtraction of the variable income.

For $x \in X$, let \tilde{x} be the constant allocation obtained from x by rearranging the allocations in an increasing order, i.e. $\{x_1, \dots, x_n\} = \{\tilde{x}_1, \dots, \tilde{x}_n\}$ and $\tilde{x}_1 \leq \dots \leq \tilde{x}_n$. A function $\phi : X \rightarrow \mathbb{R}$ is a *generalized Gini index* (GGI) if there exists $\alpha \in \mathbb{R}^n$ such that for every $x \in X$,

$$\phi(x) = \sum_{i \in I} \alpha_i \tilde{x}_i.$$

We say that a function $\phi : X \rightarrow \mathbb{R}$ *represents* society’s preferences over constant allocations on X if for every $x, y \in X$, $x \succsim_X y$ if and only if $\phi(x) \geq \phi(y)$. Notice that we do not require that α has an increasing property, i.e. $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$. We want to maintain the possibility to address the situation, as demonstrated by [Alesina and Angeletos \[2005\]](#), where a more equal society is perceived less equality than a less equal society. In fact, if we impose Pigou-Dalton

transfer principle, namely that x is a Pigou-Dalton transfer of y ² implies $x \succsim y$, then α should satisfy the increasing property.

Weymark [1981] and Ben Porath and Gilboa [1994] derived that GGI is characterized by the above axioms.

Proposition 1. *Assume that \succsim_X satisfies Axioms 1-4. Then there exists a GGI that represents it.*

The further axioms specified below will deliver our index proposed in (1). The reasons for seeking such an index were discussed in the introduction.

For $f \in F$, let f^s denote $(f_i^s)_{i \in I}$. Clearly, $f^s \in X$ for every $s \in S$.

A5. (*Monotonicity.*) For every $M \in \mathbb{M}$ and every $f, g \in M$, if $f^s \succsim_X g^s$ for all $s \in S$, then $f \succsim_M g$.

A5 says that if a contingent allocation f statewise dominates another allocation g in terms of equality, then f is regarded as a more egalitarian allocation than g ex ante. In other words, if the GG index of f^s is higher than that of g^s in every s , then f is judged as a more egalitarian allocation than g . A5 is intuitive, and the most widely used indices, for instance, Ben-Porath et al. [1997], also satisfy this property.

Let $X^e = \{(a, \dots, a) \in X : a \in \mathbb{R}\} \subset X$ be the collection of *equally distributed constant* allocation. An allocation $f \in F$ is said to be *equally distributed* if f^s is an equally distributed constant allocation for every s . Let F^e be the collection of equally distributed allocations.

A6. (*Equal uncertainty aversion.*) For every $f, g \in M$, if $f, g \in F^e$, then for all $\lambda \in (0, 1)$, $f \succsim_M g$ implies $\lambda f + (1 - \lambda)g \succsim_M g$.

This axiom states that if an equally distributed allocation f is judged to be more egalitarian than another equally distributed allocation g , then any allocation between the two allocations should be more egalitarian than g . This axiom is strictly weaker than the *Concavity* assumption of Ben-Porath et al. [1997], which is an extended version of the uncertainty aversion axiom of Gilboa and Schmeidler [1989] from an individual to a collective decision-making environment. The scope of uncertainty aversion under the concavity assumption consists of all pairs of indifferent allocations. In contrast, our axiom is strictly weaker and only applies to the indifferent pair of allocations in which every individual has identical incomes in every possible state. For arbitrary pairs of indifferent allocations, the convex combination in general will induce asymmetric impacts across individuals, which may lead to the opposite attitude toward uncertainty. Whenever individuals

²If $x, y \in X$, we say x is a Pigou-Dalton transfer of y if for some $i, j \in \{1, \dots, n\}$ we have $x_k = y_k$ for $k \notin \{i, j\}$ and $x_i + x_j = y_i + y_j$ and $|x_i - x_j| < |y_i - y_j|$.

have identical incomes in every state, such convex combinations will have symmetric impacts. Therefore, it is natural to assume that an uncertainty averse attitude applies.

We say that an allocation f *weakly egalitarian dominates* menu M if $f^s \succeq_X g^s$ for all $s \in S$ and all $g \in M$. That is, if f weakly egalitarian dominates opportunity set M , then for every possible state s , allocation f^s is more egalitarian than any other allocation g^s in set M . We say that f *nonegalitarian dominates* menu M if there does not exist a state s such that $f^s \succ_X g^s$ for all $g \in M$.

A7 (*Independence of non-egalitarian allocation.*) (INA) For every $f, g \in M$, if an allocation $h \notin M$ non-egalitarian dominates M , then $f \succeq_M g$ implies $f \succeq_{M \cup \{h\}} g$.

This axiom states that if there does not exist a state s such that h^s is strictly more egalitarian than any allocation g^s in opportunity set M , then adding h into set M does not affect the egalitarian ranking of allocations among M . This implies that if a contingent allocation by no means generates a more egalitarian allocation than M , then adding it into such an opportunity set will not change the degree of inequality loss for each contingent allocation in M . Therefore, along with the next axiom, it implies that the degree of inequality loss for each contingent allocation is solely dependent on the most egalitarian allocation in each possible state.

A8. (*Independent of Dominated Menu.*) If an allocation $h \in M \cap N$ weakly egalitarian dominates menus M, N , then $f \succeq_M g$ iff $f \succeq_N g$.

In words, A8 asserts that the egalitarian ranking between two contingent allocations should be invariant across two opportunity sets if both sets are weakly egalitarian dominated by a common contingent allocation. A7 and A8 are actually intended to pin down the attitude of society. That is, society only restricts attention to contingent allocations whose allocation at some state is the most egalitarian among the the opportunity sets.

For every $M \in \mathbb{M}$ and each number $\lambda \in [0, 1]$, the notation $\lambda M + (1 - \lambda)f$ denotes the menu generated by replacing every allocation f in M with the analog mixture. Obviously, $\lambda M + (1 - \lambda)f$ belongs to \mathbb{M} .

A9. (*Equal Independence.*) For every $f, g \in M$ and every $h \in F^e$, if $f \succeq_M g$, then for every $\lambda \in [0, 1]$, $\lambda f + (1 - \lambda)h \succeq_{\lambda M + (1 - \lambda)h} \lambda g + (1 - \lambda)h$.

A9 states that the independence axiom of the expected utility theory applies whenever the commonly mixed contingent allocation is equally distributed across individuals for every possible state. Since the mixture with equally distributed allocation does not change the relative income distribution, it is self-evident that the egalitarian ranking should not change after such a mixture. In

particular, after equal mixing, the most egalitarian allocation ex ante would remain the most egalitarian allocation among the mixed opportunity sets.

A10. (*Equally Dominated Independence.*) If M is egalitarian dominated by some $y \in X^e \cap M$, then for every $f, g \in M$ and every $x \in X^e \cap M$, $f \succsim_M g$ implies $\lambda f + (1 - \lambda)x \succsim_M \lambda g + (1 - \lambda)x$ for all $\lambda \in (0, 1)$.

A10 asserts the constant independence axiom of the maxmin expected utility theory whenever the opportunity set is egalitarian dominated by some constant allocation. Therefore, A10, along with the above axioms, implies that the inequality index should have a multiple prior function form.

2.3 Index Characterization

In this section, we apply the axioms proposed above to construct an inequality index under uncertainty.

Theorem 1. *The following two statements are equivalent:*

(i) $\{\succsim_M\}_{M \in \mathbb{M}}$ satisfies Axioms 1-10.

(ii) There exists a GGI ϕ on X and a convex and compact set $P \in \Delta(S)$ such that for every $M \in \mathbb{M}$ and $f, g \in M$,

$$f \succsim_M g \iff \min_{p \in P} \int_S \{\phi(f^s) - \max_{h \in M} \phi(h^s)\} dp(s) \geq \min_{p \in P} \int_S \{\phi(g^s) - \max_{h \in M} \phi(h^s)\} dp(s).$$

Note that the index ϕ is unique up to a positive scaling factor. Theorem 1 demonstrates that if the egalitarian preferences \succsim satisfy this set of axioms, then society would evaluate the inequality degree of constant allocation by ϕ ; possesses a set P of subjective probability measures over events; and measures the inequality of contingent allocation by taking the minimum integration over states, weighting the constant inequality losses in each state s by $\phi(f^s) - \max_{h \in M} \phi(h^s)$, with respect to the set of subjective probability measures P . This index reflects a cautious society that wants to evaluate each allocation by taking the least expectation of inequality losses.

The proof of our characterization consists of three parts. First, as we already demonstrated in Proposition 1, there exists a function ϕ that represents preferences whenever the menu consists of only certain allocations. Second, we show that if the menu contains a null allocation, then the preference relation satisfies Gilboa-Schmeidler axioms. Therefore, the preference has an MEU

representation form. Finally, for a general menu M , we can transform any act f in M into an act \hat{f} in the following way: for all states s ,

$$\phi(\hat{f}(s)) = \frac{1}{2} [\phi(f(s)) - \max_{g \in M} \phi(\hat{g}(s))].$$

Hence, our axioms imply that $f \succsim_M g \Leftrightarrow \hat{f} \succsim_{\Delta(\hat{f}, \hat{g}, 0)} \hat{g}$. Applying the result in the second step, we have the representation result for all the menus.

2.4 Characterization with Ignorance

Now, we consider the situation in which a society is not capable of formalizing any probability beliefs over states. In other words, society does not have confidence in ruling out any probability. Therefore, the set of probabilities above would be the universal set of probabilities. We will characterize this situation by introducing the following axiom.

Before formally stating the axiom, we need some new notations. For any nonempty disjoint events $E, E' \subset S$, two allocations $f, g \in F$ are said to be $\{E, E'\}$ -dual if (a) there exist $x, y \in X$ such that for all $s \in E$ and $t \in E'$, $x = f^s = g^t$ and $y = f^t = g^s$ and (b) for all $s \notin S \setminus E \cup E'$, $f^s = g^s$. That is, two allocations are $\{E, E'\}$ dual if we permute one allocation's outcomes in events E and E' ; then we obtain another allocation. Further, two menus M, M' are said to be $\{E, E'\}$ -dual if (i) for every $f \in M$, there exists $f' \in M'$ such that f, f' are $\{E, E'\}$ -dual and (ii) for every $f' \in M'$, there exists $f \in M$ such that f, f' are $\{E, E'\}$ -dual.

A11 (Ignorance.) For every $\{E, E'\}$ -dual menus M, M' and every $f, g \in M$ and $f', g' \in M'$, if f, f' and g, g' are $\{E, E'\}$ -dual, then $f \succsim_M g \Leftrightarrow f' \succsim_{M'} g'$.

This axiom states that the egalitarian ranking is invariant with respect to dual-process identical events. This axiom implies that inequality evaluation relies only on the possible outcomes and does not depend on the events associated with those outcomes. The next theorem demonstrates that if egalitarian preferences satisfy the following set of axioms, then society evaluates the inequality of each allocation only by the minimum inequality losses over all the possible states.

Theorem 2. *The following two statements are equivalent:*

(i) $\{\succsim_M\}_{M \in \mathbb{M}}$ satisfies A1-9 and A11.

(ii) There exists a GGI ϕ on X such that for every $M \in \mathbb{M}$ and $f, g \in M$,

$$f \succsim_M g \iff \min_{s \in S} \{ \phi(f^s) - \max_{h \in M} \phi(h^s) \} \geq \min_{s \in S} \{ \phi(g^s) - \max_{h \in M} \phi(h^s) \}.$$

This result characterizes an extremely pessimistic situation in which a society is incapable of determining any possible set of probabilities and, therefore, takes the universal set of probabilities as the priors. In such a situation, a society only cares about the minimum equality loss across all states. Notice that the above representation is a special case of Theorem 1, in which $P = \Delta(S)$. Therefore, it is immediate to see that A1-9 are also necessary in this characterization. However, A10 is not required here although the above representation satisfies A10. Actually, A10 plays a similar role of constant-independence in Maxmin expected utility result, which derives the existence of set of priors. In contrast, A11 requires that permutation of allocation would not affect the preference comparison. Hence, along with A6, A11 implies that the value of each allocation is evaluated by the minimum equality loss over states.

3 CONCLUSION

We would like to discuss two things to conclude our paper. The first is about the choice of the generalized Gini index ϕ with respect to constant allocations. Actually, there are many well-known indices of inequality, from absolute inequality indices of [Blackorby and Donaldson \[1980\]](#), [Yaari \[1988\]](#) to relative inequality indices of [Atkinson \[1970\]](#), [Kolm \[1969\]](#), [Sen and Foster \[1997\]](#). We select the generalized Gini index simply because it is desirable to decompose an inequality measure by the component of income. There are, of course, many extensions of the generalized Gini index, such as [Donaldson and Weymark \[1980\]](#) and [Gajdos and Weymark \[2005\]](#), which can be adapted to our index by replacing function ϕ and restating some principles accordingly.

The second is that the ways to construct the inequality measurement can be categorized into at least two types of approaches. One is the axiomatization approach, as we did in this paper, which invites society to specify the principles to compare allocations and then characterize the index based on those principles. The other is the welfare analysis approach, which first presents individual preferences over allocations and then asks the principles to aggregate individual preferences into the desired social welfare function. This approach starts from [Harsanyi \[1955\]](#) and follows up by, just to name a few among many [Fleurbaey \[2010\]](#), [Gajdos and Maurin \[2004\]](#), [Gajdos and Tallon \[2002\]](#), [Thibault Gajdos and Zoli \[2010\]](#). These approaches are by no means mutually exclusive. They are actually complementary ways to build an implementable index.

APPENDIX: PROOFS

Generalized Gini Index

In this subsection, we restate the well-known properties of Generalized Gini Index without any proof. Given a function $\phi : X \rightarrow \mathbb{R}$ from proposition 1, it is true that ϕ is additive wrt to equally distributed constant allocations. We state the fact formally below.

Property 1. *For every $x, y \in X$, we have*

- (i) *If $y \in X^e$, then $\phi(x + y) = \phi(x) + \phi(y)$;*
- (ii) *for all $\alpha > 0$, $\phi(\alpha x) = \alpha\phi(x)$;*
- (iii) *If $y \in X^e$, then for all $\beta \in \mathbb{R}$, $\phi(\beta y) = \beta\phi(y)$.*
- (iv) *If x and y are comonotonic, then $\phi(x + y) = \phi(x) + \phi(y)$.*

Proof of Theorem 1

We first prove the *sufficiency* part.

Lemma 1 (Homogeneity). *If $f \succsim_M g$, then for any $\lambda > 0$ such that $\lambda M \in \mathbb{M}$, we have $\lambda f \succsim_{\lambda M} \lambda g$.*

Proof. There are two cases we need to consider: $\lambda \leq 1$ and $\lambda > 1$. If $\lambda \leq 1$, then $\lambda f = \lambda f + (1 - \lambda)\mathbf{0}$. Therefore, Equal independence (A9) implies that $\lambda f \succsim_{\lambda M} \lambda g$. Suppose that $\lambda > 1$ and assume by contraction that $\lambda g \succ_{\lambda M} \lambda f$. Since $\frac{1}{\lambda} \in (0, 1)$, the above argument requires that $\frac{1}{\lambda} \cdot \lambda g + (1 - \frac{1}{\lambda})\mathbf{0} \succ_M \frac{1}{\lambda} \cdot \lambda f + (1 - \frac{1}{\lambda})\mathbf{0}$, which is $g \succ_M f$. This contradicts the assumption. \square

Lemma 2. *For every $x \in X$, there exists $y \in X^e$ such that $x \sim_X y$.*

Proof. Let a^* be such that $a^* \in \{x_1, x_2, \dots, x_N\}$ and $a^* \cdot \mathbf{1} \succsim_X x_i \cdot \mathbf{1}$ for all i . Similarly, let a_* be such that $a_* \in \{x_1, x_2, \dots, x_N\}$ and $x_i \cdot \mathbf{1} \succsim_X a_* \cdot \mathbf{1}$ for all i . Monotonicity implies that

$$a^* \cdot \mathbf{1} \succsim_X x \succsim_X a_* \cdot \mathbf{1}.$$

According to mixture continuity, there exists $\lambda \in (0, 1)$ such that

$$(\lambda a^* + (1 - \lambda)a_*)\mathbf{1} \sim_X x.$$

The result follows by observing that $(\lambda a^* + (1 - \lambda)a_*)\mathbf{1} \in X^e$. \square

For any $M \in \mathbb{M}$, define \bar{f}_M by

$$\phi(\bar{f}_M^s) = \max_{f \in M} \phi(f^s) \quad \text{for all } s.$$

By the above lemma, for every such \bar{f}_M , there exists $\bar{f}_M^e \in F^e$ such that $\bar{f}_M^{es} \sim_X f_M^s$ for all s . For any $f \in M$, define \hat{f}_M by

$$\hat{f}_M = \frac{1}{2}f + \frac{1}{2}(-\bar{f}_M^e).$$

Let \hat{M} be the collection of such \hat{f} .

Lemma 3. $f \succsim_M g$ iff $\hat{f}_M \succsim_{\Delta(\hat{f}_M, \hat{g}_M, \mathbf{0})} \hat{g}_M$.

Proof. (\implies) By equal independence,

$$f \succsim_M g \implies \hat{f}_M \succsim_{\frac{1}{2}M + \frac{1}{2}(-\bar{f}_M^e)} \hat{g}_M.$$

Note that for every s ,

$$\begin{aligned} \max_{f \in M} \phi(\hat{f}_M^s) &= \max_{f \in M} \phi\left(\frac{1}{2}f^s + \frac{1}{2}(-\bar{f}_M^{es})\right) \\ &= \max_{f \in M} \frac{1}{2}\phi(f^s) - \frac{1}{2}\phi(\bar{f}_M^{es}) \\ &= 0 \end{aligned}$$

Therefore, null allocation $\mathbf{0}$ non-egalitarian dominates menu $\frac{1}{2}M + \frac{1}{2}(-\bar{f}_M^e)$. INA implies that

$$\hat{f}_M \succsim_{\frac{1}{2}M + \frac{1}{2}(-\bar{f}_M^e) \cup \mathbf{0}} \hat{g}_M.$$

Since null allocation weakly egalitarian dominates menus $\frac{1}{2}M + \frac{1}{2}(-\bar{f}_M^e) \cup \mathbf{0}$ and $\Delta(\hat{f}_M, \hat{g}_M, \mathbf{0})$, Independence of Dominated Menu (A8) implies that $\hat{f}_M \succsim_{\Delta(\hat{f}_M, \hat{g}_M, \mathbf{0})} \hat{g}_M$.

(\impliedby) As we discussed above, A7 and A8 imply that

$$\hat{f}_M \succsim_{\Delta(\hat{f}_M, \hat{g}_M, \mathbf{0})} \hat{g}_M \implies \hat{f}_M \succsim_{\hat{M}} \hat{g}_M.$$

Equal Independence implies that

$$\frac{1}{2}\hat{f}_M + \frac{1}{2}\bar{f}_M^e \succsim_{\frac{1}{2}\hat{M} + \frac{1}{2}\bar{f}_M^e} \frac{1}{2}\hat{g}_M + \frac{1}{2}\bar{f}_M^e,$$

which is

$$\frac{1}{4}f \succsim_{\frac{1}{4}M} \frac{1}{4}g.$$

Because of homogeneity property we derived in Lemma 1, it is immediate that $f \succsim_M g$. \square

We denote by $\hat{F} := \cup_{M \in \mathbb{M}} \{\hat{M}\}$. Clearly, $\hat{F} \subset F$.

Lemma 4. \hat{F} is a convex set and contains null allocation.

Proof. \hat{F} contains null allocation is straightforward. We only prove that it is a convex set. Let $\hat{f}, \hat{g} \in \hat{F}$ be defined by

$$\hat{f} = \frac{1}{2}f + \frac{1}{2}(-\bar{f}_M^e) \quad \text{and} \quad \hat{g} = \frac{1}{2}g + \frac{1}{2}(-\bar{f}_N^e).$$

Take $\lambda \in (0, 1)$. Since $\bar{f}_M^e, \bar{f}_N^e \in F^e$, for every s ,

$$\phi(\lambda \bar{f}_M^{es} + (1 - \lambda) \bar{f}_N^{es}) = \lambda \phi(\bar{f}_M^{es}) + (1 - \lambda) \phi(\bar{f}_N^{es}),$$

which implies

$$\lambda \bar{f}_M^e + (1 - \lambda) \bar{f}_N^e = \bar{f}_{\lambda M + (1 - \lambda) N}^e.$$

Since $\lambda f + (1 - \lambda)g \in \lambda M + (1 - \lambda)N$,

$$\lambda \hat{f} + (1 - \lambda) \hat{g} = \frac{1}{2}(\lambda f + (1 - \lambda)g) - \frac{1}{2}(\lambda \bar{f}_M^e + (1 - \lambda) \bar{f}_N^e)$$

belongs to $\lambda M + \widehat{(1 - \lambda)N}$, henceforth, belongs to \hat{F} . \square

Lemma 5. $\hat{f}_M \succsim_{\Delta(\hat{f}_M, \hat{g}_N, \mathbf{0})} \hat{g}_N \iff \hat{f}_M \succsim_{\hat{F}} \hat{g}_N$.

Proof. Since null allocation $\mathbf{0}$ egalitarian dominates both menus $\Delta(\hat{f}_M, \hat{g}_N, \mathbf{0})$ and \hat{F} , the equivalence follows directly from Independence of Dominated menu (A8). \square

Because of the above results, it suffices to characterize the menu \hat{F} dependent preference relation. For different menu-dependent acts \hat{f}_M and \hat{g}_N , one can simply merge menus of the form $\Delta\{\hat{f}_M, \hat{g}_N, \mathbf{0}\}$ and then extend menu to \hat{F} without affecting preferences. Therefore, the menu subscript on the acts can be dropped.

Since null allocation belongs to both X^e and \hat{F} , $X^e \cap \hat{F} \neq \emptyset$. We denote by x_* a worst element in $X^e \cap \hat{F}$ in the $\succsim_{\hat{F}}$ ranking. The worst element exists because of compactness of X^e and \hat{F} .

Lemma 6. For all $f \in \hat{F}$, $\mathbf{0} \succsim_{\hat{F}} f \succsim_{\hat{F}} x_*$.

Proof. As last lemma indicates, $\mathbf{0} \in \hat{F}$. By definition of \hat{F} , if $f \in \hat{F}$, then it is obvious that $\mathbf{0} \succsim_X f^s \succsim_X x_*$ for all s . Hence, monotonicity implies that $\mathbf{0} \succsim_{\hat{F}} f \succsim_{\hat{F}} x_*$. \square

We assume, without loss of generality, that $\phi(x_*) = -1$. For any $f \in \hat{F}$, let $V(f)$ denote the function from S to \mathbb{R} , where $V(f)(s) = \phi(f^s)$. Thus, V is a mapping from \hat{F} to \mathbb{R}^S . For any $f \in \hat{F}$, there is a unique $\lambda \in [0, 1]$ such that $f \sim_{\hat{F}} \lambda \mathbf{0} + (1 - \lambda)x_*$, which is $f \sim_{\hat{F}} (1 - \lambda)x_*$, where $f \succsim_{\hat{F}} (1 - \lambda)x_*$.

Let us define a function $J : \hat{F} \rightarrow \mathbb{R}$ by $J(f) = \lambda\phi(\mathbf{0}) + (1 - \lambda)\phi(x_*) = \lambda - 1$. To close the triangle, we define a function I from the codomain of V to \mathbb{R} as follows: for $a : S \rightarrow \mathbb{R}$ with $-1 \leq a(s) \leq 0$ for all s , there is an allocation $f \in \hat{F}$ with

$$V(f)(s) = \phi(f^s) = a(s),$$

then $I(a) = J(f)$. The function I is well-defined since if for some other allocation $g \in \hat{F}$ such that $V(f)(s) = \phi(g^s) = a(s)$ for all s , then by monotonicity, $f \sim_{\hat{F}} g$, which, in turn, implies $J(g) = J(f)$. Let $\alpha \in \mathbb{R}$, we write $\bar{\alpha}$ for the constant function a such that $a(s) = \alpha$ for all s . By the definition of I , and by axioms we propose, it has the following properties:

Lemma 7. *The function I defined above satisfies the following properties in the domain of definition of I :*

- (1). $I(\bar{\alpha}) = \alpha$ for all $\alpha \in [-1, 0]$.
- (2). If $a(s) \geq b(s)$ for all s , then $I(a) \geq I(b)$.
- (3). $I(\lambda f) = \lambda I(f)$ for $\lambda > 0$.
- (4). $I(a + \bar{\alpha}) = I(a) + \alpha$.
- (5). $I(a + b) \geq I(a) + I(b)$.

Proof. (1). Let there be given $\alpha \in [-1, 0]$. Then, $\phi(-\alpha x_*) = \alpha$. By definition, $I(\bar{\alpha}) = J(-\alpha x_*) = \alpha$.

(2). We prove the monotonicity of I . Let there be given a, b such that $a(s) \geq b(s)$ for all s . By definition, there exist $f, g \in \hat{F}$ such that for all s , $V(f)(s) = \phi(f^s) = a(s)$ and $V(g)(s) = \phi(g^s) = b(s)$. Monotonicity implies that $f \succsim_{\hat{F}} g$, which is $J(f) \geq J(g)$, hence $I(a) \geq I(b)$.

(3). We prove the homogeneity of degree 1 of I . Let there be given a, b and $\lambda \in (0, 1]$ such that a, b are in the domain of I and $a = \lambda b$. We want to show that $I(a) = \lambda I(b)$. Let $g \in \hat{F}$ satisfy $\phi(g^s) = b(s)$ for all s . Define $f = \lambda g + (1 - \lambda)\mathbf{0}$. Hence for all s ,

$$\phi(f^s) = \phi(\lambda g^s) = \lambda b(s) = a(s).$$

So, $I(a) = J(f)$. Let $y \in X^e \cap \hat{F}$ satisfy $y \sim_{\hat{F}} g$. Hence $\phi(y) = J(g) = I(b)$. Since y and \hat{F} egalitarian dominated by $\mathbf{0}$. Equal dominated independence (A10) implies

$$g \sim_{\hat{F}} y \Leftrightarrow \lambda g + (1 - \lambda)\mathbf{0} \sim_{\hat{F}} \lambda y.$$

Therefore, $J(\lambda g) = J(\lambda y) = \lambda J(y) = \lambda J(a)$.

(4). We prove the constant independence of I . Let there be given a and $\alpha \in [-1, 0]$. By homogeneity, we assume wlog $2\alpha \in [-1, 0]$ and $2a$ is in the domain of I . Define $\beta = I(2a) = 2I(a)$. Let $f \in \hat{F}$ satisfy $\phi(f^s) = 2a(s)$ for all s . Therefore, $J(f) = I(2a) = \beta$. Let $y, z \in X^e \cap \hat{F}$ satisfy $\phi(y) = \beta$ and $\phi(z) = 2\gamma$. Then, $J(y) = \beta$. Since $f \sim_{\hat{F}} y$, Equal dominated independence (A10) implies,

$$\frac{1}{2}f + \frac{1}{2}z \sim_{\hat{F}} \frac{1}{2}y + \frac{1}{2}z.$$

Note that

$$I\left(\left(\frac{1}{2}f + \frac{1}{2}z\right)_{s \in S}\right) = I\left(\left(\frac{1}{2}\phi(f^s) + \frac{1}{2}\phi(z)\right)_{s \in S}\right) = I\left(a + \frac{1}{2} \cdot 2\gamma\right).$$

Also,

$$\phi\left(\frac{1}{2}y + \frac{1}{2}z\right) = I\left(\frac{1}{2}\beta + \gamma\right) = \frac{1}{2}\beta + \gamma = I(a) + \gamma.$$

Therefore, constant independence follows immediately from the above equations.

(5). We prove the superadditivity of I . Let there be given a, b and $\lambda \in (0, 1)$. We assume wlog that a, b and $\lambda a, (1 - \lambda)b$ are in the domain of I . It suffices to prove that given $I(a) = I(b)$, $I\left(\frac{1}{2}a + \frac{1}{2}b\right) \geq I(a)$. Suppose that $f, g \in \hat{F} \cap F^e$ such that for all s , $\phi(f^s) = a(s)$ and $\phi(g^s) = b(s)$. By assumption, $f \succsim_{\hat{F}} g$. By Uncertainty aversion (A10), for $\lambda \in (0, 1)$, $\lambda f + (1 - \lambda)g \succsim_{\hat{F}} g$. Hence, for all s

$$\phi(\lambda f^s + (1 - \lambda)g^s) = \lambda \phi(f^s) + (1 - \lambda)\phi(g^s) = a(s).$$

Hence, $I(\lambda a + (1 - \lambda)b) \geq I(a)$. □

Although the above properties hold only in the domain of I . However, because of homogeneity

and monotonicity, I can be extended, though homogeneity, to all of \mathbb{R}^S . Therefore, by Gilboa-Schmeidler theorem, we can get the following result.

Lemma 8 (Gilboa-Schmeidler Theorem). *There exist a closed and convex subset, say K , of $\Delta(S)$ such that for all a ,*

$$I(a) = \min_{p \in K} \int_S a(s) dp(s).$$

The conclusion of above lemme is obvious. For $f, g \in \hat{F}$,

$$f \succsim_{\hat{F}} g \Leftrightarrow J(f) \geq J(g).$$

Setting $a(s) = \phi(f^s)$ and $b(s) = \phi(g^s)$ for all s , we have by the definition of I ,

$$f \succsim_{\hat{F}} g \Leftrightarrow I(a) \geq I(b).$$

Thus, we have

$$f \succsim_{\hat{F}} g \Leftrightarrow \min_{p \in K} \int_S \phi(f^s) dp(s) \geq \min_{p \in K} \int_S \phi(g^s) dp(s).$$

The conclusion of the proof of Theorem 1 is following:

$$\begin{aligned} f \succsim_M g &\Leftrightarrow \hat{f} \succsim_{\Delta(\hat{f}, \hat{g}, \mathbf{0})} \hat{g} \\ &\Leftrightarrow \hat{f} \succsim_{\hat{F}} \hat{g} \\ &\Leftrightarrow \min_{p \in K} \int_S \phi(\hat{f}^s) dp(s) \geq \min_{p \in K} \int_S \phi(\hat{g}^s) dp(s) \\ &\Leftrightarrow \min_{p \in K} \int_S \phi\left(\frac{1}{2}f^s - \frac{1}{2}\hat{f}_M^s\right) dp(s) \geq \min_{p \in K} \int_S \phi\left(\frac{1}{2}g^s - \frac{1}{2}\hat{f}_M^s\right) dp(s) \\ &\Leftrightarrow \frac{1}{2} \min_{p \in K} \int_S \phi(f^s) - \phi(\hat{f}_M^s) dp(s) \geq \frac{1}{2} \min_{p \in K} \int_S \phi(g^s) - \phi(\hat{f}_M^s) dp(s) \\ &\Leftrightarrow \min_{p \in K} \int_S \phi(f^s) - \max_{h \in M} \phi(h^s) dp(s) \geq \min_{p \in K} \int_S \phi(g^s) - \max_{h \in M} \phi(h^s) dp(s) \end{aligned}$$

Now, we prove the *necessity* part.

A1 (Weak Order): This follows since there is a real-valued representation of \succsim_M for every M .

A2 (Mixture Continuity): It is easy to see since ϕ is a continuous function.

A3 (Constant Symmetry): This is immediate from the definition of ϕ function.

A4 (Comonotonic Translation Invariance): This follows directly from the (iv) of Property.

A5 (Monotonicity): By Proposition 1, $f^s \succsim_X g^s$ for every s implies that $\phi(f^s) \geq \phi(g^s)$ for all s .

$$\begin{aligned}
& \min_{p \in P} \int_S \{\phi(f^s) - \phi(f_M^s)\} dp(s) - \min_{p \in P} \int_S \{\phi(g^s) - \phi(f_M^s)\} dp(s) \\
&= \int_S \{\phi(f^s) - \phi(f_M^s)\} dp_f^*(s) - \int_S \{\phi(g^s) - \phi(f_M^s)\} dp_g^*(s) \\
&\geq \int_S \{\phi(g^s) - \phi(f_M^s)\} dp_f^*(s) - \int_S \{\phi(g^s) - \phi(f_M^s)\} dp_g^*(s) \\
&\geq 0.
\end{aligned}$$

Therefore, $f \succsim_M g$.

A6 (Uncertainty Aversion): Let $f, g \in M \cup F^e$ and $\lambda \in (0, 1)$. By property 1, we know that $\phi(\lambda f^s + (1 - \lambda)g^s) = \lambda\phi(f^s) + (1 - \lambda)\phi(g^s)$ for all s . Consider social allocation $\lambda f + (1 - \lambda)g$.

$$\begin{aligned}
& \min_{p \in P} \int_S \{\phi(\lambda f^s + (1 - \lambda)g^s) - \phi(f_M^s)\} dp(s) \\
&= \int_S \{\lambda\phi(f^s) + (1 - \lambda)\phi(g^s) - \phi(f_M^s)\} dp^*(s) \\
&= \lambda \int_S \{\phi(f^s) - \phi(f_M^s)\} dp^*(s) + (1 - \lambda) \int_S \{\phi(g^s) - \phi(f_M^s)\} dp^*(s) \\
&\geq \min_{p \in P} \int_S \{\phi(g^s) - \phi(f_M^s)\} dp(s)
\end{aligned}$$

A7 (Independence of non-egalitarian allocation): Let $h \notin M$ be a social allocation that non-egalitarian dominate M . Then $\phi(f_M^s) \geq \phi(h^s)$ for all s . This implies that $\phi(f_M^s) = \max_{k \in \Delta(M \cup h)} \phi(k^s)$ for all s . Therefore, $f \succsim_M g$ implies $f \succsim_{\Delta(M \cup h)} g$.

A8 (Independent of Dominated Menu): Suppose that $h \in M \cap N$ egalitarian dominates both M and N . We must have $\phi(h^s) = \phi(f_M^s) = \phi(f_N^s)$ for all s . Hence, it is clear to see that $f \succsim_M g \Leftrightarrow f \succsim_N g$.

A9 (Equal Independence): Let $f \succsim_M g$ and $h \in F^e$ and $\lambda \in (0, 1)$.

$$\begin{aligned}
& \min_{p \in P} \int_S \{\phi(\lambda f^s + (1 - \lambda)h^s) - \max_{k \in \lambda M + (1 - \lambda)h} \phi(k^s)\} dp(s) \\
&= \min_{p \in P} \int_S \{\lambda\phi(f^s) + (1 - \lambda)\phi(h^s) - \max_{k \in M} \lambda\phi(k^s) + (1 - \lambda)\phi(h^s)\} dp(s) \\
&= \lambda \min_{p \in P} \int_S \{\phi(f^s) - \phi(f_M^s)\} dp(s)
\end{aligned}$$

Similarly, we have

$$\min_{p \in P} \int_S \{ \phi(\lambda g^s + (1-\lambda)h^s) - \max_{k \in \lambda M + (1-\lambda)h} \phi(k^s) \} dp(s) = \lambda \min_{p \in P} \int_S \{ \phi(g^s) - \phi(f_M^s) \} dp(s).$$

Hence, $f \succsim_M g$ implies $\lambda f + (1-\lambda)h \succsim_{\lambda M + (1-\lambda)h} \lambda g + (1-\lambda)h$.

A10 (Equally Dominated Independence): Suppose that M is egalitarian dominated by $y \in X^e \cap M$. Let $f \succsim_M g$ and $x \in X^e \cap M$ and $\lambda \in (0, 1)$. Since $\phi(f_M^s) = \phi(y)$ for all s , we have

$$\min_{p \in P} \int_S \phi(f^s) dp(s) \geq \min_{p \in P} \int_S \phi(g^s) dp(s).$$

Therefore,

$$\begin{aligned} & \min_{p \in P} \int_S \{ \phi(\lambda f^s + (1-\lambda)x) - \phi(f_M^s) \} dp(s) \\ &= \min_{p \in P} \int_S \{ \lambda \phi(f^s) + (1-\lambda)\phi(x) - \phi(y) \} dp(s) \\ &= \lambda \min_{p \in P} \int_S \phi(f^s) dp(s) + (1-\lambda)\phi(x) - \phi(y) \\ &\geq \lambda \min_{p \in P} \int_S \phi(g^s) dp(s) + (1-\lambda)\phi(x) - \phi(y) \\ &= \min_{p \in P} \int_S \{ \phi(\lambda g^s + (1-\lambda)x) - \phi(f_M^s) \} dp(s) \end{aligned}$$

Proof of Theorem 2

Since the necessary part is similar to Theorem 1, we only prove the sufficient part. The proof is divided into a series of lemmas. It is understood that axioms A1-9 and A11 holds throughout this part.

From the proof of Theorem 1, the following observation is straightforward.

Lemma 9. Let $\hat{f}, \hat{g} \in \hat{F}$. For every menus M, N containing \hat{f}, \hat{g} , $\hat{f} \succsim_M \hat{g} \Leftrightarrow \hat{f} \succsim_N \hat{g}$.

Proof. Since every allocation in \hat{M} and \hat{N} are not egalitarian wrt $\mathbf{0}$, this result is immediate to see from the above lemma. □

Since the above result holds, the menu subscript on the preference ordering can be dropped without loss of generality.

For $f \in \hat{F}$, let f_* be the worst possible egalitarian allocation in f , which is defined by $f_* \in \{f^s : s \in S\}$ and $f^s \succsim_X f_*$ for every s . In the similar fashion, we define f^* to be the best possible

egalitarian allocation in f . Let $E_* = \{s \in S : f^s = f_*\}$ be the event that f_* will be realized and E^* be the event that f^* will be realized. For any f, g and event E , fEg is an allocation that f^s is realized if $s \in E$, otherwise g^s is realized.

Lemma 10. *Let $f \in \hat{F}$. Then, $f \sim f_*$.*

Proof. If $S = E_*$, then the result holds trivially. Suppose that E_* is a strict subset of S . Then, E_* and E^* are not empty sets. The monotonicity implies that $f_*E_*f^* \succsim f \succsim f^*E^*f_*$. Consider $f_*E_*f^*$ and switch the outcomes in events E_* and E^* . Therefore, $f_*E_*f^*$ and $f_*E^*f^*$ are $\{E_*, E^*\}$ -dual acts. Ignorance axiom implies that $f_*E_*f^* \sim f_*E^*f^*$. Similarly, consider $f^*E^*f_*$ and switch the outcomes in events E^* and $S \setminus E^*$. Therefore, $f^*E^*f_*$ and $f_*E^*f^*$ are $\{E^*, S \setminus E^*\}$ -dual acts. Again, $f^*E^*f_* \sim f_*E^*f^*$ by ignorance axiom. Therefore, transitivity requires that

$$f_*E_*f^* \sim f \sim f^*E^*f_*.$$

Take any non-empty event $E \subset S$, allocations $f_*E_*f^*, f^*Ef_*$ are $\{E_* \setminus E, E \setminus E_*\}$ -dual. Therefore, $f_*E_*f^* \sim f^*Ef_*$ by ignorance. Hence,

$$f \sim f^*Ef_*$$

for every non-empty event E . For any disjoint non-empty events E, G , we have $f_*Ef^*Gf_* \sim f_*Ef_*Gf^* \sim f_*Ef^*Gf^*$. Uncertainty aversion implies that $f_*E(\frac{f^*+f_*}{2}) \succsim f_*Ef^*$. Monotonicity implies that

$$f_*E(\frac{f^*+f_*}{2}) \sim f_*Ef^*.$$

By induction, we can repeat the above argument to get for every positive integer n ,

$$f_*E(\frac{f^*+f_*}{2^n}) \sim f_*Ef^*.$$

Since the sequence $\{2^n\}$ is dense at 0, monotonicity and continuity jointly imply that for every $\lambda \in (0, 1]$,

$$(3) \quad f_*E(\lambda f^* + (1 - \lambda)f_*) \sim f_*Ef^*$$

To get our result, we want to show the above indifference relation holds for $\lambda = 0$. We prove it by contradiction. Suppose that $f_*Ef^* \succ f_*$. Since there always exists $x \in X$ such that $f_* \succ x$,

continuity requires the existence of $\gamma \in (0, 1)$ such that

$$(\gamma f_* + (1 - \gamma)x)E(\gamma f^* + (1 - \gamma)x) \sim f_*.$$

According to Eq (1), for every $\lambda \in (0, 1]$,

$$(\gamma f_* + (1 - \gamma)x)E(\gamma f^* + (1 - \gamma)x) \sim (\gamma f_* + (1 - \gamma)x)E(\gamma f_* + (1 - \gamma)x + \gamma\lambda(f^* - f_*)).$$

But, monotonicity requires that $f_* \succ (\gamma f_* + (1 - \gamma)x)E(\gamma f_* + (1 - \gamma)x + \gamma\lambda(f^* - f_*))$ for small enough λ , which leads to a contradiction. Since $f_* \sim f_*E f^*$, by transitivity $f \sim f_*$. \square

To close our conclusion, combining the above lemmas, we get that for every $M \in \mathbb{M}$ and every $f, g \in M$,

$$\begin{aligned} f \succsim_M g &\iff \hat{f} \succsim_{\{\hat{f}, \hat{g}, 0\}} \hat{g} \\ &\iff \hat{f}_* \succsim \hat{g}_* \\ &\iff \min_s \phi(\hat{f}^s) \geq \min_s \phi(\hat{g}^s) \\ &\iff \min_s \{\phi(f^s) - \phi(f_M^s)\} \geq \min_s \{\phi(g^s) - \phi(g_M^s)\} \\ &\iff \min_s \{\phi(f^s) - \max_{h \in M} \phi(h^s)\} \geq \min_s \{\phi(g^s) - \max_{h \in M} \phi(h^s)\}. \end{aligned}$$

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