Systems of PDEs with Interconnected Bilateral Obstacles and nonlinear Neumann Boundary Conditions

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Abstract

In this article, we derive the existence and uniqueness of continuous viscosity solution of Bellman-Isaacs system of equations, associated with a multiple mode switching game in finite horizon when the state process is constrained to live in a connected bounded closed domain. We rely on systems of penalized reflected generalized BSDEs with unilateral obstacles, that provide approximation of the system of PDEs. We obtain a comparison result which gives the uniqueness and the continuity of the viscosity solution. This allows us to use Perron's method to obtain the existence.

Keywords: Bellman-Isaacs equation, Reflected Generalized Backward stochastic differential systems, Viscosity solution of PDEs, Variational inequalities, Nonlinear Neumann Boundary conditions, Switching game.

MSC Classification: 49L25 , 49N70 , 60H10 , 60H30 , 93C30.

1 Introduction

The aim of this work is to investigate existence and uniqueness of viscosity solution of the following system of variational inequalities with interconnected obstacles and nonlinear Neumann Boundary conditions: $\forall (i,j) \in \Gamma^1 \times \Gamma^2$, $\forall t \in [0,T)$,

$$\begin{cases} \min\{v^{ij}(t,x) - \max\limits_{k\in\Gamma^1, k\neq i}(v^{kj}(t,x) - \underline{g}_{ik}(t,x)); \max\{v^{ij}(t,x) - \min\limits_{l\in\Gamma^2, l\neq j}(v^{il}(t,x) + \overline{g}_{jl}(t,x)); \\ \quad -\partial_t v^{ij}(t,x) - \mathscr{L}v^{ij}(t,x) - f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma^\top(x) D_x v^{ij}(t,x))\} = 0, \ x \in D; \\ \dfrac{\partial v^{ij}}{\partial l}(t,x) + \psi^{ij}(t,x,v^{ij}(t,x)) = 0, \ x \in \partial D; \\ v^{ij}(T,x) = h^{ij}(x), \ x \in \overline{D}, \end{cases} \tag{1.1}$$

where $\Gamma^1 \times \Gamma^2$ is a finite set of modes such that $card\,(\Gamma^1 \times \Gamma^2) = \Lambda$, the generator $\mathscr{L}$ is defined by

$$\mathscr{L} = \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top) D^2_{xx\cdot} + b^\top D_{x\cdot},$$

and at any point $x \in \partial D$,

$$\frac{\partial}{\partial l} = \langle \nabla \phi(x), D_{x\cdot} \rangle = \sum_{i=1}^{d} \frac{\partial \phi}{\partial x_i}(x) \frac{\partial}{\partial x_i}.$$

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This type of inequalities turns out to be the Bellmann-Isaacs equation associated with switching game under the requirement that the state process lives in a bounded domain. In this problem, two players intervene. The first player tries to control the system in such a way to maximize its gain by switching between modes from Γ^1. The switching between a mode i and a mode k is not free and generates a cost $\underline{g}_{ik}$. The second player has the opposite aim and switches between modes in Γ^2 in order to minimize the lost. In this case, $\overline{g}_{jl}$ stands for the switching cost when the second player decides to move from mode j to mode l. In addition, h^{ij} is the payoff at time T, and f^{ij} is the running payoff on D. It should be pointed out that the presence of constraints on the state process induces in general additional costs which are represented by the function ψ^{ij} when the boundary ∂D is reached.

Problems like these ones arise for example in energy production systems and investment of capital in many economies, when randomness comes from the state process $X^{t,x}$ which evolves according to the following reflected dynamic:

$$\begin{cases} dX_s^{t,x} = b(X_s^{t,x})ds + \sigma(X_s^{t,x})dB_s + \nabla\phi(X_s^{t,x})dA_s^{t,x}, \quad s \in [t,T]; \\ X_s^{t,x} \in \bar{D} \text{ and } A_s^{t,x} = \int_t^s \chi_{\{X_r^{t,x}\in\partial D\}} dA_r^{t,x}, \, s \in [t,T]; \\ X_t^{t,x} = x, \quad \text{for} \quad s \leq t. \end{cases} \tag{1.2}$$

Here, the process $A^{t,x}$ is an increasing process and a part of the solution. The gradient $\nabla\phi(x)$ coincides with the unit normal pointing towards the interior of $D := \{\phi > 0\}$. Therefore, the quantity $\nabla\phi(X_s^{t,x})dA_s^{t,x}$ represents the actions made to keep the state process in $\overline{D}$.

An example of interest is related to the return of investment during a pandemic. More precisely, let $X_s^{t,x}$ denote the evolution of a disease in a particular bounded region $\overline{D}$. Efforts should be made in order to contain the spread of the disease and prevent from infecting the outside world. These efforts are described by $\nabla\phi(X_s^{t,x})dA_s^{t,x}$. Amid the crisis which strikes vital activities, a social planner gathers and analyses the data so that he/she can make the most effective decision to protect lives and the well being of the economy. More precisely, the social planner puts the country under different regimes in accordance with the sanitary situation, he/she chooses from a set of modes {lockdown, partial lockdown, reopening}. On the other side stands an economic agent who invests in many assets and chooses the ones that maximize his/her profitability. In this context, the economic agent and the social planner are in a game and usually exchange social services and payments. The interactions between the two players induce an instantaneous payoff that depends obviously on the mode chosen by each player. In this game, the economic agent (the first player) aims to maximize the profit which arises the question of existence of $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ that solves (1.1). On the contrary, the social planner (the second player) looks for a solution for the associated Bellmann-Isaacs equation defined as follows: $\forall (i,j) \in \Gamma^1 \times \Gamma^2$, $\forall t \in [0,T)$,

$$\begin{cases} \max\{v^{ij}(t,x) - \min\limits_{l\in\Gamma^2, l\neq j}(v^{il}(t,x) + \overline{g}_{jl}(t,x)); \min\{v^{ij}(t,x) - \max\limits_{k\in\Gamma^1, k\neq i}(v^{kj}(t,x) - \underline{g}_{ik}(t,x)); \\ \quad -\partial_t v^{ij}(t,x) - \mathscr{L}v^{ij}(t,x) - f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2}, \sigma^\top(x)D_x v^{ij}(t,x))\} = 0, \; x \in D; \\ \dfrac{\partial v^{ij}}{\partial l}(t,x) + \psi^{ij}(t,x,v^{ij}(t,x)) = 0, \; x \in \partial D; \\ v^{ij}(T,x) = h^{ij}(x), \; x \in \overline{D}. \end{cases} \tag{1.3}$$

There are many works on Bellmann-Isaacs equations for switching games when the state process is constrained to stay in a bounded domain. In [25] and [26] for example, the author has treated system of elliptic variational inequalities associated with switching game with upper and lower obstacles. The author has established existence and uniqueness of viscosity solution of the Dirichlet problem for this system on a bounded domain, when the costs are constant. Later on, Ishii and Koike [18] generalized these results to the case where the costs are no longer constant but deterministic. They proved existence and uniqueness of viscosity solutions of system of non linear second order elliptic partial differential equation of Dirichlet type. In another context, when the state process

takes values in the whole $\mathbb{R}^d$, Djehich, Hamadène and Morlais [7] have established existence and uniqueness of a continuous viscosity solution for a similar system when the utility function f^{ij} depends on the other parts of the solution and the costs are not deterministic. The authors have used probabilistic tools relying on the connection between PDEs and reflected BSDEs. Also, they have given an explicit probabilistic representation of the solution of the PDEs system. The solutions are obtained using a penalization of systems of reflected BSDEs systems with unilateral interconnected obstacles studied in several papers (see e.g. [14, 15, 16, 17] etc). Note that in [18], the PDEs system that has been considered is of Dirichlet type. In our work we consider a more general case, which is a fully non linear PDEs system with non linear boundary conditions. These conditions induce the additional costs ψ^{ij} that appear once the constraints on the state process are not satisfied. Furthermore, the utility function f^{ij} depends on the other parts of the solution and the costs are not deterministic. To the best of our knowledge, our results have not been proved yet.

The main contribution of our work is to show existence and uniqueness of a continuous viscosity solution of system (1.1). We rely on systems of generalized reflected BSDEs studied in [2], by penalizing a decreasing scheme of reflected generalized BSDEs with unilateral interconnected obstacles. Then we show the comparison between subsolutions and supersolutions which allows to use Perron's method and to obtain a solution of (1.1). By considering an increasing scheme of reflected generalized BSDEs with unilateral interconnected obstacles, and thanks to the results obtained for system (1.1), we get a unique continuous viscosity solution of the system (1.3). Note that we do not know whether the solution of (1.1) and the solution of (1.3) coincide, in which case we might get a solution for the zero-sum switching game with conditions on the boundary. This question is beyond the scope in this paper and left for future work.

The outline of this paper is as follows. In section 2, we introduce some notations and assumptions and we give some definitions related to the viscosity solution. In Section 3, we prove a comparison result of solutions of multidimentional generalized BSDEs that allows us to construct two approximation schemes of penalized reflected generalized BSDEs systems. Then, we establish their links with two sequences of PDEs systems with non linear boundary conditions. In section 4, we prove that the comparison between subsolutions and supersolutions holds. Finally, we use Perron's method in order to show the existence of a solution for system of PDEs (1.1) and as a consequence we obtain the same result for (1.3).

2 Notation and assumptions

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a fixed probability space on which is defined a standard Brownian motion $B = (B_t)_{0\leqslant t\leqslant T}$ for a fixed finite horizon $T > 0$, and let $\mathscr{F} = (\mathscr{F}_t)_{0\leq t\leq T}$ be the completed filtration of $(\sigma(B_s, 0 \leq s \leq t))_{t\leq T}$ with all $\mathbb{P}-$null sets of $\mathscr{F}_0$. Let us introduce the following spaces:

$$\begin{aligned}
\mathscr{H}^2 &= \{(\psi_t)_{0\leqslant t\leqslant T}\ \mathscr{F}_t\text{-progressively measurable process s.t. } \mathbb{E}[\int_0^T \mid \psi_t \mid^2 dt] < \infty\};\\
\mathscr{S}^2 &= \{(\psi_t)_{0\leqslant t\leqslant T}\ \mathscr{F}_t\text{-progressively measurable process s.t. } \mathbb{E}[\sup_{0\leq t\leq T} \mid \psi_t \mid^2] < \infty\};\\
\mathscr{A}^2 &= \{(K_t)_{0\leqslant t\leqslant T}\ \mathscr{F}_t\text{-adapted continuous increasing process s.t. } K_0 = 0 \text{ and } \mathbb{E}[K_T^2] < \infty\}.
\end{aligned}$$

Let D be an open bounded subset of $\mathbb{R}^d$, such that $D = \{\phi > 0\}$ and $\partial D = \{\phi = 0\}$. The function ϕ is in $C_b^2(\mathbb{R}^d)$ and $\nabla\phi(x)$ coincides with the unit normal pointing towards the interior of D. Then the interior sphere condition holds (see [21] and the references therein) *i.e.* there exists $r > 0$ such that for any $x \in \partial D$ and $y \in \overline{D}$ we have:

$$\mid y - x \mid^2 + r\langle \nabla\phi(x), y - x\rangle \geq 0. \tag{2.1}$$

Let (t,x) be in $[0,T]\times\overline{D}$ and $(X_s^{t,x},A_s^{t,x})_{t\leqslant s\leqslant T}$ the solution of the reflected SDE below:

$$\begin{cases} dX_s^{t,x} = b(X_s^{t,x})ds + \sigma(X_s^{t,x})dB_s + \nabla\phi(X_s^{t,x})dA_s^{t,x}, \quad s\in[t,T]; \\ A_s^{t,x} = \int_t^s \chi_{\{X_r^{t,x}\in\partial D\}} dA_r^{t,x}, \, s\in[t,T]; \\ X_s^{t,x} = x, \, s\leq t; \end{cases} \tag{2.2}$$

where $A^{t,x}$ is increasing, the functions $b:\mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma:\mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ are Lipschitz. The solution $(X_s^{t,x},A_s^{t,x})_{t\leqslant s\leqslant T}$ is valued in $\overline{D}\times\mathbb{R}_+$ and satisfies the following properties (see e.g. [21]):

Proposition 2.1 *For each $t\in[0,T]$, there exists a constant C such that for all $x,x'\in\overline{D}$,*

$$\mathbb{E}[\sup_{t\leqslant s\leqslant T} \mid X_s^{t,x}-X_s^{t,x'}\mid^4] \leq C\mid x-x'\mid^4,$$
$$\mathbb{E}[\sup_{t\leqslant s\leqslant T} \mid A_s^{t,x}-A_s^{t,x'}\mid^4] \leq C\mid x-x'\mid^4.$$

Moreover, for all $p\geqslant 1$, there exists a constant C_p such that for all $(s,x)\in[t,T]\times\overline{D}$,

$$\mathbb{E}(\mid A_s^{t,x}\mid^p) \leq C_p(1+t^p),$$

and for each $\mu>0$, $s\in[t,T]$, there exists $C(\mu,s)$ such that for all $x\in\overline{D}$,

$$\mathbb{E}(e^{\mu A_s^{t,x}}) \leq C(\mu,s).$$

The following assumptions are necessary for the study of the system of PDEs (1.1): $\forall(i,j)\in\Gamma^1\times\Gamma^2$

(H_1) Let $f^{ij}:[0,T]\times\overline{D}\times\mathbb{R}^\Lambda\times\mathbb{R}^{1\times d}\longrightarrow\mathbb{R}$ and $\psi^{ij}:[0,T]\times\overline{D}\times\mathbb{R}\longrightarrow\mathbb{R}$ be such that

(i) f^{ij} and ψ^{ij} are Lipschitz continuous w.r.t. $(\vec{y},z)$ and y respectively.

(ii) f^{ij} and ψ^{ij} are jointly continuous w.r.t. (t,x) uniformly in $(\vec{y},z)$ and y respectively.

(iii) $\exists\beta<0$ such that $(y-y')(\psi^{ij}(t,x,y)-\psi^{ij}(t,x,y'))\leq\beta\mid y-y'\mid^2$.

(iv) The monotonicity condition: For $(k,l)\neq(i,j)$, $y^{kl}\longrightarrow f^{ij}(t,x,\vec{y},z)$ is non-decreasing.

(H_2) Let $h^{ij}:\overline{D}\longrightarrow\mathbb{R}$ be a continuous function such that

$$\forall x\in\overline{D}, \quad \max_{k\in\Gamma^1,k\neq i}(h^{kj}(x)-\underline{g}_{ik}(T,x)) \leq h^{ij}(x) \leq \min_{l\in\Gamma^2,l\neq j}(h^{il}(x)+\overline{g}_{jl}(T,x)).$$

(H_3) For $(k,l)\in\Gamma^1\times\Gamma^2$ the switching costs $\underline{g}_{ik}$ and $\overline{g}_{jl}$ are non negative, jointly continuous in (t,x) and satisfy the non free loop property:
For any loop in $\Gamma^1\times\Gamma^2$, *i.e.* any sequence of pairs $(i_1,j_k),\ldots,(i_N,j_N)$ of $\Gamma^1\times\Gamma^2$ such that $(i_1,j_k)=(i_N,j_N)$ with $card\{(i_1,j_k),\ldots,(i_N,j_N)\}=N-1$ and for any $q=1,\ldots,N-1$ for which either $i_{q+1}=i_q$ or $j_{q+1}=j_q$ we have:

$$\forall(t,x)\in[0,T]\times\overline{D}, \quad \sum_{q=1}^{N-1}\varphi_{i_qj_q}(t,x)\neq 0,$$

where for any $q=1,\ldots,N-1$, $\varphi_{i_qj_q}(t,x) = -\underline{g}_{i_q,i_{q+1}}(t,x)\chi_{i_q\neq i_{q+1}} + \overline{g}_{j_q,j_{q+1}}\chi_{j_q\neq j_{q+1}}(t,x)$, this latter assumption implies in particular that,

$$\sum_{k=1}^{N-1}\underline{g}_{i_k,i_{k+1}}(t,x)>0, \quad \text{and} \quad \sum_{k=1}^{N-1}\overline{g}_{j_k,j_{k+1}}(t,x)>0,$$

by convention, we set $\overline{g}_{jj} = \underline{g}_{ii} = 0$.

Next, we define the notions of viscosity sub-solution and super-solution of the PDEs system (1.1), this will be done in terms of subjet and superjet, that we recall in the following definition:

Definition 2.1 *For a locally bounded function $u : [0,T] \times \overline{D} \to \mathbb{R}$, we define its lower semicontinuous envelope u_* and its upper semicontinuous enveloppe u^* as follows:*

$$u_*(t,x) = \underset{\substack{(t',x') \longrightarrow (t,x) \\ t'<T, x' \in \overline{D}}}{\underline{\lim}} u(t',x') \text{ and } u^*(t,x) = \underset{\substack{(t',x') \longrightarrow (t,x) \\ t'<T, x' \in \overline{D}}}{\overline{\lim}} u(t',x').$$

Definition 2.2 *(Subjets and Superjets)*
(**i**) *For a lower semi continuous (lsc) (resp. upper semi continuous (usc)) function $u : [0,T] \times \overline{D} \to \mathbb{R}$, we define the parabolic subjet $J^{2-}u(t,x)$ (resp. superjet $J^{2+}u(t,x)$) of u at $(t,x) \in [0,T] \times \overline{D}$, the set of triples $(p,q,M) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ s.t. for any $(t',x') \in (0,T) \times \overline{D}$, we have:*

$$\begin{aligned} u(t',x') \geq & u(t,x) + p(t'-t) + \langle q, x'-x \rangle + \frac{1}{2}\langle x'-x, M(x'-x)\rangle + o(\mid t'-t \mid + \mid x'-x \mid^2) \\ (\text{resp. } u(t',x') \leq & u(t,x) + p(t'-t) + \langle q, x'-x \rangle + \frac{1}{2}\langle x'-x, M(x'-x)\rangle + o(\mid t'-t \mid + \mid x'-x \mid^2)). \end{aligned}$$

(**ii**) *For $(t,x) \in [0,T] \times \overline{D}$, we define the parabolic limiting subjet $\overline{J}^{2-}u(t,x)$ (resp. the parabolic limiting superjet $\overline{J}^{2+}u(t,x)$) as follows:*

$$\begin{aligned} \overline{J}^{2-}u(t,x) = \{&(p,q,M) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d : \exists (t_n,x_n,p_n,q_n,M_n) \in [0,T] \times \overline{D} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \\ &\text{s.t. } (p_n,q_n,M_n) \in J^{2-}u(t_n,x_n) \text{ and } (t_n,x_n,p_n,q_n,M_n) \longrightarrow (t,x,p,q,M) \text{ as } n \to \infty\}, \\ (\text{resp. } \overline{J}^{2+}u(t,x) = \{&(p,q,M) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d : \exists (t_n,x_n,p_n,q_n,M_n) \in [0,T] \times \overline{D} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \\ &\text{s.t. } (p_n,q_n,M_n) \in J^{2+}u(t_n,x_n) \text{ and } (t_n,x_n,p_n,q_n,M_n) \longrightarrow (t,x,p,q,M) \text{ as } n \to \infty\}), \end{aligned}$$

where $\mathbb{S}^d$ is the set of symmetric real matrices of dimension d.

We give now the definition of solution in viscosity sense for system (1.1):

Definition 2.3 *(Viscosity solution)*
(**i**) *A function $\vec{v} = (v^{kl})_{(k,l) \in \Gamma^1 \times \Gamma^2} : [0,T] \times \overline{D} \longrightarrow \mathbb{R}^{\wedge}$ such that for any $(i,j) \in \Gamma^1 \times \Gamma^2$, v^{ij} is lsc (resp. usc), is called a viscosity supersolution (resp. subsolution) to (1.1), if for any $(i,j) \in \Gamma^1 \times \Gamma^2$, we have:*

$$\begin{cases} \min\{v^{ij}(t,x) - \max\limits_{k \in \Gamma^1, k \neq i}(v^{kj}(t,x) - \underline{g}_{ik}(t,x)); \max\{v^{ij}(t,x) - \min\limits_{l \in \Gamma^2, l \neq j}(v^{il}(t,x) + \overline{g}_{jl}(t,x)); \\ \quad -p - b(x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(x)M] - f^{ij}(t,x,\vec{v}(t,x),\sigma^\top(x)q)\}\} \geq 0, \ (t,x) \in [0,T) \times D, \ (p,q,M) \in \overline{J}^{2-}v^{ij}(t,x); \\ \min\{v^{ij}(t,x) - \max\limits_{k \in \Gamma^1, k \neq i}(v^{kj}(t,x) - \underline{g}_{ik}(t,x)); \max\{v^{ij}(t,x) - \min\limits_{l \in \Gamma^2, l \neq j}(v^{il}(t,x) + \overline{g}_{jl}(t,x)); \\ \quad -p - b(x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(x)M] - f^{ij}(t,x,\vec{v}(t,x),\sigma^\top(x)q)\}\} \vee \{-\langle \nabla\phi(x), q\rangle - \psi^{ij}(t,x,v^{ij}(t,x))\} \geq 0, \\ \qquad (t,x) \in [0,T) \times \partial D, \ (p,q,M) \in \overline{J}^{2-}v^{ij}(t,x); \\ v^{ij}(T,x) \geq h^{ij}(x), \quad x \in \overline{D}. \end{cases} \tag{2.3}$$

(resp.

$$\begin{cases} \min\{v^{ij}(t,x) - \max\limits_{k\in\Gamma^1,k\neq i}(v^{kj}(t,x) - \underline{g}_{ik}(t,x)); \max\{v^{ij}(t,x) - \min\limits_{l\in\Gamma^2,l\neq j}(v^{il}(t,x) + \overline{g}_{jl}(t,x)); \\ \qquad -p - b(x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(x)M] - f^{ij}(t,x,\vec{v}(t,x),\sigma^\top(x)q)\}\} \leq 0,\ (t,x)\in[0,T)\times D,\ (p,q,M)\in\overline{J}^{2+}v^{ij}(t,x); \\ \min\{v^{ij}(t,x) - \max\limits_{k\in\Gamma^1,k\neq i}(v^{kj}(t,x) - \underline{g}_{ik}(t,x)); \max\{v^{ij}(t,x) - \min\limits_{l\in\Gamma^2,l\neq j}(v^{il}(t,x) + \overline{g}_{jl}(t,x)); \\ -p - b(x)^\top q - \frac{1}{2}Tr[\sigma\sigma^\top(x)M] - f^{ij}(t,x,\vec{v}(t,x),\sigma^\top(x)q)\}\} \wedge \{-\langle\nabla\phi(x),q\rangle - \psi^{ij}(t,x,v^{ij}(t,x))\} \leq 0, \\ \qquad\qquad (t,x)\in[0,T)\times\partial D,\ (p,q,M)\in\overline{J}^{2+}v^{ij}(t,x); \\ v^{ij}(T,x) \leq h^{ij}(x), \quad x\in\overline{D}.) \end{cases} \tag{2.4}$$

(**ii**) *A locally bounded function $\vec{v} = (v^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2} : [0,T]\times\overline{D} \longrightarrow \mathbb{R}^\wedge$ is called a viscosity solution if $\vec{v}_*$ (resp. $\vec{v}^*$) is a viscosity supersolution (resp. subsolution).*

3 Approximation schemes of the solution

Let $t \leqslant T$, $n,m \geqslant 0$, and $(Y^{ij,n,m}, Z^{ij,n,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ be the solution of the following system of GBSDEs, which exists and is unique thanks to Theorem 1.6 [21]: $\forall (i,j)\in\Gamma^1\times\Gamma^2$, $\forall t\leqslant s\leqslant T$,

$$\begin{cases} Y^{ij,n,m} \in \mathscr{S}^2, Z^{ij,n,m} \in \mathscr{H}^{2,d}; \\ Y_s^{ij,n,m} = h^{ij}(X_T^{t,x}) + \int_s^T f^{ij,n,m}(r,X_r^{t,x},(Y_r^{kl,n,m})_{(k,l)\in\Gamma^1\times\Gamma^2},Z_r^{ij,n,m})dr + \int_s^T \psi^{ij}(r,X_r^{t,x},Y_r^{ij,n,m})dA_r^{t,x} \\ \qquad\qquad - \int_s^T Z_r^{ij,n,m}dB_r, \end{cases} \tag{3.1}$$

where

$$f^{ij,n,m}(s,X_s^{t,x},\vec{y},z) = f^{ij}(s,X_s^{t,x},\vec{y},z) + n(y^{ij} - \max_{k\in\Gamma^1,k\neq i}(y^{kj} - \underline{g}_{ik}(s,X_s^{t,x}))^- - m(y^{ij} - \min_{l\in\Gamma^2,l\neq j}(y^{il} + \overline{g}_{jl}(s,X_s^{t,x}))^+,$$

which is lipschitz continuous w.r.t. $(\vec{y},z)$. Then, we have the following properties:

Proposition 3.1 *For any $(i,j)\in\Gamma^1\times\Gamma^2$, and $n,m\geqslant 0$, we have:*

1. $\mathbb{P}$*-a.s* $Y^{ij,n,m+1} \leq Y^{ij,n,m} \leq Y^{ij,n+1,m}$.
2. *There exists a deterministic continuous function $v^{ij,n,m}$ such that $\forall t\leqslant T$,*
$$Y_s^{ij,n,m} = v^{ij,n,m}(s,X_s^{t,x}),\ s\in[t,T].$$
3. *For any $(t,x)\in[0,T]\times\overline{D}$, $v^{ij,n,m+1}(t,x) \leq v^{ij,n,m}(t,x) \leq v^{ij,n+1,m}(t,x)$.*

Before we proceed with the proof, we need to establish a comparison result for solutions of multidimensional generalized BSDE in a limited context. More precisely, we have:

Proposition 3.2 *Let I denote the finite set of modes $\{1,...,m\}$ and assume that the data $((\xi_i)_{i\in I},(f_i)_{i\in I},(\psi_i)_{i\in I})$ and $((\overline{\xi}_i)_{i\in I}(\overline{f}_i)_{i\in I},(\overline{\psi}_i)_{i\in I})$ satisfy the Assumptions (A1) in [2]. Then, we define Y and $\overline{Y}$ as the respective unique solutions of the following systems of generalized BSDEs: $\forall i=1,...,m$, $\forall t\leqslant T$,*

$$Y_t^i = \xi^i + \int_t^T f_i(s,Y_s,Z_s^i)ds + \int_t^T \psi_i(s,Y_s^i)dA_s - \int_t^T Z_s^i dB_s, \tag{3.2}$$

and

$$\overline{Y}_t^i = \overline{\xi}^i + \int_t^T \overline{f}_i(s,\overline{Y}_s,\overline{Z}_s^i)ds + \int_t^T \overline{\psi}_i(s,\overline{Y}_s^i)dA_s - \int_t^T \overline{Z}_s^i dB_s. \tag{3.3}$$

Also, we assume that for any $i = 1,...,m$, $f_i \leq \overline{f}_i$, $\psi_i \leq \overline{\psi}_i$ *and* $\xi^i \leq \overline{\xi}^i$. *Then for any* $i = 1,...,m$, *we have:*

$$Y^i \leq \overline{Y}^i, \mathbb{P}\text{-}a.s.$$

Proof. The equations (3.2) and (3.3) can be seen as solutions of two reflected GBSDEs systems with data . In fact, $(Y, K = 0, Z)$ satisfies the following system of reflected GBSDEs with data $((\xi^i)_{i\in I}(f_i)_{i\in I}, (\psi_i)_{i\in I}, (g_{ij})_{(i,j)\in I^2})$ where $(g_{ij})_{(i,j)\in I^2}$ stand for the costs of switching which are continuous, non negative and satisfy the non free-loop property, more precisely we have: $\forall i = 1,...,m, \forall t \leqslant T$

$$\begin{cases} Y^i \in \mathscr{S}^2, Z^i \in \mathscr{H}^{2,d}, K^i \in \mathscr{A}^2; \\ Y_t^i = \xi^i + \int_t^T f_i(s,Y_s,Z_s^i)ds + \int_t^T \psi_i(s,Y_s^i)dA_s + K_T^i - K_t^i - \int_t^T Z_s^i dB_s; \\ Y_t^i \geqslant \max_{j\neq i}(Y_t^i - g_{ij}(t)); \\ \int_0^T \{Y_t^i - \max_{j\neq i}(Y_t^i - g_{ij}(t))\}dK_t^i = 0. \end{cases} \tag{3.4}$$

It has been shown in [2], that for $i = 1,...,m$, $(Y^i, K^i = 0, Z^i)$ can be obtained as the limit of $(Y^{i,n}, K^{i,n} = 0, Z^{i,n})_{n\geqslant 0}$ which is defined as follows:

$$Y_t^{i,0} = \xi + \int_t^T f(s,Y_s^{i,0},Z_s^{i,0})ds + \int_t^T \psi(s,Y_s^{i,0})dA_s - \int_t^T Z_s^{i,0}dB_s, \ \forall t \leqslant T; \tag{3.5}$$

where $\xi = \max_{i=1,..,m} |\xi^i|$, $f(s,y,z) = \max_{i=1,..,m} f_i(s,y,...,y,z)$, $\psi(s,y) = \max_{i=1,..,m} \psi_i(s,y)$;
and for $n \geqslant 1$:

$$\begin{cases} Y^{i,n} \in \mathscr{S}^2, Z^{i,n} \in \mathscr{H}^{2,d}, K^{i,n} \in \mathscr{A}^2; \\ Y_t^{i,n} = \xi^i + \int_t^T f_i(s,Y_s^{1,n-1},...,Y_s^{i,n},...,Y_s^{m,n-1},Z_s^{i,n})ds + \int_t^T \psi_i(s,Y_s^{i,n})dA_s + K_T^{i,n} - K_t^{i,n} - \int_t^T Z_s^{i,n}dB_s, \ \forall t \leqslant T; \\ Y_t^{i,n} \geqslant \max_{j\neq i}(Y_t^{i,n} - g_{ij}(t)), \ \forall t \leqslant T; \\ \int_0^T \{Y_t^{i,n} - \max_{j\neq i}(Y_t^{i,n} - g_{ij}(t))\}dK_t^{i,n} = 0. \end{cases} \tag{3.6}$$

with $K^{i,n} = 0$ and $Y^{i,n}$ converges increasingly to Y^i. The same applies to $\overline{Y}^i$, the increasing limit of $\overline{Y}^{i,n}$ which satisfies (3.6) with data $((\overline{\xi}_i)_{i\in I}(\overline{f}_i)_{i\in I}, (\overline{\psi}_i)_{i\in I}, (g_{ij})_{(i,j)\in I^2})$ and

$$\overline{Y}_t^{i,0} = \overline{\xi} + \int_t^T \overline{f}(s,\overline{Y}^{i,0}{}_s,\overline{Z}^{i,0}{}_s)ds + \int_t^T \overline{\psi}(s,\overline{Y}^{i,0}{}_s)dA_s - \int_t^T \overline{Z}^{i,0}{}_s dB_s \ \forall t \leqslant T. \tag{3.7}$$

The equations (3.5) and (3.7) are two standard GBSDEs, then $Y^{i,0} \leqslant \overline{Y}^{i,0}$ thanks to the comparison result for solutions of GBSDEs in [21].
For $n \geqslant 1$, we assume that $Y^{i,k} \leq \overline{Y}^{i,k}$, for $k < n$.
From the assumptions, we have

$$f_i(s,X_s^{t,x},Y_s^{1,n-1},...,y,...,Y_s^{m,n-1},z) \leqslant \overline{f}_i(s,X_s^{t,x},Y_s^{1,n-1},...,y,...,Y_s^{m,n-1},z),$$

then using the monotonicity condition $(H_1)(iv)$ and since $Y^{i,n-1} \leq \overline{Y}^{i,n-1}$, we get:

$$\overline{f}_i(s,X_s^{t,x},Y_s^{1,n-1},...,y,...,Y_s^{m,n-1},z) \leqslant \overline{f}_i(s,X_s^{t,x},\overline{Y}_s^{1,n-1},...,y,...,\overline{Y}_s^{m,n-1},z).$$

Thanks to Proposition 2.1 in [2], we get $Y^{i,n} \leq \overline{Y}^{i,n}$ and we conclude by taking the limit as $n \to \infty$. $\square$

Proof of Proposition 3.1. Let $n,m \geqslant 0$ and $(Y^{n,m},Z^{n,m})$ be the solution of the GBSDEs system (3.1), then from Proposition 3.2, we deduce that:

$$Y^{ij,n,m+1} \leq Y^{ij,n,m} \leq Y^{ij,n+1,m}, \forall (i,j) \in \Gamma^1 \times \Gamma^2,$$

since $f^{ij,n,m+1} \leq f^{ij,n,m} \leq f^{ij,n+1,m}$. Moreover, it has been shown in [21], that there exists a family of deterministic continuous functions $(v^{ij,n,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ satisfying $Y_s^{ij,n,m} = v^{ij,n,m}(s,X_s^{t,x})$ since we are in a Markovian setting, then we have:

$$v^{ij,n,m+1} \leq v^{ij,n,m} \leq v^{ij,n+1,m}, \forall (i,j) \in \Gamma^1 \times \Gamma^2. \ \square$$

The system of generalized BSDEs (3.1) allows to approximate two different schemes of reflected generalized BSDEs systems, the first scheme is a decreasing sequence of generalized reflected BSDEs with interconnected lower obstacles, the other one is an increasing sequence of generalized reflected BSDEs with interconnected upper obstacles.
First, consider the following system: $\forall m \geqslant 0, \forall (i,j) \in \Gamma^1 \times \Gamma^2$,;

$$\begin{cases} \overline{Y}^{ij,m} \in \mathscr{S}^2, \overline{Z}^{ij,m} \in \mathscr{H}^{2,d} \text{ and } \overline{K}^{ij,m} \in \mathscr{A}^2; \\ \overline{Y}_s^{ij,m} = h^{ij}(X_T^{t,x}) + \int_s^T \overline{f}^{ij,m}(r,X_r^{t,x},(\overline{Y}_r^{kl,m})_{(k,l)\in\Gamma^1\times\Gamma^2},\overline{Z}_r^{ij,m})dr + \int_s^T \psi^{ij}(r,X_r^{t,x},\overline{Y}_r^{ij,m})dA_r^{t,x} + \overline{K}_T^{ij,m} - \overline{K}_s^{ij,m} \\ \qquad\qquad - \int_s^T \overline{Z}_r^{ij,m} dB_r, \quad \forall t \leqslant s \leqslant T; \\ \overline{Y}_s^{ij,m} \geqslant \max\limits_{k\in\Gamma^1,k\neq i}(\overline{Y}_s^{kj,m} - \underline{g}_{ik}(s,X_s^{t,x})), \quad \forall t \leqslant s \leqslant T; \\ \int_0^T \{\overline{Y}_s^{ij,m} - \max\limits_{k\in\Gamma^1,k\neq i}(\overline{Y}_s^{kj,m} - \underline{g}_{ik}(s,X_s^{t,x}))\} d\overline{K}_s^{ij,m} = 0. \end{cases} \tag{3.8}$$

where $\overline{f}^{ij,m}(s,X_s^{t,x},\vec{y},z) = f^{ij}(s,X_s^{t,x},\vec{y},z) - m(y^{ij} - \min\limits_{l\in\Gamma^2,l\neq j}(y^{il} + \overline{g}_{jl}(s,X_s^{t,x})))^+$, then under assumptions $(H_1)-(H_3)$ and by Theorem 2 in [2], there is a unique solution $(\overline{Y}^{ij,m},\overline{Z}^{ij,m},\overline{K}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ to system (3.8) and we have the following properties:

Proposition 3.3 *For any $(i,j) \in \Gamma^1 \times \Gamma^2$, $m \geqslant 0$ we have*

(i) $\mathbb{E}[\sup\limits_{t\leq s\leq T} |Y_s^{ij,n,m} - \overline{Y}_s^{ij,m}|^2] \longrightarrow 0$ *as* $n \to \infty$.

(ii) $\overline{Y}^{ij,m} \geq \overline{Y}^{ij,m+1}$ $\mathbb{P}-a.s.$

(iii) *There exists a deterministic continuous function $\overline{v}^{ij}$ such that*

$$\overline{Y}_s^{ij,m} = \overline{v}^{ij,m}(s,X_s^{t,x}), \forall s \in [t,T].$$

It follows that for any $(i,j) \in \Gamma^1 \times \Gamma^2$, we have:

$$\overline{v}^{ij,m}(t,x) \geq \overline{v}^{ij,m+1}(t,x), \forall (t,x) \in [0,T] \times \overline{D}.$$

Finally, $(\overline{v}^{kl,m})_{(k,l)\in\Gamma^1\times\Gamma^2}$ is the unique viscosity solution of the following system of variational inequalities with

interconnected obstacles and Neumann boundaries: $\forall (i,j)\in\Gamma^1\times\Gamma^2$, $\forall t\in[0,T)$,

$$\begin{cases}\min\{v^{ij}(t,x)-\max\limits_{k\in\Gamma^1,k\neq i}(v^{kj}(t,x)-\underline{g}_{ik}(t,x));\\ \qquad -\partial_t v^{ij}(t,x)-\mathscr{L}v^{ij}(t,x)-\overline{f}^{ij,m}(t,x,(\overline{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)D_x v^{ij}(t,x))\}=0,\ x\in D;\\ \dfrac{\partial v^{ij}}{\partial l}(t,x)+\psi^{ij}(t,x,v^{ij}(t,x))=0,\ x\in\partial D;\\ v^{ij}(T,x)=h^{ij}(x),\ x\in\overline{D}.\end{cases}\tag{3.9}$$

Proof. (i) First, note that the function $(t,x,\vec{y})\longrightarrow -m(y^{ij}-\min\limits_{l\in\Gamma^2,l\neq j}(y^{il}+\overline{g}_{jl}(s,x))^+$ has the same properties as f^{ij} for any $(i,j)\in\Gamma^1\times\Gamma^2$, then it is enough to consider the case $m=0$.
Let us show that for any $(i,j)\in\Gamma^1\times\Gamma^2$ and $n\geqslant 0$ we have $Y^{ij,n,0}\leqslant\overline{Y}^{ij,0}$. Fix $n\geqslant 0$, we define recursively the sequence:
For $k=0$ and $(i,j)\in\Gamma^1\times\Gamma^2$, $\tilde{Y}^{0,ij,n}:=\overline{Y}^{ij,0}_s$ and for $k\geqslant 1$, let $(\tilde{Y}^{k,n},\tilde{Z}^{k,n})$ be the unique solution of the following system of GBSDEs: $\forall (i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases}\tilde{Y}^{k,ij,n}\in\mathscr{S}^2,\tilde{Z}^{k,ij,n}\in\mathscr{H}^{2,d};\\ \tilde{Y}^{k,ij,n}_s=h^{ij}(X^{t,x}_T)+\int_s^T\{f^{ij}(r,X^{t,x}_r,[(\tilde{Y}^{k-1,ij,n}_r)_{(p,q)\neq(i,j)},\tilde{Y}^{k,ij,n}_r],\tilde{Z}^{k,ij,n}_r)\\ \quad +n(\tilde{Y}^{k,ij,n}_r-\max\limits_{k\in\Gamma^1,k\neq i}(\tilde{Y}^{k-1,kj,n}_r-\underline{g}_{ik}(r,X^{t,x}_r))^-\}dr+\int_s^T\psi^{ij}(r,X^{t,x}_r,\tilde{Y}^{k,ij,n}_r)dA^{t,x}_r-\int_s^T\tilde{Z}^{k,ij,n}_r dB_r,\ \forall t\leqslant s\leqslant T.\end{cases}\tag{3.10}$$

where $[(\tilde{Y}^{k-1,ij,n}_r)_{(p,q)\neq(i,j)},\tilde{Y}^{k,ij,n}_r]$ stands for the vector that has the same components as $\tilde{Y}^{k-1,n}$ except the ij-th row which is replaced by $\tilde{Y}^{k,ij,n}_r$.
On the one hand, Note that the mapping defined by

$$\varphi^{ij,n}(r,x,\vec{y},z)=f^{ij}(r,x,(y^{pq})_{(p,q)\neq(i,j)},z)+n\{y^{ij}-\max_{k\in\Gamma^1,k\neq i}(y^{ij}-\underline{g}_{ik}(r,x))\}^-$$

satisfies the assumptions of Theorem 2.1 in [2] where it has been shown that the approximation $(\tilde{Y}^{k,ij,n})_{k\geqslant 0}$ converges increasingly to $Y^{ij,n,0}$ as $k\to\infty$.
On the other hand, we can use an induction argument on k to show that $\tilde{Y}^{k,ij,n}\leq\overline{Y}^{ij,0}$, $k\geqslant 0$.
Indeed, fix $(i,j)\in\Gamma^1\times\Gamma^2$, then for $k=0$, we have $\tilde{Y}^{k,ij,n}=\overline{Y}^{ij,0}$, now assume that for $k-1$ we have $\tilde{Y}^{k-1,ij,n}\leq\overline{Y}^{ij,0}$. Then the monotonicity condition $(H_1)(iv)$ implies that:

$$f^{ij}(s,X^{t,x}_r,[(\tilde{Y}^{k-1,ij,n}_r)_{(p,q)\neq(i,j)},\overline{Y}^{ij,0}_r],z)\leq f^{ij}(r,X^{t,x}_r,[(\overline{Y}^{pq,0}_r)_{(p,q)\neq(i,j)},\overline{Y}^{ij,0}_r],z).$$

Besides $(\overline{Y}^{ij,0})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is the solution of system (3.8), then

$$\begin{aligned}\overline{Y}^{ij,0}_r&\geqslant\max_{l\in\Gamma^1,l\neq i}(\overline{Y}^{lj,0}_r-\underline{g}_{il}(r,X^{t,x}_r))\\ &\geqslant\max_{l\in\Gamma^1,l\neq i}(\tilde{Y}^{k-1,lj,n}_r-\underline{g}_{il}(r,X^{t,x}_r)).\end{aligned}$$

Thus,

$$\varphi^{ij,n}(r,X^{t,x}_r,[(\tilde{Y}^{k-1,pq,n}_r)_{(p,q)\neq(i,j)},\overline{Y}^{ij,0}_r],z)\leqslant f^{ij}(r,X^{t,x}_r,(\overline{Y}^{ij,0}_r)_{(i,j)\in\Gamma^1\times\Gamma^2},z).$$

Then, using the comparison result for standard GBSDEs, we deduce that for any $(i,j)\in\Gamma^1\times\Gamma^2$, $\tilde{Y}^{k,ij,n}\leq\overline{Y}^{ij,0}$, $\mathbb{P}$-*a.s.*, $\forall n\geqslant 0$. Finally, we take the limit w.r.t. k and we obtain $Y^{ij,n,0}\leq\overline{Y}^{ij,0}$ $\mathbb{P}$-*a.s.* It follows that $(Y^{ij,n,0})_{n\geqslant 0}$

converges increasingly to $\hat{Y}^{ij}$ and $\mathbb{E}[\sup_{t\leq s\leq T} |\hat{Y}_t^{ij}|^2] < \infty$.
In order to complete the proof, we are going to use Peng's monotonic limit theorem with the following process:

$$Y_s^{ij,n,0} + \int_0^s \psi^{ij}(r,X_r^{t,x},\hat{Y}_r^{ij})dA_r^{t,x} = Y_0^{ij,n,0} - \int_0^s f^{ij,n,0}(r,X_r^{t,x},(Y_r^{kl,n,0})_{(k,l)\in\Gamma^1\times\Gamma^2},Z_r^{ij,n,0})dr$$
$$- \int_0^s \{\psi^{ij}(r,X_r^{t,x},Y_r^{ij,n,0}) - \psi^{ij}(r,X_r^{t,x},\hat{Y}_r^{ij})\}dA_r^{t,x} + \int_0^s Z_r^{ij,n,0}dB_r.$$

As ψ^{ij} is non increasing, the process $\left(\int_0^s \{\psi^{ij}(r,X_r^{t,x},Y_r^{ij,n,0}) - \psi^{ij}(r,X_r^{t,x},\hat{Y}_r^{ij})\}dA_r^{t,x}\right)_{t\leqslant s\leqslant T}$ is increasing for any $n \geqslant 0$. Then, thanks to Proposition 2.1 and assumptions $(H_1)(i),(ii)$ we can apply Peng's monotonic limit theorem [22] to the increasing sequence $\left(Y_s^{ij,n,0} + \int_0^s \psi^{ij}(r,X_r^{t,x},\hat{Y}_r^{ij})dA_r^{t,x}\right)_{n\geqslant 0}$. This provides the existence of two processes $\hat{K}^{ij}$ and $\hat{Z}^{ij}$ such that:

(a) $\hat{K}^{ij}$ is an RCLL increasing process satisfying $\hat{K}_0^{ij} = 0$ and for any stopping time τ,

$$\int_0^\tau n\{Y_r^{ij,n,0} - \max_{k\in\Gamma^1,k\neq i}(Y_r^{kj,n,0} - \underline{g}_{ik}(r,X_r^{t,x}))\}^- dr + \int_0^\tau \psi^{ij}(r,X_r^{t,x},Y_r^{ij,n,0}) - \psi^{ij}(r,X_r^{t,x},\hat{Y}_r^{ij})dA_r^{t,x} \underset{n\to\infty}{\longrightarrow} \hat{K}_\tau^{ij}.$$

Note that $\mathbb{E}\int_0^\tau \{\psi^{ij}(r,X_r^{t,x},Y_r^{ij,n,0}) - \psi^{ij}(r,X_r^{t,x},\hat{Y}_r^{ij})\}^2 dA_r^{t,x} \leqslant M\mathbb{E}\int_0^\tau |Y_r^{ij,n,0} - \hat{Y}_r^{ij}|^2 dA_r^{t,x} \underset{n\to\infty}{\longrightarrow} 0$, which implies that $\hat{K}_\tau^{ij} = \lim_{n\to\infty}\int_0^\tau n\{Y_r^{ij,n,0} - \max_{k\in\Gamma^1,k\neq i}(Y_r^{kj,n,0} - \underline{g}_{ik}(r,X_r^{t,x}))\}^- dr$.

(b) The process $\hat{Z}^{ij}$ belongs to $\mathscr{H}^2$ and we have $\mathbb{E}[\int_0^T |Z_s^{ij,n,0} - \hat{Z}_s^{ij}|^p ds] \underset{n\to\infty}{\longrightarrow} 0$, $\forall p \in [1,2)$.

(c) For any $(i,j)\in\Gamma^1\times\Gamma^2$, the triple $(\hat{Y}^{ij},\hat{Z}^{ij},\hat{K}^{ij})$ satisfies the following system: $\forall t \leqslant s \leqslant T$

$$\begin{cases} \hat{Y}_s^{ij} = h^{ij}(X_T^{t,x}) + \int_s^T f^{ij}(r,X_r^{t,x},(\hat{Y}_r^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},\hat{Z}_r^{ij})dr + \int_s^T \psi^{ij}(r,X_r^{t,x},\hat{Y}_r^{ij})dA_r^{t,x} + \hat{K}_T^{ij} - \hat{K}_s^{ij} - \int_s^T \hat{Z}_r^{ij}dB_r, \\ \hat{Y}_s^{ij} \geqslant \max_{k\in\Gamma^1,k\neq i}(\hat{Y}_s^{kj} - \underline{g}_{ik}(s,X_s^{t,x})). \end{cases} \tag{3.11}$$

The remaining of the proof is similar to what has been done in Theorem 2.1 [2].(ii) is obtained from $Y^{ij,n,m+1} \leq Y^{ij,n,m}$ and by taking the limit as $n\to\infty$. (iii) $(\overline{Y}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is the solution for system of Generalized BSDEs with unilateral interconnected barriers, according to Theorem 3.2 in [2], there exists a family of deterministic continuous functions $(\overline{v}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ such that $\overline{Y}_s^{ij,m} = \overline{v}^{ij,m}(s,X_s^{t,x})$, $\forall s\in[t,T]$ and is a solution of (3.9). It follows from (ii), that $(\overline{v}^{ij,m})_{m\geqslant 1}$ is decreasing. □

As mentioned previously, the system of GBSDEs (3.1) provides an approximation for a system of an increasing scheme of reflected GBSDEs system, namely: $\forall n\geqslant 0$, $\forall (i,j)\in\Gamma^1\times\Gamma^2$, $\forall t\leqslant s\leqslant T$;

$$\begin{cases} \underline{Y}^{ij,n} \in \mathscr{S}^2, \underline{Z}^{ij,n} \in \mathscr{H}^{2,d} \text{ and } \underline{K}^{ij,n} \in \mathscr{A}^2; \\ \underline{Y}_s^{ij,n} = h^{ij}(X_T^{t,x}) + \int_s^T \underline{f}^{ij,n}(r,X_r^{t,x},(\underline{Y}_r^{kl,n})_{(k,l)\in\Gamma^1\times\Gamma^2},\underline{Z}_r^{ij,n})dr + \int_s^T \psi^{ij}(r,X_r^{t,x},\underline{Y}_r^{ij,n})dA_r^{t,x} + \underline{K}_T^{ij,n} - \underline{K}_s^{ij,n} \\ \qquad\qquad - \int_s^T \underline{Z}_r^{ij,n}dB_r, \quad \forall t\leqslant s\leqslant T; \\ \underline{Y}_s^{ij,n} \geqslant \min_{l\in\Gamma^2,l\neq j}(\underline{Y}_s^{il,n} + \overline{g}_{jl}(s,X_s^{t,x})) \quad \forall t\leqslant s\leqslant T; \\ \int_0^T \{\underline{Y}_s^{ij,n} - \min_{l\in\Gamma^2,l\neq j}(\underline{Y}_s^{il,n} + \overline{g}_{jl}(s,X_s^{t,x}))\}d\underline{K}_s^{ij,n} = 0, \end{cases} \tag{3.12}$$

where $\underline{f}^{ij,n}(s,X_s^{t,x},\vec{y},z)=f^{ij}(s,X_s^{t,x},\vec{y},z)+n(y^{ij}-\max_{k\in\Gamma^1,k\neq i}(y^{ij}-\underline{g}_{ik}(s,X_s^{t,x})))^-$. Therefore, by considering the equation satisfied by $(-\underline{Y}^{ij,n},-\underline{Z}^{ij,n},-\underline{K}^{ij,n})_{(i,j)\in\Gamma^1\times\Gamma^2}$, we get an analogous of the previous proposition:

Proposition 3.4 *For any $(i,j)\in\Gamma^1\times\Gamma^2$, $n\geqslant 0$ we have:*

(i) $\mathbb{E}[\sup_{t\leq s\leq T}|Y_s^{ij,n,m}-\underline{Y}_s^{ij,n}|^2]\underset{m\to\infty}{\longrightarrow}0$.

(ii) $\underline{Y}^{ij,n+1}\geq\underline{Y}^{ij,n}$ $\mathbb{P}-a.s.$.

(iii) *There exists a deterministic continuous function $\underline{v}^{ij,n}$ such that*

$$\underline{Y}_s^{ij,n}=\underline{v}^{ij,n}(s,X_s^{t,x}),\quad s\in[t,T].$$

It follows that $\underline{v}^{ij,n+1}(t,x)\geq\underline{v}^{ij,n}(t,x)$, $\forall(t,x)\in[0,T]\times\overline{D}$.

Finally, for any $n\geqslant 0$, $(\underline{v}^{kl,n})_{(k,l)\in\Gamma^1\times\Gamma^2}$ is the unique viscosity solution of the following system of variational inequalities with interconnected obstacles and Neumann boundaries: $\forall(i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases}\max\{v^{ij}(t,x)-\min_{l\in\Gamma^2,l\neq j}(v^{il}(t,x)+\overline{g}_{jl}(t,x));\\ \qquad -\partial_t v^{ij}(t,x)-\mathscr{L}v^{ij}(t,x)-\underline{f}^{ij,n}(t,x,(\underline{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)D_x v^{ij}(t,x))\}=0,\ x\in\overline{D};\\ \dfrac{\partial v^{ij}}{\partial l}(t,x)+\psi^{ij}(t,x,v^{ij}(t,x))=0,\ x\in\partial D;\\ v^{ij}(T,x)=h^{ij}(x),\ x\in\overline{D}.\end{cases}\tag{3.13}$$

Next, we suggest we recall that for any $(i,j)\in\Gamma^1\times\Gamma^2$ and $m\geqslant 0$, we have $\overline{v}^{ij,m}(t,x)\leqslant\overline{v}^{ij,0}(t,x)$, $\forall(t,x)\in[0,T]\times\overline{D}$, where $\overline{v}^{ij,0}(s,X_s^{t,x})=\overline{Y}_s^{ij,0}$, $\forall t\leqslant s\leqslant T$. Since $(\overline{v}^{ij,m})_{m\geqslant 0}$ are continuous, we can define an u.s.c. $\overline{v}^{ij}$ function as:

$$\overline{v}^{ij}(t,x)=\lim_{m\to\infty}\overline{v}^{ij,m}(t,x).$$

Similarly, we can define a l.s.c. function $\underline{v}^{ij}$ as

$$\underline{v}^{ij}(t,x):=\lim_{n\to\infty}\underline{v}^{ij,n}(t,x).$$

Then, we have the following:

Corollary 3.1 *For any $(i,j)\in\Gamma^1\times\Gamma^2$, we have:*

$$\underline{v}^{ij}\leqslant\overline{v}^{ij}\tag{3.14}$$

Moreover, the families $(\underline{v}^{kl,n})_{(k,l)\in\Gamma^1\times\Gamma^2}$ and $(\overline{v}^{kl,n})_{(k,l)\in\Gamma^1\times\Gamma^2}$ are bounded on $[0,T]\times\overline{D}$.

Proof. The comparison follows from Proposition 3.1 since $\underline{v}^{ij,n}\leqslant v^{ij,n,m}\leqslant\overline{v}^{ij,m}$, for any $m,n\geqslant 0$. Therefore, the boundedness is obtained thanks to the upper semicontinuity and the lower semicontinuity of $\overline{v}^{ij}$ and $\overline{v}^{ij}$ respectively and the compactness of $[0,T]\times\overline{D}$. □

4 Uniqueness of the solution

In this part, we show uniqueness of the viscosity solution for system (1.1), by establishing the comparison between the viscosity subsolution and the viscosity supersolution.

Theorem 4.1 *Let $(u_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ and $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ be respectively an u.s.c. subsolution and a l.s.c. supersolution to PDEs system (1.1). Then, for any $(i,j)\in\Gamma^1\times\Gamma^2$ we have:*

$$u_{ij}\leqslant v^{ij}. \tag{4.1}$$

Proof. The proof will be obtained in two steps.
Step 1: We first assume that there exists a constant $\lambda<-\Lambda.\max_{(i,j)\in\Gamma^1\times\Gamma^2}C_f^{ij}$, ($C_f^{ij}$ being the Lipschitz constant of f^{ij} for $(i,j)\in\Gamma^1\times\Gamma^2$) such that: $\forall(i,j)\in\Gamma^1\times\Gamma^2$, $\forall t,x,\vec{y}^{-ij},y,\bar{y},z\in[0,T]\times\overline{D}\times\mathbb{R}^{\Lambda+1}\times\mathbb{R}^d$,
if $y\geq\bar{y}$ then

$$f^{ij}(t,x,[\vec{y}^{-ij},y],z)-f^{ij}(t,x,[\vec{y}^{-ij},\bar{y}],z)\leq\lambda(y-\bar{y}). \tag{4.2}$$

Let $(u_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ and $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ be respectively a subsolution and a supersolution of system (1.1). Thanks to Lemma 7.6 [6], there exists a positive function $\varphi\in C^2(\overline{D})$, that satisfies

$$\langle-\nabla\phi(x),D\varphi(x)\rangle\geq 1,\ \forall x\in\partial D. \tag{4.3}$$

Then fix $(i,j)\in\Gamma^1\times\Gamma^2$ and put: $\forall(t,x)\in[0,T]\times\overline{D}$

$$u_{ij}^{\varepsilon}(t,x)=u_{ij}(t,x)-\varepsilon\varphi(x)-\frac{\varepsilon}{t}-C\quad\text{ and }\quad v_{\varepsilon}^{ij}(t,x)=v^{ij}(t,x)+\varepsilon\varphi(x)+C,$$

where, C is a positive constant that will be chosen later on.
Let us show that u_{ij}^{ε} is a subsolution to a specific PDEs system. we shall point out that $\forall x\in\overline{D}$, we have:

$$u_{ij}^{\varepsilon}(T,x)\leq u_{ij}(T,x)\leq h^{ij}(x).$$

Now, fix (t,x) in $[0,T)\times\overline{D}$ and $(p^{\varepsilon},q^{\varepsilon},M^{\varepsilon})\in\overline{J}^{2+}u_{ij}^{\varepsilon}(t,x)$. If we set $p=p^{\varepsilon}-\frac{\varepsilon}{t^2}$, $q=q^{\varepsilon}+\varepsilon D\varphi(x)$ and $M=M^{\varepsilon}+\varepsilon D^2\varphi(x)$, we can show easily that $(p,q,M)\in\overline{J}^{2+}u_{ij}(t,x)$ and we have:

$$\begin{aligned}
&\min\{u_{ij}^{\varepsilon}(t,x)-\max_{k\in\Gamma^1,k\neq i}(u_{kj}^{\varepsilon}(t,x)-\underline{g}_{ik}(t,x));\max\{u_{ij}^{\varepsilon}(t,x)-\min_{l\in\Gamma^2,l\neq j}(u_{il}^{\varepsilon}(t,x)+\overline{g}_{jl}(t,x));\\
&\quad-p^{\varepsilon}-b(x)^{\top}q^{\varepsilon}-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^{\top}(x)M^{\varepsilon}]-f^{ij}(t,x,(u_{ij}^{\varepsilon}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^{\top}(x)q^{\varepsilon})\}\}\\
&=\min\{u_{ij}(t,x)-\max_{k\in\Gamma^1,k\neq i}(u_{kj}(t,x)-\underline{g}_{ik}(t,x));\max\{u_{ij}(t,x)-\min_{l\in\Gamma^2,l\neq j}(u_{il}(t,x)+\overline{g}_{jl}(t,x));\\
&\quad-p-b(x)^{\top}q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^{\top}(x)M]-f^{ij}(t,x,(u_{ij}(t,x))_{ij},\sigma^{\top}(x)q)-\frac{\varepsilon}{t^2}+b(x)^{\top}(\varepsilon D\varphi(x))\\
&\quad+\frac{1}{2}\mathrm{Tr}[\sigma\sigma^{\top}(x)(\varepsilon D^2\varphi(x))]+f^{ij}(t,x,(u_{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^{\top}(x)q)-f^{ij}(t,x,(u_{ij}^{\varepsilon}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^{\top}(x)q^{\varepsilon})\}\}.
\end{aligned}$$

The boundedness of b, σ, φ and $D\varphi(x)$ on $\overline{D}$ implies that there exists $\kappa_1>0$ satisfying:

$$b(x)^{\top}(\varepsilon D\varphi(x))+\frac{1}{2}\mathrm{Tr}[\sigma\sigma^{\top}(x)(\varepsilon D^2\varphi(x))]\leq\varepsilon\kappa_1. \tag{4.4}$$

Then thanks to (4.2) and $(H_1)(i)$, we get:

$$\begin{aligned}
&f^{ij}(t,x,(u_{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^{\top}(x)q)-f^{ij}(t,x,(u_{ij}^{\varepsilon}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^{\top}(x)q^{\varepsilon})\\
&\quad\leq(\Lambda-1).\max_{ij}C_f^{ij}(\varepsilon\varphi(x)+\frac{\varepsilon}{t}+C)+\lambda(\varepsilon\varphi(x)+\frac{\varepsilon}{t}+C)+\max_{ij}C_f^{ij}\mid\sigma^{\top}(x)(\varepsilon D^2\varphi(x))\mid,\\
&\quad\leq(\Lambda.\max_{ij}C_f^{ij}+\lambda)(\varepsilon\varphi(x)+\frac{\varepsilon}{t}+C)-\max_{ij}C_f^{ij}(\varepsilon\varphi(x)+\frac{\varepsilon}{t})-C.\max_{ij}C_f^{ij}+\varepsilon\kappa_2\max_{ij}C_f^{ij},
\end{aligned}$$

where $\kappa_2 > 0$.
Since $\Lambda.\max_{ij} C_f^{ij} + \lambda < 0$ and $\varphi \geqslant 0$, we get:

$$b(x)^\top(\varepsilon D\varphi(x)) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)(\varepsilon D^2\varphi(x))] + f^{ij}(t,x,(u_{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q) - f^{ij}(t,x,(u^\varepsilon_{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q^\varepsilon) \leq \varepsilon\kappa_1 - C.\max_{ij} C_f^{ij} + \varepsilon\kappa_2\max_{ij} C_f^{ij}. \tag{4.5}$$

Thus, by choosing $C = \frac{\varepsilon\kappa_1}{\max_{ij} C_f^{ij}} + \varepsilon\kappa_2$, we can see that the left hand side of (4.5) is negative. It follows that:

$$\begin{aligned}
&\min\{u^\varepsilon_{ij}(t,x) - \max_{k\in\Gamma^1,k\neq i}(u^\varepsilon_{kj}(t,x) - \underline{g}_{ik}(t,x)); \max\{u^\varepsilon_{ij}(t,x) - \min_{l\in\Gamma^2,l\neq j}(u^\varepsilon_{il}(t,x) + \overline{g}_{jl}(t,x));\\
&\qquad - p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M^\varepsilon] - f^{ij}(t,x,(u^\varepsilon_{ij}(t,x))_{ij},\sigma^\top(x)q^\varepsilon)\}\}\\
&\leqslant \min\{u_{ij}(t,x) - \max_{k\in\Gamma^1,k\neq i}(u_{kj}(t,x) - \underline{g}_{ik}(t,x)); \max\{u_{ij}(t,x) - \min_{l\in\Gamma^2,l\neq j}(u_{il}(t,x) + \overline{g}_{jl}(t,x));\\
&\qquad - p - b(x)^\top q - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M] - f^{ij}(t,x,(u_{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q) - \frac{\varepsilon}{t^2}\}\}.
\end{aligned} \tag{4.6}$$

As for the condition on the boundary, we have:

$$-\langle\nabla\phi(x),q^\varepsilon\rangle - \psi^{ij}(t,x,u^\varepsilon_{ij}(t,x)) = -\langle\nabla\phi(x),q\rangle - \psi^{ij}(t,x,u_{ij}(t,x) - \varepsilon\varphi(x) - \frac{\varepsilon}{t} - C) - \varepsilon\langle-\nabla\phi(x),D\varphi(x)\rangle.$$

As ψ^{ij} is non-increasing and using (4.3), we get for $x\in\partial D$:

$$-\langle\nabla\phi(x),q^\varepsilon\rangle - \psi^{ij}(t,x,u^\varepsilon_{ij}(t,x)) + \varepsilon \leq -\langle\nabla\phi(x),q\rangle - \psi^{ij}(t,x,u_{ij}(t,x)). \tag{4.7}$$

Therefore, if $(u_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ satisfies (2.4), we obtain from (4.6): $\forall x\in D$,

$$\begin{aligned}
&\min\{u^\varepsilon_{ij}(t,x) - \max_{k\in\Gamma^1,k\neq i}(u^\varepsilon_{kj}(t,x) - \underline{g}_{ik}(t,x)); \max\{u^\varepsilon_{ij}(t,x) - \min_{l\in\Gamma^2,l\neq j}(u^\varepsilon_{il}(t,x) + \overline{g}_{jl}(t,x));\\
&\qquad - p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M^\varepsilon] - f^{ij}(t,x,(u^\varepsilon_{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q^\varepsilon)\}\} \leq 0,
\end{aligned}$$

If $x\in\partial D$, it follows from (4.6) and (4.7) that:

$$\begin{aligned}
&\min\{u^\varepsilon_{ij}(t,x) - \max_{k\in\Gamma^1,k\neq i}(u^\varepsilon_{kj}(t,x) - \underline{g}_{ik}(t,x)); \max\{u^\varepsilon_{ij}(t,x) - \min_{l\in\Gamma^1,l\neq j}(u^\varepsilon_{ij}(t,x) + \overline{g}_{jl}(t,x));\\
&\qquad - p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M^\varepsilon] - f^{ij}(t,x,(u^\varepsilon_{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q^\varepsilon) + \frac{\varepsilon}{t^2}\}\}\\
&\qquad \wedge\{-\langle\nabla\phi(x),q^\varepsilon\rangle - \psi^{ij}(t,x,u^\varepsilon_{ij}(t,x)) + \varepsilon\} \leq 0.
\end{aligned}$$

We conclude that $(u^\varepsilon_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$, satisfies the following system: $\forall(i,j)\in\Gamma^1\times\Gamma^2$, $\forall t\in[0,T)$:

$$\begin{cases}
\min\{u^\varepsilon_{ij}(t,x) - \max_{k\neq i}(u^\varepsilon_{kj}(t,x) - \underline{g}_{ik}(t,x)); \max\{u^\varepsilon_{ij}(t,x) - \min_{l\neq j}(u^\varepsilon_{il}(t,x) + \overline{g}_{jl}(t,x));\\
-p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M^\varepsilon] - f^{ij}(t,x,(u^\varepsilon_{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q^\varepsilon) + \frac{\varepsilon}{t^2}\}\} \leq 0,\\
\qquad \forall x\in D,\ (p^\varepsilon,q^\varepsilon,M^\varepsilon)\in\overline{J}^{2+}u^\varepsilon_{ij}(t,x);\\
\min\{u^\varepsilon_{ij}(t,x) - \max_{k\neq i}(u^\varepsilon_{kj}(t,x) - \underline{g}_{ik}(t,x)); \max\{u^\varepsilon_{ij}(t,x) - \min_{l\neq j}(u^\varepsilon_{il}(t,x) + \overline{g}_{jl}(t,x));\\
-p^\varepsilon - b(x)^\top q^\varepsilon - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M^\varepsilon] - f^{ij}(t,x,(u^\varepsilon_{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q^\varepsilon) + \frac{\varepsilon}{t^2}\}\}\\
\qquad \wedge\{-\langle\nabla\phi(x),q^\varepsilon\rangle - \psi^{ij}(t,x,u^\varepsilon_{ij}(t,x)) + \varepsilon\} \leq 0, \quad \forall x\in\partial D,\ (p^\varepsilon,q^\varepsilon,M^\varepsilon)\in\overline{J}^{2+}u^\varepsilon_{ij}(t,x);\\
u^\varepsilon_{ij}(T,x) \leq h^{ij}(x),\ \forall x\in\overline{D}.
\end{cases} \tag{4.8}$$

Similarly, using the supersolution property of $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ together with assumption (4.2), and assumptions $(H_1)(i),(iii)$, we show that $(v_\varepsilon^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ satisfies the following system of inequalities: $\forall(i,j)\in\Gamma^1\times\Gamma^2$, $\forall t\in[0,T)$:

$$\begin{cases}
\min\{v_\varepsilon^{ij}(t,x)-\max\limits_{k\neq i}(v_\varepsilon^{kj}(t,x)-\underline{g}_{ik}(t,x));\max\{v_\varepsilon^{ij}(t,x)-\min\limits_{l\neq j}(v_\varepsilon^{il}(t,x)+\overline{g}_{jl}(t,x));\\
\quad -p_\varepsilon-b(x)^\top q_\varepsilon-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M_\varepsilon]-f^{ij}(t,x,(v_\varepsilon^{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q_\varepsilon)\}\}\geq 0,\\
\qquad\qquad \forall x\in D,\ (p_\varepsilon,q_\varepsilon,M_\varepsilon)\in\overline{J}^{2-}v_\varepsilon^{ij}(t,x);\\
\min\{v_\varepsilon^{ij}(t,x)-\max\limits_{k\neq i}(v_\varepsilon^{kj}(t,x)-\underline{g}_{ik}(t,x));\max\{v_\varepsilon^{ij}(t,x)-\min\limits_{l\neq j}(v_\varepsilon^{il}(t,x)+\overline{g}_{jl}(t,x));-p_\varepsilon-b(x)^\top q_\varepsilon\\
-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M_\varepsilon]-f^{ij}(t,x,(v_\varepsilon^{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q_\varepsilon)\}\}\vee\{-\langle\nabla\phi(x),q_\varepsilon\rangle-\psi^{ij}(t,x,v_\varepsilon^{ij}(t,x))-\varepsilon\}\geq 0,\\
\qquad\qquad \forall x\in\partial D,\ (p_\varepsilon,q_\varepsilon,M_\varepsilon)\in\overline{J}^{2-}v_\varepsilon^{ij}(t,x);\\
v_\varepsilon^{ij}(T,x)\geq h^{ij}(x),\ \forall x\in\overline{D}.
\end{cases}\tag{4.9}$$

In the next part, we show that $u_{ij}^\varepsilon\leq v_\varepsilon^{ij}$, $\forall(i,j)\in\Gamma^1\times\Gamma^2$. We proceed by contradiction assuming that $\max\limits_{[0,T]\times\overline{D}}\max\limits_{(i,j)\in\Gamma^1\times\Gamma^2}(u_{ij}^\varepsilon-v_\varepsilon^{ij})>0$. Taking into consideration the values of $u_{ij}^\varepsilon-v_\varepsilon^{ij}$ at T, and by definition of u_{ij}^ε, we know that there exists $(\bar t,\bar x)\in(0,T)\times\overline{D}$ such that

$$\max_{(i,j)\in\Gamma^1\times\Gamma^2}(u_{ij}^\varepsilon(\bar t,\bar x)-v_\varepsilon^{ij}(\bar t,\bar x))>0.$$

According to Lemma 4.1 [14], we can always find $(i,j)\in\tilde\Gamma(\bar t,\bar x)$ such that

$$u_{ij}^\varepsilon(\bar t,\bar x)>\max_{k\neq i}(u_{kj}^\varepsilon(\bar t,\bar x)-\underline{g}_{ik}(\bar t,\bar x))\text{ and }v_\varepsilon^{ij}(\bar t,\bar x)<\min_{l\neq j}(v_\varepsilon^{il}(\bar t,\bar x)+\overline{g}_{jl}(\bar t,\bar x)),\tag{4.10}$$

where $\tilde\Gamma(\bar t,\bar x)=\{(i,j)\in\Gamma^1\times\Gamma^2/\,u_{ij}^\varepsilon(\bar t,\bar x)-v_\varepsilon^{ij}(\bar t,\bar x)=\max\limits_{(k,l)\in\Gamma^1\times\Gamma^2}(u_{kl}^\varepsilon(\bar t,\bar x)-v_\varepsilon^{kl}(\bar t,\bar x))\}$.

Let us fix $(i,j)\in\tilde\Gamma(\bar t,\bar x)$ that satisfies (4.10) and suppose first that $\bar x\in\partial D$. Let Φ_α^{ij} be the u.s.c. mapping defined on $[0,T]\times\overline{D}^2$ by:

$$\Phi_\alpha^{ij}(t,x,y)=u_{ij}^\varepsilon(t,x)-v_\varepsilon^{ij}(t,y)-\Psi_\alpha^{ij}(t,x,y),$$

where $\Psi_\alpha^{ij}(t,x,y)=\frac{\alpha}{2}\mid x-y\mid^2-\psi^{ij}(\bar t,\bar x,u_{ij}^\varepsilon(\bar t,\bar x))\langle\nabla\phi(\bar x),x-y\rangle+\mid x-\bar x\mid^4+\mid t-\bar t\mid^2$.

Let $(t_\alpha,x_\alpha,y_\alpha)$ be the maximum point of Φ_α^{ij} over $[0,T]\times\overline{D}^2$, which exists due to the upper semicontinuity of $u_{ij}^\varepsilon-v_\varepsilon^{ij}$ and the compactness of $\overline{D}$, and let M_α be defined as follows:

$$M_\alpha=\max_{(t,x,y)\in[0,T]\times\overline{D}\times\overline{D}}\Phi_\alpha^{ij}(t,x,y),$$

and note this supremum is achieved only if t_α is in $(0,T)$.
On the one hand, M_α satisfies:

$$M_\alpha\geq\max_{(t,x,y)\in[0,T]\times\overline{D}/x=y}u_{ij}^\varepsilon(t,x)-v_\varepsilon^{ij}(t,y)-\mid x-\bar x\mid^4-\mid t-\bar t\mid^2=u_{ij}^\varepsilon(\bar t,\bar x)-v_\varepsilon^{ij}(\bar t,\bar x).$$

Note that M_α is non-increasing w.r.t. α then it is finite and $\alpha\mid x_\alpha-y_\alpha\mid^2\underset{\alpha\to\infty}{\longrightarrow}0$. Thus, $(x_\alpha-y_\alpha)\underset{\alpha\to\infty}{\longrightarrow}0$ thanks to the boundedness of $\overline{D}$, which implies that $(t_\alpha,x_\alpha,y_\alpha)\underset{\alpha\to\infty}{\longrightarrow}(\hat t,\hat x,\hat x)$.

Next, recall that

$$u^{\varepsilon}_{ij}(\bar{t},\bar{x}) - v^{ij}_{\varepsilon}(\bar{t},\bar{x}) \leq M_{\alpha} \leq u^{\varepsilon}_{ij}(t_{\alpha},x_{\alpha}) - v^{ij}_{\varepsilon}(t_{\alpha},y_{\alpha}) + \psi^{ij}(\bar{t},\bar{x},u^{\varepsilon}_{ij}(\bar{t},\bar{x}))\langle \nabla\phi(\bar{x}), x_{\alpha}-y_{\alpha}\rangle.$$

By taking the superior limit in the right hand side of the previous inequality, and taking into account the upper semi-continuity of the function in the right term, we obtain:

$$\begin{aligned} u^{\varepsilon}_{ij}(\bar{t},\bar{x}) - v^{ij}_{\varepsilon}(\bar{t},\bar{x}) &\leq \overline{\lim_{\alpha\to\infty}}\,(u^{\varepsilon}_{ij}(t_{\alpha},x_{\alpha}) - v^{ij}_{\varepsilon}(t_{\alpha},y_{\alpha}) + \psi^{ij}(\bar{t},\bar{x},u^{\varepsilon}_{ij}(\bar{t},\bar{x}))\langle \nabla\phi(\bar{x}), x_{\alpha}-y_{\alpha}\rangle) \\ &\leq u^{\varepsilon}_{ij}(\hat{t},\hat{x}) - v^{ij}_{\varepsilon}(\hat{t},\hat{x}). \end{aligned}$$

On the other hand, $(\bar{t},\bar{x})$ is the maximum point of $u^{\varepsilon}_{ij} - v^{ij}_{\varepsilon}$, which gives the other inequality:

$$u^{\varepsilon}_{ij}(\bar{t},\bar{x}) - v^{ij}_{\varepsilon}(\bar{t},\bar{x}) = u^{\varepsilon}_{ij}(\hat{t},\hat{x}) - v^{ij}_{\varepsilon}(\hat{t},\hat{x}).$$

Thanks to the semi-continuity of the functions u^{ε}_{ij} and v^{ij}_{ε}, we have

$$\begin{aligned} u^{\varepsilon}_{ij}(\bar{t},\bar{x}) - v^{ij}_{\varepsilon}(\bar{t},\bar{x}) &\leq \underline{\lim_{\alpha\to\infty}}\,(u^{\varepsilon}_{ij}(t_{\alpha},x_{\alpha}) - v^{ij}_{\varepsilon}(t_{\alpha},y_{\alpha})) \\ &\leq \overline{\lim_{\alpha\to\infty}}\,(u^{\varepsilon}_{ij}(t_{\alpha},x_{\alpha}) - v^{ij}_{\varepsilon}(t_{\alpha},y_{\alpha})) \\ &\leq u^{\varepsilon}_{ij}(\bar{t},\bar{x}) - v^{ij}_{\varepsilon}(\bar{t},\bar{x}). \end{aligned}$$

Then $\lim_{\alpha\to\infty} u^{\varepsilon}_{ij}(t_{\alpha},x_{\alpha}) - v^{ij}_{\varepsilon}(t_{\alpha},y_{\alpha}) = u^{\varepsilon}_{ij}(\bar{t},\bar{x}) - v^{ij}_{\varepsilon}(\bar{t},\bar{x})$, and we have:

$$(t_{\alpha},x_{\alpha},y_{\alpha}) \underset{\alpha\to\infty}{\longrightarrow} (\bar{t},\bar{x},\bar{x}). \tag{4.11}$$

It follows that:

$$\underline{\lim_{\alpha\to\infty}}\, u^{\varepsilon}_{ij}(t_{\alpha},x_{\alpha}) \geq u^{\varepsilon}_{ij}(\bar{t},\bar{x}) - v^{ij}_{\varepsilon}(\bar{t},\bar{x}) + \underline{\lim_{\alpha\to\infty}}\, v^{ij}_{\varepsilon}(t_{\alpha},y_{\alpha}) \geq u^{\varepsilon}_{ij}(\bar{t},\bar{x}) \geq \overline{\lim_{\alpha\to\infty}}\, u^{\varepsilon}_{ij}(t_{\alpha},x_{\alpha}).$$

Thus,

$$(u^{\varepsilon}_{ij}(t_{\alpha},x_{\alpha}), v^{ij}_{\varepsilon}(t_{\alpha},y_{\alpha})) \underset{\alpha\to\infty}{\longrightarrow} (u^{\varepsilon}_{ij}(\bar{t},\bar{x}), v^{ij}_{\varepsilon}(\bar{t},\bar{x})). \tag{4.12}$$

Using this, the inequalities (4.10) and the continuity of $(\underline{g}_{ik})_{(i,k)\in\Gamma^1\times\Gamma^1}$ and $(\overline{g}_{jl})_{(j,l)\in\Gamma^2\times\Gamma^2}$ and since the other components of $(u^{\varepsilon}_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ and $(v^{ij}_{\varepsilon})_{(i,j)\in\Gamma^1\times\Gamma^2}$ are respectively u.s.c and l.s.c. too, we get:

$$u^{\varepsilon}_{ij}(t_{\alpha},x_{\alpha}) > \max_{k\neq i}(u^{\varepsilon}_{kj}(t_{\alpha},x_{\alpha}) - \underline{g}_{ik}(t_{\alpha},x_{\alpha})) \text{ and } v^{ij}_{\varepsilon}(t_{\alpha},x_{\alpha}) < \min_{l\neq j}(v^{il}_{\varepsilon}(t_{\alpha},x_{\alpha}) + \overline{g}_{jl}(t_{\alpha},x_{\alpha})). \tag{4.13}$$

Back to Ψ^{ij}_{α}, let us note that:

$$\begin{aligned} &D_x\Psi^{ij}_{\alpha}(t,x,y) = \alpha(x-y) - \psi^{ij}(\bar{t},\bar{x},u^{\varepsilon}_{ij}(\bar{t},\bar{x}))\nabla\phi(\bar{x}) + 4\,|\,x-\bar{x}\,|^2\,(x-\bar{x}), \\ &D_y\Psi^{ij}_{\alpha}(t,x,y) = -\alpha(x-y) + \psi^{ij}(\bar{t},\bar{x},u^{\varepsilon}_{ij}(\bar{t},\bar{x}))\nabla\phi(\bar{x}) \text{ and } \partial_t\Psi^{ij}_{\alpha}(t,x,y) \ = 2(t-\bar{t}). \end{aligned}$$

Relying on the definition of Ψ^{ij}_{α} and the interior sphere condition (2.1), we show that we can avoid the conditions on the boundary ∂D so we can be reduced to deal only with the equations that hold in the interior D.
In fact, we distinguish two cases. If $x_{\alpha}\in\partial D$, we have:

$$
\begin{aligned}
-\langle \nabla\phi(x_\alpha) \quad , D_x\Psi^{ij}_\alpha(t_\alpha,x_\alpha,y_\alpha)\rangle - \psi^{ij}(t_\alpha,x_\alpha,u^\varepsilon_{ij}(t_\alpha,x_\alpha)) &= -\alpha\langle \nabla\phi(x_\alpha), x_\alpha - y_\alpha\rangle \\
-4 \mid x_\alpha - \bar{x} \mid^2 \langle \nabla\phi(x_\alpha), x_\alpha - \bar{x}\rangle - \psi^{ij}(t_\alpha,x_\alpha,u^\varepsilon_{ij}(t_\alpha,x_\alpha)) + \psi^{ij}(\bar{t},\bar{x},u^\varepsilon_{ij}(\bar{t},\bar{x}))\langle \nabla\phi(\bar{x}), \nabla\phi(x_\alpha)\rangle & \\
\geq -\tfrac{\alpha}{r} \mid x_\alpha - y_\alpha \mid^2 - \tfrac{4}{r} \mid x_\alpha - \bar{x} \mid^4 + \psi^{ij}(\bar{t},\bar{x},u^\varepsilon_{ij}(\bar{t},\bar{x}))\langle \nabla\phi(\bar{x}), \nabla\phi(x_\alpha)\rangle - \psi^{ij}(t_\alpha,x_\alpha,u^\varepsilon_{ij}(t_\alpha,x_\alpha)), &
\end{aligned}
$$

where we have used the interior sphere condition (2.1). In view of the convergences (4.11) and (4.12), we know that the right hand side tends to 0 as $\alpha \to \infty$. Then for α large enough we deduce that

$$
-\langle \nabla\phi(x_\alpha), D_x\Psi^{ij}_\alpha(t_\alpha,x_\alpha,y_\alpha)\rangle - \psi^{ij}(t_\alpha,x_\alpha,u^\varepsilon_{ij}(t_\alpha,x_\alpha)) > -\varepsilon. \tag{4.14}
$$

Similarly, if $y_\alpha \in \partial D$, we get:

$$
\begin{aligned}
-\langle \nabla\phi(y_\alpha), -D_y\Psi^{ij}_\alpha(t_\alpha,x_\alpha,y_\alpha)\rangle - \psi^{ij}(t_\alpha,y_\alpha,v^{ij}_\varepsilon(t_\alpha,y_\alpha)) &= \alpha\langle \nabla\phi(y_\alpha), y_\alpha - x_\alpha\rangle \\
&\quad + \psi^{ij}(\bar{t},\bar{x},u^\varepsilon_{ij}(\bar{t},\bar{x}))\langle \nabla\phi(\bar{x}), \nabla\phi(y_\alpha)\rangle - \psi^{ij}(t_\alpha,x_\alpha,v^{ij}_\varepsilon(t_\alpha,y_\alpha)), \\
&\leq \alpha \mid x_\alpha - y_\alpha \mid^2 + \psi^{ij}(\bar{t},\bar{x},u^\varepsilon_{ij}(\bar{t},\bar{x}))\langle \nabla\phi(\bar{x}), \nabla\phi(y_\alpha)\rangle - \psi^{ij}(\bar{t},\bar{x},u^\varepsilon_{ij}(\bar{t},\bar{x}))\langle \nabla\phi(\bar{x}), \nabla\phi(\bar{x})\rangle \\
&\quad + \psi^{ij}(\bar{t},\bar{x},v^{ij}_\varepsilon(\bar{t},\bar{x})) - \psi^{ij}(t_\alpha,x_\alpha,v^{ij}_\varepsilon(t_\alpha,y_\alpha)) + \psi^{ij}(\bar{t},\bar{x},u^\varepsilon_{ij}(\bar{t},\bar{x})) - \psi^{ij}(\bar{t},\bar{x},v^{ij}_\varepsilon(\bar{t},\bar{x})) \\
&\leq \alpha \mid x_\alpha - y_\alpha \mid^2 + \psi^{ij}(\bar{t},\bar{x},u^\varepsilon_{ij}(\bar{t},\bar{x}))\langle \nabla\phi(\bar{x}), \nabla\phi(y_\alpha)\rangle - \psi^{ij}(\bar{t},\bar{x},u^\varepsilon_{ij}(\bar{t},\bar{x}))\langle \nabla\phi(\bar{x}), \nabla\phi(\bar{x})\rangle \\
&\quad + \psi^{ij}(\bar{t},\bar{x},v^{ij}_\varepsilon(\bar{t},\bar{x})) - \psi^{ij}(t_\alpha,x_\alpha,v^{ij}_\varepsilon(t_\alpha,y_\alpha)),
\end{aligned}
$$

since $\psi^{ij}(\bar{t},\bar{x},u^\varepsilon_{ij}(\bar{t},\bar{x})) - \psi^{ij}(\bar{t},\bar{x},v^{ij}_\varepsilon(\bar{t},\bar{x})) \leq 0$. Again, with the use of (4.11) and (4.12) and assumptions $(H_1)(i)$, we can see that:

$$
\begin{aligned}
\alpha \mid x_\alpha - y_\alpha \mid^2 + \psi^{ij}(\bar{t},\bar{x},u^\varepsilon_{ij}(\bar{t},\bar{x}))\langle \nabla\phi(\bar{x}), \nabla\phi(y_\alpha)\rangle - \psi^{ij}(\bar{t},\bar{x},u^\varepsilon_{ij}(\bar{t},\bar{x}))\langle \nabla\phi(\bar{x}), \nabla\phi(\bar{x})\rangle + \psi^{ij}(\bar{t},\bar{x},v^{ij}_\varepsilon(\bar{t},\bar{x})) & \\
- \psi^{ij}(t_\alpha,x_\alpha,v^{ij}_\varepsilon(t_\alpha,y_\alpha)) \underset{\alpha\to\infty}{\longrightarrow} 0, &
\end{aligned}
$$

then it can be strictly dominated by ε for α large. Finally,for α large enough, we get:

$$
-\langle \nabla\phi(y_\alpha), -D_y\Psi^{ij}_\alpha(t_\alpha,x_\alpha,y_\alpha)\rangle - \psi^{ij}(t_\alpha,y_\alpha,v^{ij}_\varepsilon(t_\alpha,y_\alpha)) < \varepsilon. \tag{4.15}
$$

Next, as $(u^\varepsilon_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a subsolution of system (4.8) and $(v^{ij}_\varepsilon)_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a supersolution for (4.9), we deduce from (4.14) and (4.15) that the conditions on the boundary of $\overline{D}$ are reduced to the following inequalities: $\forall(\bar{t},\bar{x}) \in (0,T)\times\partial D$, $\forall\alpha$ large enough,

$$
\begin{aligned}
-\partial_t\Psi^{ij}_\alpha(t_\alpha,x_\alpha,y_\alpha) - b(x_\alpha)^\top D_x\Psi^{ij}_\alpha(t_\alpha,x_\alpha,y_\alpha) - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x_\alpha)M^\varepsilon] & \\
-f^{ij}(t_\alpha,x_\alpha,(u^\varepsilon_{ij}(t_\alpha,x_\alpha))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x_\alpha)D_x\Psi^{ij}_\alpha(t_\alpha,x_\alpha,y_\alpha)) \leq 0, & \\
\text{for } (\partial_t\Psi^{ij}_\alpha(t_\alpha,x_\alpha,y_\alpha), D_x\Psi^{ij}_\alpha(t_\alpha,x_\alpha,y_\alpha), M^\varepsilon) \in \overline{J}^{2+}u^\varepsilon_{ij}(t_\alpha,x_\alpha); & \\
-\partial_t\Psi^{ij}_\alpha(t_\alpha,x_\alpha,y_\alpha) - b(y_\alpha)^\top(-D_y\Psi^{ij}_\alpha(t_\alpha,x_\alpha,y_\alpha)) - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(y_\alpha)M_\varepsilon] & \\
-f^{ij}(t_\alpha,y_\alpha,(v^{ij}_\varepsilon(t_\alpha,y_\alpha))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(y_\alpha)(-D_y\Psi^{ij}_\alpha(t_\alpha,x_\alpha,y_\alpha))) \geq 0, & \\
\text{for } (\partial_t\Psi^{ij}_\alpha(t_\alpha,x_\alpha,y_\alpha), -D_y\Psi^{ij}_\alpha(t_\alpha,x_\alpha,y_\alpha), M_\varepsilon) \in \overline{J}^{2-}v^{ij}_\varepsilon(t_\alpha,y_\alpha). &
\end{aligned}
$$

The latter inequalities meet the ones that hold in the interior D. More precisely, if $\bar{x} \in D$, the subsequence (x_α, y_α) is in $D\times D$ for α large enough, then the inequalities above hold true. Therefore, we can apply Crandall-Ishii-Lions's Lemma (Theorem 3.2 [6]) with u^ε_{ij} and v^{ij}_ε on $(0,T)\times\overline{D}$ and $\varepsilon = \frac{1}{\alpha}$ to find $(p^u_\alpha, q^u_\alpha, M^u_\alpha) \in \overline{J}^{2+}u^\varepsilon_{ij}(t_\alpha,x_\alpha)$

and $(p^v_\alpha, q^v_\alpha, M^v_\alpha) \in \overline{J}^{2-} v^{ij}_\varepsilon(t_\alpha, x_\alpha)$ such that

$$p^u_\alpha - p^v_\alpha = \partial_t \Psi^{ij}_\alpha(t_\alpha, x_\alpha, y_\alpha) = 2(t_\alpha - \bar{t}),$$
$$q^u_\alpha - q^v_\alpha = -(D_x \Psi^{ij}_\alpha(t_\alpha, x_\alpha, y_\alpha) - D_y \Psi^{ij}_\alpha(t_\alpha, x_\alpha, y_\alpha))$$

and
$-(\alpha + \| A \|)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} M^u_\alpha & 0 \\ 0 & M^v_\alpha \end{pmatrix} \leq A + \frac{1}{\alpha}A^2$, where $A = D^2_{(x,y)} \Psi^{ij}_\alpha(t_\alpha, x_\alpha, y_\alpha)$.
By replacing in the above inequalities, we obtain:

$$-p^u_\alpha - b(x_\alpha)^\top q^u_\alpha - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x_\alpha) M^u_\alpha] - f^{ij}(t_\alpha, x_\alpha, (u^\varepsilon_{ij}(t_\alpha, x_\alpha))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma^\top(x_\alpha) q^u_\alpha) \leq 0,$$
$$-p^v_\alpha - b(y_\alpha)^\top q^v_\alpha - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(y_\alpha) M^v_\alpha] - f^{ij}(t_\alpha, y_\alpha, (v^{ij}_\varepsilon(t_\alpha, y_\alpha))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma^\top(y_\alpha) q^v_\alpha) \geq 0.$$

Combining the two inequalities, gives

$$-(p^u_\alpha - p^v_\alpha) - (b(x_\alpha)^\top q^u_\alpha - b(y_\alpha)^\top q^v_\alpha) - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x_\alpha) M^u_\alpha - \sigma\sigma^\top(y_\alpha) M^v_\alpha]$$
$$- \{f^{ij}(t_\alpha, x_\alpha, (u^\varepsilon_{ij}(t_\alpha, x_\alpha))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma^\top(x_\alpha) q^u_\alpha) - f^{ij}(t_\alpha, y_\alpha, (v^{ij}_\varepsilon(t_\alpha, y_\alpha))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma^\top(y_\alpha) q^v_\alpha)\} \leq 0.$$

Thanks to assumption $(H_1)(i)$, the Lipschitz assumption on b and σ and the above convergences (4.11) and (4.12), we can find some Σ_α such that $\overline{\lim_{\alpha\longrightarrow\infty}} \Sigma_\alpha \leqslant 0$ and

$$-\{f^{ij}(t_\alpha, x_\alpha, (u^\varepsilon_{ij}(t_\alpha, x_\alpha))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma^\top(x_\alpha) q^u_\alpha) - f^{ij}(t_\alpha, x_\alpha, (v^{ij}_\varepsilon(t_\alpha, y_\alpha))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma^\top(x_\alpha) q^u_\alpha)\} \leq \Sigma_\alpha.$$

From (4.2) and $(H_1)(i)$, we obtain

$$-\lambda(u^\varepsilon_{ij}(t_\alpha, x_\alpha) - v^{ij}_\varepsilon(t_\alpha, y_\alpha)) - \sum_{(k,l)\neq(i,j)} \Theta^{kl}_\alpha (u^\varepsilon_{kl}(t_\alpha, x_\alpha) - v^{kl}_\varepsilon(t_\alpha, y_\alpha)) \leq \Sigma_\alpha,$$

where Θ^{kl}_α stands for the increment rate of f^{ij} with respect to y_{kl} for $(k,l) \neq (i,j)$. Observe that Θ^{kl}_α is nonnegative and bounded by $\max_{ij} C^{ij}_f$ the Lipschitz constant, thanks to the monotonicity condition $(H_1)(iv)$. It follows that

$$\begin{aligned} -\lambda(u^\varepsilon_{ij}(t_\alpha, x_\alpha) - v^{ij}_\varepsilon(t_\alpha, y_\alpha)) &\leq \sum_{(k,l)\neq(i,j)} \Theta^{kl}_\alpha (u^\varepsilon_{kl}(t_\alpha, x_\alpha) - v^{kl}_\varepsilon(t_\alpha, y_\alpha))^+ + \Sigma_\alpha, \\ &\leq \max_{ij} C^{ij}_f \sum_{(k,l)\neq(i,j)} (u^\varepsilon_{kl}(t_\alpha, x_\alpha) - v^{kl}_\varepsilon(t_\alpha, y_\alpha))^+ + \Sigma_\alpha. \end{aligned}$$

Taking the superior limit in both sides as $\alpha \to \infty$, and since $u^\varepsilon_{kl} - v^{kl}_\varepsilon$ is u.s.c., we obtain: $\forall (i,j) \in \Gamma^1 \times \Gamma^2$

$$\begin{aligned} -\lambda(u^\varepsilon_{ij}(\bar{t}, \bar{x}) - v^{ij}_\varepsilon(\bar{t}, \bar{x})) &\leq \max_{ij} C^{ij}_f \sum_{k\neq ij} (u^\varepsilon_{kl}(\bar{t}, \bar{x}) - v^{kl}_\varepsilon(\bar{t}, \bar{x}))^+, \\ &\leq (m-1) \max_{ij} C^{ij}_f (u^\varepsilon_{ij}(\bar{t}, \bar{x}) - v^{ij}_\varepsilon(\bar{t}, \bar{x})), \end{aligned}$$

which is contradictory since $u^\varepsilon_{ij}(\bar{t}, \bar{x}) - v^{ij}_\varepsilon(\bar{t}, \bar{x}) > 0$ and $-\lambda > m. \max_{ij} C^{ij}_f$. Thus $\forall (i,j) \in \Gamma^1 \times \Gamma^2$, $u^\varepsilon_{ij} \leq v^{ij}_\varepsilon$ on $[0,T) \times \overline{D}$. To conclude, it suffices to take the limit as $\varepsilon \to 0$.

Step 2 : General case.

For λ arbitrary in $\mathbb{R}$, assume that $(u_{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ and $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ are respectively subsolution and supersolution of the PDEs system (1.1). Then the functions defined by $(\tilde{u}_{ij}(t,x) = e^{\lambda t} u_{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2}$ and $(\tilde{v}^{ij}(t,x) =$

$e^{\lambda t}v^{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2}$ are respectively subsolution and supersolution of the following PDEs system: $\forall(i,j)\in\Gamma^1\times\Gamma^2$, $\forall t\in[0,T)$,

$$\begin{cases}\min\{\tilde{u}^{ij}(t,x)-\max\limits_{k\neq i}(\tilde{u}^{kj}(t,x)-\underline{g}_{ik}(t,x));\max\{\tilde{u}^{ij}(t,x)-\min\limits_{l\neq j}(\tilde{u}^{il}(t,x)+\overline{g}_{jl}(t,x));-\partial_t\tilde{u}^{ij}(t,x)-\mathscr{L}\tilde{u}^{ij}(t,x)\\ +\lambda\tilde{u}^{ij}(t,x)-e^{\lambda t}f^{ij}(t,x,(e^{-\lambda t}\tilde{u}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},e^{-\lambda t}\sigma^\top(x)D_x\tilde{u}^{ij}(t,x))\}=0,\ (t,x)\in[0,T)\times D;\\ \dfrac{\partial\tilde{u}^{ij}}{\partial l}+e^{\lambda t}\psi^{ij}(t,x,e^{-\lambda t}\tilde{u}^{ij}(t,x))=0,\ (t,x)\in[0,T)\times\partial D;\\ \tilde{u}^{ij}(T,x)=e^{\lambda T}h_{ij}(x),\ x\in\overline{D},\end{cases}\tag{4.16}$$

where the function F^{ij} defined by:

$$F^{ij}(t,x,\vec{y},z)=-\lambda y^{ij}+f^{ij}(t,x,e^{-\lambda t}\vec{y},e^{-\lambda t}z),\forall(i,j)\in\Gamma^1\times\Gamma^2;$$

satisfies (4.2) for λ small enough. Then, according to the first case $\tilde{u}_{ij}\leq\tilde{v}^{ij}$, then $u_{ij}\leq v^{ij}$, $\forall(i,j)\in\Gamma^1\times\Gamma^2$. □

Corollary 4.1 *If the solution of the system of PDEs (1.1) exists, it is unique and continuous.*

5 Existence of the solution

In this section, we show that the family $(\overline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ constructed in section 2 provides the unique viscosity solution for (1.1), this will be done in three steps. At first, relying on the properties of the decreasing sequence of solutions to the PDEs systems obtained in Proposition 3.3, we show that the limit $(\overline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a subsolution of the (1.1). Next, with the use of the connection between the latter sequence and the decreasing scheme of reflected GBSDEs system (3.8), we show that for each m_0, $(\overline{v}^{ij,m_0})_{ij\in\Gamma^1\times\Gamma^2}$ is a supersolution of the system (1.1). Finally, in order to conclude that $(\overline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is the unique viscosity solution of (1.1), we use Perron's method which provides a supersolution of system (1.1). This consists of showing that the supremum of subsolutions of an l.s.c. PDE problem is also a solution.

Proposition 5.1 *The family $(\overline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$, is a viscosity subsolution of the PDEs system (1.1).*

Proof. Recall that for any $(i,j)\in\Gamma^1\times\Gamma^2$, $\overline{v}^{ij}$ is usc, then $\overline{v}^{ij*}=\overline{v}^{ij}$. Moreover, as $\overline{v}^{ij,m}$ satisfies (3.9), we have:

$$\overline{v}^{ij}(T,x)=\lim_{m\to\infty}\overline{v}^{ij,m}(T,x)=h^{ij}(x),\forall x\in\overline{D}.\tag{5.1}$$

We need to prove that

$$\begin{cases}\min\{\overline{v}^{ij}(t,x)-\max\limits_{k\neq i}(\overline{v}^{kj}(t,x)-\underline{g}_{ik}(t,x));\max\{\overline{v}^{ij}(t,x)-\min\limits_{l\neq j}(\overline{v}^{il}(t,x)+\overline{g}_{jl}(t,x));\\ \quad -p-b(x)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M]-f^{ij}(t,x,(\overline{v}^{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q)\}\}\leq 0,\\ \qquad (t,x)\in[0,T)\times D,\ (p,q,M)\in\overline{J}^{2+}\overline{v}^{ij}(t,x);\\ \min\{\overline{v}^{ij}(t,x)-\max\limits_{k\neq i}(\overline{v}^{kj}(t,x)-\underline{g}_{ik}(t,x));\max\{\overline{v}^{ij}(t,x)-\min\limits_{l\neq j}(\overline{v}^{il}(t,x)+\overline{g}_{jl}(t,x));\\ \quad -p-b(x)^\top q-\frac{1}{2}\mathrm{Tr}[(\sigma\sigma^\top)(x)M]f^{ij}(t,x,(\overline{v}^{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q)\}\}\\ \qquad\wedge\{-\langle\nabla\phi(x),q\rangle-\psi^{ij}(t,x,\overline{v}^{ij}(t,x))\}\leq 0,\quad (t,x)\in[0,T)\times\partial D,\ (p,q,M)\in\overline{J}^{2+}\overline{v}^{ij}(t,x).\end{cases}\tag{5.2}$$

$\underline{1^{st}\text{case}}$: Let $(t,x) \in [0,T] \times \partial D$ and $(i,j) \in \Gamma^1 \times \Gamma^2$ and fix $(p,q,M) \in \overline{J}^{2+} \overline{v}^{ij}(t,x)$, we suppose there exists $\varepsilon > 0$ such that

$$\overline{v}^{ij}(t,x) \geq \max_{k \neq i} (\overline{v}^{kj}(t,x) - \underline{g}_{ik}(t,x)) + \varepsilon, \tag{5.3}$$

otherwise the subsolution property is satisfied. Then, there exists $m_0 \geqslant 1$ such that

$$\overline{v}^{ij,m}(t,x) \geqslant \max_{k \neq i} (\overline{v}^{kj,m}(t,x) - \underline{g}_{ik}(t,x)) + \varepsilon, \forall m \geqslant m_0;$$

which holds on a neighborhood Θ_m of (t,x) thanks to the continuity of $(\overline{v}^{ij,m})_{(i,j)\in\Gamma^1\times\Gamma^2}$ and $(\underline{g}_{ik})_{(i,k)\in\Gamma^1\times\Gamma^1}$.
Besides, according to Lemma 6.1 in [6] there exist $m_k \to \infty$ and $x_k \in \overline{D}$ such that

$$(t_k, x_k, \overline{v}^{ij,m_k}(t_k,x_k), p_k, q_k, M_k) \underset{k\to\infty}{\longrightarrow} (t,x,\overline{v}^{ij}(t,x), p, q, M), \tag{5.4}$$

and $(t_k,x_k)_{k\geqslant 1}$ can be chosen from Θ_m. Note that the subsequence $(x_k)_{k\geqslant 1}$ takes values in $\overline{D}$. Then, by extracting two subsequences that are valued in D or ∂D respectively and for which we have kept the same notation and using the subsolution property of v^{ij,m_k}, we distinguish two cases:
If $x_k \in \partial D$, we get from the PDEs system (3.9):

$$\begin{aligned}\{-p_k - b(x_k)^\top q_k - \frac{1}{2}\mathrm{Tr}[(\sigma\sigma^\top)(x_k)M_k] - \overline{f}^{ij,m_k}(t_k,x_k,(v^{ij,m_k}(t_k,x_k))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma^\top(x_k)q_k)\} \\ \wedge \{-\langle \nabla\phi(x_k), q_k\rangle - \psi^{ij}(t_k,x_k,\overline{v}^{ij,m_k}(t_k,x_k))\} \leq 0,\end{aligned} \tag{5.5}$$

where $\overline{f}^{ij,m_k}(s,X_s^{t,x},\vec{y},z) = f^{ij}(s,X_s^{t,x},\vec{y},z) - m_k(y^{ij} - \min_{l\in\Gamma^2, l\neq j}(y^{il} + \overline{g}_{jl}(s,X_s^{t,x}))^+$.
Observe that if $\{-\langle \nabla\phi(x_k), q_k\rangle - \psi^{ij}(t_k,x_k,\overline{v}^{ij,m_k}(t_k,x_k))\} \leq 0$, we use the continuity of all data, then by taking the limit as k tends to ∞, we obtain:

$$-\langle \nabla\phi(x), q\rangle - \psi^{ij}(t,x,\overline{v}^{ij}(t,x)) \leq 0 \tag{5.6}$$

Therefore, from (5.5), we get:

$$\begin{aligned}-p_k - b(x_k)^\top q_k - \frac{1}{2}\mathrm{Tr}[(\sigma\sigma^\top)(x_k)M_k] - f^{ij}(t,x,(v^{ij,m_k}(t_k,x_k))_{(i,j)\in\Gamma^1\times\Gamma^2}, \sigma^\top(x_k)q_k) \\ - m_k(\overline{v}^{ij,m_k}(t_k,x_k) - \min_{l\in\Gamma^2,l\neq j}(\overline{v}^{il,m_k}(t_k,x_k) + \overline{g}_{jl}(s,X_s^{t,x}))^+ \leq 0.\end{aligned} \tag{5.7}$$

On the one hand, thanks to the boundedness of $(t_k,x_k)_{k\geqslant 1}$ which takes values in $[0,T]\times\overline{D}$, the continuity of $(\overline{v}^{ij,m_k})_{k\geqslant 1}$, b, σ and f^{ij}, together with (5.4), the inequality (5.7) implies that:

$$\varepsilon_k := \{\overline{v}^{ij,m_k}(t_k,x_k) - \min_{l\in\Gamma^2,l\neq j}(\overline{v}^{il,m_k}(t_k,x_k) + \overline{g}_{jl}(t_k,x_k)\}^+ \underset{k\to\infty}{\longrightarrow} 0.$$

Then there exists $m_k^0 \geqslant m_0$ such that $\forall m_k \geqslant m_k^0$, we have:

$$\overline{v}^{ij,m_k}(t_k,x_k) \leqslant \min_{l\in\Gamma^2,l\neq j}(\overline{v}^{il,m_k^0}(t_k,x_k) + \overline{g}_{jl}(t_k,x_k) + \varepsilon_k, \tag{5.8}$$

since $(\overline{v}^{ij,m_k}(t,x))_{k\geqslant 1}$ is non increasing.
By taking the limit as $k\to\infty$ and in view of the convergence (5.4) as well as the continuity of $(\overline{v}^{ij,m_k^0})_{(i,j)\in\Gamma^1\times\Gamma^2}$ and $(\overline{g}_{jl})_{(j,l)\in\Gamma^2\times\Gamma^2}$, we get:

$$\overline{v}^{ij}(t,x) \leqslant \min_{l\in\Gamma^2,l\neq j}(\overline{v}^{il,m_k^0}(t,x) + \overline{g}_{jl}(t,x)).$$

Taking the limit as $m_k^0 \to \infty$, implies that

$$\overline{v}^{ij}(t,x) - \min_{l\in\Gamma^2,l\neq j}(\overline{v}^{il}(t,x) + \overline{g}_{jl}(t,x)) \leqslant 0. \tag{5.9}$$

On the other hand, there exists a subsequence $(k_l)_{l\geqslant 0}$ such that for any $(i,j)\in\Gamma^1\times\Gamma^2$, $(v^{ij,k_l}(t_{k_l},x_{k_l}))_{l\geqslant 0}$ is convergent, which exists thanks to the boundedness of $\overline{D}$ and the continuity of $(v^{ij,k_l})_{(i,j)\in\Gamma^1\times\Gamma^2}$. Then by taking the limit w.r.t. l, we get:

$$\begin{aligned}-p-b(x)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M] &\leqslant \overline{\lim_{l\to\infty}} f^{ij,k_l}(t_{k_l},x_{k_l},(v^{ij,k_l}(t_{k_l},x_{k_l}))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x_{k_l})q_{k_l})\\ &\leqslant \overline{\lim_{l\to\infty}} f^{ij,k_l}(t_{k_l},x_{k_l},(v^{ij,k_l}(t_{k_l},x_{k_l}))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x_{k_l})q_{k_l}).\end{aligned}$$

Then, as f^{ij} satisfies the assumption $(H_1)(i)$, we get:

$$-p-b(x)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M]\leqslant f^{ij}(t,x,(\overline{\lim_{l\to\infty}} v^{ij,k_l}(t_{k_l},x_{k_l}))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q).$$

Moreover, for any $(p,q)\in\Gamma^1\times\Gamma^2$, $(\overline{v}^{pq,k_l}(t,x))_{l\geqslant 1}$ is non increasing and continuous, thus:
$\forall (p,q)\neq(i,j)$

$$\overline{v}^{pq,\star}(t,x)=\overline{v}^{pq}(t,x)=\overline{\lim_{\substack{(t',x')\longrightarrow(t,x)\\ t'<T,x'\in D}}}\lim_{l\to\infty}\overline{v}^{pq,l}(t',x')\geqslant\overline{\lim_{l\to\infty}}\overline{v}^{pq,l}(t_{k_l},x_{k_l}) \text{ and } \overline{v}^{ij}(t,x)=\overline{\lim_{l\to\infty}}\overline{v}^{pq,l}(t_{k_l},x_{k_l}).$$

Then under $(H_1)(iv)$, we have:

$$-p-b(x)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M]\leqslant f^{ij}(t,x,(v^{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q). \tag{5.10}$$

From this, (5.3), (5.6) and (5.9), we conclude that:

$$\begin{aligned}&\min\{\overline{v}^{ij}(t,x)-\max_{k\neq i}(\overline{v}^{kj}(t,x)-\underline{g}_{ik}(t,x));\max\{\overline{v}^{ij}(t,x)-\min_{l\neq j}(\overline{v}^{il}(t,x)+\overline{g}_{jl}(t,x));-p-b(x)^\top q\\ &-\frac{1}{2}\mathrm{Tr}[(\sigma\sigma^\top)(x)M]-f^{ij}(t,x,(\overline{v}^{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q)\}\}\wedge\{-\langle\nabla\phi(x),q\rangle-\psi^{ij}(t,x,\overline{v}^{ij}(t,x))\}\leq 0.\end{aligned} \tag{5.11}$$

The same calculus are used if $x_k\in D$, by considering the subsolution property satisfied by v^{ij,m_k} on $[0,T]\times D$. Then we conclude that $(\overline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a viscosity subsolution of system (1.1).
2^{sd}case : If $(t,x)\in[0,T]\times D$, we can find $(t_k,x_k)\in[0,T]\times D$ and the inequality (5.7) holds and (5.9) and (5.10).

As (i,j) is arbitrary in $\Gamma^1\times\Gamma^2$, the function $(\overline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a subsolution of the PDEs system (1.1). □

Proposition 5.2 *Let m_0 be fixed in $\mathbb{N}$. Then, the family $(\overline{v}^{ij,m_0})_{ij\in\Gamma^1\times\Gamma^2}$ is a supersolution of the system (1.1).*

Proof. Fix $m_0\geqslant 0$ and recall that $(\overline{Y}^{ij,m_0},\overline{Z}^{ij,m_0},\overline{K}^{ij,m_0})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is the solution of the following system of GRB-SDEs: $\forall(i,j)\in\Gamma^1\times\Gamma^2$,

$$\begin{cases}\overline{Y}^{ij,m_0}\in\mathscr{S}^2,\overline{Z}^{ij,m_0}\in\mathscr{H}^{2,d},\overline{K}^{ij,m_0}\in\mathscr{A}^2;\\ \overline{Y}^{ij,m_0}_s=h^{ij}(X^{t,x}_T)+\int_s^T\overline{f}^{ij,m_0}(r,X^{t,x}_r,(\overline{Y}^{kl,m_0}_r)_{(k,l)\in\Gamma^1\times\Gamma^2},\overline{Z}^{ij,m_0}_r)dr+\int_s^T\psi^{ij}(r,X^{t,x}_r,\overline{Y}^{ij,m_0}_r)dA^{t,x}_r\\ \qquad\qquad+\overline{K}^{ij,m_0}_T-\overline{K}^{ij,m_0}_s-\int_s^T\overline{Z}^{ij,m_0}_r dB_r,\ \forall t\leqslant s\leqslant T,\\ \overline{Y}^{ij,m_0}_s\geqslant\max_{k\in\Gamma,k\neq i}(\overline{Y}^{kj,m_0}_s-\underline{g}_{ik}(s,X^{t,x}_s)),\ \forall t\leqslant s\leqslant T,\\ \int_0^T\{\overline{Y}^{ij,m_0}_s-\max_{k\in\Gamma^1,k\neq i}(\overline{Y}^{kj,m_0}_s-g_{ik}(s,X^{t,x}_s))\}d\overline{K}^{ij,m_0}_s=0,\end{cases} \tag{5.12}$$

where $\overline{f}^{ij,m_0}(s,X_s^{t,x},\vec{y},z) = f^{ij}(s,X_s^{t,x},\vec{y},z) - m_0(y^{ij} - \min_{l\in\Gamma^2,l\neq j}(y^{il}+\overline{g}_{jl}(s,X_s^{t,x}))^+$.

Thanks to Proposition 3.3, there exists a family of deterministic continuous functions $(\overline{v}^{ij,m_0})_{(i,j)\in\Gamma^1\times\Gamma^2}$ such that:

$$\overline{Y}_s^{ij,m_0} = \overline{v}^{ij,m_0}(s,X_s^{t,x}),\ \forall s\in[t,T],\ \forall (i,j)\in\Gamma^1\times\Gamma^2. \tag{5.13}$$

Fix $(i,j)\in\Gamma^1\times\Gamma^2$, we can see that if $(\overline{Y}^{ij,m_0},\overline{Z}^{ij,m_0},\overline{K}^{ij,m_0})$ solves (5.12) then it satisfies: $\forall t\leqslant s\leqslant T$,

$$\begin{aligned}\overline{Y}_s^{ij,m_0} = h^{ij}(X_T^{t,x}) + \int_s^T f^{ij}(r,X_r^{t,x},(\overline{Y}_r^{ij,m_0})_{(i,j)\in\Gamma^1\times\Gamma^2},\overline{Z}_r^{ij,m_0})dr + \int_s^T \psi^{ij}(r,X_r^{t,x},\overline{Y}_r^{ij,m_0})dA_r^{t,x} + \overline{K}_T^{ij,m_0} - \overline{K}_s^{ij,m_0}\\ -(\overline{K}_T^{ij,m_0^-} - \overline{K}_s^{ij,m_0^-}) - \int_s^T \overline{Z}_r^{ij,m_0}dB_r,\end{aligned}$$

where $\overline{K}_s^{ij,m_0^-} = m_0\int_0^s\{\overline{Y}_r^{ij,m_0} - \min_{l\in\Gamma^2,l\neq j}(\overline{Y}_r^{il,m_0}+\overline{g}_{jl}(r,X_r^{t,x}))\}^+ds$ which is an increasing process.

Moreover,

$$\max_{k\in\Gamma^1,k\neq i}(\overline{Y}_s^{kj,m_0} - \underline{g}_{ik}(s,X_s^{t,x})) \leqslant \overline{Y}_s^{ij,m_0} \leqslant \overline{Y}_s^{ij,m_0} \vee \min_{l\in\Gamma^2,l\neq j}(\overline{Y}_s^{il,m_0}+\overline{g}_{jl}(s,X_s^{t,x})),$$

then, we have:

$$\begin{aligned}&\int_0^T\{\overline{Y}_s^{ij,m_0} - \overline{Y}_s^{ij,m_0}\vee\min_{l\in\Gamma^2,l\neq j}(\overline{Y}_s^{il,m_0}+\overline{g}_{jl}(s,X_s^{t,x}))\}d\overline{K}_s^{ij,m_0^-}\\ &\quad= \int_0^T\{\overline{Y}_s^{ij,m_0} - \min_{l\in\Gamma^2,l\neq j}(\overline{Y}_s^{il,m_0}+\overline{g}_{jl}(s,X_s^{t,x}))\}\times\max\left(\overline{Y}_s^{ij,m_0} - \min_{l\in\Gamma^2,l\neq j}(\overline{Y}_s^{il,m_0}+\overline{g}_{jl}(s,X_s^{t,x})),0\right)ds = 0.\end{aligned}$$

Besides, thanks to (5.12) the process $\overline{Y}^{ij,m_0}$ belongs to $\mathscr{S}^2$, then since $(X_s^{t,x})_{t\leqslant s\leqslant T}$ is bounded and the costs $(\overline{g}_{jl})_{(j,l)\in\Gamma^2\times\Gamma^2}$ are continuous we have $\mathbb{E}[\overline{K}_T^{ij,m_0}{}^2] < \infty$, which shows also that:

$$\mathbb{E}(\sup_{t\leqslant s\leqslant T}|\{\max_{k\in\Gamma^1,k\neq i}(\overline{Y}_s^{kj,m_0} - \underline{g}_{ik}(s,X_s^{t,x}))\}^+|^2 + \sup_{t\leqslant s\leqslant T}|\{\overline{Y}_s^{ij,m_0}\vee\min_{l\in\Gamma^1,l\neq j}(\overline{Y}_s^{il,m_0}+\overline{g}_{il}(s,X_s^{t,x}))\}^-|^2) < \infty.$$

All this combined together show that the so-called Mokobodzki's condition holds. Therefore, according to [9] there exists a unique solution $(\mathrm{Y},\mathrm{K}^+,\mathrm{K}^-,\mathrm{Z})$ for the standard reflected GBSDE with two barriers:

$$\begin{cases}\mathrm{Y}\in\mathscr{S}^2,\mathrm{Z}\in\mathscr{H}^{2,d},\mathrm{K}^+,\mathrm{K}^-\in\mathscr{A}^2;\\ \mathrm{Y}_s = h^{ij}(X_T^{t,x}) + \int_s^T f^{ij}(r,X_r^{t,x},(\overline{Y}_r^{ij,m_0})_{(i,j)\in\Gamma^1\times\Gamma^2},\mathrm{Z}_r)dr + \int_s^T\psi^{ij}(r,X_r^{t,x},\mathrm{Y}_r)dA_r^{t,x} + \mathrm{K}_T^+ - \mathrm{K}_s^+ - (\mathrm{K}_T^- - \mathrm{K}_s^-)\\ \qquad\qquad - \int_s^T\mathrm{Z}_r dB_r,\ \forall t\leqslant s\leqslant T,\\ \max_{k\in\Gamma^1,k\neq i}(\overline{Y}_s^{kj,m_0} - \underline{g}_{ik}(s,X_s^{t,x})) \leqslant \mathrm{Y}_s \leqslant \overline{Y}_s^{ij,m_0}\vee\min_{l\in\Gamma^2,l\neq j}(\overline{Y}_s^{il,m_0}+\overline{g}_{jl}(s,X_s^{t,x})),\ \forall t\leqslant s\leqslant T,\\ \int_0^T\{\mathrm{Y}_s - \max_{k\in\Gamma^1,k\neq i}(\overline{Y}_s^{ij,m_0} - \underline{g}_{ik}(s,X_s^{t,x}))\}d\mathrm{K}_s^+ = 0.\\ \int_0^T\{\mathrm{Y}_s - \overline{Y}_s^{ij,m_0}\vee\min_{l\in\Gamma^2,l\neq j}(\overline{Y}_s^{il,m_0}+\overline{g}_{jl}(s,X_s^{t,x}))\}d\mathrm{K}_s^- = 0.\end{cases} \tag{5.14}$$

It has been shown in [9] that the deterministic continuous function u defined $[0,T]\times\overline{D}$ by $\mathrm{Y}_s = \mathrm{u}(s,X_s^{t,x}),\ \forall s\in[t,T]$ is a solution of a ralated semilinear PDE with Neumann boundary condition. More precisely, using the Feymann-Kac formula (5.13), it holds that u is a solution of the following semilinear PDE with Neumann boundary condition:

$$\begin{cases}\min\{v(t,x)-\max\limits_{k\in\Gamma^1,k\neq i}(\overline{v}^{kj,m_0}(t,x)-\underline{g}_{ik}(t,x));\max\{v(t,x)-\overline{v}^{ij,m_0}(t,x)\vee\min\limits_{l\in\Gamma^2,l\neq j}(\overline{v}^{il,m_0}(t,x)+\overline{g}_{jl}(t,x));\\ -\partial_t v(t,x)-b(x)^\top D_x v(t,x)-\frac{1}{2}\mathrm{Tr}[(\sigma\sigma^\top)(x)D^2_{xx}v(t,x)]-f^{ij}(t,x,(\overline{v}^{ij,m_0}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)D_x v(t,x))\}\}=0,\\ \qquad\qquad\forall(t,x)\in[0,T]\times D;\\ \dfrac{\partial v}{\partial l}(t,x)+\psi^{ij}(t,x,v(t,x))=0,\ \forall(t,x)\in[0,T]\times\partial D;\\ v(T,x)=h^{ij}(x),\ \forall x\in\overline{D}.\end{cases}\tag{5.15}$$

As previously mentioned, $\overline{Y}^{ij,m_0}$ satisfies the equation (5.14) then once more thanks to (5.13) and by [9], we deduce that $\overline{v}^{ij,m_0}$ is a viscosity solution of the PDE (5.15), which is in particular a supersolution of PDE (5.15) and we have:

$$\begin{cases}\min\{\overline{v}^{ij,m_0}(t,x)-\max\limits_{k\in\Gamma^1,k\neq i}(\overline{v}^{kj,m_0}(t,x)-\underline{g}_{ik}(t,x));\max\{\overline{v}^{ij,m_0}(t,x)-\overline{v}^{ij,m_0}(t,x)\vee\min\limits_{l\in\Gamma^2,l\neq j}(\overline{v}^{il,m_0}(t,x)+\overline{g}_{jl}(t,x));\\ \qquad -p-b(x)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M]-f^{ij}(t,x,(\overline{v}^{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q)\}\}\geqslant 0,\\ \qquad\qquad\forall(t,x)\in[0,T]\times D,\forall(p,q,M)\in\overline{J}^-\overline{v}^{ij,m_0}(t,x);\\ \min\{\overline{v}^{ij,m_0}(t,x)-\max\limits_{k\in\Gamma^1,k\neq i}(\overline{v}^{kj,m_0}(t,x)-\underline{g}_{ik}(t,x));\max\{\overline{v}^{ij,m_0}(t,x)-\overline{v}^{ij,m_0}(t,x)\vee\min\limits_{l\in\Gamma^2,l\neq j}(\overline{v}^{il,m_0}(t,x)+\overline{g}_{jl}(t,x));\\ \qquad -p-b(x)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M]-f^{ij}(t,x,(\overline{v}^{ij}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q)\}\}\\ \qquad\vee\{-\langle\nabla\phi(x),q\rangle-\psi^{ij}(t,x,\overline{v}^{ij,m_0}(t,x))\}\geqslant 0,\ \forall(t,x)\in[0,T]\times\partial D,\forall(p,q,M)\in\overline{J}^-\overline{v}^{ij,m_0}(t,x);\\ \overline{v}^{ij,m_0}(T,x)\geqslant h^{ij}(x),\ x\in\overline{D}.\end{cases}\tag{5.16}$$

Let $(t,x)\in[0,T]\times\partial D$, then from (5.16) and using inequality $a-(a\vee b)\leqslant a-b$, we get:

$$\min\{\overline{v}^{ij,m_0}(t,x)-\max_{k\in\Gamma^1,k\neq i}(\overline{v}^{kj,m_0}(t,x)-\underline{g}_{ik}(t,x));\max\{\overline{v}^{ij,m_0}(t,x)-\min_{l\in\Gamma^2,l\neq j}(\overline{v}^{il,m_0}(t,x)+\overline{g}_{jl}(t,x));-p-b(x)^\top q$$
$$-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M]-f^{ij}(t,x,(\overline{v}^{ij,m_0}(t,x))_{(i,j)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q)\}\}\vee\{-\langle\nabla\phi(x),q\rangle-\psi^{ij}(t,x,\overline{v}^{ij,m_0}(t,x))\}\geqslant 0.$$

If $(t,x)\in[0,T]\times D$, the same argument is used to show that the inequality on D holds.

Since (i,j) is arbitrary in $\Gamma^1\times\Gamma^2$, we conclude that $(\overline{v}^{ij,m_0})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a viscosity supersolution of (1.1). □

It has been shown in the previous parts that $(\overline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a viscosity subsolution of (1.1) and $(\overline{v}^{ij,m_0})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a viscosity supersolution of (1.1).
We introduce the following set

$$\mathscr{U}_{m_0}=\{\vec{u}=(u^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}\text{ subsolution of (1.1) and }\forall(i,j)\in\Gamma^1\times\Gamma^2,\overline{v}^{ij}\leqslant u^{ij}\leqslant\overline{v}^{ij,m_0}\}$$

which is not empty since it contains $(\overline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ and we set for $(t,x)\in[0,T]\times\overline{D}$, and $(i,j)\in\Gamma^1\times\Gamma^2$:

$${}^{m_0}v^{ij}(t,x)=\sup\{u^{ij}(t,x),\vec{u}\in\mathscr{U}_{m_0}\}.$$

Proposition 5.3 *The family* $({}^{m_0}v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *does not depend on* m_0 *and is the unique continuous viscosity solution of the system (1.1).*

Proof. The function ${}^{m_0}v^{ij}$ will be denoted v^{ij} in the sequel.
Step 1: We first show that $(v^{ij})_{ij}$ is a subsolution of the PDEs system (1.1).
Note that for given $(u^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}\in\mathscr{U}_{m_0}$

$$\bar{v}^{ij}\leqslant u^{ij}\leqslant \bar{v}^{ij,m_0},$$

then $\forall(t,x)\in[0,T]\times\overline{D}$,

$$\bar{v}^{ij}\leqslant v^{ij}\leqslant \bar{v}^{ij,m_0}.$$

For any $(i,j)\in\Gamma^1\times\Gamma^2$, v^{ij} has an upper bound since $\bar{v}^{ij}$ is u.s.c. and $\bar{v}^{ij,m_0}$ is continuous on $[0,T]\times\overline{D}$ then we can define the upper semicontinuous enveloppe v^{ij*} and we have: $\bar{v}^{ij}(t,x)\leqslant v^{ij*}(t,x)\leqslant\bar{v}^{ij,m_0}$. Moreover, from (5.1) we deduce that $v^{ij*}(T,x)=h^{ij}(x)$.
Next, fix $(i,j)\in\Gamma^1\times\Gamma^2$ and let $(\tilde{v}^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}$ be an arbitrary element of $\mathscr{U}_{m_0}$, then we have:

$$\begin{cases}\min\{\tilde{v}^{ij}(t,x)-\max\limits_{k\in\Gamma^1,k\neq i}(\tilde{v}^{kj}(t,x)-\underline{g}_{ik}(t,x));\max\{\tilde{v}^{ij}(t,x)-\min\limits_{l\in\Gamma^2,l\neq j}(\tilde{v}^{il}(t,x)+\overline{g}_{jl}(t,x));\\ \quad -p-b(x)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M]-f^{ij}(t,x,(\tilde{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q)\}\}\leqslant 0,\\ \qquad\forall(t,x)\in[0,T]\times D,\forall(p,q,M)\in\overline{J}^+\tilde{v}^{ij*}(t,x);\\ \min\{\tilde{v}^{ij}(t,x)-\max\limits_{k\in\Gamma^1,k\neq i}(\tilde{v}^{kj}(t,x)-\underline{g}_{ik}(t,x));\max\{\tilde{v}^{ij}(t,x)-\min\limits_{l\in\Gamma^2,l\neq j}(\tilde{v}^{il}(t,x)+\overline{g}_{jl}(t,x));-p-b(x)^\top q\\ \quad -\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M]-f^{ij}(t,x,(\tilde{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q)\}\wedge\{-\langle\nabla\phi(x),q\rangle-\psi^{ij}(t,x,\tilde{v}^{ij}(t,x))\}\leqslant 0,\\ \qquad\forall(t,x)\in[0,T]\times\partial D,\forall(p,q,M)\in\overline{J}^+\tilde{v}^{ij*}(t,x).\end{cases}\tag{5.17}$$

By definition, $\tilde{v}^{kl}\leqslant v^{kl}$ and $\tilde{v}^{kl*}\leqslant v^{kl*}$ for any $(k,l)\in\Gamma^1\times\Gamma^2$, which implies:

$$\begin{aligned}\tilde{v}^{ij*}(t,x)-\max_{k\in\Gamma^1,k\neq i}(v^{kj*}(t,x)-\underline{g}_{ik}(t,x))\leqslant\ &\tilde{v}^{ij*}(t,x)-\max_{k\in\Gamma^1,k\neq i}(\tilde{v}^{kj*}(t,x)-\underline{g}_{ik}(t,x)),\\ \tilde{v}^{ij*}(t,x)-\min_{l\in\Gamma^2,l\neq j}(v^{il*}(t,x)+\overline{g}_{jl}(t,x))\leqslant\ &\tilde{v}^{ij*}(t,x)-\min_{l\in\Gamma^2,l\neq j}(\tilde{v}^{il*}(t,x)+\overline{g}_{jl}(t,x)).\end{aligned}$$

Also, in view of assumption $(H2)(iv)$ we have:

$$-f^{ij}(t,x,[(v^{kl*}(t,x))_{(k,l)\neq(i,j)},\tilde{v}^{ij}(t,x)],\sigma^\top(x)q)\leqslant -f^{ij}(t,x,(\tilde{v}^{kl}(t,x))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(x)q).$$

Therefore, the inequalities below hold for any $(i,j)\in\Gamma^1\times\Gamma^2$:

$$\begin{cases}\min\{\tilde{v}^{ij*}(t,x)-\max\limits_{k\in\Gamma^1,k\neq i}(v^{kj*}(t,x)-\underline{g}_{ik}(t,x));\max\{\tilde{v}^{ij*}(t,x)-\min\limits_{l\in\Gamma^2,l\neq j}(v^{il*}(t,x)+\overline{g}_{jl}(t,x));\\ \quad -p-b(x)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M]-f^{ij}(t,x,[(v^{kl*}(t,x))_{(k,l)\neq(i,j)},\tilde{v}^{ij*}(t,x)],\sigma^\top(x)q)\}\}\leqslant 0,\\ \qquad\forall(t,x)\in[0,T]\times D,\forall(p,q,M)\in\overline{J}^+\tilde{v}^{ij*}(t,x);\\ \min\{\tilde{v}^{ij*}(t,x)-\max\limits_{k\in\Gamma^1,k\neq i}(v^{kj*}(t,x)-\underline{g}_{ik}(t,x));\max\{\tilde{v}^{ij*}(t,x)-\min\limits_{l\in\Gamma^2,l\neq j}(v^{il*}(t,x)+\overline{g}_{jl}(t,x));\\ \quad -p-b(x)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M]-f^{ij}(t,x,[(v^{kl*}(t,x))_{(k,l)\neq(i,j)},\tilde{v}^{ij*}(t,x)],\sigma^\top(x)q)\}\}\\ \qquad\wedge\{-\langle\nabla\phi(x),q\rangle-\psi^{ij}(t,x,\tilde{v}^{ij*}(t,x))\}\leqslant 0,\ \ \forall(t,x)\in[0,T]\times\partial D,\forall(p,q,M)\in\overline{J}^+\tilde{v}^{ij*}(t,x).\end{cases}\tag{5.18}$$

For any $(i,j)\in\Gamma^1\times\Gamma^2$,let $\mathscr{G}^{ij}$ be a mapping defined on $[0,T]\times\overline{D}\times\mathbb{R}\times\mathbb{R}^{d+1+d}\times\mathbb{S}^d$ by:

$$\mathscr{G}^{ij}(t,x,w,p,q,M)=\begin{cases}\min\{w-\max\limits_{k\in\Gamma^1,k\neq i}(v^{kj*}(t,x)-\underline{g}_{ik}(t,x));\max\{w-\min\limits_{l\in\Gamma^2,l\neq j}(v^{il*}(t,x)+\overline{g}_{jl}(t,x));\\ \quad -p-b(x)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M]-f^{ij}(t,x,[(v^{kl*}(t,x))_{(k,l)\neq(i,j)},w],\sigma^\top(x)q)\}\},\ x\in D;\\ \min\{w-\max\limits_{k\in\Gamma^1,k\neq i}(v^{kj*}(t,x)-\underline{g}_{ik}(t,x));\max\{w-\min\limits_{l\in\Gamma^2,l\neq j}(v^{il*}(t,x)+\overline{g}_{jl}(t,x));\\ \quad -p-b(x)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)M]-f^{ij}(t,x,[(v^{kl*}(t,x))_{(k,l)\neq(i,j)},w],\sigma^\top(x)q)\}\}\\ \qquad\qquad\wedge\{-\langle\nabla\phi(x),q\rangle-\psi^{ij}(t,x,w)\},\ x\in\partial D.\end{cases}$$

Recall that $(v^{kl*})_{(k,l)\neq(i,j)}$ is a family of upper semicontinuous functions, then the function $\mathscr{G}$ is lower semicontinuous and $\tilde{v}^{ij}$ satisfies the PDE $\mathscr{G}^{ij}\leqslant 0$, then relying on Lemma 4.2 [6], we deduce that the supremum is a solution too. Finally, v^{ij} is a solution of the following PDE:

$$\begin{cases}\min\{w(t,x)-\max\limits_{k\in\Gamma^1,k\neq i}(v^{kj*}(t,x)-\underline{g}_{ik}(t,x));\max\{w(t,x)-\min\limits_{l\in\Gamma^2,l\neq j}(v^{il*}(t,x)+\overline{g}_{jl}(t,x));-\partial_t w(t,x)\\ \quad -b(x)^\top D_x w(t,x)-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)D_{xx}w(t,x)]-f^{ij}(t,x,[(v^{kl*}(t,x))_{(k,l)\neq(i,j)},w(t,x)],\sigma^\top(x)D_x w(t,x))\}\}=0,\\ \qquad\qquad\forall(t,x)\in[0,T]\times D,\\ -\langle\nabla\phi(x),D_x w(t,x)\rangle-\psi^{ij}(t,x,w(t,x))\}=0,\ \forall(t,x)\in[0,T]\times\partial D,\\ w(T,x)=h^{ij}(x),\ \forall x\in\overline{D}.\end{cases}\tag{5.19}$$

Since (i,j) is arbitrary in $\Gamma^1\times\Gamma^2$, $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a subsolution of the PDEs system (1.1).

Step 2: Recall that for (i,j) in $\Gamma_1\times\Gamma_2$, $\overline{v}_*^{ij}\leqslant v_*^{ij}\leqslant\overline{v}_*^{ij,m_0}$ and by corollary 4.1 $\underline{v}^{ij}=\underline{v}_*^{ij}$. This implies that

$$\underline{v}^{ij}\leqslant v_*^{ij}\leqslant\overline{v}^{ij,m_0},$$

since $\underline{v}^{ij}$ and $\overline{v}^{ij,m_0}$ are l.s.c. and continuous respectively. Then, as $\underline{v}^{ij,n}$ is a solution of (3.13) and from (5.1) it follows that $v_*^{ij}(T,x)=h^{ij}(x)$.

Assume that $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is not a supersolution to (1.1), which means that there exists $(i,j)\in\Gamma^1\times\Gamma^2$ such that v^{ij} does not satisfy the supersolution property, then there exists $(t_0,x_0)\in[0,T)\times\overline{D}$ such that:

$$\begin{aligned}F^{ij}(t_0,x_0,v_*^{ij}(t_0,x_0),p,q,M):=\min\{&v_*^{ij}(t_0,x_0)-\max_{k\in\Gamma^1,k\neq i}(v_*^{kj}(t_0,x_0)-\underline{g}_{ik}(t_0,x_0));\\ &\max\{v_*^{ij}(t_0,x_0)-\min_{l\in\Gamma^2,l\neq j}(v_*^{il}(t_0,x_0)+\overline{g}_{jl}(t_0,x_0));-p-b(x_0)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x_0)M]\\ &-f^{ij}(t_0,x_0,(v_*^{kl}(t_0,x_0))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(x_0)q)\}\}<0,\ \text{if } x_0\in D;\end{aligned}$$

and

$$\begin{aligned}F^{ij}(t_0,x_0,v_*^{ij}(t_0,x_0),p,q,M)\wedge B^{ij}(t_0,x_0,v_*^{ij}(t_0,x_0),q):=\min\{&v_*^{ij}(t_0,x_0)-\max_{k\in\Gamma^1,k\neq i}(v_*^{kj}(t_0,x_0)-\underline{g}_{ik}(t_0,x_0));\\ &\max\{v_*^{ij}(t_0,x_0)-\min_{l\in\Gamma^2,l\neq j}(v_*^{il}(t_0,x_0)+\overline{g}_{jl}(t_0,x_0));-p-b(x_0)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x_0)M]\\ &-f^{ij}(t_0,x_0,(v_*^{kl}(t_0,x_0))_{(k,l)\in\Gamma^1\times\Gamma^2},\sigma^\top(x_0)q)\}\wedge\{-\langle\nabla\phi(x_0),q\rangle-\psi^{ij}(t_0,x_0,v_*^{ij}(t_0,x_0))\}<0,\ \text{if } x_0\in\partial D,\end{aligned}$$

for some (p,q,M) that can be chosen in $J^-v_*^{ij}(t_0,x_0)$ thanks to the continuity of all data. Next, note that the function defined on $\mathbb{R}\times\mathbb{S}$ by:

$$(w,X)\mapsto\begin{cases}F^{ij}(t_0,x_0,w,p,q,X),\ \text{if } x_0\in D\\ F^{ij}(t_0,x_0,w,p,q,X)\wedge B^{ij}(t_0,x_0,w,q),\ \text{if } x_0\in\partial D,\end{cases}$$

is continuous, then there exist $\delta,\gamma>0$ such that

$$\begin{cases}F^{ij}(t_0,x_0,v_*^{ij}(t_0,x_0)+\delta,p,q,M-2\gamma)<0,\ \text{if } x_0\in D;\\ F^{ij}(t_0,x_0,v_*^{ij}(t_0,x_0)+\delta,p,q,M-2\gamma)\vee B^{ij}(t_0,x_0,v_*^{ij}(t_0,x_0)+\delta,q)\\ \qquad\leqslant F^{ij}(t_0,x_0,v_*^{ij}(t_0,x_0)+\delta,p,q,M-2\gamma)\wedge B^{ij}(t_0,x_0,v_*^{ij}(t_0,x_0)+\delta,q)<0,\ \text{if } x_0\in\partial D.\end{cases}\tag{5.20}$$

Now, let us define:

$$u_{\delta,\gamma}(t,x)=v_*^{ij}(t_0,x_0)+\delta+\langle q,x-x_0\rangle+p(t-t_0)+\frac{1}{2}\langle(M-2\gamma)(x-x_0),x-x_0\rangle,$$

and

$$\mathscr{G}(t,x)=\begin{cases}\min\{u_{\delta,\gamma}(t,x)-\max\limits_{k\in\Gamma^1,k\neq i}(v^{kj*}(t,x)-\underline{g}_{ik}(t,x));\max\{u_{\delta,\gamma}(t,x)-\min\limits_{l\in\Gamma^2,l\neq j}(v^{il*}(t,x)+\overline{g}_{jl}(t,x));\\ -p-b(x)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)(M-2\gamma)]-f^{ij}(t,x,[(v^{kl*}(t,x))_{(k,l)\neq(i,j)},u_{\delta,\gamma}(t,x)],\sigma^\top(x)q)\}\},\ \text{if } x\in D;\\ \min\{u_{\delta,\gamma}(t,x)-\max\limits_{k\in\Gamma^1,k\neq i}(v^{kj*}(t,x)-\underline{g}_{ik}(t,x));\max\{u_{\delta,\gamma}(t,x)-\min\limits_{l\in\Gamma^2,l\neq j}(v^{il*}(t,x)+\overline{g}_{jl}(t,x));\\ -p-b(x)^\top q-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)(M-2\gamma)]-f^{ij}(t,x,[(v^{kl*}(t,x))_{(k,l)\neq(i,j)},u_{\delta,\gamma}(t,x)],\sigma^\top(x)q)\}\}\\ \qquad\wedge\{-\langle\nabla\phi(x),q\rangle-\psi^{ij}(t,x,u_{\delta,\gamma}(t,x))\},\ \text{if } x\in\partial D.\end{cases}$$

Note that $u_{\delta,\gamma}(t_0,x_0)=v_*^{ij}(t_0,x_0)+\delta$, it follows from (5.20) that $\mathscr{G}(t_0,x_0)<0$. Besides, $u_{\delta,\gamma}$ is continuous and $(v_*^{kl})_{(k,l)\neq(i,j)}$ is a family of lower semicontinuous functions, then the function $\mathscr{G}$ is upper semicontinuous. Then there exists $\eta>0$ such that $\mathscr{G}(t,x)\leqslant 0$ on a neighborhood

$$\mathrm{V}_\mu^{t_0}\times\mathrm{V}_\mu^{x_0}=\{t\in[0,T)/|t-t_0|<\mu/2\}\times\{x\in D/|x-x_0|<\mu/2\}.$$

Thus $u_{\delta,\gamma}$ satisfies the following inequalities:

$$\begin{cases}\min\{Q(t,x)-\max\limits_{k\in\Gamma^1,k\neq i}(v_*^{kj}(t,x)-\underline{g}_{ik}(t,x));\max\{Q(t,x)-\min\limits_{l\in\Gamma^2,l\neq j}(v_*^{il}(t,x)+\overline{g}_{jl}(t,x));-\partial_t Q(t,x)\\ \quad -b(x)^\top D_xQ(t,x)-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)D_{xx}Q(t,x)]-f^{ij}(t,x,[(v_*^{kl}(t,x))_{(k,l)\neq(i,j)},Q(t,x)],\sigma^\top(x)D_xQ(t,x))\}\}\leqslant 0,\\ \qquad\qquad(t,x)\in\mathrm{V}_\mu^{t_0}\times\mathrm{V}_\mu^{x_0},\\ \min\{Q(t,x)-\max\limits_{k\in\Gamma^1,k\neq i}(v_*^{kj}(t,x)-\underline{g}_{ik}(t,x));\max\{Q(t,x)-\min\limits_{l\in\Gamma^2,l\neq j}(v_*^{il}(t,x)+\overline{g}_{jl}(t,x));-\partial Q_t(t,x)\\ \quad -b(x)^\top D_xQ(t,x)-\frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x)D_{xx}Q(t,x)]-f^{ij}(t,x,[(v_*^{kl}(t,x))_{(k,l)\neq(i,j)},D_{xx}Q(t,x)],\sigma^\top(x)D_xQ(t,x))\}\}\\ \qquad\wedge\{-\langle\nabla\phi(x),D_xQ(t,x)\rangle-\psi^{ij}(t,x,Q(t,x))\}\leqslant 0,\ (t,x)\in\mathrm{V}_\mu^{t_0}\times\partial\mathrm{V}_\mu^{x_0}.\end{cases}\tag{5.21}$$

Again, as $v_*^{kl} \leqslant v^{kl*}$, $\forall (k,l) \in \Gamma^1 \times \Gamma^2$, we use the same arguments as in step 1 we deduce that $u_{\delta,\gamma}$ satisfies:

$$\begin{cases} \min\{Q(t,x) - \max\limits_{k\in\Gamma^1,k\neq i}(v^{kj*}(t,x) - \underline{g}_{ik}(t,x)); \max\{Q(t,x) - \min\limits_{l\in\Gamma^2,l\neq j}(v^{il*}(t,x) + \overline{g}_{jl}(t,x)); -\partial_t Q(t,x) \\ \quad -b(x)^\top D_x Q(t,x) - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x) D_{xx} Q(t,x)] - f^{ij}(t,x,[(v^{kl*}(t,x))_{(k,l)\neq(i,j)}, Q(t,x)], \sigma^\top(x) D_x Q(t,x))\}\} \leqslant 0, \\ \qquad\qquad (t,x) \in \mathrm{V}_\mu^{t_0} \times \mathrm{V}_\mu^{x_0}, \\ \min\{Q(t,x) - \max\limits_{k\in\Gamma^1,k\neq i}(v^{kj*}(t,x) - \underline{g}_{ik}(t,x)); \max\{Q(t,x) - \min\limits_{l\in\Gamma^2,l\neq j}(v^{il*}(t,x) + \overline{g}_{jl}(t,x)); -\partial_t Q(t,x) \\ \quad -b(x)^\top D_x Q(t,x) - \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(x) D_{xx} Q(t,x)] - f^{ij}(t,x,[(v^{kl*}(t,x))_{(k,l)\neq(i,j)}, Q(t,x)], \sigma^\top(x) D_x Q(t,x))\}\} \\ \qquad\qquad \wedge\{-\langle \nabla\phi(x), D_x Q(t,x)\rangle - \psi^{ij}(t,x,Q(t,x))\} \leqslant 0, \ (t,x) \in \mathrm{V}_\mu^{t_0} \times \partial\mathrm{V}_\mu^{x_0}. \end{cases} \tag{5.22}$$

Therefore, as $(p,q,M) \in J^- v_*^{ij}(t_0,x_0)$, we have:

$$\begin{aligned} v^{ij}(t,x) \geqslant v_*^{ij}(t,x) &\geqslant v_*^{ij}(t_0,x_0) + \langle q, x-x_0\rangle + p(t-t_0) + \frac{1}{2}\langle M(x-x_0), x-x_0\rangle + o(|t-t_0| + |x-x_0|^2) \\ &> u_{\delta,\gamma}(t,x) \end{aligned}$$

for $\delta < \frac{\mu^2\gamma}{4}$. Consider the function

$$\tilde{u}^{ij}(t,x) = \begin{cases} \max\{u_{\delta,\gamma}(t,x), v^{ij}(t,x)\}, \ (t,x) \in \mathrm{V}_\mu^{t_0} \times \overline{\mathrm{V}}_\mu^{x_0} \\ v^{ij}(t,x), \ (t,x) \in [0,T) \times \overline{D} \backslash \mathrm{V}_\mu^{t_0} \times \overline{\mathrm{V}}_\mu^{x_0}. \end{cases}$$

It should be mentioned that (5.19) is a lower semicontinuous problem, it follows from Lemma 4.2 [6] that $u_{\delta,\gamma}$ is a subsolution of the PDEs system (5.19). Then, as $\tilde{u}^{ij} \geqslant v^{ij}$ and using the monotonicity condition $(H_1)(iv)$, it holds that $[(v^{kl}(t,x))_{(k,l)\neq(i,j)}, \tilde{v}^{ij}(t,x)]$ is a subsolution of the PDEs system (1.1). Next, from Theorem 4.1 we have $[(v^{kl})_{(k,l)\neq(i,j)}, \tilde{u}^{ij}] \in \mathscr{U}_{m_0}$.
Finally, in view of the definition of v^{ij*} there exists a sequence $(t_n, x_n, v^{ij}(t_n,x_n))_{n\geqslant 1}$ that converges to $(t_0,x_0,v_*^{ij}(t_0,x_0))$, this implies that

$$\lim_n (\tilde{u}^{ij} - v^{ij})(t_n,x_n) = (u_{\delta,\gamma} - v_*^{ij})(t_0,x_0) > 0, \tag{5.23}$$

which contradicts the fact that v^{ij} is the supremum. Therefore, $(v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is a supersolution of the PDEs system (1.1) and we conclude using Corollary 4.1 that $({}^{m_0}v^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ does not depend on m_0. □

Theorem 5.1 *The family of functions $(\overline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is the unique continuous viscosity solution of the PDEs system (1.1).*

Proof. Recall that $\overline{v}^{ij} = \lim\limits_{m\to\infty} \overline{v}^{ij,m}$ and $\overline{v}^{ij} \leqslant v^{ij} \leqslant \overline{v}^{ij,m_0}$, then taking the limit as $m_0 \to \infty$ we get $\overline{v}^{ij} = v^{ij}$. The result follows from Proposition 5.3. □

Analogously, thanks to Proposition 3.4 and Theorem 5.1, we have:

Theorem 5.2 *The family of functions $(\underline{v}^{ij})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is the unique continuous viscosity solution of the PDEs system(1.3).*

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