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Orbit configuration spaces and the homotopy groups of the pair $(\prod_1^n M, F_n(M))$ for M either \mathbb{S}^2 or $\mathbb{R}P^2$

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Abstract

Let $n \geq 1$, and let $\iota_n: F_n(M) \longrightarrow \prod_1^n M$ be the natural inclusion of the n^{th} configuration space of M in the n -fold Cartesian product of M with itself. In this paper, we study the map ι_n , the homotopy fibre I_n of ι_n and its homotopy groups, and the induced homomorphisms $(\iota_n)_{\#k}$ on the k^{th} homotopy groups of $F_n(M)$ and $\prod_1^n M$ for all $k \geq 1$, where M is the 2-sphere \mathbb{S}^2 or the real projective plane $\mathbb{R}P^2$. It is well known that the group $\pi_k(I_n)$ is the homotopy group $\pi_{k+1}(\prod_1^n M, F_n(M))$ for all $k \geq 0$. If $k \geq 2$, we show that the homomorphism $(\iota_n)_{\#k}$ is injective and diagonal, with the exception of the case $n = k = 2$ and $M = \mathbb{S}^2$, where it is anti-diagonal. We then show that I_n has the homotopy type of $K(R_{n-1}, 1) \times \Omega(\prod_1^{n-1} \mathbb{S}^2)$, where R_{n-1} is the $(n-1)^{\text{th}}$ Artin pure braid group if $M = \mathbb{S}^2$, and is the fundamental group G_{n-1} of the $(n-1)^{\text{th}}$ orbit configuration space of the open cylinder $\mathbb{S}^2 \setminus \{\tilde{z}_0, -\tilde{z}_0\}$ with respect to the action of the antipodal map of \mathbb{S}^2 if $M = \mathbb{R}P^2$, where $\tilde{z}_0 \in \mathbb{S}^2$. This enables us to describe the long exact sequence in homotopy of the homotopy fibration $I_n \longrightarrow F_n(M) \xrightarrow{\iota_n} \prod_1^n M$ in geometric terms, and notably the image of the boundary homomorphism $\pi_{k+1}(\prod_1^n M) \longrightarrow \pi_k(I_n)$. From this, if $M = \mathbb{S}^2$ and $n \geq 3$ (resp. $M = \mathbb{R}P^2$ and $n \geq 2$), we show that $\text{Ker}((\iota_n)_{\#1})$ is isomorphic to the quotient of R_{n-1} by the square of its centre, as well as to an iterated semi-direct product of free groups with the subgroup of order 2 generated by the centre of $P_n(M)$ that is reminiscent of the combing operation for the Artin pure braid groups, as well as decompositions obtained in [GG5].

2010 AMS Subject Classification: 20F36 (primary); 55P15 Classification of homotopy type; 55Q40 Homotopy groups of spheres; 55R80 Discriminantal varieties, configuration spaces; 55R05 Fibre spaces

1 Introduction

Let M be a connected surface, perhaps with boundary, and either compact, or with a finite number of points removed from the interior of the surface, and let $\prod_1^n M = M \times \cdots \times M$ denote the n -fold Cartesian product of M with itself. The n^{th} configuration space of M is defined by:

$$F_n(M) = \left\{ (x_1, \dots, x_n) \in \prod_1^n M \mid x_i \neq x_j \text{ for all } 1 \leq i, j \leq n, i \neq j \right\}.$$

It is well known that the fundamental group $\pi_1(F_n(M))$ of $F_n(M)$ is isomorphic to the pure braid group $P_n(M)$ of M on n strings [FaN, FoN], and if M is the 2-disc then $P_n(M)$ is the Artin pure braid group P_n . Let $\iota_n: F_n(M) \rightarrow \prod_1^n M$ denote the inclusion map, and for $k \geq 0$, let $(\iota_n)_{\#k}: \pi_k(F_n(M)) \rightarrow \pi_k(\prod_1^n M)$ be the induced homomorphism of the corresponding homotopy groups. If no confusion is possible, we shall often just write ι in place of ι_n , and $\iota_{\#}$ or $(\iota)_{\#}$ if $k = 1$. The homomorphism $\iota_{\#}: P_n(M) \rightarrow \pi_1(\prod_1^n M)$ was studied by Birman in 1969 [Bi], and if M is a compact surface without boundary different from the 2-sphere \mathbb{S}^2 and the real projective plane $\mathbb{R}P^2$, Goldberg showed that $\text{Ker}(\iota_{\#})$ is equal to the normal closure in $P_n(M)$ of the image of the homomorphism $j_{\#}: P_n \rightarrow P_n(M)$ induced by the inclusion $j: \mathbb{D}^2 \rightarrow M$ of a topological disc \mathbb{D}^2 in M [Go]. In [GG5], we extended this result to \mathbb{S}^2 and $\mathbb{R}P^2$, and in the case of $\mathbb{R}P^2$, we proved that $\iota_{\#}$ coincides with Abelianisation, so that $\text{Ker}(\iota_{\#})$ is equal to the commutator subgroup $\Gamma_2(P_n(\mathbb{R}P^2))$ of $P_n(\mathbb{R}P^2)$.

The aim of this paper is to study further the maps ι_n and the induced homomorphisms $(\iota_n)_{\#k}$ in more detail in the cases where $M = \mathbb{S}^2$ or $\mathbb{R}P^2$, and to determine the homotopy type of the homotopy fibre of ι_n . Following [A, pages 91 and 108], recall that if $f: (X, x_0) \rightarrow (Y, y_0)$ is a map of (pointed) topological spaces and I denotes the unit interval then the *mapping path* of f , defined by:

$$E_f = \left\{ (x, \lambda) \in X \times Y^I \mid f(x) = \lambda(0) \right\} \quad (1)$$

has the same homotopy type as X , the map $r_f: E_f \rightarrow X$ given by $r_f(x, \lambda) = x$ being a homotopy equivalence, and the map $p_f: E_f \rightarrow Y$ defined by $p_f(x, \lambda) = \lambda(1)$ is a fibration whose fibre is the *homotopy fibre* I_f of f defined by:

$$I_f = \left\{ (x, \lambda) \in X \times Y^I \mid f(x) = \lambda(0) \text{ and } \lambda(1) = y_0 \right\}. \quad (2)$$

We will refer to the sequence of maps $I_f \rightarrow X \xrightarrow{f} Y$ as the *homotopy fibration* of f , where the map $I_f \rightarrow X$ is the composition of the inclusion $I_f \rightarrow E_f$ by r_f . More details may be found in [A, Ha, W]. In what follows, we denote the homotopy fibre of ι_n by I_n . We shall determine the homotopy type of I_n in the cases where M is either \mathbb{S}^2 or $\mathbb{R}P^2$. This leads to a better understanding of the long exact sequence in homotopy of the fibration p_{ι_n} , as well as an alternative interpretation of $P_n(M)$ in terms of exact sequences. Additional motivation for our study comes from the following two facts:

(a) up to a shift in index, the homotopy groups of I_n coincide with those of the pair $(\prod_1^n M, F_n(M))$. More precisely, by [A, Definition 4.5.3] and the paragraph that precedes it, $\pi_{k-1}(I_n) = \pi_k((\prod_1^n M, F_n(M)))$ for all $k \geq 0$. If one uses the classical definition of the relative homotopy groups of a pair, the two definitions may be seen to be equivalent using the paragraph that follows [A, Definition 4.5.3].

(b) the higher homotopy groups of $F_n(M)$ are known to be isomorphic to those of \mathbb{S}^2 or \mathbb{S}^3 (see Theorem 5), and so such exact sequences bring in to play these homotopy groups.

In the rest of the introduction, let M be \mathbb{S}^2 or $\mathbb{R}P^2$. The paper is organised as follows. In Section 2, we show in Lemma 7 that for all $n, k \geq 2$, the homomorphism $(\iota_n)_{\#k}$ is injective, and in Propositions 9 and 10, we prove that this homomorphism is diagonal, with the exception of the case $M = \mathbb{S}^2$ and $n = k = 2$, where it is anti-diagonal. The aim of Section 3 is to prove Theorem 1 stated below that describes the homotopy type of I_n . In Section 3.1, we define the framework and much of the notation that will be used in the rest of the paper. In the case of $\mathbb{R}P^2$, this description makes use of certain generalisations of configuration spaces, which we now define. Let $\tilde{z}_0 \in \mathbb{S}^2$, let $C = \mathbb{S}^2 \setminus \{\tilde{z}_0, -\tilde{z}_0\}$ denote the open cylinder, and let $\tau: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ denote the antipodal map defined by $\tau(x) = -x$ for all $x \in \mathbb{S}^2$ as well as its restriction to C . If $n \geq 1$, the n^{th} orbit configuration space of C with respect to the group $\langle \tau \rangle$, which is a subspace of $F_n(C)$, is defined by:

$$F_n^{\langle \tau \rangle}(C) = \left\{ (x_1, \dots, x_n) \in \prod_1^n M \mid x_i \notin \{x_j, \tau(x_j)\} \text{ for all } 1 \leq i, j \leq n, i \neq j \right\}.$$

Orbit configuration spaces were defined and studied in [CX], but some examples already appeared in [Fa, FVB]. Let $G_n = \pi_1(F_n^{\langle \tau \rangle}(C))$ denote the fundamental group of $F_n^{\langle \tau \rangle}(C)$. It is known that $F_n^{\langle \tau \rangle}(C)$ is a space of type $K(G_n, 1)$ [C]. We first determine the homotopy type of the homotopy fibre I_n .

THEOREM 1. *Let $n \geq 2$ and let $M = \mathbb{S}^2$ or $\mathbb{R}P^2$. Then the homotopy fibre I_n of the map $\iota_n: F_n(M) \rightarrow \prod_1^n M$ has the homotopy type of:*

(a) $F_{n-1}(\mathbb{D}^2) \times \Omega(\prod_1^{n-1} \mathbb{S}^2)$ if $M = \mathbb{S}^2$, or equivalently of $K(P_{n-1}, 1) \times \Omega(\prod_1^{n-1} \mathbb{S}^2)$, where $\Omega(\prod_1^{n-1} \mathbb{S}^2)$ denotes the loop space of $\prod_1^{n-1} \mathbb{S}^2$.

(b) $F_{n-1}^{\langle \tau \rangle}(C) \times \Omega(\prod_1^{n-1} \mathbb{S}^2)$ if $M = \mathbb{R}P^2$, or equivalently of $K(G_{n-1}, 1) \times \Omega(\prod_1^{n-1} \mathbb{S}^2)$.

Theorem 1 will be proved in Section 3.2. The basic idea of the proof is to ‘replace’ ι_n by a map that is null homotopic and whose homotopy fibre has the same homotopy type as that of ι_n . The fact that this map is null homotopic implies that its homotopy fibre has the homotopy type of the homotopy fibre I_c of a constant map (see Remarks 14(b)), I_c being the product space that appears in the statement of Theorem 1. Another important tool is the relation between the homotopy fibres of fibre spaces and certain subspaces (see the Appendix, and Lemma 28 in particular). Taking the long exact sequence in homotopy of the homotopy fibration of ι_n and using the results of Section 2, we obtain the following exact sequences, where for $k \geq 1$, $\partial_{n,k}: \pi_k(\prod_1^n M) \rightarrow \pi_{k-1}(I_n)$ denotes the boundary homomorphism on the level of π_k associated to the homotopy fibration:

$$I_n \longrightarrow F_n(M) \xrightarrow{\iota_n} \prod_1^n M. \quad (3)$$

Let $R_{n-1} = P_{n-1}$ if $M = \mathbb{S}^2$, and $R_{n-1} = G_{n-1}$ if $M = \mathbb{R}P^2$.

COROLLARY 2. *Let $n \geq 2$, and let $M = \mathbb{S}^2$ or $\mathbb{R}P^2$.*

(a) Suppose that $k \geq 3$ (resp. $M = \mathbb{S}^2$ and $n = k = 2$). Then we have the following split short exact sequence of Abelian groups:

$$1 \longrightarrow \pi_k(F_n(M)) \xrightarrow{(\iota_n)_{\#k}} \pi_k\left(\prod_1^n M\right) \xrightarrow{\partial_{n,k}} \pi_{k-1}(I_n) \longrightarrow 1, \quad (4)$$

where the homomorphism $(\iota_n)_{\#k}$ is diagonal (resp. anti-diagonal), and $\pi_{k-1}(I_n)$ is isomorphic to $\pi_{k-1}(\Omega(\prod_1^{n-1} \mathbb{S}^2))$. Up to isomorphism, the short exact sequence (4) may also be written as:

$$1 \longrightarrow \pi_k(M) \longrightarrow \prod_1^n \pi_k(M) \longrightarrow \prod_1^{n-1} \pi_k(M) \longrightarrow 1.$$

(b) Suppose that $k = 2$, and that $n \geq 3$ if $M = \mathbb{S}^2$. Then we have the following exact sequence:

$$1 \longrightarrow \pi_2\left(\prod_1^n M\right) \xrightarrow{\partial_{n,2}} \pi_1(I_n) \longrightarrow P_n(M) \xrightarrow{(\iota_n)_{\#1}} \pi_1\left(\prod_1^n M\right) \longrightarrow 1, \quad (5)$$

where $\pi_1(I_n)$ is isomorphic to $R_{n-1} \times \pi_1(\Omega(\prod_1^{n-1} \mathbb{S}^2))$. Up to isomorphism, the exact sequence (5) may also be written as:

$$\begin{aligned} 1 \longrightarrow \mathbb{Z}^n \longrightarrow P_{n-1} \oplus \mathbb{Z}^{n-1} \longrightarrow P_n(\mathbb{S}^2) \longrightarrow 1 & \quad \text{if } M = \mathbb{S}^2 \\ 1 \longrightarrow \mathbb{Z}^n \longrightarrow G_{n-1} \times \mathbb{Z}^{n-1} \longrightarrow P_n(\mathbb{R}P^2) \xrightarrow{(\iota_n)_{\#1}} \mathbb{Z}_2^n \longrightarrow 1 & \quad \text{if } M = \mathbb{R}P^2. \end{aligned}$$

In the case $M = \mathbb{S}^2$, the short exact sequence does not split.

We can make more explicit the kernel and the image of the homomorphisms that appear in the exact sequences of Corollary 2. In order to understand better the homotopy fibration (3) and these exact sequences, we need to study the boundary homomorphism $\partial_{n,k}$, the case $k = 2$ being the most complicated due to the appearance of G_{n-1} in $\pi_1(I_n)$ if $M = \mathbb{R}P^2$, and to make use of the structure of $F_n^{(\tau)}(C)$ and G_n . With this in mind, in Section 3.3, we derive a presentation of G_n (see Proposition 19), as well as the fact that the centre of G_n is infinite cyclic (this result may be found in [BS, Satz 7.2] or [CP, Proposition 6.2]), generated by an element Θ_n that is somewhat similar in nature to the full-twist braid of $P_n(M)$ (see Proposition 20).

If $M = \mathbb{S}^2$ (resp. $\mathbb{R}P^2$), let $n_0 = 3$ (resp. $n_0 = 2$). From now on, we denote the boundary homomorphism $\partial_{n,2}$ by ∂_n . By (5), the image of ∂_n is a free Abelian group of rank n . The aim of Section 4 is to describe this image more precisely, and to visualise it more clearly in terms of the direct product $R_{n-1} \times \pi_1(\Omega(\prod_1^{n-1} \mathbb{S}^2))$, which is the fundamental group of I_c , via an explicit isomorphism between $\pi_1(I_n)$ and $\pi_1(I_c)$. The main difficulty that we face is to determine explicit homotopy equivalences between I_n and I_c . In Section 4.1, we introduce geometric representatives of generators of $\pi_1(\Omega(M))$, that we use to describe certain elements of $\pi_1(I_{n_0})$ in Lemma 22, and also to fix a basis $\mathcal{B} = (\tilde{\lambda}_{\tilde{x}_0}, \tilde{\lambda}_{\tilde{x}_1}, \dots, \tilde{\lambda}_{\tilde{x}_{n-3}}, \tilde{\lambda}_{\tilde{z}_0}, \tilde{\lambda}_{-\tilde{z}_0})$ (resp. $\mathcal{B} = (\lambda_{x_0}, \lambda_{x_1}, \dots, \lambda_{x_{n-2}}, \lambda_{z_0})$) of $\pi_2(\prod_1^n M)$ in equation (46) (see Section 4.2). Here, $\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{n-2}, \tilde{z}_0$ are certain basepoints of \mathbb{S}^2 that are defined in Section 3.1, and $x_0, x_1, \dots, x_{n-2}, z_0$ are their projections in $\mathbb{R}P^2$ under the universal covering map from \mathbb{S}^2 to $\mathbb{R}P^2$. In Lemma 24(a) and (b), we construct explicit homotopy equivalences between the intermediate homotopy fibres defined in Section 3.1 whose composition yields a homotopy equivalence between I_c and I_n . Here we also require Corollary 30 that is proved in the Appendix. This enables us to prove in Lemma 24(d) that the images by ∂_n of the first $n - 1$ elements of \mathcal{B} may be identified via the induced isomorphism between $\pi_1(I_c)$ and $\pi_1(I_n)$ with the corresponding factor

of $\pi_1(\Omega(\prod_1^{n-1} \mathbb{S}^2))$, and allows us to describe clearly these images viewed as elements of $\pi_1(I_c)$.

The elucidation of the image by ∂_n of the remaining element $\tilde{\lambda}_{-\tilde{z}_0}$ (resp. λ_{z_0}) of \mathcal{B} if $M = \mathbb{S}^2$ (resp. if $M = \mathbb{R}P^2$) is much more complicated. The computation has been carried out in [GG7, Sections 5.3 and 5.4], but is rather long. In Section 4.2, we describe the image of ∂_n by bringing into play an element $\hat{\tau}_n$ of $\pi_1(I_n)$ in Proposition 25 (this element was also present in [GG7], but it is put to use here more directly). This has several advantages. First, under the given isomorphism between $\pi_1(I_c)$ and $\pi_1(I_n)$, the description of the image of ∂_n viewed as a subgroup of $\pi_1(I_c)$ becomes simpler compared to that obtained by studying $\partial(\mathcal{B})$. Secondly, $\hat{\tau}_n$ corresponds to a generator of the centre of R_{n-1} , namely the full-twist braid of P_{n-1} if $M = \mathbb{S}^2$, and to Θ_{n-1}^{-1} if $M = \mathbb{R}P^2$. Thirdly, under the projection of $\pi_1(M)$ to $P_n(M)$, $\hat{\tau}_n$ is sent to the full-twist braid of $P_n(M)$ that plays an important rôle in the structure of this group. The main result of Section 4 is the following theorem.

THEOREM 3.

$$\text{Im}(\partial_n) = \begin{cases} \langle \partial_n(\tilde{\lambda}_{\tilde{x}_0}), \dots, \partial_n(\tilde{\lambda}_{\tilde{x}_{n-3}}), \partial_n(\tilde{\lambda}_{\tilde{z}_0}), \hat{\tau}_n^2 \rangle & \text{if } M = \mathbb{S}^2 \\ \langle \partial_n(\lambda_{x_0}), \dots, \partial_n(\lambda_{x_{n-2}}), \hat{\tau}_n^2 \rangle & \text{if } M = \mathbb{R}P^2. \end{cases}$$

We underline the fact that the basis of $\text{Im}(\partial_n)$ given by Theorem 3 is coherent with the decomposition $R_{n-1} \times \pi_1(\Omega(\prod_1^{n-1} \mathbb{S}^2))$ of $\pi_1(I_c)$, in the sense that under the given isomorphism between $\pi_1(I_c)$ and $\pi_1(I_n)$, the first $n - 1$ elements correspond to generators of the factors of $\pi_1(\Omega(\prod_1^{n-1} \mathbb{S}^2))$ by Lemma 24(d), and $\hat{\tau}_n^2$ corresponds to a generator of the centre of R_{n-1} by Remarks 26(b). Theorem 3 enables us to reprove several isomorphisms involving $P_n(\mathbb{S}^2)$, and to improve the description of $\Gamma_2(P_n(\mathbb{R}P^2))$ given in [GG5] as follows.

PROPOSITION 4. *Let $M = \mathbb{S}^2$ or $\mathbb{R}P^2$, and let $n \geq n_0$.*

(a) *If $M = \mathbb{S}^2$ then there are isomorphisms:*

$$\text{Ker}((\iota_n)_{\#1}) = P_n(\mathbb{S}^2) \cong P_{n-1} / \langle \Delta_{n-1}^4 \rangle \cong \mathbb{F}_{n-2} \rtimes (\mathbb{F}_{n-3} \rtimes (\cdots \rtimes (\mathbb{F}_3 \rtimes \mathbb{F}_2) \cdots)) \times \mathbb{Z}_2. \quad (6)$$

(b) *If $M = \mathbb{R}P^2$ then there are isomorphisms:*

$$\text{Ker}((\iota_n)_{\#1}) = \Gamma_2(P_n(\mathbb{R}P^2)) \cong G_{n-1} / \langle \Theta_{n-1}^2 \rangle \cong \mathbb{F}_{2n-3} \rtimes (\mathbb{F}_{2n-5} \rtimes (\cdots \rtimes (\mathbb{F}_5 \rtimes \mathbb{F}_3) \cdots)) \times \mathbb{Z}_2. \quad (7)$$

In both cases, the \mathbb{Z}_2 -factor of $\text{Ker}((\iota_n)_{\#1})$ corresponds to the subgroup $\langle \Delta_n^2 \rangle$ of $P_n(M)$.

The result of part (a) is in agreement with [GG5, equation (2)].

Finally, in an Appendix, we prove Proposition 29 that relates the homotopy fibres of fibre spaces and certain subspaces, and that implies that one of the maps that appears in between I_c and I_n is indeed a homotopy equivalence. This result seems to be well known to the experts in the field, but we were not able to find a proof in the literature.

The paper [GG7] is an extended version of this manuscript. As well as containing more detailed proofs of Propositions 19 and 20, as mentioned above, the image of the image by ∂_n of the element $\tilde{\lambda}_{-\tilde{z}_0}$ (resp. λ_{z_0}) of \mathcal{B} if $M = \mathbb{S}^2$ (resp. if $M = \mathbb{R}P^2$) is determined (see Remarks 26(a)), and a few other results that are not relevant to this paper are given there. With respect to [GG7], the approach taken in Section 4.2 of this paper allows us to describe the image of ∂_n in Theorem 3 and to obtain Proposition 4 more directly.

Some of the other results and constructions of this paper have since been generalised in [GGG]. More precisely, if X is a topological manifold without boundary of dimension at least three, under certain conditions, the homotopy type of the homotopy fibre of the inclusion map $\iota_n: F_n(X) \longrightarrow \prod_1^n X$ was determined, and was used to study the cases where either the universal covering of X is contractible, or X is an orbit space \mathbb{S}^k/G of a tame, free action of a Lie group G on the k -sphere \mathbb{S}^k . A description of the long exact sequence in homotopy of the homotopy fibration of ι_n , similar to that of Corollary 2, was given in the case $X = \mathbb{S}^k/G$, where the group G is finite and k is odd, and the boundary homomorphism was described in detail. The authors have also written a survey that summarises the current situation, and that includes some questions and open problems about graph and (orbit) configuration spaces [GG6].

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2 Properties of ι_n and $(\iota_n)_{\#k}$

In this section, we determine some properties of ι_n and of $(\iota_n)_{\#k}$, where $k \in \mathbb{N}$. If X and Y are topological spaces, and $f, g: X \longrightarrow Y$ are maps between X and Y , we write $f \simeq g$ if f and g are homotopic, and we denote the homotopy class of f by $[f]$.

We first state the following description of the homotopy type of the universal covering of the configuration spaces of \mathbb{S}^2 and $\mathbb{R}P^2$.

THEOREM 5 ([BCP, FZ], [GG3, Proposition 10(b)]). *Let $M = \mathbb{S}^2$ or $\mathbb{R}P^2$, and let $n \in \mathbb{N}$. Then the universal covering $\widetilde{F_n(M)}$ of $F_n(M)$ has the homotopy type of \mathbb{S}^2 if $M = \mathbb{S}^2$ and $n \leq 2$ or if $M = \mathbb{R}P^2$ and $n = 1$, and has the homotopy type of the 3-sphere \mathbb{S}^3 otherwise.*

REMARK 6. If $M = \mathbb{S}^2$ (resp. $M = \mathbb{R}P^2$), then by [FVB, Corollary, page 244] (resp. [VB, Corollary, page 82]), $\pi_2(F_n(M))$ is trivial for all $n \geq 3$ (resp. $n \geq 2$).

Let $M = \mathbb{S}^2$ or $\mathbb{R}P^2$ and let $n \geq 2$. For $i \in \{1, \dots, n\}$, let $p_i: F_n(M) \longrightarrow M$ and $\tilde{p}_i: \prod_1^n M \longrightarrow M$ denote the respective projections onto the i^{th} coordinate. If $1 \leq i < j \leq n$, let $\alpha_{i,j}: F_n(M) \longrightarrow F_2(M)$ and $\tilde{\alpha}_{i,j}: \prod_1^n M \longrightarrow \prod_1^2 M$ denote the respective projections onto the i^{th} and j^{th} coordinates. Observe that the maps p_i , \tilde{p}_i , $\alpha_{i,j}$ and $\tilde{\alpha}_{i,j}$ are fibrations, and that:

$$p_i = \tilde{p}_i \circ \iota_n \text{ and } \iota_2 \circ \alpha_{i,j} = \tilde{\alpha}_{i,j} \circ \iota_n. \quad (8)$$

Let $1 \leq i < j \leq n$. As maps from $F_n(M)$ to M , we have $\tilde{p}_i \circ \iota_n = p_1 \circ \alpha_{i,j}$ and $\tilde{p}_j \circ \iota_n = \tilde{p}_2 \circ \iota_2 \circ \alpha_{i,j}$, and using (8), we obtain the following commutative diagram:

$$\begin{array}{ccc}
 F_n(M) & \xrightarrow{\alpha_{i,j}} & F_2(M) \\
 \downarrow \iota_n & \searrow p_i & \swarrow p_1 \\
 & & M \\
 & \nearrow \tilde{p}_i & \nwarrow \tilde{p}_1 \\
 \prod_1^n M & \xrightarrow{\tilde{\alpha}_{i,j}} & \prod_1^2 M \\
 \downarrow \tilde{p}_j & & \downarrow \tilde{p}_2 \\
 M & \xlongequal{\quad} & M
 \end{array}
 \tag{9}$$

In all of what follows, let $\tau: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ denote the antipodal map defined by $\tau(x) = -x$ for all $x \in \mathbb{S}^2$, and let $\pi: \mathbb{S}^2 \rightarrow \mathbb{R}P^2$ denote the universal covering.

LEMMA 7. Let $M = \mathbb{S}^2$ or $\mathbb{R}P^2$, and let $n \geq 2$.

- (a) Let $1 \leq i \leq n$. If either $k \geq 3$, or $k = n = 2$ and $M = \mathbb{S}^2$, the homomorphism $(p_i)_{\#k}: \pi_k(F_n(M)) \rightarrow \pi_k(M)$ is an isomorphism.
(b) Let $k \geq 2$. Then the homomorphism $(\iota_n)_{\#k}: \pi_k(F_n(M)) \rightarrow \pi_k(\prod_1^n M)$ is injective.

REMARK 8. Let n and k be as in Lemma 7(a). Theorem 5 implies that $\pi_k(F_n(M)) \cong \pi_k(M)$, and the lemma provides an explicit isomorphism.

Proof of Lemma 7. Let $n, k \geq 2$.

- (a) Let $1 \leq i \leq n$. Then $p_i: F_n(M) \rightarrow M$ is a Fadell-Neuwirth fibration whose fibre, which is an Eilenberg-Mac Lane space of type $K(\pi, 1)$, may be identified with $F_{n-1}(M \setminus \{x_i\})$. The result follows by taking the long exact sequence in homotopy of this fibration (note that the fibre is contractible if $M = \mathbb{S}^2$ and $n = 2$).
(b) If $k \geq 3$, or $k = n = 2$ and $M = \mathbb{S}^2$ then the statement is a consequence of part (a) and (8). So assume that $k = 2$ and that $n \neq 2$ if $M = \mathbb{S}^2$. The result is a consequence of Remark 6. \square

As we shall now see, the homomorphism $(\iota_n)_{\#k}: \pi_k(F_n(M)) \rightarrow \pi_k(\prod_1^n M)$ is diagonal for most values of n and k , with the exception of one case when it is anti-diagonal.

PROPOSITION 9. Let $n, k \geq 2$.

- (a) If $n = k = 2$ then the homomorphism $(\iota_2)_{\#2}: \pi_2(F_2(\mathbb{S}^2)) \rightarrow \pi_2(\prod_{i=1}^2 \mathbb{S}^2)$ is the anti-diagonal homomorphism that sends a generator of $\pi_2(F_2(\mathbb{S}^2))$ to $(1, -1)$ or to $(-1, 1)$ depending on the choice of orientation of \mathbb{S}^2 .
(b) If $n \geq 2$ and $k \geq 3$ then $(\iota_n)_{\#k}: \pi_k(F_n(\mathbb{S}^2)) \rightarrow \pi_k(\prod_1^n \mathbb{S}^2)$ is a diagonal homomorphism.

Proof. Let $n = k = 2$ (resp. $n \geq 2$ and $k \geq 3$). In part (a) (resp. (b)), we shall show that $(\iota_n)_{\#k}: \pi_k(F_n(\mathbb{S}^2)) \rightarrow \pi_k(\prod_1^n \mathbb{S}^2)$ is an anti-diagonal (resp. diagonal) homomorphism. To do so, it suffices to prove that $(\tilde{p}_1)_{\#2} \circ (\iota_n)_{\#2} = -(\tilde{p}_2)_{\#2} \circ (\iota_n)_{\#2}$ (resp. $(\tilde{p}_i)_{\#k} \circ (\iota_n)_{\#k} = (\tilde{p}_j)_{\#k} \circ (\iota_n)_{\#k}$ for all $1 \leq i < j \leq n$), which is equivalent to showing that $(p_1)_{\#2} = -(p_2)_{\#2}$ (resp. $(p_i)_{\#k} = (p_j)_{\#k}$).

- (a) Suppose that $n = k = 2$. Thus $\pi_2(F_2(\mathbb{S}^2)) \cong \pi_2(\mathbb{S}^2) \cong \mathbb{Z}$, and for $i = 1, 2$, p_i is a homotopy equivalence, with homotopy inverse $s_i: \mathbb{S}^2 \rightarrow F_2(\mathbb{S}^2)$, where $s_1(x) =$

$(x, -x)$ and $s_2(x) = (-x, x)$ [GG4, page 859]. Since $p_1 \circ s_1 = \text{Id}_{\mathbb{S}^2}$ and $p_2 \circ s_1 = \tau$, we have $(s_1)_{\#2} = ((p_1)_{\#2})^{-1}$, $(p_2)_{\#2} \circ ((p_1)_{\#2})^{-1} = \tau_{\#2}$, and $(p_2)_{\#2} = \tau_{\#2} \circ (p_1)_{\#2} = -(p_1)_{\#2}$ since τ is of degree -1 . Now $\pi_2(F_2(\mathbb{S}^2)) \cong \pi_2(\mathbb{S}^2) \cong \mathbb{Z}$, and once we have fixed an orientation of \mathbb{S}^2 and a generator of $\pi_2(F_2(\mathbb{S}^2))$, $(\iota_2)_{\#2}$ sends this generator to $(1, -1)$ or to $(-1, 1)$ using (8) and Lemma 7(a).

(b) Let $k \geq 3$. To compare the homomorphisms $(p_i)_{\#k}: \pi_k(F_n(\mathbb{S}^2)) \rightarrow \pi_k(\mathbb{S}^2)$, where $i \in \{1, \dots, n\}$, we analyse the elements $(p_i)_{\#k}([\alpha])$ of $\pi_k(\mathbb{S}^2)$ for maps $\alpha: \mathbb{S}^k \rightarrow F_n(\mathbb{S}^2)$. We split the proof into two cases.

(i) First suppose that $n = 2$. Let $\beta = p_1 \circ \alpha$, and let s_1 be as in the proof of (a). Then $s_1 \circ \beta = s_1 \circ p_1 \circ \alpha \simeq \alpha$. Since β is a map from \mathbb{S}^k to \mathbb{S}^2 , it follows from the long exact sequence in homotopy of the Hopf fibration $\mathbb{S}^1 \rightarrow \mathbb{S}^3 \xrightarrow{H} \mathbb{S}^2$ that β factors through the Hopf map $H: \mathbb{S}^3 \rightarrow \mathbb{S}^2$, and so there exists $\beta_1: \mathbb{S}^k \rightarrow \mathbb{S}^3$ such that $\beta = H \circ \beta_1$. We thus have the following diagram:

$$\begin{array}{ccc} \mathbb{S}^k & \xrightarrow{\alpha} & F_2(\mathbb{S}^2) \\ & \searrow \beta & \uparrow s_1 \\ & & \mathbb{S}^2 \\ & \downarrow \beta_1 & \uparrow p_1 \\ \mathbb{S}^3 & \xrightarrow{H} & \mathbb{S}^2 \end{array}$$

that is commutative, with the exception of the paths involving s_1 that are commutative up to homotopy. Using the homotopy equivalences of part (a), for $i = 1, 2$, we have:

$$p_i \circ \alpha \simeq p_i \circ s_1 \circ \beta = p_i \circ s_1 \circ H \circ \beta_1 \simeq \tau^{i-1} \circ H \circ \beta_1.$$

But by a result of Hopf (see [A, page 269, Lemma 8.2.5] or [Ho, Satz II and IIIb']), $\tau \circ H \simeq H$, and so $p_1 \circ \alpha = \beta \simeq p_2 \circ \alpha$. Thus $(p_1 \circ \alpha)_{\#k} = (p_2 \circ \alpha)_{\#k}$ for all maps $\alpha: \mathbb{S}^k \rightarrow F_2(\mathbb{S}^2)$, hence $(p_1)_{\#k} = (p_2)_{\#k}$, from which we conclude that $(\iota_2)_{\#k}$ is a diagonal homomorphism.

(ii) Now suppose that $n \geq 3$. To prove that $(\iota_n)_{\#k}$ is diagonal, it suffices to show that the homotopy class $[p_i \circ \alpha]$ in $\pi_k(\mathbb{S}^2)$ of the map $p_i \circ \alpha: \mathbb{S}^k \rightarrow \mathbb{S}^2$ does not depend on i , or equivalently that $[p_j \circ \alpha] = [p_1 \circ \alpha]$ in $\pi_k(\mathbb{S}^2)$ for all $2 \leq j \leq n$. As we saw in the previous paragraph, $(\iota_2)_{\#k}: \pi_k(F_2(\mathbb{S}^2)) \rightarrow \pi_k(\prod_{i=1}^2 \mathbb{S}^2)$ is a diagonal homomorphism, in particular, $(\tilde{p}_1 \circ \iota_2)_{\#k} = (\tilde{p}_2 \circ \iota_2)_{\#k}$. Using this and taking $i = 1$ and $2 \leq j \leq n$ in the commutative diagram (9), for any map $\alpha: \mathbb{S}^k \rightarrow F_n(\mathbb{S}^2)$, we have:

$$[p_1 \circ \alpha] = [\tilde{p}_1 \circ \iota_2 \circ \alpha_{1,j} \circ \alpha] = [\tilde{p}_2 \circ \iota_2 \circ \alpha_{1,j} \circ \alpha] = [p_j \circ \alpha],$$

and thus $(p_1)_{\#k} = (p_j)_{\#k}$ as required. \square

The following proposition is the analogue of Proposition 9 for $\mathbb{R}P^2$.

PROPOSITION 10. *Let $n, k \geq 2$. Then $(\iota_n)_{\#k}: \pi_k(F_n(\mathbb{R}P^2)) \rightarrow \pi_k(\prod_1^n \mathbb{R}P^2)$ is a diagonal homomorphism.*

REMARK 11. Let $M = \mathbb{S}^2$ (resp. $M = \mathbb{R}P^2$). In the cases not covered by Lemma 7(a), namely $k = 2$ and $n \geq 3$ (resp. $n \geq 2$), $(\iota_n)_{\#2}: \pi_2(F_n(M)) \rightarrow \pi_2(\prod_1^n M)$ is trivially diagonal by Remark 6, but the homomorphism $(p_i)_{\#k}$ of Lemma 7(a) is not an isomorphism.

Proof of Proposition 10. Let $k, n \geq 2$. We consider four cases.

(a) If $k = 2$ then $\pi_2(F_n(\mathbb{R}P^2)) = 1$ by Remark 6, and hence $(\iota_n)_{\#2}$ is the trivial (diagonal) homomorphism.

(b) Let $n = 2$ and $k = 3$. By (8), it suffices to show that $(p_1)_{\#3} = (p_2)_{\#3}$. With Theorem 5 in mind, let γ (resp. ρ) denote a generator of the infinite cyclic group $\pi_3(F_3(\mathbb{R}P^2))$ (resp. $\pi_3(\mathbb{R}P^2)$). By Lemma 7(a), $(p_l)_{\#3}: \pi_3(F_3(\mathbb{R}P^2)) \rightarrow \pi_3(\mathbb{R}P^2)$ is an isomorphism for all $1 \leq l \leq 3$, so there exists $\varepsilon_l \in \{1, -1\}$ such that $(p_l)_{\#3}(\gamma) = \varepsilon_l \rho$. Hence there exist $1 \leq i < j \leq 3$ and $\varepsilon \in \{1, -1\}$ such that $(p_i)_{\#3}(\gamma) = (p_j)_{\#3}(\gamma) = \varepsilon \rho$. Consider (9), where we take $n = 3$. The map $\alpha_{i,j}$ is a Fadell-Neuwirth fibration, whose fibre may be identified with $F_1(\mathbb{R}P^2 \setminus \{x_i, x_j\})$, and as in the proof of Lemma 7(a), we see that $(\alpha_{i,j})_{\#3}: \pi_k(F_3(\mathbb{R}P^2)) \rightarrow \pi_k(F_2(\mathbb{R}P^2))$ is an isomorphism, thus $\gamma' = (\alpha_{i,j})_{\#3}(\gamma)$ is a generator of the infinite cyclic group $\pi_3(F_2(\mathbb{R}P^2))$. Taking the commutative diagram (9) on the level of π_3 , we see that:

$$\begin{aligned} \varepsilon \rho &= (p_i)_{\#3}(\gamma) = (p_1 \circ \alpha_{i,j})_{\#3}(\gamma) = (p_1)_{\#3}(\gamma') \text{ and} \\ \varepsilon \rho &= (p_j)_{\#3}(\gamma) = (p_2 \circ \alpha_{i,j})_{\#3}(\gamma) = (p_2)_{\#3}(\gamma'), \end{aligned}$$

from which it follows that $(p_1)_{\#3} = (p_2)_{\#3}$ as required.

(c) Suppose that $n = 2$ and $k \geq 4$. Let $h: F_2(\mathbb{R}P^2) \rightarrow F_2(\mathbb{R}P^2)$ be the universal covering map, and with Theorem 5 in mind, let $f: \mathbb{S}^3 \rightarrow F_2(\mathbb{R}P^2)$ and $g: F_2(\mathbb{R}P^2) \rightarrow \mathbb{S}^3$ be homotopy equivalences. If $i \in \{1, 2\}$ then $p_i \circ h \circ f: \mathbb{S}^3 \rightarrow \mathbb{R}P^2$ lifts to a map $\hat{p}_i: \mathbb{S}^3 \rightarrow \mathbb{S}^2$, and we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{S}^3 & \xrightarrow{\hat{p}_i} & \mathbb{S}^2 \\ \downarrow h \circ f & & \downarrow \pi \\ F_2(\mathbb{R}P^2) & \xrightarrow{p_i} & \mathbb{R}P^2. \end{array} \quad (10)$$

From Lemma 7(a) and the proof of case (b) above, $(p_i)_{\#k}: \pi_k(F_2(\mathbb{R}P^2)) \rightarrow \pi_k(\mathbb{R}P^2)$ is an isomorphism and $(p_i)_{\#3}(\gamma') = \varepsilon \rho$. Considering the commutative diagram (10) on the level of π_3 , $(h \circ f)_{\#3}$ and $\pi_{\#3}$ are also isomorphisms, and it follows that $(\hat{p}_i)_{\#3} = (\pi_{\#3})^{-1} \circ (p_i)_{\#3} \circ (h \circ f)_{\#3}$ is an isomorphism that is independent of i . Hence \hat{p}_i is homotopic to the Hopf map H , and on the level of π_k , we have $(p_i)_{\#k} = \pi_{\#k} \circ H_{\#k} \circ ((h \circ f)_{\#k})^{-1}$ for all $k \geq 3$. We conclude that $(p_1)_{\#k} = (p_2)_{\#k}$ as required.

(d) Finally, if $n, k \geq 3$, it suffices to replace \mathbb{S}^2 by $\mathbb{R}P^2$ in case (b)(ii) of the proof of Proposition 9, and to make use of the fact proved in cases (b) and (c), that for $n = 2$ and all $k \geq 3$, $(\iota_2)_{\#k}$ is a diagonal homomorphism. \square

REMARK 12. To prove the result of Proposition 10 in the case $n = 2$ and $k = 3$, one may make use of [DG, Proposition 2.13] or [K, Theorem 1.17] to show that the pair of maps $(\pi \circ H, \pi \circ H): \mathbb{S}^3 \rightarrow \mathbb{R}P^2 \times \mathbb{R}P^2$ may be deformed to a pair of coincidence-free maps, and so up to homotopy, factors through $F_2(\mathbb{R}P^2)$.

3 The homotopy type of the homotopy fibre of ι_n for $M = \mathbb{S}^2, \mathbb{R}P^2$

The aim of this section is to prove Theorem 1 concerning the homotopy type of the homotopy fibre of the map $\iota_n: F_n(M) \rightarrow \prod_1^n M$, where $M = \mathbb{S}^2$ or $\mathbb{R}P^2$. In Section 3.1, we define much of the basic notation that will be used in the rest of the paper, and we advise the reader to refer to this section frequently during the rest of the manuscript. In Section 3.2, we shall prove Theorem 1 and Corollary 2, and in Section 3.3, we give a

presentation of G_n .

3.1 Generalities and notation

Let X be a topological space, and let $x_0 \in X$ be a basepoint, and let $I = [0, 1]$. If l is a path in X parametrised by $t \in I$ then let l^{-1} denote its inverse, in other words $l^{-1}(t) = l(1 - t)$ for all $t \in I$, and if $x \in X$ then let c_x denote the constant path at x . If l, l' are paths in X such that $l(1) = l'(0)$, then $l * l'$ will denote their concatenation. Let ΩX denote the loop space of (X, x_0) , and if $\omega \in \Omega X$ then let $[\omega]$ denote the associated element of $\pi_1(X, x_0)$. If Y is a topological space and $f: X \rightarrow Y$ is a map, then we shall always take the basepoint of E_f and I_f defined in equations (1) and (2) to be $(x_0, c_{f(x_0)})$.

We define the following notation that will be used throughout the rest of the manuscript. Let \mathbb{S}^2 be the unit sphere in \mathbb{R}^3 , let $\tilde{z}_0 = (0, 0, 1)$, let \mathbb{D} be the open disc $\mathbb{S}^2 \setminus \{-\tilde{z}_0\}$, let C be the open cylinder $\mathbb{S}^2 \setminus \{\tilde{z}_0, -\tilde{z}_0\}$, and let $\tau: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be the antipodal map. We equip \mathbb{S}^2 with spherical coordinates (θ, φ) , where $\theta \in [0, 2\pi)$ is the longitude whose prime meridian passes through \tilde{x}_0 and $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the latitude. Let $n_0 = 3$ (resp. $n_0 = 2$) if $M = \mathbb{S}^2$ (resp. $\mathbb{R}P^2$), let $n \geq n_0$, and for $j \in \{0, 1, \dots, n - n_0\}$, let $\tilde{x}_j = (0, \frac{\pi}{2}(\frac{j}{j+1}))$ and $x_j = \pi(\tilde{x}_j)$, where $\pi: \mathbb{S}^2 \rightarrow \mathbb{R}P^2$ denotes the universal covering map. Let $M = \mathbb{S}^2$ (resp. $M = \mathbb{R}P^2$), and let $\mathbb{M} = \mathbb{D}$ (resp. $\mathbb{M} = C$). We take the basepoint $W_n = (w_1, \dots, w_n)$ of both $F_n(M)$ and $\prod_1^n M$ to be $(\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{n-3}, \tilde{z}_0, -\tilde{z}_0)$ (resp. $(x_0, x_1, \dots, x_{n-2}, z_0)$), where $z_0 = \pi(\tilde{z}_0)$, and the basepoint of both $F_{n-1}(M \setminus \{w_n\})$ and $\prod_{i=1}^{n-1} M$ to be $W'_n = (w_1, \dots, w_{n-1})$. Let $G = \{\text{Id}_{\mathbb{D}}\}$ (resp. $G = \langle \tau | C \rangle$). Then $F_{n-1}^G(\mathbb{M})$ is the usual configuration space $F_{n-1}(\mathbb{D})$ of \mathbb{D} (resp. the orbit configuration space $F_{n-1}^{\langle \tau \rangle}(C)$ of C) that we equip with the basepoint $W''_n = (w''_1, \dots, w''_{n-1})$, where $W''_n = (\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{n-3}, \tilde{z}_0)$ (resp. $W''_n = (\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{n-2})$). So as not to overload the notation, we shall omit these basepoints in much of what follows, and we shall identify $\pi_1(\prod_1^n \Omega M, (c_{w_1}, \dots, c_{w_n}))$ with $\pi_2(\prod_1^n M, W_n)$ in the usual manner.

Let E_n and I_n (resp. E'_n and I'_n) denote the mapping path and homotopy fibre respectively of the inclusion $\iota_n: F_n(M) \hookrightarrow \prod_1^n M$ (resp. $\iota'_n: F_{n-1}(M \setminus \{w_n\}) \hookrightarrow \prod_{i=1}^{n-1} M$) that are defined using (1) and (2) by:

$$E_n = \left\{ (x, \lambda) \in F_n(M) \times \prod_1^n M \mid \lambda(0) = x \right\}$$

$$I_n = \{ (x, \lambda) \in E_n \mid \lambda(1) = W_n \} \tag{11}$$

$$E'_n = \left\{ (y, \mu) \in F_{n-1}(M \setminus \{w_n\}) \times \prod_{i=1}^{n-1} M \mid \mu(0) = y \right\}$$

$$I'_n = \{ (y, \mu) \in E'_n \mid \mu(1) = W'_n \}. \tag{12}$$

The basepoints are chosen in accordance with the convention of the first paragraph. Let $j_n: I_n \rightarrow E_n$ and $j'_n: I'_n \rightarrow E'_n$ denote the associated inclusions. By [A, Proposition 3.5.8(2) and Remark 3.5.9(1)], the maps $f_n: F_n(M) \rightarrow E_n$ and $g_n: E_n \rightarrow F_n(M)$ (resp. $f'_n: F_{n-1}(M \setminus \{w_n\}) \rightarrow E'_n$ and $g'_n: E'_n \rightarrow F_{n-1}(M \setminus \{w_n\})$) defined by $f_n(x) = (x, c_{\iota_n(x)})$ and $g_n(x, \lambda) = x$ (resp. $f'_n(y) = (y, c_{\iota'_n(y)})$ and $g'_n(y, \mu) = y$) are mutual homotopy equivalences. Let $\alpha_n: E'_n \rightarrow E_n$ be defined by $\alpha_n(y, \mu) = (y, w_n, \mu, c_{w_n})$, let

$\widehat{\alpha}_n: \prod_{i=1}^{n-1} M \longrightarrow \prod_1^n M$ be defined by $\widehat{\alpha}_n(y) = (y, w_n)$, and let

$$\widehat{\alpha}_n \Big|_{F_{n-1}(M \setminus \{w_n\})} : F_{n-1}(M \setminus \{w_n\}) \longrightarrow F_n(M)$$

denote its restriction to the corresponding configuration spaces. We define the map $\widehat{\iota}_n: E_n \longrightarrow \prod_1^n M$ (resp. $\widehat{\iota}'_n: E'_n \longrightarrow \prod_{i=1}^{n-1} M$) by $\widehat{\iota}_n(x, \lambda) = \lambda(1)$ (resp. $\widehat{\iota}'_n(y, \mu) = \mu(1)$). Note that $\widehat{\iota}_n \circ f_n = \iota_n$, and that α_n restricts to a map from I'_n to I_n that we denote by α'_n . Applying [A, pages 91 and 108], we obtain the following commutative diagram:

$$\begin{array}{ccccc} I'_n & \xrightarrow{j'_n} & E'_n & \xrightarrow{\widehat{\iota}'_n} & \prod_{i=1}^{n-1} M \\ \downarrow \alpha'_n & & \downarrow \alpha_n & & \downarrow \widehat{\alpha}_n \\ I_n & \xrightarrow{j_n} & E_n & \xrightarrow{\widehat{\iota}_n} & \prod_1^n M, \end{array} \quad (13)$$

where the rows are fibrations. Let $d'_n: \prod_{i=1}^{n-1} \Omega M \longrightarrow I'_n$ (resp. $d_n: \prod_1^n \Omega M \longrightarrow I_n$) denote the boundary map corresponding to the upper (resp. lower) fibration of (13), and for all $k \geq 1$, let:

$$\partial'_{n,k} = (d'_n)_{\#k}: \pi_k \left(\prod_{i=1}^{n-1} M \right) \longrightarrow \pi_{k-1}(I'_n) \quad (\text{resp. } \partial_{n,k} = (d_n)_{\#k}: \pi_k \left(\prod_1^n M \right) \longrightarrow \pi_{k-1}(I_n))$$

denote the induced boundary homomorphism on the level of π_k . We recall the construction of these maps in the following lemma that will be used in Section 4.

LEMMA 13. *With the above notation, we have $d'_n(\omega) = (w_1, \dots, w_{n-1}, \omega)$ (resp. $d_n(\omega) = (w_1, \dots, w_n, \omega)$) for all $\omega \in \prod_{i=1}^{n-1} \Omega M$ (resp. $\omega \in \prod_1^n \Omega M$).*

Proof. The statement of the lemma is a consequence of the general construction of the boundary map of a fibration, which we now recall. Let $f: (X, x_0) \longrightarrow (Y, y_0)$ be a map between based topological spaces, let I_f denote the homotopy fibre of f , let $EY = \{ \lambda \in Y^I \mid \lambda(1) = y_0 \}$ be the path space of Y , and let $\Omega Y \longrightarrow EY \xrightarrow{P_0} Y$ be the path space fibration [A, page 85], where $P_0(\lambda) = \lambda(0)$. By [A, Proposition 3.3.12 and Definition 3.3.13], we have the following pullback square:

$$\begin{array}{ccc} I_f & \xrightarrow{u} & EY \\ \downarrow v & & \downarrow P_0 \\ X & \xrightarrow{f} & Y, \end{array}$$

where $u(x, \lambda) = \lambda$ and $v(x, \lambda) = x$ for all $(x, \lambda) \in I_f$, v is the principal fibration induced by f , and u induces a homeomorphism between the fibre F_v of v and ΩY whose inverse $u': \Omega Y \longrightarrow F_v$ is defined by $u'(\omega) = (x_0, \omega)$ for all $\omega \in \Omega Y$. The boundary map $d: \Omega Y \longrightarrow I_f$ is then defined by $d = i \circ u'$, where $i: F_v \longrightarrow I_f$ is inclusion, so $d(\omega) = (x_0, \omega)$ for all $\omega \in \Omega Y$. The result then follows by taking $X = F_{n-1}(M \setminus \{w_n\})$, $Y = \prod_{i=1}^{n-1} M$ and $f = \iota'_n$ (resp. $X = F_n(M)$, $Y = \prod_1^n M$ and $f = \iota_n$). \square

The following remarks will play an important rôle in the proof of Theorem 1.

REMARKS 14.

(a) Let $p: E \longrightarrow B$ be a fibration with fibre F , let E' be a subspace of E , and suppose that the restriction $p' = p|_{E'}: E' \longrightarrow B$ is also a fibration, with fibre F' . Let $\iota: E' \longrightarrow E$ denote inclusion, and let $\iota' = \iota|_{F'}: F' \longrightarrow F$ denote the restriction of ι to F' . The inclusions of the fibres in the corresponding total spaces induce an explicit homotopy

equivalence between the homotopy fibres I_l and $I_{l'}$ of ι and ι' respectively. Although this result seems to be folklore and is well known to the experts in the field, we were unable to locate a proof in the literature, and so we give one in Proposition 29 in the Appendix of this paper. A consequence of this proposition is that the map $\alpha'_n: I'_n \rightarrow I_n$ is a homotopy equivalence (see Corollary 30).

(b) Suppose that $f: X \rightarrow Y$ is null homotopic, and let $c: X \rightarrow Y$ denote the constant map that sends the whole of X onto y_0 . By [A, Proposition 3.3.17], I_f and I_c have the same homotopy type. On the other hand, $I_c = \{(x, \lambda) \in X \times Y^I \mid \lambda(0) = \lambda(1) = y_0\} = X \times \Omega Y$. Therefore I_f has the same homotopy type as $X \times \Omega Y$.

Let $M = \mathbb{S}^2$ (resp. $M = \mathbb{R}P^2$), and let $\iota_c: F_{n-1}^G(\mathbb{M}) \rightarrow \prod_{i=1}^{n-1} \mathbb{S}^2$ be the constant map that sends every point of $F_{n-1}^G(\mathbb{M})$ to W_n'' , which we take to be the basepoint of $\prod_{i=1}^{n-1} \mathbb{S}^2$ at this level. By Remarks 14(b), its homotopy fibre is given by:

$$I_c = F_{n-1}^G(\mathbb{M}) \times \Omega \left(\prod_{i=1}^{n-1} \mathbb{S}^2 \right), \quad (14)$$

which we equip with the basepoint W_c in accordance with the convention given in the first paragraph of this section. Let I_n'' be the homotopy fibre of the inclusion map $l_n'': F_{n-1}^G(\mathbb{M}) \rightarrow \prod_{i=1}^{n-1} \mathbb{S}^2$ defined by $l_n''(u) = u$ for all $u \in F_{n-1}^G(\mathbb{M})$, and also equipped with the basepoint W_c . So:

$$I_n'' = \left\{ (u, \mu) \in F_{n-1}^G(\mathbb{M}) \times \left(\prod_{i=1}^{n-1} \mathbb{S}^2 \right)^I \mid \mu(0) = u \text{ and } \mu(1) = W_n'' \right\}. \quad (15)$$

If $M = \mathbb{S}^2$, then $l_n'' = l_n'$, so $I_n'' = I_n'$ by (12). Let:

$$h_n: I_c \rightarrow F_{n-1}^G(\mathbb{M}) \text{ and } h_n'': I_n'' \rightarrow F_{n-1}^G(\mathbb{M})$$

denote the projections onto the first factor, and let $\hat{\pi}: \mathbb{S}^2 \rightarrow M$ be the identity $\text{Id}_{\mathbb{S}^2}$ (resp. the universal covering $\pi: \mathbb{S}^2 \rightarrow \mathbb{R}P^2$). Then $\hat{\pi}$ restricts to a map from \mathbb{D} (resp. from C) to $M \setminus \{w_n\}$ that we also denote by $\hat{\pi}$. Let $\alpha_\pi: I_n'' \rightarrow I_n'$ be defined by:

$$\alpha_\pi(y_0, \dots, y_{n-2}, l_0, \dots, l_{n-2}) = (\hat{\pi}(y_0), \dots, \hat{\pi}(y_{n-2}), \hat{\pi} \circ l_0, \dots, \hat{\pi} \circ l_{n-2}). \quad (16)$$

Note that if $M = \mathbb{S}^2$ then α_π is the identity. Let $p: \prod_{i=1}^{n-1} \mathbb{S}^2 \rightarrow \prod_{i=1}^{n-1} M$ denote the identity (resp. the 2^{n-1} -fold universal covering), and let $p': F_{n-1}^G(\mathbb{M}) \rightarrow F_{n-1}(M \setminus \{w_n\})$ denote the restriction of p to $F_{n-1}^G(\mathbb{M})$. If $\alpha_c: I_c \rightarrow I_n''$ is a pointed map that satisfies $h_n'' \circ \alpha_c(u, \mu) = u$ for all $(u, \mu) \in I_c$, using the definitions, one may check that the following diagram of pointed spaces and maps is commutative:

$$\begin{array}{ccccc} I_c & \xrightarrow{h_n} & F_{n-1}^G(\mathbb{M}) & \xrightarrow{\iota_c} & \prod_{i=1}^{n-1} \mathbb{S}^2 \\ \downarrow \alpha_c & & \downarrow \text{Id}_{F_{n-1}^G(\mathbb{M})} & & \downarrow \text{Id}_{\prod_{i=1}^{n-1} \mathbb{S}^2} \\ I_n'' & \xrightarrow{h_n''} & F_{n-1}^G(\mathbb{M}) & \xrightarrow{l_n''} & \prod_{i=1}^{n-1} \mathbb{S}^2 \\ \downarrow \alpha_\pi & & \downarrow p' & & \downarrow p \\ I_n' & \xrightarrow{g_n' \circ j_n'} & F_{n-1}(M \setminus \{w_n\}) & \xrightarrow{l_n'} & \prod_{i=1}^{n-1} M \\ \downarrow \alpha'_n & & \downarrow \hat{\alpha}_n|_{F_{n-1}(M \setminus \{w_n\})} & & \downarrow \hat{\alpha}_n \\ I_n & \xrightarrow{g_n \circ j_n} & F_n(M) & \xrightarrow{l_n} & \prod_{i=1}^n M, \end{array} \quad (17)$$

with the possible exception of the upper right-hand square. In the proof of Theorem 1,

we shall see that this square commutes up to homotopy (see Remark 18), and in Section 4.2, we exhibit a map α_c that satisfies the above condition (see equation (43)).

To end this section, we state and prove the following lemma that will be used to analyse the upper part of the long exact sequence in homotopy of the homotopy fibration of ι_n .

LEMMA 15. *Let A and B be Abelian groups, and let $n \geq 2$. Assume that $\Theta: A \rightarrow \bigoplus_{i=1}^n B$ is an injective homomorphism, and suppose that there exists $i \in \{1, \dots, n\}$ such that the composition $\pi_i \circ \Theta: A \rightarrow B$ is an isomorphism, where $\pi_i: \bigoplus_{i=1}^n B \rightarrow B$ is projection onto the i^{th} factor. Then the following short exact sequence:*

$$1 \longrightarrow A \xrightarrow{\Theta} \bigoplus_{i=1}^n B \xrightarrow{\zeta} (\bigoplus_{i=1}^n B) / \text{Im}(\Theta) \longrightarrow 1 \quad (18)$$

splits, where $\zeta: \bigoplus_{i=1}^n B \rightarrow (\bigoplus_{i=1}^n B) / \text{Im}(\Theta)$ denotes the canonical projection.

Proof. Since the groups that appear in the short exact sequence (18) are Abelian, the existence of a section for ζ is equivalent to that of a section for Θ . If $g: B \rightarrow A$ is inverse of the isomorphism $\pi_i \circ \Theta$ then the map $h: \bigoplus_{i=1}^n B \rightarrow A$ given by $h = g \circ \pi_i$ is a homomorphism and defines a section for Θ . \square

3.2 Proof of Theorem 1

In this section, we prove Theorem 1, from which we shall deduce Corollary 2. Lemma 16 appears in [C, Theorem 2.1.3, part 4]. It will play an important rôle in the proof of Theorem 1(b), and is interesting in its own right. For this reason, we give an alternative proof of this result.

LEMMA 16. *For all $n \in \mathbb{N}$, the space $F_n^{\langle \tau \rangle}(C)$ is a $K(G_n, 1)$, and G_n is isomorphic to an iterated semi-direct product of the form $\mathbb{F}_{2n-1} \rtimes (\mathbb{F}_{2n-3} \rtimes (\dots \rtimes (\mathbb{F}_3 \rtimes \mathbb{Z}) \dots))$, where for $i = 1, \dots, n-1$, \mathbb{F}_{2i+1} is a free group of rank $2i+1$.*

REMARK 17. This description of G_n is similar to that of Artin combing for the Artin pure braid groups. The actions may be determined using Proposition 19 in Section 3.3.

Proof of Lemma 16. If $n = 1$ then $F_1^{\langle \tau \rangle}(C) = C$, and the result is clear. So suppose that $n \geq 2$. The proof is similar to that for the configuration spaces of the disc, and makes use of the Fadell-Neuwirth fibrations. The map $q_n: F_n^{\langle \tau \rangle}(C) \rightarrow F_{n-1}^{\langle \tau \rangle}(C)$ defined by forgetting the last coordinate is well defined, and as in [FaN], may be seen to be a locally-trivial fibration (see also [Fa, Theorem 2.2] and [CX, Lemma 4]). The fibre over the basepoint $W_n'' = (\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{n-2})$ of $F_n^{\langle \tau \rangle}(C)$ may be identified with the set $C \setminus \{\pm \tilde{x}_0, \dots, \pm \tilde{x}_{n-2}\}$. Furthermore, q_n admits a section s_n given by choosing (for example) a point sufficiently close to \tilde{z}_0 in a continuous manner (see [Fa, Theorem 3.2] and Remark 21 for two different possibilities for s_n). Since the fibre has the homotopy type of a wedge of circles, taking the long exact sequence in homotopy of the fibration q_n , we see that $(q_n)_{\#k}: \pi_k(F_n^{\langle \tau \rangle}(C)) \rightarrow \pi_k(F_{n-1}^{\langle \tau \rangle}(C))$ is an isomorphism for all $k \geq 3$ and is injective if $k = 2$, and that there exists a split short exact sequence:

$$1 \longrightarrow \pi_1(C \setminus \{\pm \tilde{x}_0, \dots, \pm \tilde{x}_{n-2}\}) \hookrightarrow G_n \begin{array}{c} \xrightarrow{(q_n)_{\#1}} \\ \xleftarrow{(s_n)_{\#1}} \end{array} G_{n-1} \longrightarrow 1. \quad (19)$$

Induction on n then shows that $\pi_k(F_n^{\langle \tau \rangle}(C))$ is trivial for all $k \geq 2$. The first part of the statement then follows. By induction on n , equation (19), the isomorphism $G_1 \cong \mathbb{Z}$ and the fact that $\pi_1(C \setminus \{\pm \tilde{x}_0, \dots, \pm \tilde{x}_{n-2}\}) \cong \mathbb{F}_{2n-1}$, we obtain the isomorphism given in the second part of the statement. \square

Proof of Theorem 1. Let $n \geq 2$. We make use of the notation and basepoints of Section 3.1. By Corollary 30, I_n and I'_n have the same homotopy type.

(a) Suppose first that $M = \mathbb{S}^2$. We claim that the map $l'_n: F_{n-1}(\mathbb{S}^2 \setminus \{w_n\}) \hookrightarrow \prod_{i=1}^{n-1} \mathbb{S}^2$ is null homotopic. To see this, observe that for $1 \leq i \leq n-1$, the following diagram is commutative:

$$\begin{array}{ccc} F_{n-1}(\mathbb{S}^2 \setminus \{w_n\}) & \xrightarrow{l'_n} & \prod_1^{n-1} \mathbb{S}^2 \\ \downarrow p_i & & \downarrow \tilde{p}_i \\ \mathbb{S}^2 \setminus \{w_n\} & \xrightarrow{l'_2} & \mathbb{S}^2, \end{array}$$

where p_i and \tilde{p}_i are the projections of Section 2. The contractibility of $\mathbb{S}^2 \setminus \{w_n\}$ implies that the composition $l'_2 \circ p_i$ is null homotopic, so the composition $\tilde{p}_i \circ l'_n$ is null homotopic for all $1 \leq i \leq n-1$, and hence l'_n is null homotopic, which proves the claim. The first part of Theorem 1(a) follows by applying Remarks 14(b) to l'_n and using (14). The equivalence with the second part is clear using the fact that $F_{n-1}(\mathbb{D})$ is an Eilenberg-Mac Lane space of type $K(P_{n-1}, 1)$ [FaN].

(b) Let $M = \mathbb{R}P^2$. Considering $F_{n-1}(\mathbb{R}P^2 \setminus \{w_n\})$ to be a subspace of $\prod_1^{n-1} \mathbb{R}P^2$ via the inclusion l'_n , one may check that $F_{n-1}^{\langle \tau \rangle}(C) = p^{-1}(F_{n-1}(\mathbb{R}P^2 \setminus \{w_n\}))$, and that the map $p': F_{n-1}^{\langle \tau \rangle}(C) \rightarrow F_{n-1}(\mathbb{R}P^2 \setminus \{w_n\})$ is also a regular 2^{n-1} -fold covering map. Now consider the middle right-hand commutative square of (17). We claim that I'_n and I''_n have the same homotopy type. To see this, first note that since p is the universal covering map and $(l'_n)_{\#1}$ is surjective, it follows using standard properties of covering spaces that $p'_{\#1}: \pi_1(F_{n-1}^{\langle \tau \rangle}(C)) \rightarrow \text{Ker}((l'_n)_{\#1})$ is an isomorphism. Consider the homotopy fibrations:

$$I''_n \xrightarrow{h''_n} F_{n-1}^{\langle \tau \rangle}(C) \xrightarrow{l''_n} \prod_1^{n-1} \mathbb{S}^2 \text{ and } I'_n \xrightarrow{g'_n \circ j'_n} F_{n-1}(\mathbb{R}P^2 \setminus \{w_n\}) \xrightarrow{l'_n} \prod_1^{n-1} \mathbb{R}P^2 \quad (20)$$

of the second and third rows of (17), and for $k \in \mathbb{N}$, let $\partial''_{n,k}: \pi_k(\prod_{i=1}^{n-1} \mathbb{S}^2) \rightarrow \pi_{k-1}(I''_n)$ denote the boundary homomorphism corresponding to the first of these. Using the commutativity of these two rows, we obtain a map $\zeta: I''_n \rightarrow I'_n$ of homotopy fibres given by $\zeta(x, \lambda) = (p'(x), p \circ \lambda)$ for all $(x, \lambda) \in I''_n$ that satisfies $p' \circ h''_n = (g'_n \circ j'_n) \circ \zeta$, and taking the long exact sequence in homotopy of the homotopy fibrations (20), we obtain the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} \dots \rightarrow \pi_k(I''_n) & \xrightarrow{(h''_n)_{\#k}} & \pi_k(F_{n-1}^{\langle \tau \rangle}(C)) & \xrightarrow{(l''_n)_{\#k}} & \pi_k(\prod_1^{n-1} \mathbb{S}^2) & \xrightarrow{\partial''_{n,k}} & \pi_{k-1}(I''_n) \rightarrow \dots \\ \zeta_{\#k} \downarrow & & \downarrow p'_{\#k} & & \downarrow p_{\#k} & & \downarrow \zeta_{\#(k-1)} \\ \dots \rightarrow \pi_k(I'_n) & \xrightarrow{(g'_n \circ j'_n)_{\#k}} & \pi_k(F_{n-1}(\mathbb{R}P^2 \setminus \{w_n\})) & \xrightarrow{(l'_n)_{\#k}} & \pi_k(\prod_1^{n-1} \mathbb{R}P^2) & \xrightarrow{\partial'_{n,k}} & \pi_{k-1}(I'_n) \rightarrow \dots \end{array} \quad (21)$$

Now $F_{n-1}(\mathbb{R}P^2 \setminus \{w_n\})$ is an Eilenberg-Mac Lane space of type $K(\pi, 1)$, so by Lemma 16, $F_{n-1}^{\langle \tau \rangle}(C)$ is too. Since the homomorphism $p_{\#k}$ is an isomorphism for all $k \geq 2$, it follows from (21) that $\zeta_{\#k}$ is an isomorphism for all $k \geq 2$. Studying the tail of (21), and using

the surjectivity of $(l'_n)_{\#1}$ and the fact that $F_{n-1}^{\langle \tau \rangle}(C)$ and $F_{n-1}(\mathbb{R}P^2 \setminus \{w_n\})$ are path connected, we deduce that $\pi_0(I'_n)$ and $\pi_0(I''_n)$ consist of a single point, so $\zeta_{\#0}$ is a bijection. Finally, from the remaining part of the long exact sequence in homotopy, we have:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_2(\prod_1^{n-1} \mathbb{S}^2) & \xrightarrow{\partial''_{n,2}} & \pi_2(I''_n) & \xrightarrow{(h''_n)_{\#1}} & \pi_1(F_{n-1}^{\langle \tau \rangle}(C)) & \longrightarrow & 1 \\ & & \downarrow p_{\#2} & & \zeta_{\#1} \downarrow & & \downarrow p'_{\#1} & & \downarrow \\ 1 & \longrightarrow & \pi_2(\prod_1^{n-1} \mathbb{R}P^2) & \xrightarrow{\partial'_{n,2}} & \pi_1(I'_n) & \xrightarrow{(g'_n \circ j'_n)_{\#1}} & \pi_1(F_{n-1}(\mathbb{R}P^2 \setminus \{w_n\})) & \xrightarrow{(l'_n)_{\#1}} & \pi_1(\prod_1^{n-1} \mathbb{R}P^2) \longrightarrow 1, \end{array}$$

and since $p'_{\#1}$ is an isomorphism, we obtain the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_2(\prod_1^{n-1} \mathbb{S}^2) & \xrightarrow{\partial''_{n,2}} & \pi_2(I''_n) & \xrightarrow{(h''_n)_{\#1}} & \pi_1(F_{n-1}^{\langle \tau \rangle}(C)) & \longrightarrow & 1 \\ & & \cong \downarrow p_{\#2} & & \zeta_{\#1} \downarrow & & \cong \downarrow p'_{\#1} & & \\ 1 & \longrightarrow & \pi_2(\prod_1^{n-1} \mathbb{R}P^2) & \xrightarrow{\partial'_{n,2}} & \pi_1(I'_n) & \xrightarrow{(g'_n \circ j'_n)_{\#1}} & \text{Ker}((l'_n)_{\#1}) & \longrightarrow & 1. \end{array}$$

The 5-Lemma implies that $\zeta_{\#1}$ is an isomorphism. We conclude that $\zeta_{\#k}$ is an isomorphism for all $k \geq 0$. Now by [A, pages 204–5] or [M, page 247], the homotopy fibre I_f of a map $f: X \rightarrow Y$ between two topological spaces X and Y is the standard homotopy pullback of the spaces and maps $* \rightarrow Y \xleftarrow{f} X$, and by [M, Lemma 36], if X and Y each have the homotopy type of a CW-complex, then I_f has the homotopy type of a CW-complex. Applying this to the maps l'_n and l''_n , we conclude that I'_n and I''_n each have the homotopy type of a CW-complex. Whitehead's theorem then implies that they have the same homotopy type, which proves the claim.

It remains to determine the homotopy type of I''_n . Let $1 \leq i \leq n-1$, and consider the following commutative diagram:

$$\begin{array}{ccc} F_{n-1}^{\langle \tau \rangle}(C) & \xrightarrow{l''_n} & \prod_1^{n-1} \mathbb{S}^2 \\ p_i \Big|_{F_{n-1}^{\langle \tau \rangle}(C)} \downarrow & & \downarrow \tilde{p}_i \\ C & \xrightarrow{l''_2} & \mathbb{S}^2, \end{array}$$

where p_i and \tilde{p}_i are the projections of Section 2. The composition $l''_2 \circ p_i|_{F_{n-1}^{\langle \tau \rangle}(C)}$ is null homotopic because C has the homotopy type of a circle and \mathbb{S}^2 is simply connected, so the composition $\tilde{p}_i \circ l''_2$ is null homotopic, from which it follows that l''_n is also null homotopic. So I''_n has the homotopy type of $F_{n-1}^{\langle \tau \rangle}(C) \times \Omega(\prod_1^{n-1} \mathbb{S}^2)$ by Remarks 14(b) and (14), which proves the first part of the statement of part (b). The second part then follows from Lemma 16. \square

REMARK 18. In the course of the proof of Theorem 1, we showed that:

- (a) I'_n and I''_n have the same homotopy type.
- (b) l''_n is null homotopic (recall that $l''_n = l'_n$ if $M = \mathbb{S}^2$). From this, it follows that:
 - (i) the upper right-hand square of the diagram (17) commutes up to homotopy, and so the same is true for the whole diagram.
 - (ii) I''_n and I_c have the same homotopy type by [A, Proposition 3.3.17].

In Lemma 24, we shall exhibit explicit homotopy equivalences between I_c , I'_n and I''_n .

We now prove Corollary 2.

Proof of Corollary 2. We make use of the notation of Section 3.1. Let $M = \mathbb{S}^2$ or $\mathbb{R}P^2$, and let $n \geq 2$ and $k \geq 1$. Taking the long exact sequence in homotopy of the lower fibration of (13), and using the fact that $\hat{\iota}_n \circ f_n = \iota_n$, we obtain the following long exact sequence:

$$\dots \longrightarrow \pi_k(I_n) \xrightarrow{(g_n \circ j_n)_{\#k}} \pi_k(F_n(M)) \xrightarrow{(\iota_n)_{\#k}} \pi_k \left(\prod_1^n M \right) \xrightarrow{\partial_{n,k}} \pi_{k-1}(I_n) \longrightarrow \dots \quad (22)$$

(a) Suppose that $k \geq 3$ and $n \geq 2$ (resp. $M = \mathbb{S}^2$ and $n = k = 2$). Then $(\iota_n)_{\#k}$ and $(\iota_n)_{\#(k-1)}$ are injective by Lemma 7(b) (resp. $(\iota_2)_{\#2}$ is injective by Lemma 7(b) and $P_2(\mathbb{S}^2)$ is trivial), from which we obtain the short exact sequence (4), where $\pi_{k-1}(I_n)$ has been replaced by $\pi_{k-1}(\Omega(\prod_1^{n-1} \mathbb{S}^2))$ via the homotopy equivalences of Theorem 1. We have also used the fact that $F_{n-1}(\mathbb{D}^2)$ and $F_{n-1}^{(\tau)}(C)$ are Eilenberg-Mac Lane spaces of type $K(\pi, 1)$ by [FaN] and Lemma 16 (resp. $P_1(\mathbb{D}^2)$ is trivial). Proposition 9(b) and Proposition 10 (resp. Proposition 9(a)) show that $(\iota_n)_{\#k}$ is diagonal (resp. anti-diagonal). Further, if $i \in \{1, \dots, n\}$, by (8) and Lemma 7(a), $(\tilde{p}_i)_{\#k} \circ (\iota_n)_{\#k}: \pi_k(F_n(M)) \longrightarrow \pi_k(M)$ is an isomorphism, and taking $A = \pi_k(F_n(M))$, $B = \pi_k(M)$, $\Theta = (\iota_n)_{\#k}$ and $\pi_i = (\tilde{p}_i)_{\#k}$ in Lemma 15, it follows that the short exact sequence (4) splits. This proves the first part of the statement in each case. The application of Theorem 5 and standard facts about homotopy groups yields the second short exact sequence of the statement of part (a).

(b) Let $k = 2$ and $n \geq 3$. Since $\pi_2(F_n(M))$ is trivial by Theorem 5, we obtain the exact sequence (5) by applying once more Theorem 1 to equation (22). The fact that $\pi_1(I_n)$ is isomorphic to $R_{n-1} \times \pi_1(\Omega(\prod_1^{n-1} \mathbb{S}^2))$ is a consequence of the homotopy equivalences of Theorem 1. If $M = \mathbb{S}^2$, the sequence becomes short exact, but does not split because the middle group is torsion free and the quotient has torsion. The remaining exact sequences in the statement then follow using standard results about homotopy groups. \square

3.3 Some properties of the group G_n

Let $n \geq 1$. Once more, we make use of the notation of Section 3.1. As we saw in Lemma 16, the orbit configuration space $F_n^{(\tau)}(C)$ is a $K(\pi, 1)$, and its fundamental group G_n is a repeated semi-direct product of free groups. A presentation of G_n is given in [C, Theorem 1.4.3]. In this section, we give an alternative presentation of G_n in Proposition 19 that allows us to show that its centre is infinite cyclic in Proposition 20.

We identify C with the open annulus $(0, 1) \times \mathbb{S}^1$ in the complex plane \mathbb{C} , where we identify \tilde{z}_0 with the origin in \mathbb{C} , and up to this identification, if $z = re^{i\theta} \in C$ then we take its antipodal point z' to be $(1-r)e^{i(\theta+\pi)}$. Let v_1, \dots, v_n be base points in C that lie on the negative part of the x -axis, and let v'_1, \dots, v'_n be the corresponding antipodal points (see Figure 1). The map $q_{n+1}: F_{n+1}^{(\tau)}(C) \longrightarrow F_n^{(\tau)}(C)$ defined by forgetting the last coordinate is well defined, and as in [FaN], may be seen to be a locally-trivial fibration (see also [Fa, Theorem 2.2] and [CX, Lemma 4]). The map $s_{n+1}: F_n^{(\tau)}(C) \longrightarrow F_{n+1}^{(\tau)}(C)$ defined by $s_{n+1}(y_1, \dots, y_n) = (y_1, \dots, y_n, y_{n+1})$, where y_{n+1} is a point sufficiently close to \tilde{z}_0 and lying on the negative x -axis, is a section for q_{n+1} (see for example the proof of [GG1, Theorem 6]). The corresponding antipodal point y'_{n+1} lies on the positive x -axis close to the outer boundary of C , and the induced homomorphism $(s_{n+1})_{\#}: G_n \longrightarrow G_{n+1}$ is a section for $(q_{n+1})_{\#}$. This section is not unique. Another useful choice of section will

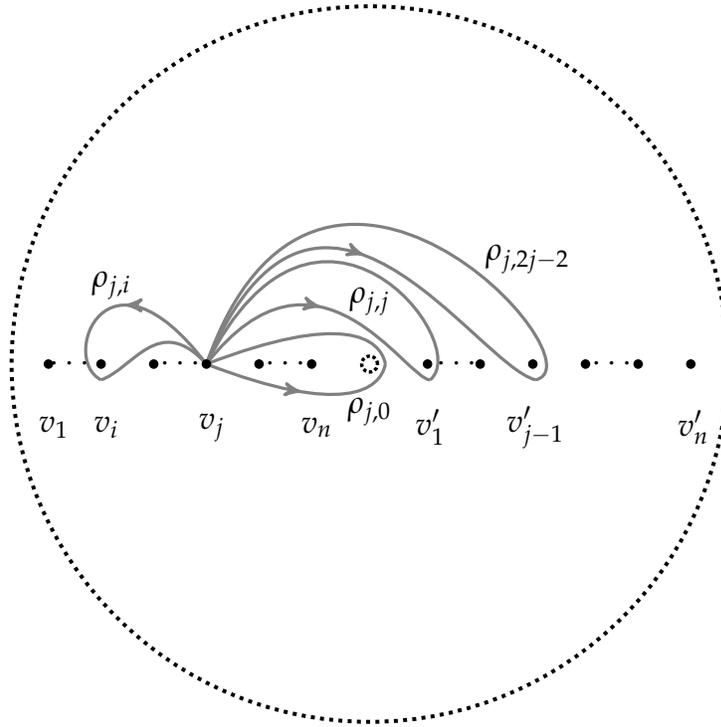


Figure 1: Generators of $\pi_1(C \setminus \{v_1, v'_1, \dots, v_{j-1}, v'_{j-1}\}, v_j)$.

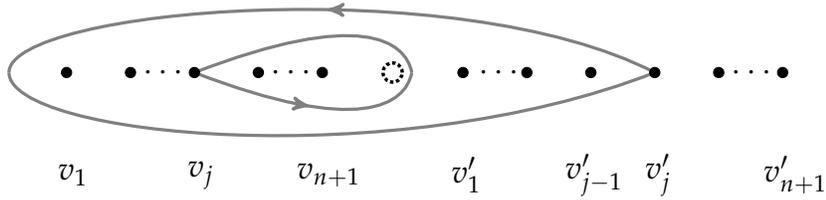


Figure 2: The generator $\rho_{j,0}$.

be described in Remark 21.

If $j \in \mathbb{N}$, the free group $\pi_1(C \setminus \{v_1, v'_1, \dots, v_{j-1}, v'_{j-1}\}, v_j)$ of rank $2j - 1$ admits a basis $\{\rho_{j,i} \mid 0 \leq i \leq 2j - 2\}$, where for $0 \leq i \leq 2j - 2$, we represent the generators $\rho_{j,i}$ by the loops depicted in Figure 1. If $j \leq i \leq 2j - 2$ then $\rho_{j,i}$ is represented by a loop based at v_j that wraps round the point v'_{i-j+1} . If $j = n$ then via the identification of the fibre of q_{n+1} over (v_1, \dots, v_n) with $C \setminus \{v_1, v'_1, \dots, v_n, v'_n\}$, this group is isomorphic to $\text{Ker}((q_{n+1})_\#)$. However, we have to take into account the fact that we are working in the orbit configuration space. If $(y_1, \dots, y_{n+1}) \in F_{n+1}^{\langle \tau \rangle}(C)$ then for all $1 \leq i < j \leq n + 1$, y_i should not only be different from y_j , but it must also avoid y'_j . We thus interpret the fibre as an ‘honest’ subspace of $F_{n+1}^{\langle \tau \rangle}(C)$, and for each $0 \leq i \leq 2j - 2$, we will represent the element $\rho_{j,i}$ by a pair of antipodal loops (see Figures 2–4). With our choice of section, if $j \leq n$, the image of $\rho_{j,i}$ ($\rho_{j,i}$ being considered as an element of G_j) by the homomorphism $(s_n \circ \dots \circ s_{j+1})_\#$ is the element $\rho_{j,i}$ of G_n . Using induction on n , it follows from this and the fact that G_n is isomorphic to the internal semi-direct product $\text{Ker}((q_n)_\#) \rtimes (s_n)_\#(G_{n-1})$ that the group G_n is generated by the set $\bigcup_{j=1}^n \{\rho_{j,i} \mid 0 \leq i \leq 2j - 2\}$. We will shortly give a presentation of G_n , but we first define several elements that will serve to simplify the statement:

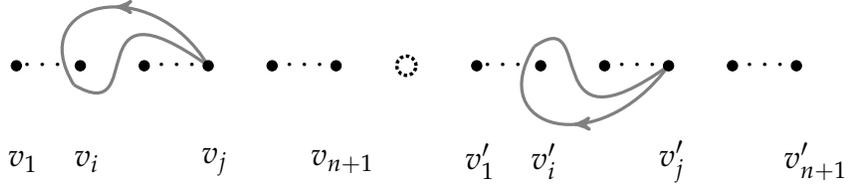


Figure 3: The generator $\rho_{j,i}$, $1 \leq i < j$.

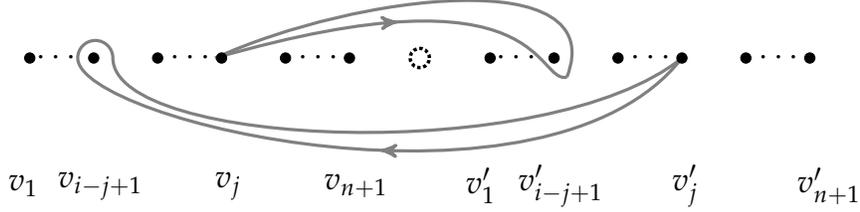


Figure 4: The generator $\rho_{j,i}$, $j \leq i \leq 2j - 2$.

- (a) for all $1 \leq j < k \leq n$, $D_{j,k} = \rho_{k,j}\rho_{k,j+1} \cdots \rho_{k,k-1}$ (see Figure 5).
 (b) for all $1 \leq j < k \leq n$, $C_{k,j} = \rho_{k,0}^{-1}D_{j+1,k}^{-1}\rho_{k,j}D_{j+1,k}\rho_{k,0}$ ($D_{j+1,k}^{-1}\rho_{k,j}D_{j+1,k}$ is the mirror image of $\rho_{k,j}^{-1}$ with respect to the horizontal axis), see Figure 6.
 (c) for all $1 \leq k \leq n$ and $k \leq m < q \leq 2k - 2$, $E_{k,m,q} = \rho_{k,m}\rho_{k,m+1} \cdots \rho_{k,q}$. The element $E_{k,m,q}$ is represented by the loop (and its antipode) based at v_k that winds successively around the points $v'_{m-k+1}, v'_{m-k+2}, \dots, v'_{q-k+1}$ (see Figure 7).

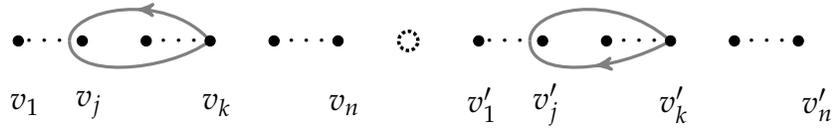


Figure 5: The element $D_{j,k}$, $1 \leq j < k \leq n$.

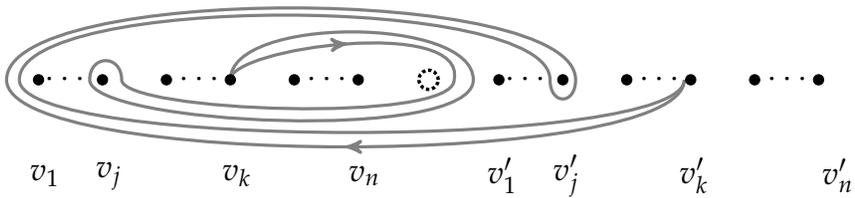


Figure 6: The element $C_{k,j}$, $1 \leq j < k \leq n$.

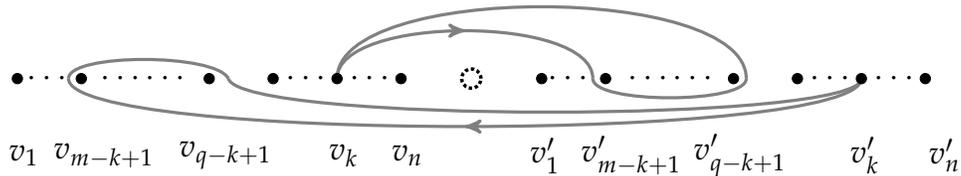


Figure 7: The element $E_{k,m,q}$, $1 \leq k \leq n$ and $k \leq m < q \leq 2k - 2$.

PROPOSITION 19. Let $n \in \mathbb{N}$. The following constitutes a presentation of the group G_n :

generating set: $\bigcup_{j=1}^n \{\rho_{j,i} \mid 0 \leq i \leq 2j - 2\}$.

relations: let $1 \leq j < k \leq n$. Then:

(I)

$$\rho_{j,0}\rho_{k,l}\rho_{j,0}^{-1} = \begin{cases} \rho_{k,l} & \text{if } l = 0 \text{ or } j < l < k \\ C_{k,j}\rho_{k,l}C_{k,j}^{-1} & \text{if } 1 \leq l < j \\ & \text{or if } k \leq l \leq k + j - 2 \\ C_{k,j} & \text{if } l = j \\ C_{k,j}E_{k,k,k+j-2}^{-1}\rho_{k,0}^{-1}D_{1,k}^{-1}\rho_{k,l}D_{1,k}\rho_{k,0}E_{k,k,k+j-2}C_{k,j}^{-1} & \text{if } l = k + j - 1 \\ C_{k,j}\rho_{k,k+j-1}\rho_{k,l}\rho_{k,k+j-1}^{-1}C_{k,j}^{-1} & \text{if } k + j \leq l \leq 2k - 2. \end{cases}$$

(II) for all $1 \leq i < j$, $\rho_{j,i}\rho_{k,l}\rho_{j,i}^{-1} = \rho_{k,l}$ if $0 \leq l \leq i - 1$, $j + 1 \leq l \leq k + i - 2$, $k + i \leq l \leq k + j - 2$ or $k + j \leq l \leq 2k - 2$, and

$$\rho_{j,i}\rho_{k,l}\rho_{j,i}^{-1} = \begin{cases} \rho_{k,j}^{-1}\rho_{k,i}\rho_{k,j} & \text{if } l = i \\ [\rho_{k,j}^{-1}, \rho_{k,i}^{-1}]\rho_{k,l}[\rho_{k,i}^{-1}, \rho_{k,j}^{-1}] & \text{if } i < l < j \\ \rho_{k,j}^{-1}\rho_{k,i}^{-1}\rho_{k,j}\rho_{k,i}\rho_{k,j} & \text{if } l = j \\ E_{k,k+i-1,k+j-1}^{-1}E_{k,k+i,k+j-2}^{-1}\rho_{k,k+i-1}E_{k,k+i,k+j-2}E_{k,k+i-1,k+j-1}^{-1} & \text{if } l = k + i - 1 \\ E_{k,k+i,k+j-2}^{-1}E_{k,k+i-1,k+j-2}\rho_{k,k+j-1}E_{k,k+i-1,k+j-2}^{-1}E_{k,k+i,k+j-2} & \text{if } l = k + j - 1 \end{cases}$$

in the remaining cases.

(III) for all $j \leq i \leq 2j - 2$, $\rho_{j,i}\rho_{k,l}\rho_{j,i}^{-1} = [\rho_{k,j}^{-1}, \rho_{k,k+i-j}^{-1}]\rho_{k,l}[\rho_{k,k+i-j}^{-1}, \rho_{k,j}^{-1}]$ if $1 \leq l \leq i - j$, $i - j + 2 \leq l \leq j - 1$, $k + i - j + 1 \leq l \leq k + j - 2$ or $k + j \leq l \leq 2k - 2$,

$$\rho_{j,i}\rho_{k,i-j+1}\rho_{j,i}^{-1} = [\rho_{k,j}^{-1}, \rho_{k,k+i-j}^{-1}]D_{i-j+1,k}\rho_{k,0}E_{k,k,k+j-1}E_{k,k,k+j-2}^{-1}C_{k,i-j+1}E_{k,k,k+j-2} \cdot \\ E_{k,k,k+j-1}^{-1}\rho_{k,0}^{-1}D_{i-j+1,k}^{-1}[\rho_{k,k+i-j}^{-1}, \rho_{k,j}^{-1}],$$

which is the case $l = i - j + 1$,

$$\rho_{j,i}\rho_{k,k+j-1}\rho_{j,i}^{-1} = [\rho_{k,j}^{-1}, \rho_{k,i-j+1}^{-1}]E_{k,k,k+j-2}^{-1}C_{k,i-j+1}E_{k,k,k+j-2}\rho_{k,k+j-1}E_{k,k,k+j-2}^{-1}C_{k,i-j+1}^{-1} \cdot \\ E_{k,k,k+j-2}[\rho_{k,i-j+1}^{-1}, \rho_{k,j}^{-1}],$$

which is the case $l = k + j - 1$, and

$$\rho_{j,i}\rho_{k,l}\rho_{j,i}^{-1} = \begin{cases} \rho_{k,l} & \text{if } l = 0 \text{ or } j + 1 \leq l \leq k + i - j - 1 \\ \rho_{k,j}^{-1}\rho_{k,k+i-j}\rho_{k,j}\rho_{k,k+i-j}\rho_{k,j} & \text{if } l = j \\ \rho_{k,j}^{-1}\rho_{k,k+i-j}\rho_{k,j} & \text{if } l = k + i - j, \end{cases}$$

in the remaining cases.

Proof. The proof is by induction on n . If $n = 1$ then $G_1 = \langle \rho_{1,0} \rangle \cong \mathbb{Z}$, and the given presentation is valid. So suppose that the result holds for some $n \geq 1$. As we saw above, G_{n+1} is generated by $\bigcup_{j=1}^{n+1} \{\rho_{j,i} \mid 0 \leq i \leq 2j - 2\}$. Applying standard techniques to obtain a presentation of the group extension G_{n+1} using the short exact sequence (19) whose kernel is a free group [J, Proposition 1, page 139], the defining relations are of two types:

(i) the images under $(s_{n+1})\#$ of the relators of the presentation of G_n , rewritten as elements of $\text{Ker}((p_{n+1})\#)$.

(ii) the elements of the form xyx^{-1} , rewritten as elements of $\text{Ker}((p_{n+1})\#)$, where

$$x \in \{\rho_{n+1,l} \mid 0 \leq l \leq 2n\} \text{ and } y \in \bigcup_{j=1}^n \{\rho_{j,i} \mid 0 \leq i \leq 2j-2\}.$$

Since $(s_{n+1})\#(\rho_{j,i}) = \rho_{j,i}$ for all $1 \leq j \leq n$ and $0 \leq i \leq 2j-2$, the relators of G_n are sent to relators of G_{n+1} under $(s_{n+1})\#$, and so the relations of (i) above are those of G_n , but considered as relations in G_{n+1} . This gives rise to all of the relations (I)–(III) for G_{n+1} with $1 \leq j < k \leq n$. It remains to analyse the relations emanating from (ii), which correspond to the relations (I)–(III) for G_{n+1} with $1 \leq j \leq n$ and $k = n+1$. We obtain the following types of relation:

- (I) conjugation of $\rho_{n+1,l}$ by the generator $\rho_{j,0}$ of Figure 2, for $1 \leq j \leq n$ and $0 \leq l \leq 2n$.
- (II) conjugation of $\rho_{n+1,l}$ by the generator $\rho_{j,i}$ of Figure 3, for $1 \leq i < j \leq n$ and $0 \leq l \leq 2n$.
- (III) conjugation of $\rho_{n+1,l}$ by the generator $\rho_{j,i}$ of Figure 4, for $1 \leq j \leq n$ and $j \leq i \leq 2j-2$ and $0 \leq l \leq 2n$.

Using these figures, for all $0 \leq l \leq 2n$ and $0 \leq i \leq 2j-2$, the conjugate of $\rho_{n+1,l}$ by $\rho_{j,i}$ may be rewritten as an element of $\text{Ker}((p_{n+1})\#)$. Although $\rho_{n+1,l}$ is also represented by two loops, the intersections between the two pairs of loops are symmetric, and geometrically it suffices to consider (and remove) the intersections between the two representative loops of $\rho_{j,i}$ with one representative loop of $\rho_{n+1,l}$. The verifications are left to the reader. \square

If $n \in \mathbb{N}$, let $\Theta_n = \rho_{1,0} \cdots \rho_{n,0}$. We now show that the centre $Z(G_n)$ of G_n is infinite cyclic, generated by Θ_n .

PROPOSITION 20. *Let $n \in \mathbb{N}$. Then $Z(G_n) = \langle \Theta_n \rangle$.*

Proof. We outline the proof, leaving the details to the reader (full computations may be found in [GG7, Proposition 20]). Let $n \in \mathbb{N}$, and let $\Theta = \Theta_n$. In what follows, (I)–(III) refer to the corresponding relations of Proposition 19. We start by showing that Θ commutes with each generator $\rho_{k,l}$ of G_n , where $1 \leq k \leq n$ and $0 \leq l \leq 2k-2$, which will imply that $\Theta \in Z(G_n)$. We consider three cases.

(a) Suppose that $l = 0$. By (I), the generators of the form $\rho_{m,0}$, $1 \leq m \leq n$, commute pairwise, so $\rho_{k,0}$ commutes with Θ for all $1 \leq k \leq n$, which proves the result in this case.

(b) Suppose that $1 \leq l < k$. By (I) and (II), $\rho_{k,l}$ and $\rho_{m,0}$ commute if $m < l$ or if $k < m$, and using the fact that the $\rho_{m,0}$, $1 \leq m \leq n$, commute pairwise, we see that:

$$\Theta \rho_{k,l} \Theta^{-1} = \rho_{k,0} \rho_{l,0} \cdots \rho_{k-1,0} \rho_{k,l} \rho_{k-1,0}^{-1} \cdots \rho_{l,0}^{-1} \rho_{k,0}^{-1}. \quad (23)$$

By (I),

$$\rho_{j,0} \rho_{k,l} \rho_{j,0}^{-1} = \begin{cases} C_{k,j} \rho_{k,l} C_{k,j}^{-1} & \text{if } 1 \leq l < j \\ C_{k,l} & \text{if } l = j, \end{cases}$$

and using the fact that $C_{k,m}$ commutes with $\rho_{q,0}$ for all $1 \leq q < m < k$, we see that $\rho_{l,0} \cdots \rho_{k-1,0} \rho_{k,l} \rho_{k-1,0}^{-1} \cdots \rho_{l,0}^{-1} = C_{k,k-1} \cdots C_{k,l+1} C_{k,l} C_{k,l+1}^{-1} \cdots C_{k,k-1}^{-1}$. Since $\rho_{k,m} \cdots \rho_{k,k-1} = D_{m,k}$, for all $1 \leq m < k$, we have $\rho_{k,0}^{-1} \rho_{k,m} \cdots \rho_{k,k-1} \rho_{k,0} = \rho_{k,0}^{-1} D_{m,k} \rho_{k,0}$. Using this relation, the definitions of $C_{k,m-1}$ and $D_{m,k}$ and the relation $C_{k,k-1} = \rho_{k,0}^{-1} \rho_{k,k-1} \rho_{k,0} =$

$\rho_{k,0}^{-1}D_{k,k-1}\rho_{k,0}$, we may prove by reverse induction on m that for all $1 \leq m < k$,

$$C_{k,k-1} \cdots C_{k,m+1}C_{k,m} = \rho_{k,0}^{-1}\rho_{k,m} \cdots \rho_{k,k-1}\rho_{k,0} = \rho_{k,0}^{-1}D_{m,k}\rho_{k,0}. \quad (24)$$

Combining the above equalities, we conclude that Θ commutes with $\rho_{k,l}$ if $1 \leq l < k$.

(c) Suppose that $k \leq l \leq 2k - 2$. By (III), $\rho_{k,l}$ commutes with $\rho_{j,0}$ for all $k + 1 \leq j \leq n$ and by (I), the $\rho_{m,0}$, $1 \leq m \leq n$, commute pairwise, so:

$$\begin{aligned} \Theta\rho_{k,l}\Theta^{-1} &= \rho_{1,0} \cdots \rho_{n,0}\rho_{k,l}\rho_{n,0}^{-1} \cdots \rho_{1,0}^{-1} = \rho_{1,0} \cdots \rho_{k,0}\rho_{k,l}\rho_{k,0}^{-1} \cdots \rho_{1,0}^{-1} \\ &= \rho_{1,0} \cdots \rho_{l-k+1,0}\rho_{k,0}(\rho_{l-k+2,0} \cdots \rho_{k-1,0}\rho_{k,l}\rho_{k-1,0}^{-1} \cdots \rho_{l-k+2,0}^{-1})\rho_{k,0}^{-1}\rho_{l-k+1,0}^{-1} \cdots \rho_{1,0}^{-1}. \end{aligned} \quad (25)$$

By (I), we have $\rho_{m,0}\rho_{k,l}\rho_{m,0}^{-1} = C_{k,m}\rho_{k,l}C_{k,m}^{-1}$ for all $l - k + 2 \leq m \leq k - 1$, and $C_{k,m}\rho_{s,0} = \rho_{s,0}C_{k,m}$ for all $1 \leq s < m$, using also the relation:

$$C_{k,m} = \rho_{k,0}^{-1}\rho_{k,k-1}^{-1} \cdots \rho_{k,m+1}^{-1}\rho_{k,m}\rho_{k,m+1} \cdots \rho_{k,k-1}\rho_{k,0}.$$

Hence $\rho_{l-k+2,0} \cdots \rho_{k-1,0}\rho_{k,l}\rho_{k-1,0}^{-1} \cdots \rho_{l-k+2,0}^{-1} = \rho_{k,0}^{-1}D_{l-k+2,k}\rho_{k,0}\rho_{k,l}\rho_{k,0}^{-1}D_{l-k+2,k}^{-1}\rho_{k,0}$. Therefore:

$$\Theta\rho_{k,l}\Theta^{-1} = \rho_{l-k+1,0}D_{l-k+2,k}\rho_{k,0}(\rho_{1,0} \cdots \rho_{l-k,0}\rho_{k,l}\rho_{l-k,0}^{-1} \cdots \rho_{1,0}^{-1})\rho_{k,0}^{-1}D_{l-k+2,k}^{-1}\rho_{l-k+1,0}^{-1}, \quad (26)$$

by (25), since $D_{l-k+2,k} = \rho_{k,l-k+2} \cdots \rho_{k,k-1}$, so $D_{l-k+2,k}$ commutes with $\rho_{s,0}$ for all $1 \leq s \leq l - k + 1$ by (I). Now from equation (24), we obtain:

$$\begin{aligned} C_{k,l-k} \cdots C_{k,1} &= (C_{k,k-1} \cdots C_{k,l-k+1})^{-1} \cdot C_{k,k-1} \cdots C_{k,l-k+1} \cdot C_{k,l-k} \cdots C_{k,1} \\ &= \rho_{k,0}^{-1}D_{l-k+1,k}^{-1}D_{1,k}\rho_{k,0}, \end{aligned} \quad (27)$$

and by (I), we have:

$$\rho_{m,0}\rho_{k,q}\rho_{m,0}^{-1} = C_{k,m}\rho_{k,k+m-1}\rho_{k,q}\rho_{k,k+m-1}^{-1}C_{k,m}^{-1} \quad \text{if } 1 \leq m \leq q - k \text{ and } q \leq 2k - 2. \quad (28)$$

So for all $1 \leq m \leq l - k$,

$$\begin{aligned} \rho_{m,0}(\rho_{k,m+k}\rho_{k,m+k+1} \cdots \rho_{k,l-1}\rho_{k,l}\rho_{k,l-1}^{-1} \cdots \rho_{k,m+k+1}^{-1}\rho_{k,m+k}^{-1})\rho_{m,0} &= \\ C_{k,m}\rho_{k,m+k-1}\rho_{k,m+k}\rho_{k,m+k+1} \cdots \rho_{k,l-1}\rho_{k,l}\rho_{k,l-1}^{-1} \cdots \rho_{k,m+k+1}^{-1}\rho_{k,m+k}^{-1}\rho_{k,m+k-1}^{-1}C_{k,m}^{-1}. \end{aligned} \quad (29)$$

Using equations (27) and (29) and the relation $C_{k,m}\rho_{s,0} = \rho_{s,0}C_{k,m}$ for all $1 \leq s < m$, one may check that:

$$\begin{aligned} \rho_{1,0} \cdots \rho_{l-k,0}\rho_{k,l}\rho_{l-k,0}^{-1} \cdots \rho_{1,0}^{-1} &= \rho_{k,0}^{-1}D_{l-k+1,k}^{-1}D_{1,k}\rho_{k,0}E_{k,k,l-1}\rho_{k,l}E_{k,k,l-1}^{-1}\rho_{k,0}^{-1}D_{1,k}^{-1} \\ &\quad D_{l-k+1,k}\rho_{k,0}. \end{aligned} \quad (30)$$

Combining (26) and (30), and using the relation $D_{l-k+2,k}D_{l-k+1,k}^{-1} = \rho_{k,l-k+1}^{-1}$, we obtain:

$$\Theta\rho_{k,l}\Theta^{-1} = \rho_{l-k+1,0}(\rho_{k,l-k+1}^{-1}D_{1,k}\rho_{k,0}E_{k,k,l-1}\rho_{k,l}E_{k,k,l-1}^{-1}\rho_{k,0}^{-1}D_{1,k}^{-1}\rho_{k,l-k+1})\rho_{l-k+1,0}^{-1}. \quad (31)$$

In (31), it remains to conjugate by $\rho_{l-k+1,0}$. Using (I), we have:

$$\rho_{l-k+1,0}\rho_{k,m}\rho_{l-k+1,0}^{-1} = \begin{cases} \rho_{k,m} & \text{if } m = 0 \text{ or if } l - k + 2 \leq m \leq k - 1 \\ C_{k,l-k+1}\rho_{k,m}C_{k,l-k+1}^{-1} & \text{if } 1 \leq m \leq l - k \text{ or if } k \leq m \leq l - 1 \\ C_{k,l-k+1} & \text{if } m = l - k + 1, \end{cases} \quad (32)$$

and

$$\rho_{l-k+1,0}\rho_{k,m}\rho_{l-k+1,0}^{-1} = C_{k,l-k+1}E_{k,k,l-1}^{-1}\rho_{k,0}^{-1}D_{1,k}^{-1}\rho_{k,l}D_{1,k}\rho_{k,0}E_{k,k,l-1}C_{k,l-k+1}^{-1} \quad \text{if } m = l. \quad (33)$$

From the definition of $C_{k,l-k+1}$, we have $\rho_{k,0}C_{k,l-k+1}\rho_{k,0}^{-1} = D_{l-k+2,k}^{-1}\rho_{k,l-k+1}D_{l-k+2,k}$, and together with (31)–(33), one may show that:

$$\Theta\rho_{k,l}\Theta^{-1} = D_{1,k} \cdot D_{1,k}^{-1}\rho_{k,l}D_{1,k} \cdot D_{1,k}^{-1} = \rho_{k,l}.$$

We have thus shown that Θ commutes with each of the generators $\rho_{k,l}$, of G_n , where $1 \leq k \leq n$ and $0 \leq l \leq 2k - 2$, and so we conclude that $\Theta \in Z(G_n)$ for all $n \in \mathbb{N}$.

It remains to prove that $Z(G_n) = \langle \Theta_n \rangle$. The proof is similar to that for P_n . If $n = 1$ then $G_1 = \langle \Theta_1 \rangle$ from Proposition 19, so the result holds. Assume by induction that $Z(G_{n-1}) = \langle \Theta_{n-1} \rangle$ for some $n \geq 2$, let $x \in Z(G_n)$, and consider the short exact sequence (19). Since $(q_n)_\#$ is surjective, $(q_n)_\#(x) \in Z(G_{n-1})$, and so by induction, there exists $l \in \mathbb{Z}$ such that $(q_n)_\#(x) = \Theta_{n-1}^l$. From the definition of q_n , we have $(q_n)_\#(\rho_{i,0}) = \rho_{i,0}$ if $1 \leq i \leq n-1$ and $(q_n)_\#(\rho_{n,0}) = 1$, thus $(q_n)_\#(\Theta_n) = \Theta_{n-1}$. Since $\Theta_n \in Z(G_n)$, $x\Theta_n^{-l} \in \text{Ker}((q_n)_\#) \cap Z(G_n)$, so $x\Theta_n^{-l} \in Z(\text{Ker}((q_n)_\#))$. Thus $x\Theta_n^{-l}$ is trivial because $\text{Ker}((q_n)_\#)$ is a free group of rank $2n-1$. It follows that $x \in \langle \Theta_n \rangle$, and this completes the proof of the proposition. \square

REMARK 21. As we already mentioned, the section $(s_{n+1})_\#: G_n \rightarrow G_{n+1}$ gives rise to a decomposition of G_n as an iterated semi-direct product of the form $\mathbb{F}_{2n-1} \rtimes (\mathbb{F}_{2n-3} \rtimes (\cdots \rtimes (\mathbb{F}_3 \rtimes \mathbb{Z}) \cdots))$. We may choose a different section for $(q_{n+1})_\#$ so that $\langle \Theta_n \rangle$ appears as a direct factor of G_n as follows. Let $s'_{n+1}: F_n^{\langle \tau \rangle}(C) \rightarrow F_{n+1}^{\langle \tau \rangle}(C)$ be the map defined by $s'_{n+1}(y_1, \dots, y_n) = (y_1, \dots, y_n, y_{n+1})$, where the point $y_{n+1} \in C$ belongs to the segment between y_n and \tilde{z}_0 and is sufficiently close to y_n . The induced homomorphism $(s'_{n+1})_\#: G_n \rightarrow G_{n+1}$ sends $\rho_{n,0}$ to $\rho_{n,0}\rho_{n+1,0}$, and Θ_n to Θ_{n+1} . As above, G_n may be decomposed as an iterated semi-direct product of free groups, but by Proposition 20, it is of the form:

$$G_n \cong \mathbb{F}_{2n-1} \rtimes (\mathbb{F}_{2n-3} \rtimes (\cdots \rtimes (\mathbb{F}_5 \rtimes \mathbb{F}_3) \cdots)) \times \mathbb{Z}, \quad (34)$$

where the \mathbb{Z} -factor is generated by Θ_n . We will make use of this section to prove Proposition 4 at the end of Section 4.2.

4 The boundary homomorphism for $M = \mathbb{S}^2$ and $\mathbb{R}P^2$

In all of this section, M will be \mathbb{S}^2 or $\mathbb{R}P^2$. Set $n_0 = 3$ (resp. $n_0 = 2$) if $M = \mathbb{S}^2$ (resp. $M = \mathbb{R}P^2$), and let $n \geq n_0$. By Theorem 5, $\pi_2(F_n(M))$ is trivial. Considering the tail of the exact sequence (22) and using the homotopy equivalence $g_n: E_n \rightarrow F_n(M)$, we obtain the following exact sequence:

$$1 \rightarrow \pi_2\left(\prod_1^n M\right) \xrightarrow{\partial_n} \pi_1(I_n) \xrightarrow{(g_n \circ j_n)_\#} P_n(M) \xrightarrow{(\iota_n)_\#} \pi_1\left(\prod_1^n M\right) \rightarrow 1, \quad (35)$$

where $\partial_n = \partial_{n,2}$ is the associated boundary homomorphism. If $M = \mathbb{S}^2$ then the sequence (35) is short exact. If $M = \mathbb{R}P^2$ then the homomorphism $(\iota_n)_\#$ is Abelianisation [GG5, Proposition 8], and $\text{Im}((g_n \circ j_n)_\#) = \text{Ker}((\iota_n)_\#) = \Gamma_2(P_n(\mathbb{R}P^2))$. So in both cases, to understand completely the exact sequence (35), we need to describe $\text{Ker}((g_n \circ j_n)_\#)$ (which is equal to $\text{Im}(\partial_n)$). This is one of the aims of this section. In Section 4.1, we describe explicit generators of $\pi_1(\Omega(\mathbb{S}^2))$ and $\pi_1(\Omega(\mathbb{R}P^2))$ that will be used in the rest of the paper. In Section 4.2, we exhibit homotopy equivalences between I_c and I''_n , and I''_n and I'_n in Lemma 24, and we prove Theorem 3 that describes the image of ∂_n , and which is the main result of this section. Using this theorem and results about $P_{n-1}/\langle \Delta_{n-1}^4 \rangle$ and $\Gamma_2(P_n(\mathbb{R}P^2))$ obtained in previous papers [GG2, GG5], we prove Proposition 4 that shows that $\text{Ker}((\iota_n)_\#)$ may be written as an iterated semi-direct product. We shall make use of the notation defined in Section 3.1. Recall that $\pi: \mathbb{S}^2 \rightarrow \mathbb{R}P^2$ denotes the universal covering, and that $\tau: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is the antipodal map.

4.1 Geometric representatives of generators of $\pi_1(\Omega(M))$

We follow the notation of Section 3.1. For $i \in \{1, 2, 3\}$, let $p_i: \mathbb{R}^3 \rightarrow \mathbb{R}$ denote projection onto the i^{th} coordinate. Taking $n = n_0$, the basepoint W_{n_0} is $(\tilde{x}_0, \tilde{z}_0, -\tilde{z}_0)$ (resp. (x_0, z_0)) if $M = \mathbb{S}^2$ (resp. $M = \mathbb{R}P^2$). If $u \in \{\tilde{x}_0, \pm\tilde{z}_0\}$, we shall identify $\pi_1(\Omega(\mathbb{S}^2), c_u)$ with $\pi_2(\mathbb{S}^2, u)$, and $\pi_1(\Omega(\mathbb{R}P^2), c_{\pi(u)})$ with $\pi_2(\mathbb{R}P^2, \pi(u))$ in the usual way. In this section, we define explicit geometric representatives of generators of $\pi_1(\Omega(\mathbb{S}^2), u)$ and of $\pi_1(\Omega(\mathbb{R}P^2), c_{\pi(u)})$, and we prove two lemmas that shall be used in what follows.

Let Π_0 be the tangent plane to \mathbb{S}^2 at \tilde{x}_0 , and let D be the straight line defined by $x = 1$ and $y = 0$ contained in Π_0 . For $t \in I$, let Π_t be the plane given by rotating Π_0 about D through an angle $t\pi$, and let $\tilde{\omega}_{\tilde{x}_0,t} = \mathbb{S}^2 \cap \Pi_t$. We choose the orientation and parametrisation so that:

- (Xa) for all $s \in (0, 1)$, $p_2(\tilde{\omega}_{\tilde{x}_0,t}(s)) < 0$ if $t \in (0, \frac{1}{2})$, and $p_2(\tilde{\omega}_{\tilde{x}_0,t}(s)) > 0$ if $t \in (\frac{1}{2}, 1)$.
- (Xb) for all $t \in (0, 1)$, $p_3(\tilde{\omega}_{\tilde{x}_0,t}(s)) < 0$ if $s \in (0, \frac{1}{2})$ and $p_3(\tilde{\omega}_{\tilde{x}_0,t}(s)) > 0$ if $s \in (\frac{1}{2}, 1)$.
- (Xc) for all $t, s \in I$, $p_i(\tilde{\omega}_{\tilde{x}_0,t}(s)) = p_i(\tilde{\omega}_{\tilde{x}_0,t}(1-s))$ for $i = 1, 2$, and $p_3(\tilde{\omega}_{\tilde{x}_0,t}(s)) = -p_3(\tilde{\omega}_{\tilde{x}_0,t}(1-s))$.
- (Xd) for all $t, s \in I$, $p_2(\tilde{\omega}_{\tilde{x}_0,t}(s)) = -p_2(\tilde{\omega}_{\tilde{x}_0,1-t}(s))$, and $p_i(\tilde{\omega}_{\tilde{x}_0,t}(s)) = p_i(\tilde{\omega}_{\tilde{x}_0,1-t}(s))$ for $i = 1, 3$.
- (Xe) for all $t \in I$, $p_1(\tilde{\omega}_{\tilde{x}_0,t}(s)) > 0$ if $s \in [0, \frac{1}{4})$, and $p_1(\tilde{\omega}_{\tilde{x}_0,\frac{1}{2}}(s)) < 0$ if $s \in (\frac{1}{4}, \frac{1}{2}]$.

For each $t \in I$, $\tilde{\omega}_{\tilde{x}_0,t}$ is a loop based at \tilde{x}_0 , $\tilde{\omega}_{\tilde{x}_0,0} = \tilde{\omega}_{\tilde{x}_0,1} = c_{\tilde{x}_0}$, and for each $\tilde{w} \in \mathbb{S}^2 \setminus \{\tilde{x}_0\}$, there exist unique values $t, s \in (0, 1)$ for which $\tilde{\omega}_{\tilde{x}_0,t}(s) = \tilde{w}$. Observe also that:

- from condition (Xb), $p_3(\tilde{\omega}_{\tilde{x}_0,t}(\frac{1}{2})) = 0$ for all $t \in I$, so the loop $(\tilde{\omega}_{\tilde{x}_0,t}(\frac{1}{2}))_{t \in I}$ is the great circle lying in the xy -plane.
- by conditions (Xc) and (Xe), for all $t \in I$, $p_1(\tilde{\omega}_{\tilde{x}_0,t}(s)) > 0$ if $s \in [0, \frac{1}{4}) \cup (\frac{3}{4}, 1]$, and $p_1(\tilde{\omega}_{\tilde{x}_0,\frac{1}{2}}(s)) < 0$ if $s \in (\frac{1}{4}, \frac{3}{4})$.

Consider the rotation R about the straight line of equation $x = z$ and $y = 0$ by an angle π . The matrix of R in the standard basis of \mathbb{R}^3 is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. For all $t \in I$, let $\tilde{\omega}_{\tilde{z}_0,t} = R(\tilde{\omega}_{\tilde{x}_0,t})$. It follows from the above conditions on $\tilde{\omega}_{\tilde{x}_0,t}$ that:

- (Za) for all $s \in (0, 1)$, $p_2(\tilde{\omega}_{\tilde{z}_0,t}(s)) > 0$ if $t \in (0, \frac{1}{2})$, and $p_2(\tilde{\omega}_{\tilde{z}_0,t}(s)) < 0$ if $t \in (\frac{1}{2}, 1)$.
- (Zb) for all $t \in (0, 1)$, $p_1(\tilde{\omega}_{\tilde{z}_0,t}(s)) < 0$ if $s \in (0, \frac{1}{2})$ and $p_1(\tilde{\omega}_{\tilde{z}_0,t}(s)) > 0$ if $s \in (\frac{1}{2}, 1)$.
- (Zc) for all $t, s \in I$, $p_i(\tilde{\omega}_{\tilde{z}_0,t}(s)) = p_i(\tilde{\omega}_{\tilde{z}_0,t}(1-s))$ for $i = 2, 3$, and $p_1(\tilde{\omega}_{\tilde{z}_0,t}(s)) = -p_1(\tilde{\omega}_{\tilde{z}_0,t}(1-s))$.
- (Zd) for all $t, s \in I$, $p_2(\tilde{\omega}_{\tilde{z}_0,t}(s)) = -p_2(\tilde{\omega}_{\tilde{z}_0,1-t}(s))$, and $p_i(\tilde{\omega}_{\tilde{z}_0,t}(s)) = p_i(\tilde{\omega}_{\tilde{z}_0,1-t}(s))$ for $i = 1, 3$.
- (Ze) for all $t \in I$, $p_3(\tilde{\omega}_{\tilde{z}_0,t}(s)) > 0$ if $s \in [0, \frac{1}{4})$, and $p_3(\tilde{\omega}_{\tilde{z}_0,\frac{1}{2}}(s)) < 0$ if $s \in (\frac{1}{4}, \frac{1}{2}]$.

In particular:

- by condition (Zb), $p_1(\tilde{\omega}_{\tilde{z}_0,t}(\frac{1}{2})) = 0$, so the loop $(\tilde{\omega}_{\tilde{z}_0,t}(\frac{1}{2}))_{t \in I}$ is the great circle lying in the yz -plane.
- by conditions (Zc) and (Ze), for all $t \in I$, $p_3(\tilde{\omega}_{\tilde{z}_0,t}(s)) > 0$ if $s \in [0, \frac{1}{4}) \cup (\frac{3}{4}, 1]$, and $p_3(\tilde{\omega}_{\tilde{z}_0,\frac{1}{2}}(s)) < 0$ if $s \in (\frac{1}{4}, \frac{3}{4})$.

The loop $\tilde{\omega}_{u,t}$ is illustrated in Figure 8 for $u \in \{\tilde{x}_0, \tilde{z}_0\}$ and small values of t . We also set:

$$\tilde{\omega}_{-u,t} = -\tilde{\omega}_{u,1-t}. \quad (36)$$

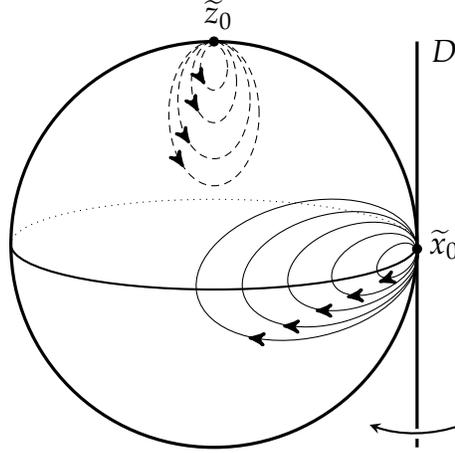


Figure 8: The loops $\tilde{\omega}_{\tilde{x}_0,t}$ and $\tilde{\omega}_{\tilde{z}_0,t}$ for small values of t . The dotted lines indicate that the loops are on the 'hidden' side of the sphere.

If $u \in \{\tilde{x}_0, \tilde{z}_0, -\tilde{z}_0\}$, then $(\tilde{\omega}_{u,t})_{t \in I}$ is a geometric representative of a generator, which we shall denote by $\tilde{\lambda}_u$, of $\pi_1(\Omega(\mathbb{S}^2), c_u)$. If $u \in \{\tilde{x}_0, \tilde{z}_0\}$, the fact that the covering map π induces an isomorphism on the π_2 -level implies that $(\pi \circ \tilde{\omega}_{u,t})_{t \in I}$ is a geometric representative of a generator of $\pi_1(\Omega(\mathbb{R}P^2), c_{\pi(u)})$ that we denote by $\lambda_{\pi(u)}$. Note that:

$$[(\pi \circ \tilde{\omega}_{-\tilde{z}_0,t})_{t \in I}] = [(\pi \circ (-\tilde{\omega}_{\tilde{z}_0,1-t}))_{t \in I}] = [(\pi \circ \tilde{\omega}_{\tilde{z}_0,t}^{-1})_{t \in I}] = -\lambda_{\pi(\tilde{z}_0)} \quad (37)$$

using (36). If $M = \mathbb{S}^2$ (resp. $M = \mathbb{R}P^2$), each direct factor of $\pi_1(\prod_1^{n_0} \Omega(M), W_{n_0})$ will be identified with the corresponding group $\pi_1(\Omega(\mathbb{S}^2), c_u)$ (resp. $\pi_1(\Omega(\mathbb{R}P^2), c_{\pi(u)})$). If $u \in \{\tilde{x}_0, \tilde{z}_0, -\tilde{z}_0\}$ (resp. $u \in \{\tilde{x}_0, \tilde{z}_0\}$), let $\tilde{\delta}_u = \partial_{n_0}(\tilde{\lambda}_u)$ (resp. $\delta_{\pi(u)} = \partial_{n_0}(\lambda_{\pi(u)})$) in $\pi_1(I_{n_0})$.

LEMMA 22. *With the above notation:*

- (a) if $M = \mathbb{S}^2$, the loop $(\tilde{x}_0, \tilde{z}_0, -\tilde{z}_0, \tilde{\omega}_{\tilde{x}_0,t}, c_{\tilde{z}_0}, c_{-\tilde{z}_0})_{t \in I}$ (resp. $(\tilde{x}_0, \tilde{z}_0, -\tilde{z}_0, c_{\tilde{x}_0}, \tilde{\omega}_{\tilde{z}_0,t}, c_{-\tilde{z}_0})_{t \in I}$, $(\tilde{x}_0, \tilde{z}_0, -\tilde{z}_0, c_{\tilde{x}_0}, c_{\tilde{z}_0}, \tilde{\omega}_{-\tilde{z}_0,t})_{t \in I}$) is a geometric representative in I_{n_0} of $\tilde{\delta}_{\tilde{x}_0}$ (resp. of $\tilde{\delta}_{\tilde{z}_0}$, $\tilde{\delta}_{-\tilde{z}_0}$).
- (b) if $M = \mathbb{R}P^2$, the loop $(x_0, z_0, \pi \circ \tilde{\omega}_{\tilde{x}_0,t}, c_{z_0})_{t \in I}$ (resp. $(x_0, z_0, c_{x_0}, \pi \circ \tilde{\omega}_{\pm\tilde{z}_0,t})_{t \in I}$) is a geometric representative in I_{n_0} of δ_{x_0} (resp. of $\pm\delta_{z_0}$).

Proof. The result is a consequence of the application of the construction given in the proof of Lemma 13 to the lower fibration of (13), and equation (37). \square

LEMMA 23. *Let $u \in \{\tilde{x}_0, \tilde{z}_0\}$. Then in $\pi_1(\Omega(\mathbb{S}^2), c_u)$, $[(\tilde{\omega}_{u,t})_{t \in I}] = [(\tilde{\omega}_{u,1-t}^{-1})_{t \in I}]$.*

Proof. First suppose that $u = \tilde{x}_0$, and for $\theta \in [0, \pi]$, let $R'_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$. Then for all $t \in I$, $R'_\theta(\tilde{\omega}_{\tilde{x}_0,t}) = \tilde{\omega}_{\tilde{x}_0,t}$, and using conditions (Xc) and (Xd), for all $t, s \in I$, we have:

$$\begin{aligned} R'_\pi(\tilde{\omega}_{\tilde{x}_0,t}(s)) &= \begin{pmatrix} p_1(\tilde{\omega}_{\tilde{x}_0,t}(s)) \\ -p_2(\tilde{\omega}_{\tilde{x}_0,t}(s)) \\ -p_3(\tilde{\omega}_{\tilde{x}_0,t}(s)) \end{pmatrix} = \begin{pmatrix} p_1(\tilde{\omega}_{\tilde{x}_0,1-t}(s)) \\ p_2(\tilde{\omega}_{\tilde{x}_0,1-t}(s)) \\ -p_3(\tilde{\omega}_{\tilde{x}_0,1-t}(s)) \end{pmatrix} = \begin{pmatrix} p_1(\tilde{\omega}_{\tilde{x}_0,1-t}(1-s)) \\ p_2(\tilde{\omega}_{\tilde{x}_0,1-t}(1-s)) \\ p_3(\tilde{\omega}_{\tilde{x}_0,1-t}(1-s)) \end{pmatrix} \\ &= \tilde{\omega}_{\tilde{x}_0,1-t}(1-s) = \tilde{\omega}_{\tilde{x}_0,1-t}^{-1}(s). \end{aligned}$$

So $R'_\pi(\tilde{\omega}_{\tilde{x}_0,t}) = \tilde{\omega}_{\tilde{x}_0,1-t}^{-1}$ from which we deduce the result for \tilde{x}_0 using the homotopy $(R'_\theta)_{\theta \in [0, \pi]}$. If $u = \tilde{z}_0$, by the first part, $R \circ R'_0(\tilde{\omega}_{\tilde{x}_0,t}) = R(\tilde{\omega}_{\tilde{x}_0,t}) = \tilde{\omega}_{\tilde{z}_0,t}$ and $R \circ R'_\pi(\tilde{\omega}_{\tilde{x}_0,t}) = R(\tilde{\omega}_{\tilde{x}_0,1-t}^{-1}) = \tilde{\omega}_{\tilde{z}_0,1-t}^{-1}$ for all $t \in I$, and the result then follows as in the case of \tilde{x}_0 . \square

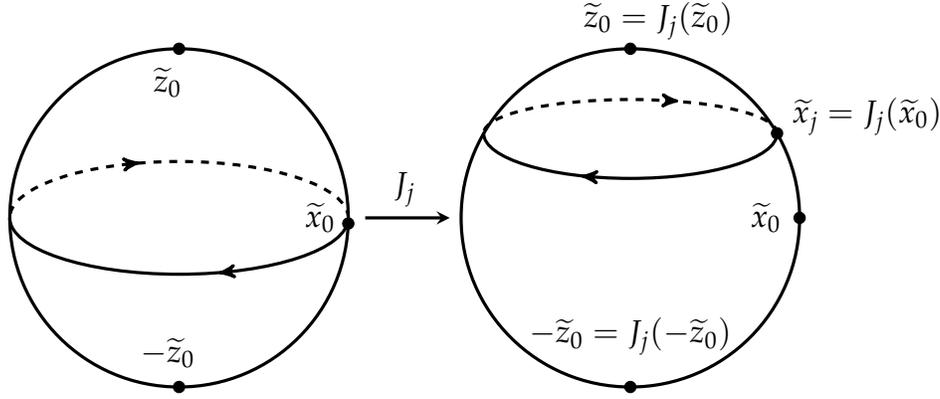


Figure 9: The homeomorphism J_j .

4.2 Construction of homotopy equivalences and the homomorphisms of the long exact sequence in homotopy

Once more, the notation will be that defined in Section 3.1. We start by introducing some other notation. Let $n \geq n_0$, and let $j \in \{0, 1, \dots, n - n_0\}$. Let $J_j: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be the orientation-preserving homeomorphism defined by:

$$J_j(\theta, \varphi) = \begin{cases} \left(\theta, \varphi + \left(\frac{\pi}{2} - \varphi\right) \left(\frac{j}{j+1}\right) \right) & \text{if } \varphi \geq 0 \\ \left(\theta, \varphi + \left(\frac{\pi}{2} + \varphi\right) \left(\frac{j}{j+1}\right) \right) & \text{if } \varphi < 0. \end{cases} \quad (38)$$

Note that $J_0 = \text{Id}_{\mathbb{S}^2}$, $J_j(\tilde{x}_0) = \tilde{x}_j$, J_j sends the equator ($\varphi = 0$) to the circle of equation $\varphi = \frac{\pi}{2} \left(\frac{j}{j+1}\right)$, leaves the longitudes invariant, and fixes the poles (see Figure 9). If $t \in I$, let $h_{j,t}: I \rightarrow \mathbb{S}^2$ be the path defined by:

$$h_{j,t}(r) = \tilde{\omega}_{\tilde{x}_j,t} \left(\frac{1+r}{2} \right) \text{ for all } r \in I, \text{ where } \tilde{\omega}_{\tilde{x}_j,t} = J_j \circ \tilde{\omega}_{\tilde{x}_0,t}. \quad (39)$$

The subspace \mathbb{M} of \mathbb{D} defined in Section 3.1 is contractible. Let us define two explicit contractions, $H_1: \mathbb{M} \times I \rightarrow \mathbb{D}$ of \mathbb{M} onto \tilde{x}_0 , and $H_2: \mathbb{D} \times I \rightarrow \mathbb{D}$ of \mathbb{D} onto \tilde{z}_0 as follows.

(a) If $v \in \mathbb{M}$, let $v = (\theta, \varphi)$. To define the homotopy H_1 , we separate the cases $\varphi \geq 0$ and $-\pi/2 < \varphi < 0$:

(i) suppose that $\varphi \geq 0$. If $v = \tilde{x}_0$ then set $H_1(v, r) = \tilde{x}_0$ for all $r \in I$. If $v \neq \tilde{x}_0$ then there exist $t \in (0, 1)$ and $s \in [\frac{1}{2}, 1)$, both unique, such that $\tilde{\omega}_{\tilde{x}_0,t}(s) = v$. We then set:

$$H_1(v, r) = \tilde{\omega}_{\tilde{x}_0,t}(s(1-r) + r) \text{ for all } r \in I. \quad (40)$$

(ii) suppose that $-\pi/2 < \varphi < 0$. Notice that $(\theta, 0) = \tilde{\omega}_{\tilde{x}_0, (2\pi-\theta)/2\pi}(\frac{1}{2})$, and that $\min\left(\frac{-2\varphi}{\pi}, \frac{1}{2}\right) \in (0, 1)$. We set:

$$H_1(v, r) = \begin{cases} \left(\theta, \varphi \left(1 - \frac{r}{\min\left(\frac{-2\varphi}{\pi}, \frac{1}{2}\right)}\right) \right) & \text{if } 0 \leq r \leq \min\left(\frac{-2\varphi}{\pi}, \frac{1}{2}\right) \\ \tilde{\omega}_{\tilde{x}_0, (2\pi-\theta)/2\pi} \left(\frac{1+r - 2\min\left(\frac{-2\varphi}{\pi}, \frac{1}{2}\right)}{2(1 - \min\left(\frac{-2\varphi}{\pi}, \frac{1}{2}\right))} \right) & \text{if } \min\left(\frac{-2\varphi}{\pi}, \frac{1}{2}\right) < r \leq 1. \end{cases}$$

One may check that H_1 is continuous, and that $H_1(v, 0) = v$ and $H_1(v, 1) = \tilde{x}_0$ for all $v \in \mathbb{M}$.

Geometrically, H_1 contracts \mathbb{M} onto \tilde{x}_0 , first along the longitudes in the lower hemi-

sphere, then along the paths $(\tilde{\omega}_{\tilde{x}_0,t}(s))_{s \in [\frac{1}{2}, 1]}$ in the upper hemisphere.

(b) If $v \in \mathbb{D}$, let $v = (\theta, \varphi)$. The homotopy H_2 of \mathbb{D} is defined by $H_2(v, r) = H_2((\theta, \varphi), r) = (\theta, (1-r)\varphi + \frac{r\pi}{2})$. Geometrically, H_2 contracts \mathbb{D} onto \tilde{z}_0 along the longitudes.

If $v \in \mathbb{M}$ (resp. $v \in \mathbb{D}$), we define $h_v^{(1)} : I \rightarrow \mathbb{D}$ (resp. $h_v^{(2)} : I \rightarrow \mathbb{D}$) by:

$$h_v^{(1)}(r) = H_1(v, r) \text{ and } h_v^{(2)}(r) = H_2(v, r). \quad (41)$$

Observe that $h_v^{(1)}$ (resp. $h_v^{(2)}$) is a path that joins v to \tilde{x}_0 (resp. to \tilde{z}_0) via the homotopy H_1 (resp. H_2). Let $\Psi_j : \mathbb{M} \times I \rightarrow \mathbb{D}$ be the homotopy defined by $\Psi_j(\tilde{x}, t) = J_j(H_1(J_j^{-1}(\tilde{x}), t))$

using (38), and for $\tilde{x} \in \mathbb{M}$, let $\psi_{\tilde{x}}^{(j)} : I \rightarrow \mathbb{D}$ be the path defined by $\psi_{\tilde{x}}^{(j)}(r) = \Psi_j(\tilde{x}, r)$ that joins \tilde{x} to \tilde{x}_j . Note that $\psi_{\tilde{x}}^{(0)} = h_{\tilde{x}}^{(1)}$, and that for all $r \in I$:

$$\begin{aligned} \psi_{\tilde{\omega}_{\tilde{x}_j,t}(\frac{1}{2})}^{(j)}(r) &= \Psi_j(\tilde{\omega}_{\tilde{x}_j,t}(\frac{1}{2}), r) = J_j(H_1(J_j^{-1}(\tilde{\omega}_{\tilde{x}_j,t}(\frac{1}{2})), r)) \\ &= J_j(H_1(\tilde{\omega}_{\tilde{x}_0,t}(\frac{1}{2}), r)) = J_j(\tilde{\omega}_{\tilde{x}_0,t}(\frac{1+r}{2})) \text{ by equations (39) and (40)} \\ &= h_{j,t}(r) \text{ by equation (39)}. \end{aligned} \quad (42)$$

Let $\alpha_c : I_c \rightarrow I_n''$ be the pointed map defined by:

$$\alpha_c(y_1, \dots, y_{n-1}, \omega_1, \dots, \omega_{n-1}) = (y_1, \dots, y_{n-1}, \hat{\psi}_{y_1}^{(1)} * \omega_1, \dots, \hat{\psi}_{y_{n-1}}^{(n-1)} * \omega_{n-1}), \quad (43)$$

where for all $y \in \mathbb{M}$ and $i = 1, \dots, n-1$:

$$\hat{\psi}_y^{(i)} = \begin{cases} h_y^{(2)} & \text{if } M = \mathbb{S}^2 \text{ and } i = n-1 \\ \psi_y^{(i-1)} & \text{otherwise.} \end{cases} \quad (44)$$

Note that α_c satisfies $h_n'' \circ \alpha_c(u, \mu) = u$ for all $(u, \mu) \in I_c$, and so diagram (17) is commutative up to homotopy using Remark 18. From the definition of $\alpha_n' : I_n' \rightarrow I_n$, (16) and (43), it follows that:

$$\begin{aligned} \alpha_n' \circ \alpha_\pi \circ \alpha_c(y_1, \dots, y_{n-1}, \omega_1, \dots, \omega_{n-1}) &= \\ &(\hat{\pi}(y_1), \dots, \hat{\pi}(y_{n-1}), w_n, \hat{\pi} \circ (\hat{\psi}_{y_1}^{(1)} * \omega_1), \dots, \hat{\pi} \circ (\hat{\psi}_{y_{n-1}}^{(n-1)} * \omega_{n-1}), c_{w_n}). \end{aligned} \quad (45)$$

Recall that $\hat{\pi}$ was defined just before equation (16).

The homomorphism $J_j : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ defined by equation (38) and the covering map $\pi : \mathbb{S}^2 \rightarrow \mathbb{R}P^2$ induce isomorphisms on the π_2 -level, so by equation (39), $(\hat{\pi} \circ \tilde{\omega}_{\tilde{x}_j,t})_{t \in I}$ is a geometric representative of a generator of $\pi_1(\Omega(\mathbb{S}^2), c_{\tilde{x}_j})$ (resp. of $\pi_1(\Omega(\mathbb{R}P^2), c_{x_j})$) that we denote by $\tilde{\lambda}_{\tilde{x}_j}$ (resp. by λ_{x_j}). Identifying $\pi_1(\Omega(\mathbb{S}^2), c_{\tilde{x}_j})$ (resp. $\pi_1(\Omega(\mathbb{R}P^2), c_{x_j})$) with $\pi_2(\mathbb{S}^2, \tilde{x}_j)$ (resp. $\pi_2(\mathbb{R}P^2, x_j)$) as usual, we may thus take:

$$(\tilde{\lambda}_{\tilde{x}_0}, \tilde{\lambda}_{\tilde{x}_1}, \dots, \tilde{\lambda}_{\tilde{x}_{n-3}}, \tilde{\lambda}_{\tilde{z}_0}, \tilde{\lambda}_{-\tilde{z}_0}) \text{ (resp. } (\lambda_{x_0}, \lambda_{x_1}, \dots, \lambda_{x_{n-2}}, \lambda_{z_0})) \quad (46)$$

to be a basis \mathcal{B} of the free Abelian group $\pi_2(\prod_1^n \mathbb{S}^2)$ (resp. of $\pi_2(\prod_1^n \mathbb{R}P^2)$). For each $v \in \{\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_{n-3}, \tilde{z}_0, -\tilde{z}_0\}$ (resp. $v \in \{x_0, x_1, \dots, x_{n-2}, z_0\}$), let

$$\tilde{\delta}_v = \partial_n(\tilde{\lambda}_v) \text{ (resp. } \delta_v = \partial_n(\lambda_v)). \quad (47)$$

By Remark 18 and the proof of Theorem 1, the homotopy fibres I_c , I_n'' and I_n' have the same homotopy type. We are now able to exhibit explicit homotopy equivalences between them, which will allow us to describe the images of certain elements of $\pi_1(I_c)$ by the induced isomorphisms. Recall that the maps $j_n : I_n \rightarrow E_n$ and $g_n : E_n \rightarrow F_n(M)$ were defined just after equation (12).

LEMMA 24.

- (a) The map $\alpha_c: I_c \longrightarrow I_n''$ defined in (43) is a homotopy equivalence.
(b) The map $\alpha_\pi: I_n'' \longrightarrow I_n'$ defined in (16) is a homotopy equivalence.
(c) With the notation of (45),

$$g_n \circ j_n \circ \alpha_n' \circ \alpha_\pi \circ \alpha_c(y_1, \dots, y_{n-1}, \omega_1, \dots, \omega_{n-1}) = (\widehat{\pi}(y_1), \dots, \widehat{\pi}(y_{n-1}), w_n).$$

- (d) For all $i = 1, \dots, n-1$, $(\alpha_n' \circ \alpha_\pi \circ \alpha_c)_\#(\widetilde{\lambda}_{w_i}'') = \begin{cases} \widetilde{\delta}_{w_i} = \partial_n(\widetilde{\lambda}_{w_i}) & \text{if } M = \mathbb{S}^2 \\ \delta_{w_i} = \partial_n(\lambda_{w_i}) & \text{if } M = \mathbb{R}P^2. \end{cases}$

Proof.

- (a) Let $\beta_c: I_n'' \longrightarrow I_c$ be defined by:

$$\beta_c(v_1, \dots, v_{n-1}, \gamma_1, \dots, \gamma_{n-1}) = (v_1, \dots, v_{n-1}, (\widehat{\psi}_{v_1}^{(1)})^{-1} * \gamma_1, \dots, (\widehat{\psi}_{v_{n-1}}^{(n-1)})^{-1} * \gamma_{n-1}).$$

Then $\alpha_c \circ \beta_c \simeq \text{Id}_{I_n''}$ and $\beta_c \circ \alpha_c \simeq \text{Id}_{I_c}$, so α_c and β_c are mutual homotopy inverses between I_c and I_n'' .

- (b) If $M = \mathbb{S}^2$ then α_π is the identity, and the result is clear. So suppose that $M = \mathbb{R}P^2$. Define $\beta_\pi: I_n' \longrightarrow I_n''$ by:

$$\beta_\pi(v_1, \dots, v_{n-1}, l_1, \dots, l_{n-1}) = (\widetilde{v}_1, \dots, \widetilde{v}_{n-1}, \widetilde{l}_1, \dots, \widetilde{l}_{n-1}),$$

where for $i = 1, \dots, n-1$, $\widetilde{l}_i: I \longrightarrow \mathbb{S}^2$ is the lift of the path $l_i: I \longrightarrow \mathbb{R}P^2$ by the covering map π such that $\widetilde{l}_i(1) = w_i''$, and \widetilde{v}_i is given by $\widetilde{v}_i = \widetilde{l}_i(0)$. The map β_π is well defined, since if $(v_1, \dots, v_{n-1}) \in F_{n-1}(\mathbb{R}P^2 \setminus \{z_0\})$ then $(\widetilde{v}_1, \dots, \widetilde{v}_{n-1}) \in F_{n-1}^{(\tau)}(C)$. The fact that π is a covering map implies that $\alpha_\pi \circ \beta_\pi = \text{Id}_{I_n'}$ and $\beta_\pi \circ \alpha_\pi = \text{Id}_{I_n''}$, so α_π and β_π are mutual homotopy inverses between I_n' and I_n'' .

- (c) This follows directly from (45) and the definition of the maps g_n and j_n given in Section 3.1.

- (d) Let $i \in \{1, \dots, n-1\}$. Then:

$$\widetilde{\lambda}_{w_i}'' = [(w_1'', \dots, w_{n-1}'', c_{w_1}'', \dots, c_{w_{i-1}}'', \widetilde{\omega}_{w_i'', t}, c_{w_{i+1}}'', \dots, c_{w_{n-1}}'')_{t \in I}] \quad (48)$$

in $\pi_1(I_c)$. For all $j = 1, \dots, n-1$, $\widehat{\pi}(w_j'') = w_j$, and $\widehat{\psi}_{w_j}^{j-1} = c_{w_j}$ using (44) and the definition of Ψ_{j-1} . It follows from (47), (48) and Lemma 13 that:

$$\begin{aligned} (\alpha_n' \circ \alpha_\pi \circ \alpha_c)_\#(\widetilde{\lambda}_{w_i}'') &= [(w_1, \dots, w_{n-1}, c_{w_1}, \dots, c_{w_{i-1}}, \widehat{\pi} \circ \widetilde{\omega}_{w_i'', t}, c_{w_{i+1}}, \dots, c_{w_{n-1}})_{t \in I}] \\ &= \begin{cases} \partial_n(\widetilde{\lambda}_{w_i}) = \widetilde{\delta}_{w_i} & \text{if } M = \mathbb{S}^2 \\ \partial_n(\lambda_{w_i}) = \delta_{w_i} & \text{if } M = \mathbb{R}P^2, \end{cases} \end{aligned}$$

as required. \square

The following proposition relates the elements that appear in (46) to an element $\widehat{\tau}_n$ of $\pi_1(I_n)$ that projects to the full twist braid under the homomorphism $(g_n \circ j_n)_\#$.

PROPOSITION 25. *Let $M = \mathbb{S}^2$ or $\mathbb{R}P^2$, let $n \geq n_0$, let $\tau_n: I \longrightarrow I_c$ be the loop in I_c based at W_c defined by:*

$$\tau_n(t) = \begin{cases} (\widetilde{\omega}_{\widetilde{x}_0, t}(\frac{1}{2}), \widetilde{\omega}_{\widetilde{x}_1, t}(\frac{1}{2}), \dots, \widetilde{\omega}_{\widetilde{x}_{n-3}, t}(\frac{1}{2}), \widetilde{z}_0, c_{\widetilde{x}_0}, c_{\widetilde{x}_1}, \dots, c_{\widetilde{x}_{n-3}}, c_{\widetilde{z}_0}) & \text{if } M = \mathbb{S}^2 \\ (\widetilde{\omega}_{\widetilde{x}_0, t}(\frac{1}{2}), \widetilde{\omega}_{\widetilde{x}_1, t}(\frac{1}{2}), \dots, \widetilde{\omega}_{\widetilde{x}_{n-2}, t}(\frac{1}{2}), c_{\widetilde{x}_0}, c_{\widetilde{x}_1}, \dots, c_{\widetilde{x}_{n-2}}) & \text{if } M = \mathbb{R}P^2, \end{cases} \quad (49)$$

and let $\widehat{\tau}_n$ be the element of $\pi_1(I_n)$ defined by $\widehat{\tau}_n = [\alpha_n' \circ \alpha_\pi \circ \alpha_c(\tau_n)]$. Then $(g_n \circ j_n)_\#(\widehat{\tau}_n) = \Delta_n^2$, the full-twist braid of $P_n(M)$, and there exist unique integers $m_0, m_1, \dots, m_{n-1} \in \mathbb{Z}$ such that:

$$\widehat{\tau}_n^2 = \begin{cases} \partial_n(m_0 \widetilde{\lambda}_{\widetilde{x}_0} + \dots + m_{n-3} \widetilde{\lambda}_{\widetilde{x}_{n-3}} + m_{n-2} \widetilde{\lambda}_{\widetilde{z}_0} + m_{n-1} \widetilde{\lambda}_{-\widetilde{z}_0}) & \text{if } M = \mathbb{S}^2 \\ \partial_n(m_0 \lambda_{x_0} + \dots + m_{n-2} \lambda_{x_{n-2}} + m_{n-1} \lambda_{z_0}) & \text{if } M = \mathbb{R}P^2. \end{cases} \quad (50)$$

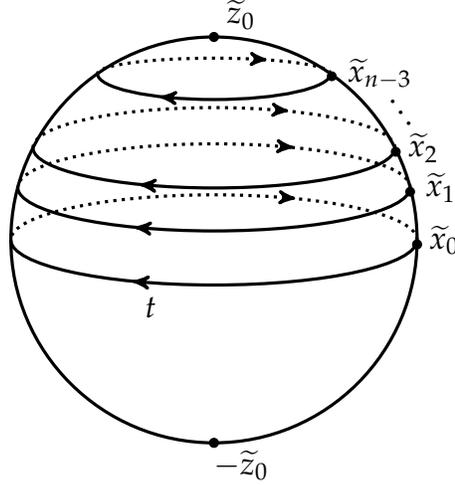


Figure 10: The full twist braid Δ_n^2 of $P_n(\mathbb{S}^2)$, which is equal to $(g_n \circ j_n)_\#(\widehat{\tau}_n)$.

Proof. Let τ_n be as defined in equation (49), and for $t \in I$, let:

$$(y_1(t), \dots, y_{n-1}(t)) = \begin{cases} (\tilde{\omega}_{\tilde{x}_0, t}(\frac{1}{2}), \tilde{\omega}_{\tilde{x}_1, t}(\frac{1}{2}), \dots, \tilde{\omega}_{\tilde{x}_{n-3}, t}(\frac{1}{2}), \tilde{z}_0) & \text{if } M = \mathbb{S}^2 \\ (\tilde{\omega}_{\tilde{x}_0, t}(\frac{1}{2}), \tilde{\omega}_{\tilde{x}_1, t}(\frac{1}{2}), \dots, \tilde{\omega}_{\tilde{x}_{n-2}, t}(\frac{1}{2})) & \text{if } M = \mathbb{R}P^2. \end{cases} \quad (51)$$

Then $\tau_n(t) = (y_1(t), \dots, y_{n-1}(t), c_{y_1(0)}, \dots, c_{y_{n-1}(0)})$, and using (45), we have:

$$\begin{aligned} \widehat{\tau}_n &= [\alpha'_n \circ \alpha_\pi \circ \alpha_c \circ \tau_n] \\ &= [(\widehat{\pi}(y_1(t)), \dots, \widehat{\pi}(y_{n-1}(t)), w_n, \widehat{\pi} \circ \widehat{\psi}_{y_1(t)}^{(1)}, \dots, \widehat{\pi} \circ \widehat{\psi}_{y_{n-1}(t)}^{(n-1)}, c_{w_n})_{t \in I}]. \end{aligned} \quad (52)$$

By (44), (52), Lemma 24(c) and the fact that $h_{\tilde{z}_0}^{(2)} = c_{\tilde{z}_0}$, we see that:

$$(g_n \circ j_n)_\#(\widehat{\tau}_n) = \begin{cases} \left[\left(\tilde{\omega}_{\tilde{x}_0, t}(\frac{1}{2}), \tilde{\omega}_{\tilde{x}_1, t}(\frac{1}{2}), \dots, \tilde{\omega}_{\tilde{x}_{n-3}, t}(\frac{1}{2}), \tilde{z}_0, -\tilde{z}_0 \right)_{t \in I} \right] & \text{if } M = \mathbb{S}^2 \\ \left[\left(\pi(\tilde{\omega}_{\tilde{x}_0, t}(\frac{1}{2})), \pi(\tilde{\omega}_{\tilde{x}_1, t}(\frac{1}{2})), \dots, \pi(\tilde{\omega}_{\tilde{x}_{n-2}, t}(\frac{1}{2})), z_0 \right)_{t \in I} \right] & \text{if } M = \mathbb{R}P^2. \end{cases} \quad (53)$$

Using the constructions of Section 4.1 and equation (39), this element may be seen to be a geometric representative of Δ_n^2 in $P_n(M)$ (see Figure 10 for the case $M = \mathbb{S}^2$). In particular, $\widehat{\tau}_n$ is non trivial, and $\widehat{\tau}_n^2 \in \text{Im}(\partial_n)$ by exactness of (35). With respect to the basis of $\pi_2(\prod_1^n M)$ given by (46), the existence and uniqueness of the integers m_0, m_1, \dots, m_{n-1} given in (50) also follow from the exactness of (35). \square

REMARKS 26.

(a) In [GG7, Theorem 3], we showed that the coefficients of (50) are given by $m_0 = \dots = m_{n-2} = 1$, and $m_{n-1} = -1$ (resp. $m_{n-1} = 1$) if $M = \mathbb{S}^2$ (resp. if $M = \mathbb{R}P^2$).

(b) Let $M = \mathbb{S}^2$ (resp. $M = \mathbb{R}P^2$). Writing the group $\pi_1(I_c)$ as the direct product $R_{n-1} \times \pi_1(\Omega(\prod_1^{n-1} \mathbb{S}^2))$ by (14), $[\tau_n]$ may be regarded using (49) as the element Δ_{n-1}^2 of P_{n-1} with respect to the choice of orientation of the loops given in Figure 10 (resp. as the element Θ_{n-1}^{-1} of G_{n-1} , the inverse being due to the fact that in Figure 1, we have chosen the opposite orientation of the loops). By definition, under the isomorphism $(\alpha'_n \circ \alpha_\pi \circ \alpha_c)_\#: \pi_1(I_c) \rightarrow \pi_1(I_n)$ given by Lemma 24 and Corollary 30, $[\tau_n]$ is sent to $\widehat{\tau}_n$, and by (53), $(g_n \circ j_n)_\#(\widehat{\tau}_n) = \Delta_n^2$ in $P_n(M)$.

(c) A useful algebraic section involving the group R_n given in the statement of Corollary 2(b) may be obtained in a straightforward manner as follows. Let $M = \mathbb{S}^2$ (resp. $M = \mathbb{R}P^2$), let $n_0 = 3$ (resp. $n_0 = 2$), let $n \geq n_0 - 1$, and let $R_n = P_n$ (resp. $R_n = G_n$).

With the notation of Section 3.1, consider the homomorphism $p: R_n \longrightarrow R_{n_0-1}$ induced by the projection $F_n^G(\mathbb{M}) \longrightarrow F_{n_0-1}^G(\mathbb{M})$ that forgets the last $n - n_0 + 1$ coordinates. Then R_{n_0-1} is the infinite cyclic group $P_2 = \langle \Delta_2^2 \rangle$ (resp. $G_1 = \langle \rho_{1,0} \rangle$). Thus p admits a section, and $R_n \cong \text{Ker}(p) \rtimes R_{n_0-1}$. Further, $p(\Delta_n^2) = \Delta_2^2$ (resp. $p(\Theta_n) = \Theta_1$), and since the centre of R_n is equal to $\langle \Delta_n^2 \rangle$ (resp. to $\langle \Theta_n \rangle$), we conclude that R_n may be written as an internal direct product $\text{Ker}(p) \times Z(R_n)$. Since $F_n^G(\mathbb{M})$ is a space of type $K(R_n, 1)$, $\text{Ker}(p)$ is torsion free.

To end this section, we prove Theorem 3 and Proposition 4.

Proof of Theorem 3. If $M = \mathbb{S}^2$ (resp. $\mathbb{R}P^2$), let $e_{i+1} = \partial_n(\tilde{\lambda}_{\tilde{x}_i})$ for $i = 0, \dots, n-3$, $e_{n-1} = \partial_n(\tilde{\lambda}_{\tilde{z}_0})$ and $e_n = \partial_n(\tilde{\lambda}_{-\tilde{z}_0})$ (resp. $e_{i+1} = \partial_n(\lambda_{x_i})$ for $i = 0, \dots, n-2$ and $e_n = \partial_n(\lambda_{z_0})$). By exactness of (35), (e_1, \dots, e_n) is a basis of the free subgroup $\text{Im}(\partial_n)$ of $\pi_1(I_n)$, and using also Proposition 25, we have $\langle e_1, \dots, e_{n-1}, \hat{\tau}_n^2 \rangle \subset \text{Im}(\partial_n)$. Consider the isomorphism $(\alpha'_n \circ \alpha_\pi \circ \alpha_c)_\#: \pi_1(I_c) \longrightarrow \pi_1(I_n)$ given by Lemma 24 and Corollary 30. For $i = 1, \dots, n$, let f_i be the unique element of $\pi_1(I_c)$ such that $(\alpha'_n \circ \alpha_\pi \circ \alpha_c)_\#(f_i) = e_i$, and let V_1 and V be the subgroups of $\pi_1(I_c)$ for which $(\alpha'_n \circ \alpha_\pi \circ \alpha_c)_\#(V_1) = \langle e_1, \dots, e_{n-1}, \hat{\tau}_n^2 \rangle$ and $(\alpha'_n \circ \alpha_\pi \circ \alpha_c)_\#(V) = \text{Im}(\partial_n)$. Then $V_1 \subset V$, and by Lemma 24(d) and the decomposition $\pi_1(I_c) = R_{n-1} \times \pi_1(\Omega(\prod_1^{n-1} \mathbb{S}^2))$ of Corollary 2 as an internal direct product induced by (14), we have $\pi_1(\Omega(\prod_1^{n-1} \mathbb{S}^2)) = \langle f_1, \dots, f_{n-1} \rangle \subset V_1 \subset V$. Further $(\alpha'_n \circ \alpha_\pi \circ \alpha_c)_\#([\tau_n]) = \hat{\tau}_n$ by Proposition 25, so $[\tau_n]^2 \in V_1$. By Remarks 26(b), $[\tau_n]$ may be viewed as an element of the factor R_{n-1} , namely Δ_{n-1}^2 if $M = \mathbb{S}^2$, and Θ_{n-1}^{-1} if $M = \mathbb{R}P^2$. Since $\pi_1(\Omega(\prod_1^{n-1} \mathbb{S}^2)) = \langle f_1, \dots, f_{n-1} \rangle \subset V \subset \pi_1(I_c)$, there exists an element $\gamma \in R_{n-1}$ such that $(f_1, \dots, f_{n-1}, \gamma)$ is a basis of V . The fact that $[\tau_n]^2 \in V_1 \cap R_{n-1} \subset V$ implies that there exists $k \in \mathbb{Z}$ such that $[\tau_n]^2 = \gamma^k$, where γ and $[\tau_n]$ belong to R_{n-1} . Since $\langle [\tau_n] \rangle = Z(R_{n-1})$, it follows from Remarks 26(c) that $\gamma \in \{[\tau_n], [\tau_n]^{-1}, [\tau_n]^2, [\tau_n]^{-2}\}$. If $\gamma \in \{[\tau_n], [\tau_n]^{-1}\}$ then $\hat{\tau}_n \in \text{Im}(\partial_n)$, and thus $\hat{\tau}_n \in \text{Ker}((g_n \circ j_n)_\#)$ by exactness of (35), which contradicts the fact that $(g_n \circ j_n)_\#(\hat{\tau}_n) = \Delta_n^2$ in $P_n(M)$ by (53). Hence $\gamma \in \{[\tau_n]^2, [\tau_n]^{-2}\}$, from which we conclude that $V \subset V_1$. So $V_1 = V$, and thus $\langle e_1, \dots, e_{n-1}, \hat{\tau}_n^2 \rangle = \text{Im}(\partial_n)$ as required. \square

Proof of Proposition 4. Let $M = \mathbb{S}^2$ (resp. $M = \mathbb{R}P^2$), and let $n \geq n_0$. By exactness of (35),

$$\text{Ker}((\iota_n)_\#) = \text{Im}((g_n \circ j_n)_\#) \cong \pi_1(I_n) / \text{Im}(\partial_n), \quad (54)$$

where $\text{Ker}((\iota_n)_\#) = P_n(\mathbb{S}^2)$ (resp. $\text{Ker}((\iota_n)_\#) = \Gamma_2(P_n(\mathbb{R}P^2))$) by [GG5, equation (2) and Proposition 1(a)(i)], and the isomorphism of (54) is induced by the homomorphism $(g_n \circ j_n)_\#$. From Theorem 3 and (47), we have:

$$\text{Im}(\partial_n) = \langle \tilde{\delta}_{\tilde{x}_0}, \tilde{\delta}_{\tilde{x}_0}, \dots, \tilde{\delta}_{\tilde{x}_{n-3}}, \tilde{\delta}_{\tilde{z}_0}, \hat{\tau}_n^2 \rangle \quad (\text{resp. } \text{Im}(\partial_n) = \langle \delta_{x_0}, \delta_{x_1}, \dots, \delta_{x_{n-2}}, \hat{\tau}_n^2 \rangle).$$

Making use of the isomorphism $(\alpha'_n \circ \alpha_\pi \circ \alpha_c)_\#: \pi_1(I_c) \longrightarrow \pi_1(I_n)$ and the notation of the proof of Theorem 3, we have:

$$\pi_1(I_n) / \text{Im}(\partial_n) \cong \pi_1(I_c) / \langle f_1, \dots, f_{n-1}, [\tau_n^2] \rangle, \quad (55)$$

the isomorphism being induced by $(\alpha'_n \circ \alpha_\pi \circ \alpha_c)_\#$. Since f_1, \dots, f_{n-1} are generators of the respective factors of $\pi_1(\Omega(\prod_1^{n-1} \mathbb{S}^2))$ in $\pi_1(I_c)$, it follows from (14) and (55) that $\pi_1(I_n) / \text{Im}(\partial_n) \cong R_{n-1} / \langle [\tau_n^2] \rangle$. Using Remarks 26(b), if $M = \mathbb{S}^2$ then $[\tau_n]$ may be interpreted as the full-twist braid Δ_{n-1}^2 of P_{n-1} , and so $\pi_1(I_n) / \text{Im}(\partial_n) \cong P_{n-1} / \langle \Delta_{n-1}^4 \rangle \cong \mathbb{F}_{n-2} \rtimes (\mathbb{F}_{n-3} \rtimes (\dots \rtimes (\mathbb{F}_3 \rtimes \mathbb{F}_2) \dots)) \times \mathbb{Z}_2$ using [GG2, Proposition 8] and the Artin

combing operation. We thus obtain the isomorphisms of (6). Now suppose that $M = \mathbb{R}P^2$. By Remarks 26(b), $[\tau_n]$ may be interpreted as the element Θ_{n-1}^{-1} of G_{n-1} , and the isomorphisms of (7) follow from (54), the fact that $\text{Ker}((\iota_n)_\#) = \Gamma_2(P_n(\mathbb{R}P^2))$ and equation (34). \square

REMARK 27. Let $M = \mathbb{R}P^2$, let $n \in \mathbb{N}$, and let $K_n = \text{Ker}((\iota_n)_\#)$. In [GG5, Proposition (1)(a) and Theorem 3], we showed that $(\iota_n)_\#: P_n(\mathbb{R}P^2) \rightarrow \pi_1(\prod_1^n \mathbb{R}P^2)$ is Abelianisation, so $K_n = \Gamma_2(P_n)$, and that $K_n \cong L_n \times \mathbb{Z}_2$, where $L_n \cong \mathbb{F}_{2^{n-3}} \rtimes (\mathbb{F}_{2^{n-5}} \rtimes \cdots \rtimes (\mathbb{F}_5 \rtimes \mathbb{F}_3) \cdots)$. Since the basepoints involved are not the same, we cannot compare L_n directly with the corresponding factor of (7). However, one may show that the map $g_n \circ j_n \circ \alpha'_n \circ \alpha_\pi \circ \alpha_c: I_c \rightarrow F_n(\mathbb{R}P^2)$ induces a surjective homomorphism $G_{n-1} \rightarrow K_n$, from which it should be possible to make explicit the factors $\mathbb{F}_{2^{i+1}}$, where $i = 1, \dots, n-2$, as subgroups of K_n . Independently of Proposition 4, if $p: P_{n+1}(\mathbb{R}P^2) \rightarrow P_n(\mathbb{R}P^2)$ is the projection defined geometrically by forgetting the last string, one may show using the results of [GG5] that the homomorphism $p|_{K_{n+1}}: K_{n+1} \rightarrow K_n$ splits by making use of the fact that the restriction $p|_{L_{n+1}}: L_{n+1} \rightarrow L_n$ splits.

Appendix

As we mentioned in Remarks 14, the map $\alpha'_n: I'_n \rightarrow I_n$ defined in Section 3.1 is a homotopy equivalence. This follows from Proposition 29 given below relating the homotopy fibres of fibre spaces and certain subspaces, and is a consequence of Lemma 28. This lemma seems to be folklore, and we were not able to find a proof in the literature, although the statement appears in [N, Section 3.2, Exercises(2), page 99]. We are grateful to Michael Crabb for proposing proofs of Lemma 28 and Proposition 29. In this section, we will denote the homotopy fibre of a map $f: X \rightarrow Y$ by $I(f)$ rather than I_f .

We first fix some notation and recall the homotopy fibres that we will analyse. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be pointed maps. By (2), we have:

$$\begin{aligned} I(f) &= \left\{ (x, \rho) \in X \times Y^I \mid \rho(0) = f(x), \rho(1) = *_{Y} \right\}, \\ I(g) &= \left\{ (y, \gamma) \in Y \times Z^I \mid \gamma(0) = g(y), \gamma(1) = *_{Z} \right\} \text{ and} \\ I(g \circ f) &= \left\{ (x, \gamma) \in X \times Z^I \mid \gamma(0) = g(f(x)), \gamma(1) = *_{Z} \right\}. \end{aligned}$$

Let $f_*: I(g \circ f) \rightarrow I(g)$ be the map defined by $f_*(x, \gamma) = (f(x), \gamma)$. Then:

$$I(f_*) = \left\{ (x, \gamma, \Phi) \in X \times Z^I \times I(g)^I \mid \Phi(0) = f_*(x, \gamma), \Phi(1) = *_{I(g)} \right\}.$$

If $\Phi: I \rightarrow I(g)$ belongs to $I(g)^I$ then there exists $(\rho, \Gamma) \in (Y \times Z^I)^I$ such that for all $s \in I$, $\Phi(s) = (\rho(s), \Gamma(s, \cdot)) \in I(g)$, $\Gamma(s, 0) = g(\rho(s))$ and $\Gamma(s, 1) = *_{Z}$ for all $s \in I$. Hence:

$$\begin{aligned} I(f_*) &= \left\{ (x, \gamma, \rho, \Gamma) \in X \times Z^I \times (Y \times Z^I)^I \mid \gamma(0) = g(f(x)), \gamma(1) = *_{Z}, \right. \\ &\quad \left. \Gamma(s, 0) = g(\rho(s)) \text{ and } \Gamma(s, 1) = *_{Z} \text{ for all } s \in I, \right. \\ &\quad \left. (f(x), \gamma) = (\rho(0), \Gamma(0, \cdot)), (\rho(1), \Gamma(1, \cdot)) = (*_{Y}, c_{*_{Z}}) \right\}. \end{aligned}$$

Therefore $\gamma(t) = \Gamma(0, t)$ for all $t \in I$, and thus $\gamma(1) = \Gamma(0, 1) = *_{Z}$ and $\gamma(0) = \Gamma(0, 0) = g(\rho(0)) = g(f(x))$, which implies that two of the defining conditions of $I(f_*)$ may be

removed, and γ is redundant. Hence we may identify $I(f_*)$ with the set:

$$\left\{ (x, \rho, \Gamma) \in X \times (Y \times Z^I)^I \mid f(x) = \rho(0), \rho(1) = *Y, \Gamma(1, t) = *Z \text{ for all } t \in I, \right. \\ \left. \Gamma(s, 0) = g(\rho(s)) \text{ and } \Gamma(s, 1) = *Z \text{ for all } s \in I \right\}. \quad (56)$$

LEMMA 28. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be pointed maps, and let $f_*: I(g \circ f) \rightarrow I(g)$ be the map induced by f on the level of homotopy fibres. Then $I(f_*)$ is homotopy equivalent to $I(f)$.

Proof. With the identification of $I(f_*)$ with the set given in (56), let $\psi: I(f_*) \rightarrow I(f)$ be defined by $\psi(x, \rho, \Gamma) = (x, \rho)$ and $\varphi: I(f) \rightarrow I(f_*)$ by $\varphi(x, \rho) = (x, \rho, \Gamma')$, where $\Gamma'(s, t) = g(\rho(\max(s, t)))$ for all $s, t \in I$. Note that φ is well defined since $f(x) = \rho(0)$, $\rho(1) = *Y$, $\Gamma'(s, 0) = g(\rho(s))$ and $\Gamma'(s, 1) = \Gamma'(1, t) = g(\rho(1)) = *Z$ for all $s, t \in I$. Clearly $\psi \circ \varphi = \text{Id}_{I(f)}$. It thus remains to prove that $\varphi \circ \psi \simeq \text{Id}_{I(f_*)}$, or equivalently, that there exists a homotopy in $I(f_*)$ that takes (x, ρ, Γ) to (x, ρ, Γ') , where Γ' is as defined above. It is clear that there exists a homotopy $H_u: I \times I \rightarrow I \times I$, where $u \in I$, for which H_0 is the identity, $H_1(s, t) \in I \times \{0\}$, $H_u(s, 0) = (s, 0)$, and $H_u(A) \subset A$ for all $u, s, t \in I$, where $A = I \times \{0\} \cup \{0\} \times I$. For example, we may take:

$$H_u(s, t) = \begin{cases} ((1 - 2u)s + 2u \max(s, t), t) & \text{if } 0 \leq u \leq \frac{1}{2} \\ (\max(s, t), 2(1 - u)t) & \text{if } \frac{1}{2} < u \leq 1. \end{cases}$$

Let $u, s, t \in I$. Since $H_{\frac{1}{2}}(s, t) = (\max(s, t), t)$ and $H_{\frac{1}{2}}(0, t) = (t, t)$, we see that H_u is continuous. For $u, s, t \in I$, let $\Gamma_u(s, t) = \Gamma(H_u(s, t))$. Then $\Gamma_0 = \Gamma$, $\Gamma(t, 0) = g(\rho(t))$, and $\Gamma_1(s, t) = \Gamma(\max(s, t), 0) = g(\rho(\max(s, t))) = \Gamma'(s, t)$, so $\Gamma_1 = \Gamma'$. It remains to show that $(x, \rho, \Gamma_u) \in I(f_*)$ for all $u \in I$. This is the case, since $f(x) = \rho(0)$, $\rho(1) = *Y$, $\Gamma_u(s, 0) = \Gamma(H_u(s, 0)) = \Gamma(s, 0) = g(\rho(s))$, $\Gamma_u(s, 1) = \Gamma(H_u(s, 1)) = *Z$ and $\Gamma_u(1, t) = \Gamma(H_u(1, t)) = *Z$ using the properties of H_u given above. It follows that $\varphi \circ \psi \simeq \text{Id}_{I(f_*)}$, which proves the result. \square

PROPOSITION 29. Let $p: E \rightarrow B$ be a fibration, and let E_0 be a subspace of E such that the restriction $p_0 = p|_{E_0}: E_0 \rightarrow B$ is also a fibration. Let F (resp. F_0) denote the fibre of p (resp. p_0) over a base point in B , let $\iota: E_0 \rightarrow E$, $\iota_0: F_0 \rightarrow F$, $j: F \rightarrow E$, and $j_0: F_0 \rightarrow E_0$ denote the respective inclusions, where $\iota_0 = \iota|_{F_0}$, and let $\alpha': I(\iota_0) \rightarrow I(\iota)$ be the map defined by $\alpha'(x, \gamma) = (j_0(x), j \circ \gamma)$ for all $(x, \gamma) \in I(\iota_0)$. Then α' is a homotopy equivalence between $I(\iota_0)$ and $I(\iota)$.

Proof. First note that we have the following commutative diagram:

$$\begin{array}{ccccc} I(\iota_0) & \longrightarrow & F_0 & \xrightarrow{\iota_0} & F \\ \downarrow \alpha' & & \downarrow j_0 & & \downarrow j \\ I(\iota) & \longrightarrow & E_0 & \xrightarrow{\iota} & E \\ & & \downarrow p_0 & & \downarrow p \\ & & B & \xlongequal{\quad} & B. \end{array} \quad (57)$$

Let $x_0 \in F_0$ be a basepoint that we propagate to F, E_0, E and B using this diagram. In particular, let $b_0 = p_0 \circ j_0(x_0)$ be the basepoint of B . The map $\alpha': I(\iota_0) \rightarrow I(\iota)$ defined in the statement is well defined, since if $(x, \gamma) \in I(\iota_0)$ then $\iota_0(x) = \gamma_0$ and $\gamma(1) = \iota_0(x_0)$, so $\alpha'(x, \gamma) = (j_0(x), j \circ \gamma) \in E_0 \times E^I$, and we have $j \circ \gamma(0) = j \circ \iota_0(x) = \iota(j_0(x))$, and $j \circ \gamma(1) = j \circ \iota_0(x_0) = \iota(j_0(x_0))$, which is the basepoint of E .

Taking $X = E_0$, $Y = E$, $Z = B$, $f = \iota$ and $g = p$ in the statement and proof of Lemma 28, and noting that $p_0 = p \circ \iota$, we obtain a map $\iota_*: I(p_0) \rightarrow I(p)$, and the map $\psi: I(\iota_*) \rightarrow I(\iota)$ defined in the first line of the proof of Lemma 28 is a homotopy equivalence. We will exhibit a homotopy equivalence $h: I(\iota_0) \rightarrow I(\iota_*)$ for which $\alpha' = \psi \circ h$, which will prove the result. Since p and p_0 are fibrations, the maps $g: F \rightarrow I(p)$ and $g_0: F_0 \rightarrow I(p_0)$ defined by $g(x) = (j(x), c_{b_0})$ for all $x \in F$ and $g_0(y) = (j_0(y), c_{b_0})$ for all $y \in F_0$ are homotopy equivalences [A, Proposition 3.5.10 and Remark 3.5.11]. Consider the following diagram:

$$\begin{array}{ccccc} I(\iota_0) & \longrightarrow & F_0 & \xrightarrow{\iota_0} & F \\ \downarrow h & & \downarrow g_0 & & \downarrow g \\ I(\iota_*) & \longrightarrow & I(p_0) & \xrightarrow{\iota_*} & I(p). \end{array}$$

Using the commutativity of (57), it is straightforward to check that the right-hand square is commutative, and we thus obtain an induced map $h: I(\iota_0) \rightarrow I(\iota_*)$ defined by $h(x, \gamma) = (g_0(x), g \circ \gamma)$ for all $(x, \gamma) \in I(\iota_0)$ that is a homotopy equivalence because g and g_0 are homotopy equivalences. If $c': I \rightarrow B^I$ is the path defined by $c'(t) = c_{b_0}$ for all $t \in I$, then using the definition of ψ , for all $(x, \gamma) \in I(\iota_0)$, we have:

$$\psi \circ h(x, \gamma) = \psi(g_0(x), g \circ \gamma) = \psi((j_0(x), c_{b_0}), (j \circ \gamma, c')) = (j_0(x), j \circ \gamma) = \alpha'(x, \gamma).$$

So $\alpha' = \psi \circ h$, and hence α' is a homotopy equivalence as required. \square

We end this paper with two corollaries of Proposition 29.

COROLLARY 30. *The map $\alpha'_n: I'_n \rightarrow I_n$ defined in Section 3.1 is a homotopy equivalence.*

Proof. It suffices to take the first two rows (resp. the fibrations p_0 and p) of (57) to be the bottom two rows (the fibrations p_n and \tilde{p}_n of Section 2) of (17), and to apply Proposition 29. \square

COROLLARY 31. *For all $k \geq 1$, the inclusion $j: F \rightarrow E$ and its restriction $j|_{F_0}: F_0 \rightarrow E_0$ induce an isomorphism $j_{\#k}: \pi_k(F, F_0) \rightarrow \pi_k(E, E_0)$.*

Proof. Recall from [A, Definition 4.5.3] and the paragraph that precedes it that the relative homotopy group $\pi_k(E, E_0)$ (resp. $\pi_k(F, F_0)$) is the homotopy group $\pi_{k-1}(I(\iota))$ (resp. $\pi_{k-1}(I(\iota_0))$) of the homotopy fibre of ι (resp. ι_0) for all $k \geq 1$. So for all $k \geq 2$, the map of pairs $j: (F, F_0) \rightarrow (E, E_0)$ induces a homomorphism $j_{\#k}: \pi_k(F, F_0) \rightarrow \pi_k(E, E_0)$ that coincides with the isomorphism $\alpha'_{\#(k-1)}: \pi_{k-1}(I(\iota_0)) \rightarrow \pi_{k-1}(I(\iota))$ given by Proposition 29 (observe that if $k = 1$ then as in [A, Definition 4.5.3], we refer to the map $j_{\#1}: \pi_1(F, F_0) \rightarrow \pi_1(E, E_0)$ as a homomorphism, and we know that it is also a bijection). \square

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