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A new H_∞ Observer for continuous LPV systems affected by Unknown Inputs

L. Meyer¹

Abstract—The main contribution of this paper is the development of a H_∞ based state observer for noisy Linear Parameter Varying (LPV) systems affected by Unknown Inputs (UI). The observer is constructed in order to be unbiased (in particular the state estimation error is independent of the UI) and with a minimum L_2 transfer between the perturbations (assumed to be with finite energy) and the estimation errors. Contrary to equivalent observers developed in the literature, the present one relaxes the classical rank condition widely used for the decoupling of the UI. In order to do so, High Order Sliding Modes (HOSM) differentiator are used. Finally, an example illustrates the theoretical contribution.

Index Terms— H_∞ observer, Unknown Input, Linear Parameter Varying systems, High Order Sliding Mode differentiator, Sobolev Space

I. INTRODUCTION

State estimation is of high importance in engineering as the behavior of any controlled system is based on its state knowledge. To that purpose, observers are of great interests and widely used. Among them, Kalman filtering is an observer in stochastic framework that minimizes the variance of the estimation error [1], Luenberger observers are dedicated for the deterministic frame [2], and H_∞ observers minimizes the L_2 transfer between the noises and the estimation error, which can be design using Linear Matrix Inequalities (LMI) formulation [3] [4].

The focus on linear systems (Linear Time Invariant (LTI) or Linear Time Varying (LTV) systems) is of first interest, as such systems can benefit of powerful mathematical results, while being a good approximation for a lot of real systems.

Beyond LTI and LTV systems, another class of systems is of great importance: the Linear Parameter Varying (LPV) systems where matrices depends on a bounded parameter. This kind of system is of great importance as they are closer to non linear systems than the other classical kind of linear systems (LTI, LTV), whereas most of linear mathematical techniques can still be applied on their study [5][6].

Among the systems studied in control theory, systems with unknown inputs are of particular interest. Indeed, a lot of real systems evolves according to a law in which some inputs are partially or totally unknown. The challenge is thus to obtain a good estimation of the state without any *a priori* knowledge on this unknown input.

The present paper deals with LPV systems in the presence of an Unknown Input (UI), such that the following system:

$$\begin{cases} \dot{x} &= A_\rho x + D_\rho d + F_\rho w \\ y &= C_\rho x + E_\rho d + v \\ \psi &= C_\rho x + E_\rho d \end{cases}, \quad (1)$$

where $x \in \mathbb{R}^{n_x}$ is the state of the system, $d \in \mathbb{R}^{n_d}$ denotes the unknown input, $y \in \mathbb{R}^{n_y}$ is the output, and $\psi \in \mathbb{R}^{n_\psi}$ is the free-noise output. $w \in \mathbb{R}^{n_w}$ and $v \in \mathbb{R}^{n_v}$ are respectively the state and measurement noises, that are assumed to be bounded. A_ρ , C_ρ , F_ρ , D_ρ and E_ρ are LPV-matrices with appropriate dimensions, $\rho \in \Omega$ being the parameter. $\Omega \subset \mathbb{R}^{n_\rho}$ is assumed to be convex and compact.

Systems such as (1) have already been studied in continuous or discrete-time framework in particular cases. In discrete-time, with LTI matrices and without any noises, [7] has dealt with state estimation. [8] and [9] have solved the problem without the presence of any unknown input in the measurement equation (i.e. $E = 0$ is assumed), and still with constant matrices, and without any noises. In this simple discrete-time cases (LTI systems without any noise), the former papers have used generalized rank condition on system matrices in order to decouple the unknown input from the state estimation error, which leads to time-delay observations. In continuous-time framework, [10] has proposed a state observer for Linear Time-Delay Time-Invariant Systems without any nose, and without any unknown input in the measurement equation. In the same way [11] has proposed a functional observer for Linear Time-Delay Time-Invariant descriptor systems.

[12] has dealt with continuous LTI systems without UI in measurement equation, and has proposed a dual state and unknown input H_∞ observer. In a similar framework, [13] has proposed a state observer based on an augmented form of a state observer for descriptor systems. However in that two lasts paper, the needed condition is the classical rank condition $rank(CD) = rank(D)$, and no solution has been proposed if that condition is not satisfied. [14] has treated the case of noise-free continuous-time Linear Parameter Varying (LPV) systems, with a relaxed rank condition and the use of the derivatives of the outputs. However no noise has been considered.

The aim and the main novelty of the present paper is to unify and generalized in a certain sense all of the previous works. Indeed, in this paper, E is not assumed to be equal

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to 0, and besides the classical rank condition $rank(CD) = rank(E)$ is relaxed in order to develop a more general observer. Furthermore, instead of limiting our approach to LTI systems, this work focuses on LTV/LPV systems. The problem raised is the construction of a state observer that minimizes the L_2 norm of the impact of the noises on the estimation error. In order to solve it, the use of successive derivatives of the measurement equation is needed. To do so, an High Order Sliding Mode (HOSM) differentiator is used, and Sobolev spaces are introduced, so that the differences between the successive true noise-free output derivatives and their estimations are bounded.

The paper is divided as follow. Section II recalls some results on Sobolev spaces and HOSM differentiator. The H_∞ problem is stated in section III. Then, section IV focuses on the derivation of the state observer. Finally, an academic example illustrating the theoretical contribution is proposed in section V.

II. PRELIMINARIES

A. Notations

The set of real matrices with $n \times p$ elements is denoted by $\mathcal{M}_{n,p}(\mathbb{R})$. The subset of invertible matrices of $\mathcal{M}_{n,n}(\mathbb{R})$ is denoted by $GL_n(\mathbb{R})$. $I_n \in \mathcal{M}_{n,n}(\mathbb{R})$ (I when there is no ambiguity) denotes the identity matrix and $0_{n,p} \in \mathcal{M}_{n,p}(\mathbb{R})$ (0 when there is no ambiguity) denotes the null matrix with n rows and p columns. $\mathbf{1}_{n,p} \in \mathcal{M}_{n,p}(\mathbb{R})$ ($\mathbf{1}$ when there is no ambiguity) denotes the matrix whose all elements equal to 1.

For any matrix $A \in \mathcal{M}_{n,p}(\mathbb{R})$, $tr(A)$ denotes its trace, A^T its transpose, and A^\dagger its pseudo-inverse. For any square matrix $A \in \mathcal{M}_{n,n}(\mathbb{R})$, $det(A)$ denotes its determinant, $\lambda(A)$ denotes the vector of its eigenvalues, and $\lambda_{min}(A)$ (resp. $\lambda_{min}^*(A)$, $\lambda_{max}(A)$) denotes the minimum (resp. minimum non null, when it exists, maximum) singular value of A . For any invertible matrix $A \in GL_n(\mathbb{R})$, A^{-1} denotes its inverse.

If $A = A^T \in \mathcal{M}_{n,n}(\mathbb{R})$ is a symmetric matrix, then $A > 0$ (resp. $A < 0$) means that A is a positive (resp. negative) definite matrix, and $A \geq 0$ (resp. $A \leq 0$) means that A is a nonnegative (res. nonpositive) matrix.

For a vector v , $\|v\|_2$ denotes its euclidean norm. For a matrix $A \in \mathcal{M}_{n,n}$, the norm of the maximum is $\|A\|_{max} = \max_{i=1..n, j=1..n} |A_{i,j}|$, and the induced L_2 norm is $\|A\|_2 = \sqrt{\max_{i=1..n} \lambda_i(A^T A)}$. Note that for any square matrix $A \in \mathcal{M}_{n,n}(\mathbb{R})$, $\|A\|_{max} \leq \|A\|_2 \leq n\|A\|_{max}$.

Let consider that x is a function of time. $\|x\|_2$ denotes its euclidean norm defined by $\|x\|_2 = \sqrt{\int_{t=0}^{+\infty} x(t)^T x(t) dt}$, and $\|x\|_\infty$ denotes its infinity norm defined by $\|x\|_\infty = \sup\{\|x(t)\|_2, t \geq 0\}$.

For $n \geq 1$ and $r \geq 1$, let set $U_{n,r} = \begin{bmatrix} I_n & 0_{n,r \times n} \end{bmatrix}$.

B. On the parameter

Throughout all this paper, the following assumption on the parameter ρ holds.

Assumption 1: All Parameter Varying matrices depend **continuously** on the parameter $\rho \in \Omega$.

Remark 1: Ω being a compact set, assumption 1 implies that for each matrix M_ρ , the subset $\{M_\rho, \rho \in \Omega\}$ is also a compact set (this is due to Heine theorem which states that the image of a compact set by a continuous function is a compact set), and thus is bounded. In particular, the sum, the difference or any product of such matrices stay bounded. The inverse of such an LPV matrix stays bounded if for all possible values of the parameter, the inverse exists.

C. Sobolev Spaces and HOSM differentiator

The main difficulty in continuous observation of systems affected by UI is the derivation of the outputs. Indeed, the derivative is needed in order to decouple the state estimation error from the UI (so that the estimation is guaranteed regardless of the UI variations). If the output is noise-free, there is no difficulty to do that, and any differentiator can be used. However, as it is the case in this paper, the presence of noise in the output is a big difficulty while making the derivation of it. This justify the following assumption.

Assumption 2: The noises w (resp. v) are assumed to be $s-1$ (resp. s) times differentiable, and all their respective derivatives are be bounded: $|w| \leq \bar{w}$, $|\dot{w}| \leq \bar{\dot{w}}$, ..., $|w^{(s-1)}| \leq \bar{w}^{(s-1)}$, $|v| \leq \bar{v}$, $|\dot{v}| \leq \bar{\dot{v}}$, ..., $|v^{(s)}| \leq \bar{v}^{(s)}$, where \bar{w} , $\bar{\dot{w}}$, ..., $\bar{w}^{(s-1)}$, \bar{v} , $\bar{\dot{v}}$ and $\bar{v}^{(s)}$ are positive constants.

In a more formal way, and as in [15], the previous assumption can be stated under the frame of Sobolev spaces. A Sobolev space is a normed vector space of functions with a norm, as well as its derivatives up to a given order are part of \mathcal{L}_p for a given p .

Definition 1: Let be s and p two integers (possibly ∞). The Sobolev space $\mathcal{W}_n^{s,p}$ is defined by:

$$\mathcal{W}_n^{s,p} = \{z : [0, \infty] \rightarrow \mathbb{R}^n \mid \frac{\partial^i z}{\partial t^i} \in \mathcal{L}_p^n([0, \infty]), \forall i = 0, \dots, s\}. \quad (2)$$

The associated norm is the following:

$$\|z\|_{s,p}^n = [\sum_{i=0}^s (\|\frac{\partial^i z(t)}{\partial t^i}\|_{\mathcal{L}_p^n}^p)]^{1/p} = (\sum_{i=0}^s \int_0^{+\infty} \|\frac{\partial^i z(t)}{\partial t^i}\|^p dt)^{1/p} \quad (3)$$

Now, assumption 2 can be stated as:

Assumption 3: There exist an integer s such that $w \in \mathcal{W}_{n_w}^{s-1, \infty}$, $v \in \mathcal{W}_{n_v}^{s, \infty}$ and $\rho \in \mathcal{W}_{n_\rho}^{s, \infty}$.

Remark 2: In order to complete remark 1 it is worth noting that all matrices depending continuously of ρ or one of its successive derivative till the order s are compact. And it is also the case of the sum, the difference, the product or the inverse (provided that it exists) of such matrices.

Besides the previous assumption, the output, as well as the Unknown Input are also assumed to be s times differentiable (without bounded assumption here).

Assumption 4: The output y as well as the unknown input d in 1 are both s times differentiable.

In a practical point of view, even if the noises are bounded, the derivation remains a problem, as classical method of

derivation (as Euler differentiator) can introduce other errors. The use of the following differentiator introduced by [16] enables to tackle this problem.

The calculation of the successive derivatives of y in the presence of the noise v can be done thanks to the following High Order Sliding Mode differentiator detailed in [16]:

$$\begin{cases} \dot{q}_0 = v_0, & v_0 = -\lambda_0|q_0 - y(t)|^{\frac{s}{s+1}} \text{sign}(q_0 - y(t)) + q_1 \\ \dot{q}_i = v_i, & i = 1, \dots, s-1 \\ v_i = -\lambda_i|q_i - v_{i-1}|^{\frac{s-i}{s-i+1}} \text{sign}(q_i - v_{i-1}) + q_{i+1} \\ \dot{q}_s = -\lambda_s \text{sign}(q_s - v_{s-1}) \end{cases} \quad (4)$$

where $\lambda, k = 0, \dots, s$ are positive constants to be tuned (a procedure is proposed in [16]).

The values $q, k = 1..s$, solutions of (4), are approximations of the successive derivatives of the free-output ψ . Due to the presence of the noise, the differentiation suffer from some estimation error (error between q and $\psi^{(k)}$). An upper bound of this error is given by the following theorem due to [16].

Theorem 1: [16] Let $y = \psi + v : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a s times continuously differentiable signal, with $|v| \leq \bar{v}$, then there exist $0 \leq T < \infty$, and some constants $\mu > 0, k = 0, \dots, s$ (dependent on $\lambda, k = 0, \dots, s$ only) such that for all $t \geq T$:

$$|q(t) - \psi^{(k)}(t)| \leq \mu|v(t)|^{\frac{s-k+1}{s+1}}, \quad k = 0, \dots, s. \quad (5)$$

The previous theorem claims that the errors on the calculation of the successive derivatives of ψ with (4) are bounded. Using the bounds given by that theorem, it comes for each $k = 0, \dots, s, t \geq T$:

$$q(t) = \psi^{(k)}(t) + \beta(t), \quad (6)$$

where $|\beta(t)| \leq \bar{\beta} := \mu\bar{v}^{\frac{s-k+1}{s+1}}$ for any $t \geq T$.

D. Some useful relationships

Before establishing the main results, let state some useful relationships. The first one is the compact form of the expression of the output, as well as its successive derivative in a unique vector. It can be established (the calculation is straightforward using the previous notations) that for all $r \geq 1$ and $s \leq r$, if $y^{(s)}$ is under the form:

$$y^{(s)} = \mathcal{A}_{r,\rho}^s x + \mathcal{D}_{r,\rho}^s \tilde{d}_s + \mathcal{F}_{r,\rho}^s \tilde{w}_s + \tilde{v}_s, \quad (7)$$

where $\tilde{d}_s = [d \ \dot{d} \ \dots \ d^{(s)}]^T$, $\tilde{w}_s = [v \ \dot{v} \ \dots \ v^{(s-1)}]^T$ and $\tilde{v}_s = [\beta_0 \ \beta_1 \ \dots \ \beta_s]^T$, and with some appropriate matrices $\mathcal{A}_{r,\rho}^s, \mathcal{D}_{r,\rho}^s$ and $\mathcal{F}_{r,\rho}^s$, then $y^{(s+1)}$ is under the form (a straightforward calculation leads to that result):

$$y^{(s+1)} = \mathcal{A}_{r,\rho}^{s+1} x + \mathcal{D}_{r,\rho}^{s+1} \tilde{d}_{s+1} + \mathcal{F}_{r,\rho}^{s+1} \tilde{w}_{s+1} + \tilde{v}_{s+1}, \quad (8)$$

where $\mathcal{A}_{r,\rho}^s, \mathcal{D}_{r,\rho}^s$ and $\mathcal{F}_{r,\rho}^s$ are defined recursively by:

$$\begin{aligned} \mathcal{A}_{r,\rho}^{s+1} &= \mathcal{A}_{r,\rho}^s + \mathcal{A}_{r,\rho}^s A_\rho, \text{ with } \mathcal{A}_{r,\rho}^0 = C_\rho, \\ \mathcal{D}_{r,\rho}^{s+1} &= [\mathcal{A}_{r,\rho}^s D_\rho \quad \mathcal{D}_{r,\rho}^s] + [\dot{\mathcal{D}}_{r,\rho}^s \quad 0_{n_y, n_d}], \text{ with } \mathcal{D}_{r,\rho}^0 = E_\rho, \text{ and} \\ \mathcal{F}_{r,\rho}^{s+1} &= [\mathcal{A}_{r,\rho}^s F_\rho \quad \mathcal{F}_{r,\rho}^s] + [\dot{\mathcal{F}}_{r,\rho}^s \quad 0_{n_y, n_w}], \text{ with } \mathcal{F}_{r,\rho}^0 = 0_{n_y, n_w}. \end{aligned}$$

Therefore, it comes that:

$$\tilde{y}_r = \mathcal{A}_{r,\rho} x + \mathcal{D}_{r,\rho} \tilde{d}_r + \mathcal{F}_{r,\rho} \tilde{w}_r + \tilde{v}_r, \quad (9)$$

$$\text{where } \tilde{y}_r = \begin{bmatrix} q_0 \\ q_1 \\ \dots \\ q_r \end{bmatrix}, \text{ and } \mathcal{A}_{r,\rho} = \begin{bmatrix} \mathcal{A}_{r,\rho}^0 \\ \mathcal{A}_{r,\rho}^1 \\ \mathcal{A}_{r,\rho}^2 \\ \dots \\ \mathcal{A}_{r,\rho}^r \end{bmatrix}, \mathcal{D}_{r,\rho} = \begin{bmatrix} \mathcal{D}_{r,\rho}^0 & 0_{n_y, r n_d} \\ \mathcal{D}_{r,\rho}^1 & 0_{n_y, (r-1)n_d} \\ \mathcal{D}_{r,\rho}^2 & 0_{n_y, (r-2)n_d} \\ \dots & \dots \\ \mathcal{D}_{r,\rho}^r & \dots \end{bmatrix}$$

$$\text{and } \mathcal{F}_{r,\rho} = \begin{bmatrix} \mathcal{F}_{r,\rho}^0 & 0_{n_y, r n_w} \\ \mathcal{F}_{r,\rho}^1 & 0_{n_y, (r-1)n_w} \\ \mathcal{F}_{r,\rho}^2 & 0_{n_y, (r-2)n_w} \\ \dots & \dots \\ \mathcal{F}_{r,\rho}^r & \dots \end{bmatrix}.$$

Besides, and according to equation (6), the following inequalities give another result for any time $t \geq T$ (with T given by theorem 1):

$$\begin{cases} \tilde{w}_r \leq \bar{W}_r \\ \tilde{v}_r \leq \bar{V}_r \end{cases}, \quad (10)$$

where $\bar{W}_r = [\bar{w} \ \bar{\dot{w}} \ \dots \ \bar{w}^{(r-1)}]^T$ and $\bar{V}_r = [\bar{\beta}_0 \ \bar{\beta}_1 \ \dots \ \bar{\beta}_r]^T$.

III. PROBLEM STATEMENT

Throughout that paper, it is assumed that ρ as well as its derivatives are perfectly known.

It is worth noting that there is no a priori knowledge on the unknown input d , even in terms of boundedness: it can be any disturbance, bounded or not, e.g. any unbounded noise or perturbation, or any unknown parameter of the system. However, the state estimation will be bounded nonetheless thanks to the main results of this paper.

Remark 3: The results of this paper are presented for LPV systems such as system (1). However, the main results of the paper (theorems 3 and 4) are still true for LTV systems (and therefore LTI systems), that is if matrices $A_{\rho(t)}$ (resp. $N_{\rho(t)}$, etc.) are replaced by matrices A_t (resp. B_t , etc.) depending on time, providing that they are bounded.

Let consider the following observer for system (1):

$$\dot{\hat{x}}(t) = M_\rho \hat{x}(t) + K_\rho \tilde{y}_r(t), \quad (11)$$

$r > 0$ being a positive integer, and M_ρ and K_ρ are gain matrices.

Note that (11) uses $\tilde{y}_r(t)$ which is a concatenation of the successive derivatives of the output, which is known provided that assumptions (3) holds, and provided that one uses the differentiation method proposed in 4.

Remark 4: One of the aims of the work is also to find the minimal integer r , as this is linked to the number of derivation of $y(t)$ to perform.

Problem 1: The aim of this paper is to design an unbiased H_∞ observer, i.e. to determine the appropriate integer $r > 0$, and to determine matrices M_ρ and K_ρ such that the following conditions are satisfied (with the state estimation error denoted by $e = x - \hat{x}$):

- 1) The state observer is unbiased, that is, if the noises are null, $w = 0$ and $v = 0$, and if $e_0 = 0$, then $e = 0$ for all $t \geq 0$. In particular, the state estimation error e is independent from unknown input d .
- 2) The state observer is stable, that is, if $w = 0$ and $v = 0$, then $e \rightarrow 0$, whatever e_0 is.
- 3) The L_2 transfer between noises and state estimation error is minimized:

$$\sup_{\omega|0 < \|\omega\|_2 < \infty} \frac{\|e\|_2}{\|\Gamma\omega\|_2} < 1, \quad (12)$$

where Γ is a diagonal matrix with the minimal possible trace (such that the previous condition holds).

The following lemma gives a sufficient condition in order to establish the L_2 attenuation between the noise and the state estimation error.

Lemma 2: Let consider the following dynamics equation:

$$\dot{x} = f(x, w), \quad (13)$$

where x is a vector depending on time, w the perturbation (or noise), and f is any continuous function. Let Γ be a diagonal matrix with positive diagonal (the desired L_2 attenuation). If there exist a Lyapunov function $V(t) \geq 0$, such that for all $t > 0$:

$$x^T(t)x(t) - w(t)^T \Gamma^T \Gamma w(t) + \dot{V}(x) < 0, \quad (14)$$

the following statement holds:

$$\sup_{\omega|0 < \|\omega\|_2 < \infty} \frac{\|e\|_2}{\|\Gamma\omega\|_2} < 1, \quad (15)$$

where: $\|s\|_{L_2} = \sqrt{\int_0^{+\infty} s(t)^T s(t) dt}$.

According to problem 1 and lemma 2 the H_∞ criteria and its results works for bounded perturbations. This justify a posteriori the introduction of Sobolev spaces and HOSM differentiator.

The following assumption is needed in order to establish the main results of the paper. It enables to decouple the unknown input from the state estimation error (as it will be seen in the proof of theorem 3).

Assumption 5: There exist an integer $r > 0$ such that the following rank condition is satisfied:

$$\text{rank} \left(\begin{pmatrix} \mathcal{D}_{r,\rho} \\ D_\rho U_{n_d,r} \end{pmatrix} \right) = \text{rank}(\mathcal{D}_{r,\rho}). \quad (16)$$

IV. MAIN RESULTS

The aim of this section is to derive LMI conditions for Unbiased State Observer with minimum L_2 -transfer for system (1).

A. Unbiased State Observer

The following theorem gives the gain values of the observer defined by equation (11) in order to make it unbiased.

Theorem 3: Let assumption 5 hold, and let consider the minimal integer $r \geq 1$ whose existence is given by that

assumption. Then equation (11) defines an unbiased observer for system (1) with gains given by:

$$\begin{cases} M_\rho &= A_\rho - K_\rho \mathcal{A}_{r,\rho} \\ K_\rho &= G_\rho + X_\rho H_\rho \end{cases}, \quad (17)$$

where

$$\begin{cases} G_\rho &= D_\rho U_{n_d,r} \mathcal{D}_{r,\rho}^+ \\ H_\rho &= I_{(r+1)n_y} - \mathcal{D}_{r,\rho} \mathcal{D}_{r,\rho}^+ \end{cases}. \quad (18)$$

where $X_\rho \in \mathcal{M}_{n_x, (r+1)n_y}$ is an arbitrary matrix. Besides, in that case, the state estimation error $e = x - \hat{x}_k$ follows the dynamics given by:

$$\begin{aligned} \dot{e} &= (A_\rho - G_\rho \mathcal{A}_{r,\rho} - X_\rho H_\rho \mathcal{A}_{r,\rho})e \\ &+ (F_\rho U_{n_w,r} - G_\rho \mathcal{F}_{r,\rho} - X_\rho H_\rho \mathcal{F}_{r,\rho})\tilde{w}_r - (G_\rho + X_\rho H_\rho)\tilde{v}_r. \end{aligned} \quad (19)$$

Remark 5: Matrix X_ρ defined in theorem 3 will be determined later in order to minimize the L_2 transfer.

Proof: Let consider the estimation error dynamics $e = x - \hat{x}$. Its dynamics is given by:

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} \\ &= A_\rho x + D_\rho d + F_\rho v - M_\rho \hat{x} - K_\rho \tilde{y}_r \\ &= A_\rho x + D_\rho d + F_\rho w - M_\rho \hat{x} - K_\rho \mathcal{A}_{r,\rho} x \\ &\quad - K_\rho \mathcal{D}_{r,\rho} \tilde{d}_r - K_\rho \mathcal{F}_{r,\rho} \tilde{w}_r - K_\rho \tilde{v}_r \\ &= (A_\rho - K_\rho \mathcal{A}_{r,\rho})e + (A_\rho - K_\rho \mathcal{A}_{r,\rho} - M_\rho)\hat{x}_\rho \\ &\quad + (D_\rho U_{n_d,r} - K_\rho D_{r,\rho})\tilde{C}_r + (F_\rho U_{n_w,r} - K_\rho \mathcal{F}_{r,\rho})\tilde{w}_r - K_\rho \tilde{v}_r \end{aligned} \quad (20)$$

In order to have an unbiased observer (i.e. $e = 0$ when there is no noise ($v = w = 0, \forall t \geq T$) and no initial error ($e(0) = 0$), it is necessary and sufficient to have:

$$\begin{cases} 0 &= A_\rho - K_\rho \mathcal{A}_{r,\rho} - M_\rho \\ 0 &= D_\rho U_{n_d,r} - K_\rho \mathcal{D}_{r,\rho} \end{cases} \quad (21)$$

and thus:

$$M_\rho = A_\rho - K_\rho \mathcal{A}_{r,\rho}, \quad (22)$$

and K_ρ has to satisfy $K_\rho \mathcal{D}_{r,\rho} = D_\rho U_{n_d,r}$. And thanks to assumption 5, this last equation has a solution for K_k , whose general form is given by:

$$K_\rho = G_\rho + X_\rho H_\rho, \quad (23)$$

with $G_\rho = D_\rho U_{n_d,r} \mathcal{D}_{r,\rho}^+$, $H_\rho = I_{(r+1)n_y} - \mathcal{D}_{r,\rho} \mathcal{D}_{r,\rho}^+$ and X_ρ can be any matrix (see remark 5).

Finally, the observer can be written as:

$$\dot{\hat{x}} = (A_\rho - G_\rho \mathcal{A}_{r,\rho} - X_\rho H_\rho \mathcal{A}_{r,\rho})\hat{x} + (G_\rho + X_\rho H_\rho)\tilde{y}_{r,\rho} \quad (24)$$

and the error dynamic is given by:

$$\begin{aligned} \dot{e} &= (A_\rho - G_\rho \mathcal{A}_{r,\rho} - X_\rho H_\rho \mathcal{A}_{r,\rho})e \\ &+ (F_\rho U_{n_w,r} - G_\rho \mathcal{F}_{r,\rho} - X_\rho H_\rho \mathcal{F}_{r,\rho})\tilde{w}_r - (G_\rho + X_\rho H_\rho)\tilde{v}_r. \end{aligned} \quad (25)$$

■

B. Unbiased H_∞ State Observer

In theorem 3, a gain X has been introduced. This gain will low be derived so that the L_2 -transfer from the noises to the state observer can be minimized.

Let consider the integer r used in theorem 3, and let rewrite the error equation given by this theorem:

$$\dot{e} = \bar{A}_\rho e + \bar{B}_\rho \omega, \quad (26)$$

where:

$$\begin{cases} \bar{A}_\rho &= A_\rho - G_\rho \mathcal{A}_{r,\rho} - X_\rho H_\rho \mathcal{A}_{r,\rho} \\ \bar{B}_\rho &= \begin{bmatrix} F_\rho U_{n_w,r} - G_\rho \mathcal{F}_{r,\rho} & -G_\rho \end{bmatrix} + X_\rho \begin{bmatrix} -H_\rho \mathcal{F}_{r,\rho} & -H_\rho \end{bmatrix} \\ \omega &= \begin{bmatrix} \tilde{w}_r \\ \tilde{v}_r \end{bmatrix} \end{cases} \quad (27)$$

Theorem 4: Let assumptions 2 and 5 hold, and let consider the notations introduced in theorem 3. Let $\Gamma = \text{diag}[\gamma_w \dots \gamma_w \gamma_v \dots \gamma_v]$ ($r+1$ times γ_w and $r+1$ times γ_v) be the L_2 attenuation gain matrix, with $\gamma_w > 0$, $\gamma_v > 0$. System (19) is stable and, under the condition $e(0) = 0$, satisfies the minimal L_2 criteria transfer from the time T given by theorem 1, if there exist a matrix function $P(t) = P(t)^T > 0$, such that:

$$\begin{bmatrix} I + \bar{A}_\rho^T P + P \bar{A}_\rho + \dot{P} & P \bar{B}_\rho \\ \bar{B}_\rho^T P & -\Gamma^T \Gamma \end{bmatrix} < 0, \quad \forall t > T. \quad (28)$$

Proof: Let consider the following Lyapunov function:

$$V(t) = e(t)^T P(t) e(t). \quad (29)$$

It comes that $V(t) \geq 0$ for all k , and with the theorem's assumption that $e(0) = 0$, it comes that $V(0) = 0$. Then, according to lemma 2 the H_∞ filtering problem has a solution with the Γ attenuation if:

$$e^T e - \omega^T \Gamma^T \Gamma \omega + \dot{V} < 0. \quad (30)$$

Then, thanks to equation (26), it comes:

$$\dot{V} = \dot{e}^T P e + e^T P \dot{e} + e^T \dot{P} e = \begin{bmatrix} e \\ \omega \end{bmatrix}^T \begin{bmatrix} \bar{A}_\rho^T P + P \bar{A}_\rho + \dot{P} & P \bar{B}_\rho \\ \bar{B}_\rho^T P & 0 \end{bmatrix} \begin{bmatrix} e \\ \omega \end{bmatrix}. \quad (31)$$

Thus, equation (30) can be rewritten as:

$$\begin{bmatrix} e \\ \omega \end{bmatrix}^T \mathcal{P} \begin{bmatrix} e \\ \omega \end{bmatrix} < 0, \quad (32)$$

where $\mathcal{P} = \begin{bmatrix} I + \bar{A}_\rho^T P + P \bar{A}_\rho + \dot{P} & P \bar{B}_\rho \\ \bar{B}_\rho^T P & -\Gamma^T \Gamma \end{bmatrix}$. Finally, the H_∞ filtering problem has a solution if $\mathcal{P}(t) < 0$ for all $t \geq 0$, which is the condition given by the theorem's statement. ■

C. Linear Matrix Inequality condition

In order to be able to use the previous result in practical calculation of the gain matrix X_ρ , condition (28) needs to be rewrite under an Linear Matrix Inequality (LMI) condition on X_ρ . This is the aim of the following theorem. Before its statement, let set the following notations:

$$\begin{cases} \bar{A}_{1,\rho} &= A_\rho - G_\rho \mathcal{A}_{r,\rho} \\ \bar{A}_{2,\rho} &= -H_\rho \mathcal{A}_{r,\rho} \\ \bar{B}_{1,\rho} &= \begin{bmatrix} F_\rho U_{n_w,r} - G_\rho \mathcal{F}_{r,\rho} & -G_\rho \end{bmatrix} \\ \bar{B}_{2,\rho} &= \begin{bmatrix} -H_\rho \mathcal{F}_{r,\rho} & -H_\rho \end{bmatrix} \end{cases}, \quad (33)$$

which leads to:

$$\begin{cases} \bar{A}_\rho &= \bar{A}_{1,\rho} + X_\rho \bar{A}_{2,\rho} \\ \bar{B}_\rho &= \bar{B}_{1,\rho} + X_\rho \bar{B}_{2,\rho} \end{cases}. \quad (34)$$

Theorem 5: Under the assumptions and notations of theorem 4, the minimal L_2 transfer problem of system (1) can be written as:

$$\min_{\Gamma > 0} \text{trace}(\Gamma), \quad (35)$$

under the existence of matrix functions $P(t) = P(t)^T > 0 \in \mathcal{M}_{n_x}$ and $V(t) \in \mathcal{M}_{n_x, (r+1)n_y}$ satisfying the following LMI condition:

$$\begin{bmatrix} I + \bar{A}_{1,\rho}^T P + \bar{A}_{2,\rho}^T V_\rho^T + P \bar{A}_{1,\rho} + V_\rho \bar{A}_{2,\rho} + \dot{P} & P \bar{B}_{1,\rho} + V_\rho \bar{B}_{2,\rho} \\ \bar{B}_{1,\rho}^T P + \bar{B}_{2,\rho}^T V_\rho^T & -\Gamma^T \Gamma \end{bmatrix} < 0, \quad (36)$$

Finally, X_ρ is obtained by setting: $X_\rho = P^{-1} V_\rho$.

Proof: Equation (28) can be written as:

$$\begin{bmatrix} I + \bar{A}_{1,\rho}^T P + \bar{A}_{2,\rho}^T X_\rho^T P + P \bar{A}_{1,\rho} + P X_\rho \bar{A}_{2,\rho} + \dot{P} & P \bar{B}_{1,\rho} + P X_\rho \bar{B}_{2,\rho} \\ \bar{B}_{1,\rho}^T P + \bar{B}_{2,\rho}^T X_\rho^T P & -\Gamma^T \Gamma \end{bmatrix} < 0, \quad (37)$$

and the theorem's statement is obtained by setting $V_\rho = P X_\rho$. ■

V. ILLUSTRATIVE EXAMPLE

Let consider the following LTI example:

$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -2 \end{bmatrix} x(t) + \begin{bmatrix} -0.1 \\ 2 \\ 0 \end{bmatrix} d(t) + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} d(t) + v(t) \end{cases}, \quad (38)$$

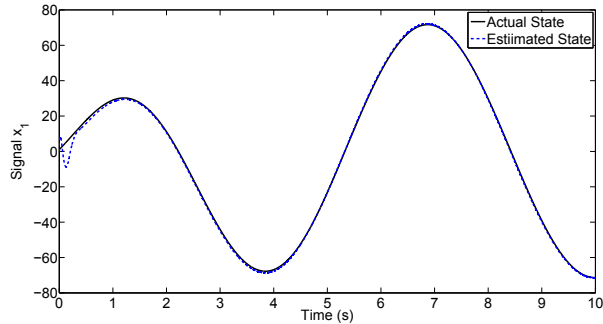
where

- the simulation is launched from $t = 0s$ to $t = 10s$,
- the perturbations are taken equal to $w(t) = 0.5 \cos(4t + 0.4)$ and $v(t) = 0.8 \sin(1.2t + 0.7) \begin{bmatrix} 1 & -1 \end{bmatrix}^T$,
- the unknown input is set to $d = \begin{bmatrix} 12 \cos(t) & -4 \sin(t) \end{bmatrix}^T$.

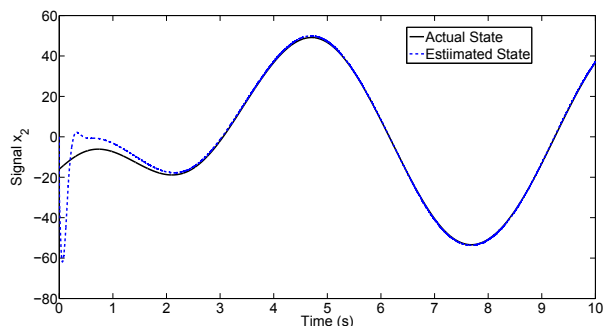
It can be noticed that the classical rank condition $\text{rank}(CD) = \text{rank}(D)$ does not hold in that case. More precisely, $r = 2$ is the minimal integer such that the condition given in Assumption 5 is satisfied.

The estimations of the successive output derivatives are obtained using the HOSM differentiator presented in equations (4) with $s = 2$, $\lambda_0 = 2.0L^{1/3}$, $\lambda_1 = 1.5L^{1/2}$, $\lambda_2 = 1.1L$, according to [17] and [16], where L is an estimation of the upper bound of the norm of the $(s + 1)$ -th derivative of Ψ : $\Psi^{(s+1)} < L$. In this example, $L = 200000$ is taken.

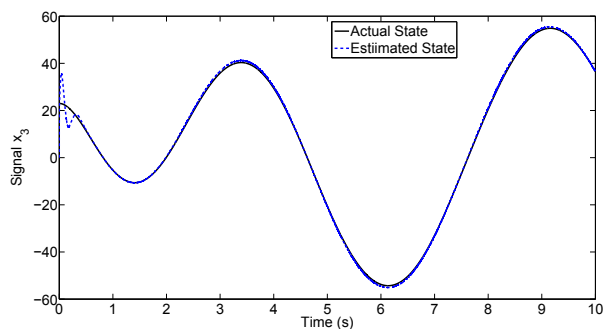
Finally, the results given by the observer proposed in the present paper are given on figures 1a to 1c.



(a) x_1 state estimation



(b) x_2 state estimation



(c) x_3 state estimation

Fig. 1: State estimations

VI. CONCLUSIONS

The main contribution of this paper is to propose a state observer for noisy LTV/LPV systems affected by unknown inputs in both state and measurement equations. The present

work relaxes the classical rank condition needed for the construction of the observer. This relaxation leads to the use of high order derivative that are obtained via sliding mode techniques in order to keep the noise's impact bounded for any order of differentiation. An example illustrates the theoretical contribution of the paper. In a future work, the present results may be applied in a in real-world scenario.

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