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When Self-Generated Gradients interact with Expansion by Cell Division and Diffusion. Analysis of a Minimal Model.

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Abstract

We investigate a minimal model for cell propagation involving migration along self-generated signaling gradients and cell division, which has been proposed in an earlier study. The model consists in a system of two coupled parabolic diffusion-advection-reaction equations. Because of a discontinuous advection term, the Cauchy problem should be handled with care. We first establish existence and uniqueness locally in time through the reduction of the problem to the well-posedness of an ODE, under a monotonicity condition on the signaling gradient. Then, we carry out an asymptotic analysis of the system. All positive and bounded traveling waves of the system are computed and an explicit formula for the minimal wave speed is deduced. An analysis on the inside dynamics of the wave establishes a dichotomy between pushed and pulled waves depending on the strength of the advection. We identified the minimal wave speed as the biologically relevant speed, in a weak sense, that is, the solution propagates slower, respectively faster, than the minimal wave speed, up to time extraction. Finally, we extend the study to a hyperbolic two-velocity model with persistence.

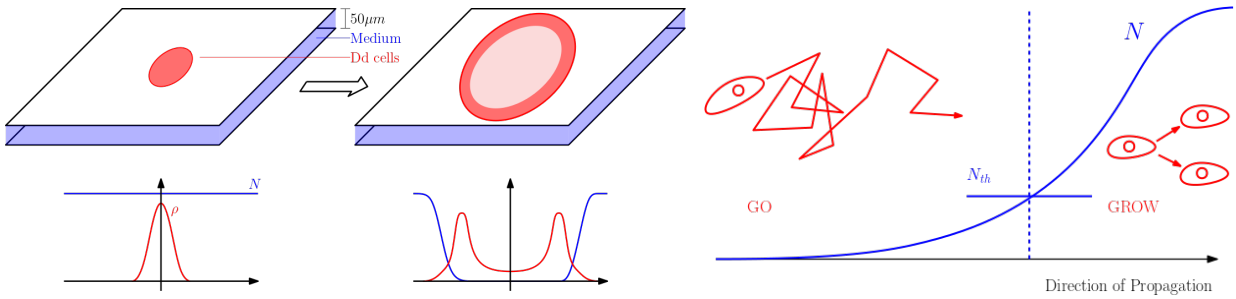
1 Introduction

In this paper, we are mainly concerned with the investigation of spreading properties for a one-dimensional parabolic system of two diffusion-advection-reaction equations, with $t \geq 0, x \in \mathbb{R}$,

$$\partial_t \rho - \partial_{xx}\rho + \partial_x(\chi \mathrm{sign}(\partial_x N)\mathbb{1}_{N \leq N_{\mathrm{th}}}\rho) = \mathbb{1}_{N > N_{\mathrm{th}}}\rho \tag{1a}$$

$$\partial_t N - D\partial_{xx}N = -\rho N. \tag{1b}$$

Here $\rho(t,x)$ describes a cell population subject to diffusion, with a diffusion constant normalized to 1, and either growth or advection depending on its position. The switch between growth and advection is mediated by the value $N(t,x)$ of a chemical nutrient field: for a given threshold value N_{th}, if $N > N_{\mathrm{th}}$, the population ρ is subject to growth with constant rate normalized to 1, and if $N \leq N_{\mathrm{th}}$, the population is subject to advection with constant speed $\chi > 0$ in the direction of the gradient $\partial_x N$. This advection speed results from biases in individual cell trajectories, which are averaged at the macroscopic level. The chemical nutrient N undergoes a reaction-diffusion equation through a simple consumption term $-\rho N$, with the consumption rate per cell normalized to 1. All along the article, we work in the setting, where N is increasing in space, $\lim_{x\to-\infty} N(t,x) = 0$ and $\lim_{x\to+\infty} N(t,x) = 1 > N_{\mathrm{th}}$, by normalizing



(a) (b)

Figure 1: (a): A schematic representation of the experiment carried out in [15]. Cells are confined between two narrowly spaced plates and quickly consume available oxygen, so that the colony experiences self-induced hypoxic conditions. This, in turn, triggers outward migration of the colony under the form of a ring expanding at constant speed over long periods of time. (b): A cartoon representation of the 'Go or Grow' hypothesis. Cells switch between two behaviors, depending on the level of oxygen. When oxygen concentration is above some threshold N_{th}, cells divide and follow Brownian trajectories (this is the 'Grow' behavior). In contrast, when oxygen concentration drops below N_{th}, cells stop dividing and follow a biased Brownian motion towards higher levels of oxygen (this is the 'Go' behavior).

the limit to 1. Under these conditions, the cell population propagates from left to right. Furthermore, we introduce the (unique) position of the threshold $\bar{x}(t)$, such that:

$$N(t, \bar{x}(t)) = N_{\text{th}}. \tag{2}$$

In Equation (1a), the advection term is discontinuous, but the flux should still be continous. Hence in particular at the interface $\bar{x}(t)$ the flux must be continuous. More precisely, consider a weak solution to Equation (1a), which is continuous and sufficiently regular on either side of the interface $\bar{x}(t)$. By a Rankine-Hugoniot type argument, ρ satisfies the following C^1-jump relation at the interface $\bar{x}(t)$:

$$\partial_x \rho \left(t, \bar{x}(t)^+\right) - \partial_x \rho \left(t, \bar{x}(t)^-\right) = -\chi \rho(t, \bar{x}(t)). \tag{3}$$

A typical initial datum $(\rho^0, N^0) \in L^\infty(\mathbb{R})^2$ for System (1) satisfies nonnegativity, *i.e.* $\rho^0, N^0 \geq 0$. In addition, we assume that ρ^0 satisfies the C^1-jump relation (3) and is bounded by an exponentially decreasing function at $x = +\infty$. Furthermore, $\partial_x N^0 > 0$, $\lim_{x\to-\infty} N^0(x) = 0$ and $\lim_{x\to+\infty} N^0(x) = 1 > N_{\text{th}}$.

System (1) was introduced in [15] by the author and collaborators as a minimal model for cell collective behaviour triggered by a self-generated gradient. In this study, the following emerging behavior of *Dictyostelium discoideum* cells (*Dd* cells in short) in hypoxic conditions was observed: when a colony of *Dd* cells is confined between two narrowly spaced plates, *Dd* cells form a dense ring moving outwards. After a brief transitory phase, the ring of cells moves at constant speed and constant density over the time course of the experiment (see Figure 1a). The authors emitted the hypothesis that the quick consumption of oxygen by *Dd* cells exposes them to hypoxia, *i.e.* lack of oxygen, and in turn induces *aerotaxis*, *i.e.* a bias in the individual trajectories of *Dd* cells towards higher oxygen concentrations, leading to a macroscopic outward motion. We refer to [8, 15] (see also [41]) for a biological discussion on the hypotheses under which the observed phenomenon may arise. The scientific approach in [15] leading to this minimal model can be described as follows: (i) Experimentally, it was observed that cells exhibit various individual behaviors accross the colony. (ii) In the model, two particular behaviors were retained as an alternative: either cell division, or migration towards oxygen. (iii) It was postulated that the transition between the two behaviors depends on a single threshold (see

Figure 1b). Indeed, it is for instance well known that *Dd* cells do not have enough energy to divide, when oxygen is lacking. The term 'Go or Grow' was coined to describe this dichotomy, by analogy with a similar mechanism in the modeling of glioma cells [27], which nevertheless is of another nature, as it describes a density-dependent rather than an oxygen-dependent switch between diffusion and cell division.

The mechanism of cell division in System (1) resembles to some extent to standard reaction-diffusion models, among which the classical Fisher/Kolmogorov-Petrovsky-Piskunov [4, 19, 31] is a prototype. The F/KPP model describes in particular expansion of a cell population undergoing cell division and diffusion. However, while in standard reaction-diffusion models, growth is limited by a density-mediated mechanism (*e.g.* a quadratic saturation term in the F/KPP case), here it is limited via the dependence on the chemical nutrient N, which leads to a similar regulatory mechanism: the more cells divide, the more they consume the chemical nutrient N, the more their growth is limited.

Interestingly, this minimal model is sufficient to describe the propagation of a wave of cells, in a context of self-generated oxygen gradients. In fact, System (1) exhibits explicit traveling wave solutions and the minimal wave speed can be computed explicitly (see [15] and also Section 3). This gives rise to the following formula for the minimal wave speed,

$$\sigma^* = \begin{cases} \chi + \frac{1}{\chi} & \text{if } \chi > 1 \\ 2 & \text{if } \chi \leq 1 \end{cases} . \tag{4}$$

Thus, there are two different regimes: in the regime $\chi \leq 1$, which we call the *small bias* regime, the speed corresponds to the well-known F/KPP speed, $\sigma_{F/KPP} := 2$, whereas in the regime of *large bias*, $\chi > 1$, the wave speed $\chi + \frac{1}{\chi}$ is greater than $\sigma_{F/KPP}$. The threshold is reached when the advection speed χ is equal to half the F/KPP speed $\sigma_{F/KPP}$: there, the two expressions coincide.

System (1) combines two distinct propagation phenomena, one being aerotaxis of cells triggered by the self-generated gradient, and the other being expansion by division-diffusion, such as described by the F/KPP model. Biologically, it is therefore relevant to ask how these two propagation phenomena combine with each other (see [8, 15]). Since the collective propagation speed σ^* is always higher than the advection speed χ, cell division has a net positive effect on the propagation of cells. In parallel, in the regime of large bias $\chi > 1$, aerotaxis has also in turn a net positive effect on the propagation speed, compared to mere expansion by division-diffusion, which is in agreement with the findings by [16]. In that case, we may refer to the wave as an aerotactic wave. Yet, in the regime of small bias $\chi \leq 1$, the propagation is driven by division-diffusion, and aerotaxis does not contribute to the wave speed: in that case, we refer to the propagation as a F/KPP wave.

Recently, numerous works, among which [16, 35, 50, 51], have investigated propagation of cells under self-generated gradients and shown its biological relevance as an efficient migration strategy. Notably in [16], the authors observe that the the combination of *chemotaxis* (the response to chemical gradients, which is very analogous to aerotaxis in a mathematical setting) and cell division, leads to an enhanced expansion.

Other works have investigated chemotactic waves in *Escherichia coli* bacteria (and references therein [12, 46]). These works have proposed a description at a mesoscopic scale, through a kinetic model, and at a macroscopic scale, through a parabolic model, analogous to some extent with System (1). However, the main difference is that in order to sustain the propagation of the wave, two attractants are required in [12, 46], whereas here the single attractant N is sufficient. Additionally in the case of *E. coli* cell division is negligible, while here it plays a key ingredient. We refer to [13] for a discussion on different models of propagation through self-generated signaling gradients.

In parallel, the Keller-Segel model [29, 30] has been widely used in order to give a description of cells undergoing chemotaxis. In [38], a variation on this model was proposed by adding a density-dependent growth term to the model. This Keller-Segel model with growth term has been the subject of numerous investigations in recent years, among which the works [10, 34, 38, 44, 45]. These types of

models combine chemotaxis and cell division and exhibit traveling waves under some conditions on the parameters. Nevertheless, chemotactic self-aggregation (in the aformentioned studies) and aerotaxis (in the present study) lead to biases in opposite directions at the edge of expansion front. Recently, in the works [26, 28], the authors have investigated the case of *negative chemotaxis*, where the bias induced by chemotaxis is in the same direction than the propagation induced by division-diffusion and thus bears a similarity to the aerotactic advection term. In [28], the author was able to obtain bounds on the propagation speed. Since the considered model is different from ours, this result cannot be directly compared to our explicit speed Formula (4), but in the regime of small negative chemotaxis, the wave speed exactly agrees, *i.e.* $\sigma^* = 2$, and the propagation is caused by division-diffusion. In the regime of large negative chemotaxis, the wave speed increases, which is also in agreement with our findings.

Interestingly, Formula (4) coincides with the formula for the wave speed obtained in the monostable cubic reaction-diffusion equation with reaction term $f(u) = u(1-u)(1+2\chi^2 u)$ [5, 25] or the Burgers-FKPP equation of the form $\partial_t u - \partial_{xx} u + 2\chi u \partial_x u = u(1-u)$ [3]. We also mention a class of free boundary problems introduced in [6], that is linked to the large-population limit of the N-Branching Brownian Motion [17]. The authors of [6] show that the following free boundary problem for $(u, \mu_t) \in C(\mathbb{R}_+ \times \mathbb{R}) \times C(\mathbb{R}_+)$ sufficiently regular admits Formula (4) as minimal wave speed:

$$\begin{aligned} &\partial_t u - \partial_{xx} u = u, \text{ for } x > \mu_t \\ &u(t, \mu_t) = 1 \text{ and } \partial_x u(t, \mu_t) = -\chi \end{aligned}$$

In [23, 42], the authors have investigated the inside dynamics of traveling waves in reaction-diffusion equations. They have proposed a new characterization of the categorization between *pushed* and *pulled* waves. A pushed wave is subject to a significant contribution from the overall population to the net propagation, whereas a pulled wave is driven by growth and diffusion of the population at the edge of the front with negligible contribution from the overall population. In particular, it was shown in [23] that in the monostable cubic reaction-diffusion equation with reaction term $f(u) = u(1-u)(1+2\chi^2 u)$ a transition from a pulled to a pushed nature of the wave exists at $\chi = 1$. The same dichotomy was observed at play in System (1), essentially via numerical simulations [15]: in the case of small bias $\chi \leq 1$, the traveling wave is pulled. In contrast, in the case of large bias $\chi > 1$, the traveling wave is pushed. A transition from pulled to pushed waves has recently been of interest in the works [21, 22, 40], but in these studies they are due to structural modifications of the system, *i.e.* the reaction term goes from a monostable structure to a bistable structure. However here, as well as in the case of the monostable cubic case and the Burgers-FKPP equation [3], it is a transition induced by the relative size of the parameters without changing the nature of the stable states.

Of note, the definitions of pulled and pushed waves can vary in the literature. The historical definition as proposed in [47] (see also [43, 52]) is based on the criterion whether the minimal speed σ^* is equal to the speed of the linearized front around the steady state 0 (pulled), or greater than this speed (pushed). The definition proposed in [23, 42] is based in turn on the inside dynamics of the traveling waves. In this paper, we follow the latter definition and more precisely the study of the inside dynamics of ρ, that we believe is more appropriate to the study of a system of equations, such as System (1). However, both definitions coincide in our case, since linearizing System (1) around its leading edge yields a constant chemical nutrient field $N \equiv 1$ and the linear F/KPP equation on ρ, which gives rise to a front traveling at speed $\sigma = 2$.

In this paper, we are mainly concerned with the study of the parabolic System (1), that can be viewed as a minimal model on a macroscopic scale including self-generated signaling gradients and cell division. However, following the discovery of the run and tumble motion in *E. coli* [7], it may be relevant to model collective motion of micro-organisms on a mesoscopic scale through kinetic transport equations, see for instance [1, 2, 14, 39, 48]. In particular in [12], the author has investigated the existence of traveling waves in *E. coli* populations propagating in microchannels. In *Dd* cells, persistent

motion is observed as well [15]. Therefore, it is very natural to propose a kinetic model for their study, involving free transport and reorientations of cells. Hence, by analogy with the aforementioned studies, we propose the following kinetic model:

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) = \lambda\left(M(v;N,\partial_x N)\rho - f\right) + r(N)\rho \tag{5a}$$
$$\partial_t N - D\partial_{xx}N = -\rho N, \tag{5b}$$

where $v \in V$, a compact subset of $\mathbb{R}$ and $\rho(t,x) = \frac{1}{|V|}\int_V f(t,x,v)dx$. Equation (5a) describes the evolution of the mesoscopic density of cells, that undergo cell division and persistent motion: cells move with velocity v and at a constant rate λ cells reorient themselves according to the probability distribution described by the Maxwellian $M(v;N,\partial_x N)$. In addition, cells divide with rate $r(N)$, depending on the ambient oxygen level, and the new cells have a velocity that is drawn from the uniform distribution on V. In particular, we can adapt the 'Go or Grow' hypothesis to Equation (5a): for $N < N_{\text{th}}$, set cell division to zero and a fixed Maxwellian distribution with mean $\chi\text{sign}(\partial_x N)$; for $N > N_{\text{th}}$, set a fixed nonzero cell division rate and a fixed Maxwellian with zero mean.

Whilst the general study of System (5) is postponed to future investigations, we analyze in the present study the two-velocity case, which we will refer to as the two-velocity system with persistence. In fact, we consider $V = \{\pm\varepsilon^{-1}\}$, with rescaled velocity and System (5) becomes a system of two hyperbolic equations for $f^{\pm}(t,x) := f(t,x,\pm\varepsilon^{-1})$ and a parabolic equation for N, with $t \geq 0, x \in \mathbb{R}$,

$$\partial_t f^+ + \varepsilon^{-1}\partial_x f^+ = \varepsilon^{-2}\left(M(+\varepsilon^{-1};N,\partial_x N)\rho - f^+\right) + \mathbb{1}_{N>N_{\text{th}}}\rho \tag{6a}$$
$$\partial_t f^- - \varepsilon^{-1}\partial_x f^- = \varepsilon^{-2}\left(M(-\varepsilon^{-1};N,\partial_x N)\rho - f^-\right) + \mathbb{1}_{N>N_{\text{th}}}\rho \tag{6b}$$
$$\partial_t N - D\partial_{xx}N = -\rho N, \tag{6c}$$

where $\rho := \frac{f^+ + f^-}{2}$ and,

$$M(\pm\varepsilon^{-1};N,\partial_x N) = \begin{cases} 1 & \text{if } N > N_{\text{th}} \\ 1 \pm \varepsilon\chi & \text{if } N \leq N_{\text{th}} \text{ and } \partial_x N \geq 0 \\ 1 \mp \varepsilon\chi & \text{if } N \leq N_{\text{th}} \text{ and } \partial_x N < 0 \end{cases},$$

with $\chi < \varepsilon^{-1}$ and, up to a scale of units, $r(N) = \mathbb{1}_{N>N_{\text{th}}}$, $\lambda = \varepsilon^{-2}$ and the mean of the Maxwellian distribution equal to $\pm\varepsilon\chi$, when $N \leq N_{\text{th}}$.

The parabolic System (1) and the two-velocity System with persistence (6) are linked through the so-called parabolic scaling limit. Indeed, taking the limit $\varepsilon \to 0$ of Equations (6) leads, at least in a formal sense, to Equations (1). We refer to [14] for a rigorous derivation in a general chemotaxis model without cell division.

Results and Strategies of Proof

The present article contains on the one hand an analysis of the well-posedness of System (1) locally in time and on the other hand an asymptotic analysis of System (1), including: the computation of traveling wave solutions, the study of the inside dynamics of the traveling waves, as well as a weak characterization of the asymptotic behavior of the spreading speed in the Cauchy problem. We also characterize traveling wave solutions for the two-velocity System with persistence (6). All along the article we work in the setting where N is increasing in space and cells propagate from left to right. Concerning the well-posedness we give a non-optimal criterion on the initial datum N^0 under which monotonicity of N is preserved locally in time and globally in space. However, for the rest of the article, monotonicity of N is a restrictive assumption in our study.

In Section 2, we prove an existence and uniqueness result for the parabolic System (1) locally in time under the assumption that N is initally monotonic. Because of the discontinuity of the advection

coefficients, involving the coupling with the signaling gradient, direct application of Banach's Fixed point Theorem seems not directly applicable. To circumvent this delicate issue, we use the monotonicity of N and the definition of the threshold position $\bar{x}(t)$ (see Equation (2)) and apply Banach's Fixed Point Theorem to the curve $\bar{x}(\,\cdot\,)$. Our strategy relies on an endpoint estimate of N in $W^{3,\infty}$, which is out of the range of textbook estimates (to the best of our knowledge), that we achieve by a careful handling of the singularity at the interface. This reduction to an equation on the dynamics of a curve, coupled with a PDE, is reminiscent of studies in one-dimensional free boundary problems (see, *e.g.* Chapter 3 in [18] on the Stefan problem, [33] in the context of front propagation, or [37] in the context of mutation-selection dynamics in evolutionary biology).

In Section 3, we exhibit all positive and bounded traveling wave solutions for Equations (1), which propagate from left to right, *i.e.* all stationary solutions in the frame $(t, z) = (t, x - \sigma t)$, with $\sigma \geq 0$. This completes the preliminary analysis performed in [15]. In that case, Equation (1a) reduces to a piecewise constant second-order differential equation. All traveling wave profiles of ρ are a concatenation of a constant profile on the left side and an exponentially decreasing profile on the right side. In both the small and large bias regimes, there exists a minimal velocity σ^* for traveling waves, given by Formula (4). For each $\sigma \geq \sigma^*$, there exists an associated wave profile (ρ^σ, N^σ), whose exponential decay at $z = +\infty$ is slower than the decay of the profile associated to the minimal velocity σ^*, which will be a crucial observation for Section 5.

In Section 4, we investigate the inside dynamics of the traveling waves. We introduce the formalism of neutral fractions [23, 42] and extent it to System (1). The methodology consists in studying the evolution of a fraction $\nu = \frac{\rho}{\rho^\sigma}$ of the traveling wave, relative to the stationary dynamics in the moving frame prescribed by the traveling wave solution ρ^σ. This gives rise to a linear parabolic equation,

$$\partial_t \nu + L\nu = 0.$$

In the case of large bias ($\chi > 1$), the elliptic operator L is self-adjoint in a weighted L^2-space and has the following spectral properties: 0 is a simple eigenvalue, whose eigenspace is spanned by the constants, and the operator L has a spectral gap. This leads to the conclusion that every neutral fraction converges exponentially to a constant in a weighted L^2-norm, which constitues the signature of a pushed wave, according to [23, 42]. In contrast, in the regime of small bias ($\chi \leq 1$), under the condition that $\nu_0 \rho^{\sigma^*}$ is square-integrable at $z = +\infty$, the solution ν converges to 0, which is the signature of a pulled wave. To do so, we use an energy method in an L^2-setting to show uniform convergence to 0 on intervals of the form $[a, +\infty)$.

In Section 5, we give a weak description of the asymptotic behavior of solutions to System (1). In fact, if we define the instantaneous spreading speed to be $\dot{\bar{x}}(t)$, we show that for initial conditions bounded above by a multiple of ρ^{σ^*} and under the technical assumption that $\dot{\bar{x}} \in L^\infty(\mathbb{R}_+)$, we have that,

$$\liminf_{t \to +\infty} \dot{\bar{x}}(t) \leq \sigma^*, \text{ and } \limsup_{t \to +\infty} \dot{\bar{x}}(t) \geq \sigma^*.$$

In other words, for an inital cell profile, whose exponential decay is faster than ρ^{σ^*}, up to a time extraction, the cell profile spreads either slower or quicker than the minimal wave speed σ^*. Hence, an important conclusion of this Section is that for biologically relevant initial conditions (*e.g.* a profile, whose support is bounded above), the only reasonable candidate for convergence to a traveling wave profile, is the one associated to the minimal wave speed σ^*. However, convergence in a proper sense to the traveling wave profile remains an open problem, a major difficulty being notably the lack of a suitable comparison principle.

Finally, in Section 6, we compute all subsonic traveling wave solutions, *i.e.* $\sigma < \varepsilon^{-1}$, for System (6). Subsonic traveling wave solutions exist if and only if $\varepsilon^{-1} > 1$. The structure of these solutions then follows *mutatis mutandis* the structure of the solutions for the parabolic System (1). In particular

we find a similar expression for the minimal wave speed,

$$\sigma^* = \frac{1}{1+\varepsilon^2} \cdot \begin{cases} \chi + \frac{1}{\chi} & \text{if } \chi \in (1, \varepsilon^{-1}) \\ 2 & \text{if } \chi \leq 1 \end{cases} . \tag{7}$$

The proof consists in solving piecewise constant linear differential equations. Furthermore, the velocity formula for $\chi \leq 1$ coincides with the velocity of traveling waves in two-velocity models with a reaction term, but without advection (see for instance [9, 24]).

2 Existence and Uniquess of Solutions for the Parabolic Model

In this Section, we establish existence and uniqueness locally in time for the parabolic System (1), under certain conditions. The main difficulty to prove such a result stems from the singular advection term $\partial_x \left(\chi \text{sign}(\partial_x N) \mathbb{1}_{N \leq N_{\text{th}}} \rho\right)$ in Equation (1a). Therefore, we will work in the framework, where N is increasing and $N(t, \cdot\,) = N_{\text{th}}$ admits a unique solution $\bar{x}(t)$. In this framework, System (1) is equivalent to the following simpler System:

$$\partial_t \rho - \partial_{xx} \rho + \partial_x (\chi \mathbb{1}_{x \leq \bar{x}(t)} \rho) = \mathbb{1}_{x > \bar{x}(t)} \rho \tag{8a}$$

$$\partial_t N - D \partial_{xx} N = -\rho N \tag{8b}$$

$$N(t, \bar{x}(t)) = N_{\text{th}}. \tag{8c}$$

In order to prove existence and uniqueness of the solution $\bar{x}(t)$, it suffices to require that $\partial_x N > 0$. Nevertheless, the property that $\partial_x N(t, \cdot\,) > 0$ is in general not implied by the sole condition that $\partial_x N^0 > 0$. In fact, it is possible to exhibit an initial configuration (ρ^0, N^0) where N^0 is monotonic, but nearly constant, and ρ^0 is sufficiently localized, so that the concentration $N(t, x)$ is no longer monotonic after some time $t > 0$, simply because of strong depletion around a spatial location. To circumvent this issue, we will here simply study System (8) and at the end of Section, we give a simple criterion, far from being optimal, on the inital data (ρ^0, N^0) such that the property $\partial_x N > 0$ is conserved for small time. Hence the solution of System (8), (ρ, N) is in fact a solution of (1).

The strategy of proof consists in applying a fixed point mapping to the curve $t \mapsto \bar{x}(t)$, *i.e.* the unique solution to $N(t, \cdot\,) = N_{\text{th}}$. More precisely, the main steps consist in (i) given the curve $\bar{x}(\,\cdot\,)$, solving Equations (8a,8b). (ii) Given the solution $(\rho[\bar{x}], N[\bar{x}])$, we show existence and uniqueness of a solution to Equation $N[\bar{x}](t, \cdot\,) = N_{\text{th}}$, that we denote $\bar{y}(\,\cdot\,)$. We then show that the solution $\bar{y}(\,\cdot\,)$ satisfies the following ODE:

$$\begin{cases} \dot{\bar{y}}(t) = -\frac{\partial_t N[\bar{x}](t, \bar{y}(t))}{\partial_x N[\bar{x}](t, \bar{y}(t))} \\ \bar{y}(0) = \bar{x}(0) \end{cases} . \tag{9}$$

In fact, ODE (9) is equivalent to Equation $N(t, \cdot\,) = N_{\text{th}}$. To show well-posedness of ODE (9), we need to obtain enough regularity on N: $\partial_t N[\bar{x}]$ and $(\partial_x N[\bar{x}])^{-1}$ should be locally Lipschitz in space. But from standard parabolic theory and the fact that the time derivative is expected to have the same regularity as the double space derivative, this roughly corresponds to $\partial_{xx} N[\bar{x}]$ being locally Lipschitz in space (and $\partial_x N[\bar{x}](t, \bar{y}(t))$ uniformly bounded away from 0). Hence the required regularity of N is $W^{3,\infty}$ in space. We shall see that this regularity holds true, but it is an endpoint case. (iii) Finally, the aim is to show that the mapping $\bar{x}(\,\cdot\,) \mapsto \bar{y}(\,\cdot\,)$ is a contraction and there exists a unique solution to the Cauchy problem:

$$\begin{cases} \dot{\bar{x}}(t) = -\frac{\partial_t N[\bar{x}](t, \bar{x}(t))}{\partial_x N[\bar{x}](t, \bar{x}(t))} \\ \bar{x}(\,\cdot\,)|_{t=0} = \bar{x}(0) \end{cases} .$$

However, because of the regularity requirement on N, the mapping $\bar{x} \mapsto \bar{y}$ becomes only a contraction, as will be seen, in a space that controls also the time derivative $\dot{\bar{x}}$. This deviates from standard Picard-Lindelöf theory of integration for ODEs, where contraction in L^∞-norm is sufficient and is obtained through a local in time integration of the ODE. Furthermore, from the ODE (9) it becomes clear that $\bar{x} \mapsto \bar{y}$ is at best merely Lipschitz continuous in the $W^{1,\infty}$-norm and not a contraction. In order to circumvent this issue, we will consider the mapping $\bar{x} \mapsto \bar{y}$ in an $W^{1,p}$-norm (with $p < \infty$) and therefore by a local in time integration of the ODE (9), the mapping $\bar{x}(\,\cdot\,) \mapsto \bar{y}(\,\cdot\,)$ becomes a contraction in that given norm.

Next, we introduce some notations and basic facts, before moving on to the statement of Theorem 2.1.

The evolution operator of the heat equation $e^{t\mu\partial_{xx}}$ (here $\mu = 1$ and $\mu = D$ will be of interest) on the real-line are defined as follows:

$$e^{t\mu\partial_{xx}} f(x) = \frac{1}{\sqrt{4\pi\mu t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4\mu t}} f(y) dy.$$

The operator $e^{t\mu\partial_{xx}}$ satisfies the following well-known functional inequalities, as a consequence of Young's convolutional inequality. For $1 \leq p \leq q \leq \infty$:

$$\left\| e^{t\mu\partial_{xx}} f \right\|_q \leq C t^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_p \,, \tag{10}$$

$$\left\| e^{t\mu\partial_{xx}} \partial_x f \right\|_q \leq C t^{-\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}+1\right)} \|f\|_p \,. \tag{11}$$

In addition, we consider Hölder spaces $C^{k,\alpha}(\mathbb{R})$, with $\alpha \in (0,1)$ and the norm $\|\cdot\|_{C^{k,\alpha}(\mathbb{R})} = \sum_{i=0}^k \left\|\partial_x^i \cdot\right\|_\infty + [\partial_x^k \cdot\,]_\alpha$, where $[f]_\alpha := \sup_{x,y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$. From Real Interpolation Theory with the K-method (see Chapter 1 in [36]), we know that $C^{k,\alpha}(\mathbb{R}) = \left(C^k(\mathbb{R}), C^{k+1}(\mathbb{R})\right)_{\alpha,\infty}$, where $C^k(\mathbb{R})$ is the space of functions k-times differentiable with bounded derivatives, equipped with its usual $W^{k,\infty}$-norm. This leads to the following bounds:

$$\left\| e^{t\mu\partial_{xx}} f \right\|_{C^{0,\alpha}(\mathbb{R})} \leq C t^{-\frac{\alpha}{2}} \|f\|_\infty \,, \tag{12}$$

$$\left\| e^{t\mu\partial_{xx}} f \right\|_{C^{1,\alpha}(\mathbb{R})} \leq C t^{-\frac{1+\alpha}{2}} \|f\|_\infty \,, \tag{13}$$

$$\left\| \partial_x e^{t\mu\partial_{xx}} f \right\|_\infty \leq C t^{\frac{\alpha-1}{2}} \|f\|_{C^{0,\alpha}(\mathbb{R})} \,, \tag{14}$$

$$\left\| \partial_{xx} e^{t\partial_{xx}} f \right\|_\infty \leq C t^{\frac{\alpha}{2}-1} \|f\|_{C^{0,\alpha}(\mathbb{R})} \,. \tag{15}$$

Set $B(A) := B_{W^{1,p}([0,T])}(A) = \left\{\bar{y}(\,\cdot\,) \in W^{1,p}([0,T]) \middle| \|y\|_{W^{1,p}} \leq A\right\}$, with the norm $\|y\|_{W^{1,p}} = \|y\|_p + \|\dot{y}\|_p$ and $p \in (4,\infty)$. Given a curve $\bar{x} \in B(A)$, we consider System (8) in the moving frame of reference $(t,z) = (t, x - \bar{x}(t))$, which yields:

$$\partial_t \tilde{\rho} - \dot{\bar{x}}(t)\partial_z \tilde{\rho} - \partial_{zz}\tilde{\rho} + \partial_z(\chi \mathbb{1}_{z\leq 0}\tilde{\rho}) = \mathbb{1}_{z>0}\tilde{\rho} \tag{16a}$$

$$\partial_t \tilde{N} - \dot{\bar{x}}(t)\partial_z \tilde{N} - D\partial_{zz}\tilde{N} = -\tilde{\rho}\tilde{N}. \tag{16b}$$

Of note, throughout this Section $\tilde{\rho}, \tilde{N}$ denotes the solutions in the moving frame, in order to easily distinguish between for instance ρ and $\tilde{\rho}$. Without loss of generality, we also suppose that $\bar{x}(0) = 0$, so that $\tilde{\rho}^0 = \rho^0$ and $\tilde{N}^0 = N^0$. Thereafter, we will however drop the diacritical mark $\tilde{}$ and systematically designate by ρ the solution in the moving frame.

As already observed in the Introduction, continuity of the flux in Equation (16a) leads by a Rankine-Hugoniot type of argument to a C^1-jump relation (3), which in the moving frame can simply be rewritten as:

$$\partial_z \tilde{\rho}(t, 0^+) - \partial_z \tilde{\rho}(t, 0^-) = -\chi \tilde{\rho}(t, 0).$$

In fact, it is furthermore possible to factorize $\tilde{\rho}$ under the form vU, with a function U such that the factorization precisely cancels out the C^1-jump relation at $z = 0$. For instance, we can choose $U(z) = \begin{cases} 1 & \text{if } z \leq 0 \\ e^{-\chi z} & \text{if } z > 0 \end{cases}$. Notice that U exactly satisfies the C^1-jump relation and that we obtain the following Equation on v:

$$\partial_t v - \partial_{zz} v - \beta(t, z) \partial_z v - \gamma(t, z) v = 0, \tag{17}$$

where $\beta(t, z) := \dot{\bar{x}} - \chi \mathbb{1}_{z \leq 0} - 2\chi \mathbb{1}_{z>0}$ and $\gamma(t, z) := \chi \left(\left(\chi + \frac{1}{\chi} \right) - \dot{\bar{x}} \right) \mathbb{1}_{z>0}$. Under this circumstance, v will be of higher regularity, *i.e.* $C^{1,\alpha}$. Of note, the particular choice of U is arbitrary in this Section, although as will be seen in Section 3, U corresponds to the traveling wave profile in the case $\chi \geq 1$. In fact, one could consider other candidates for U, but for the sake of simplicity we restrict ourselves to this particular choice.

Let us now move to the statement of a well-posedness theorem locally in time under the condition that N^0 is increasing.

Theorem 2.1. *Let $p \in (4, \infty)$, $\alpha \in \left(0, 1 - \frac{2}{p}\right)$ and $\alpha' \in \left(\frac{2}{p}, 1 - \frac{2}{p}\right)$. Suppose that $N^0 \in W^{3,\infty}(\mathbb{R})$ and $\frac{\rho^0}{U} \in C^{1,\alpha}(\mathbb{R})$. Additionally suppose that $\partial_x N^0 > 0$ and that $N^0(\,\cdot\,) = N_{th}$ admits a (unique) solution, for $x = 0$. For a certain $\zeta > 0$, denote $\underline{m}$ a lower bound of $\partial_x N^0$ on the interval $[-\zeta, +\zeta]$.*

Given $A > 0$ big enough (depending on $D, \chi, p, \alpha, \alpha', \left\| \frac{\rho^0}{U} \right\|_{C^{1,\alpha}}, \|N^0\|_{W^{3,\infty}}, \underline{m}, \zeta$) there exists a small enough $T > 0$, such that for any curve $\bar{x} \in B(A)$, there exists a unique solution $(\tilde{\rho}, \tilde{N})$ to System (16). Moreover, $\tilde{\rho} \in L^\infty([0, T], W^{1,\infty}(\mathbb{R}))$, $v = \frac{\tilde{\rho}}{U} \in L^\infty([0, T], C^{1,\alpha}(\mathbb{R}))$ and $\tilde{N} \in L^\infty([0, T], W^{3,\infty}(\mathbb{R}))$.

Furthermore, there exists a unique curve $\bar{x} \in B(A)$, such that the solution $(\tilde{\rho}, \tilde{N})$ to System (16) satisfies in addition the condition $\tilde{N}(t, 0) = N_{th}$, or in the static frame $N(t, \bar{x}(t)) = N_{th}$, for $t \in [0, T]$. In other terms, (ρ, N) in the static frame is the unique solution to System (8).

Proof. We divide the proof of Theorem 2.1 into several steps:

1. a) We fix a curve $\bar{x}(\,\cdot\,) \in W^{1,p}([0, T])$ and construct the unique (mild) solution $v \in L^\infty\left([0, T], C^{1,\alpha}(\mathbb{R})\right)$ to Equation (17). Furthermore the map $\bar{x} \in B(A) \mapsto v \in L^\infty([0, T], C^{1,\alpha}(\mathbb{R}))$ is Lipschitz continuous.
   b) Given v and thus $\tilde{\rho}$, we construct the unique (mild) solution $\tilde{N} \in L^\infty\left([0, T], C^{2,\alpha'}(\mathbb{R})\right)$ to Equation (16b). Furthermore the map $\bar{x} \in B(A) \mapsto \tilde{N} \in L^\infty([0, T], C^{2,\alpha'}(\mathbb{R}))$ is Lipschitz continuous.

2. We show that $\tilde{N} \in L^\infty([0, T], W^{3,\infty}(\mathbb{R}))$. This regularity is an improvement from the more standard regularity result obtained in Step 1b) and is crucial for the rest of the proof. We carry out estimates on explicit computations and finally we refer to the Remark at the end of the proof of Theorem 2.1 for a brief argument to show that the obtained regularity is borderline.

3. For $t \in [0, T], N(t)$ (in the static frame) admits a unique solution to the equation $N(t, \,\cdot\,) = N_{\text{th}}$ that we denote by $\bar{y}(t)$. Furthermore $\bar{y}(\,\cdot\,)$ satisfies ODE (9) and the regularity obtained on N leads to well-posedness of ODE (9).

4. For $A > 0$ big enough the map $\bar{x} \mapsto \bar{y}$ maps from $B(A)$ into itself and is furthermore a contraction. We conclude by Banach's Fixed Point Theorem and obtain well-posedness.

Conventions. For the sake of clarity, throughout the proof we will make use of the following conventions. C will represent constants that depend on $D, \chi, p, \alpha, \alpha', \underline{m}, \zeta$. In order to simplify the presentations of the inequalities, we suppose that $A, \left\|\frac{\rho^0}{U}\right\|_{C^{1,\alpha}}, \left\|N^0\right\|_{W^{3,\infty}}, \left\|N^0\right\|_{C^{2,\alpha'}} > 1$ in order to use freely for instance the bounds $1 + A \leq 2A$ or $\left\|\frac{\rho^0}{U}\right\|_{C^{1,\alpha}} \leq \left\|\frac{\rho^0}{U}\right\|^2_{C^{1,\alpha}}$. In parallel, we suppose that $T < 1$, in order to use freely bounds of the type $|t-s|^{-c} \leq |t-s|^{-d}$, for $t, s \in [0, T]$, when $0 < c < d$.

Step 1a: Existence and uniqueness of a (mild) solution v to Equation (17).
Consider the affine map $F : L^\infty\left([0,T], C^{1,\alpha}(\mathbb{R})\right) \to L^\infty\left([0,T], C^{1,\alpha}(\mathbb{R})\right)$, for $t \in [0,T]$:

$$F[u](t) = e^{t\partial_{zz}}\left(\frac{\rho^0}{U}\right) + \int_0^t e^{(t-s)\partial_{zz}}\left(\beta(s)\partial_z u(s) + \gamma(s)u(s)\right) ds. \tag{18}$$

As $\frac{\rho^0}{U} \in C^{1,\alpha}(\mathbb{R})$, we have that $e^{t\partial_{zz}}\left(\frac{\rho^0}{U}\right) \in L^\infty\left([0,T], C^{1,\alpha}(\mathbb{R})\right)$.
We will now show that the second term in (18) is in $L^\infty\left([0,T], C^{1,\alpha}(\mathbb{R})\right)$. Notice that $\beta(s,z)\partial_z u(s,z) = \partial_z\left(\beta(s,z)u(s,z)\right) + \chi u(s,0)\delta_0$. Hence by using Bounds (10,11), the fact that $\left\|e^{(t-s)\partial_{zz}}\delta_0\right\|_\infty \leq \frac{C}{\sqrt{t-s}}$ and that $|\beta| \leq C(1 + |\dot{\bar{x}}(s)|)$, we have that:

$$\begin{aligned}
\left\|\int_0^t e^{(t-s)\partial_{zz}}\left(\beta(s)\partial_z u(s)\right) ds\right\|_\infty &\leq C\int_0^t \left(\frac{\|\beta(s)u(s)\|_\infty}{\sqrt{t-s}} + \frac{|u(s,0)|}{\sqrt{t-s}}\right) ds \\
&\leq C\int_0^t \frac{(1 + |\dot{\bar{x}}(s)|)\,\|u(s)\|_\infty}{\sqrt{t-s}} ds \\
&\leq C\,\|u\|_\infty \int_0^t \frac{(1 + |\dot{\bar{x}}(s)|)}{\sqrt{t-s}} ds \\
&\leq C\,\|u\|_\infty \left(1 + \|\dot{\bar{x}}\|_p\right) T^{\frac{p-2}{2p}} \\
&\leq C\,\|u\|_\infty\, AT^{\frac{p-2}{2p}},
\end{aligned}$$

where we have used Hölder's inequality, $p > 2$ in order to guarantee the integrability of $s \mapsto (t-s)^{-\frac{p}{2(p-1)}}$ and the convention that $T < 1$ and $A > 1$. In a slightly easier manner, we have also that:

$$\left\|\partial_z \int_0^t e^{(t-s)\partial_{zz}}\left(\beta(s)\partial_z u(s)\right) ds\right\|_\infty \leq C\,\|\partial_z u\|_\infty\, AT^{\frac{p-2}{2p}}.$$

Finally by using Bound (13):

$$\left[\partial_z \int_0^t e^{(t-s)\partial_{zz}}\left(\beta(s)\partial_z u(s)\right) ds\right]_\alpha \leq C\,\|\partial_z u\|_\infty\, AT^{\frac{p(1-\alpha)-2}{2p}},$$

where integrability of $\left\|s \mapsto (t-s)^{-\frac{\alpha+1}{2}}\right\|_{\frac{p}{p-1}}$ is guaranteed by the condition $\alpha < 1 - \frac{2}{p}$. This yields the bound:

$$\left\|\int_0^t e^{(t-s)\partial_{zz}}\left(\beta(s)\partial_z u(s)\right) ds\right\|_{L^\infty([0,T], C^{1,\alpha}(\mathbb{R}))} \leq C\,\|u\|_{L^\infty([0,T], W^{1,\infty}(\mathbb{R}))}\, AT^{\frac{p(1-\alpha)-2}{2p}},$$

where by the conventions, we have used that $T^{\frac{p-2}{2p}} < T^{\frac{p(1-\alpha)-2}{2p}}$. The remaining term in (18) admits a similar bound. In fact:

$$\left\| \int_0^t e^{(t-s)\partial_{zz}} \left(\gamma(s)u(s)\right) ds \right\|_{L^\infty([0,T],C^{1,\alpha}(\mathbb{R}))} \leq C \left\|u\right\|_{L^\infty([0,T]\times\mathbb{R})} AT^{\frac{p(1-\alpha)-2}{2p}}.$$

Hence F maps $L^\infty([0,T],C^{1,\alpha}(\mathbb{R}))$ into itself.

Furthermore, if we choose T small enough (depending on A), for instance such that the Lipschitz constant of F becomes $\frac{1}{2}$, then F is a contraction and by Banach's Fixed Point Theorem Equation (16a) admits a unique solution v. Furthermore, we have the bound:

$$\begin{aligned} \|v\| = \|F[v]\| \leq \|F[v]-F[0]\| + \|F[0]\| &\leq \frac{\|v\|}{2} + \|F[0]\| \\ \implies \|v\|_{L^\infty([0,T],C^{1,\alpha}(\mathbb{R}))} &\leq 2\left\|\frac{\rho^0}{U}\right\|_{C^{1,\alpha}(\mathbb{R})}. \end{aligned} \tag{19}$$

It remains to show that the map $\bar{x} \mapsto v$ is Lipschitz continuous. Given $\bar{x}_1, \bar{x}_2 \in B(A)$, consider the two corresponding functions $v_1, v_2 \in L^\infty([0,T],C^{1,\alpha}(\mathbb{R}))$, as well as the two corresponding maps F_1, F_2, and set $w := v_1 - v_2$:

$$\begin{aligned} w(t) &= \int_0^t e^{(t-s)\partial_{zz}} \left(\beta_1\partial_z w + (\beta_1-\beta_2)\partial_z v_2 + \gamma_1 w + (\gamma_1-\gamma_2)v_2\right) ds \\ &= \left(F_1[w] - e^{t\partial_{zz}}\left(\frac{\rho^0}{U}\right)\right) + \int_0^t e^{(t-s)\partial_{zz}} \left((\beta_1-\beta_2)\partial_z v_2 + (\gamma_1-\gamma_2)v_2\right) ds. \end{aligned}$$

First of all, by using that $e^{t\partial_{zz}}\left(\frac{\rho^0}{U}\right) = F_1[0]$ and by recalling that by the choice of T the Lipschitz constant of F_1 is $\frac{1}{2}$, we have:

$$\begin{aligned} \left\|F_1[w] - e^{t\partial_{zz}}\left(\frac{\rho^0}{U}\right)\right\|_{L^\infty([0,T],C^{1,\alpha}(\mathbb{R}))} &= \|F_1[w] - F_1[0]\|_{L^\infty([0,T],C^{1,\alpha}(\mathbb{R}))} \\ &\leq \frac{1}{2}\|w\|_{L^\infty([0,T],C^{1,\alpha}(\mathbb{R}))}. \end{aligned}$$

We have that $\beta_1 - \beta_2 = \dot{\bar{x}}_1 - \dot{\bar{x}}_2$ and $\gamma_1 - \gamma_2 = (\dot{\bar{x}}_1 - \dot{\bar{x}}_2)\chi\mathbb{1}_{z\geq 0}$, which leads for $t \in [0,T]$ to:

$$\begin{aligned} &\left\| \int_0^t e^{(t-s)\partial_{zz}} \left((\beta_1-\beta_2)\partial_z v_2 + (\gamma_1-\gamma_2)v_2\right) ds \right\|_{C^{1,\alpha}(\mathbb{R})} \\ =&\left\| \int_0^t e^{(t-s)\partial_{zz}} (\dot{\bar{x}}_1(s) - \dot{\bar{x}}_2(s))(\partial_z v_2 + \chi\mathbb{1}_{z\geq 0}v_2) ds \right\|_{C^{1,\alpha}(\mathbb{R})} \\ \leq& C\int_0^t \frac{|\dot{\bar{x}}_2(s) - \dot{\bar{x}}_1(s)| \left\|\partial_z v_2(s) + \chi\mathbb{1}_{z\geq 0}v_2(s)\right\|_\infty}{(t-s)^{\frac{\alpha+1}{2}}} ds \\ \leq& C\left\|v_2\right\|_{L^\infty([0,T],C^{1,\alpha}(\mathbb{R}))} \int_0^t \frac{|\dot{\bar{x}}_2(s) - \dot{\bar{x}}_1(s)|}{(t-s)^{\frac{\alpha+1}{2}}} ds \\ \leq& C\left\|\frac{\rho^0}{U}\right\|_{C^{1,\alpha}(\mathbb{R})} \left\|\dot{\bar{x}}_2 - \dot{\bar{x}}_1\right\|_p T^{\frac{p(1-\alpha)-2}{2p}}, \end{aligned}$$

where the last bound is an application of Hölder's inequality. Hence we have that:

$$\|w\|_{L^\infty([0,T],C^{1,\alpha}(\mathbb{R}))} \leq \frac{1}{2}\|w\|_{L^\infty([0,T],C^{1,\alpha}(\mathbb{R}))} + C\left\|\frac{\rho^0}{U}\right\|_{C^{1,\alpha}(\mathbb{R})} \left\|\dot{\bar{x}}_2 - \dot{\bar{x}}_1\right\|_p T^{\frac{p(1-\alpha)-2}{2p}}.$$

And finally we establish that the map $\bar{x} \mapsto v$ is Lipschitz continuous with:

$$\|w\|_{L^\infty([0,T],C^{1,\alpha}(\mathbb{R}))} \leq CT^{\frac{p(1-\alpha)-2}{2p}} \left\|\frac{\rho^0}{U}\right\|_{C^{1,\alpha}(\mathbb{R})} \|\dot{\bar{x}}_2 - \dot{\bar{x}}_1\|_p \tag{20}$$

Step 1b: Existence and uniqueness of a (mild) solution $\tilde{N}$ to Equation (16b).

Consider the map $G : L^\infty([0,T], C^{2,\alpha'}(\mathbb{R})) \to L^\infty([0,T], C^{2,\alpha'}(\mathbb{R}))$, with $\alpha' \in (\frac{2}{p}, 1 - \frac{2}{p})$. For $t \in [0,T]$:

$$G[u](t) = e^{tD\partial_{zz}} N^0 + \int_0^t e^{(t-s)D\partial_{zz}} \left(\dot{\bar{x}}(s)\partial_z u(s) - v(s)Uu(s)\right) ds. \tag{21}$$

We proceed as before and treat explicitly only the following two terms:

$$\begin{aligned}
\left[\partial_{zz} \int_0^t e^{D(t-s)\partial_{zz}} v(s)Uu(s)ds\right]_{\alpha'} &\leq C \int_0^t (t-s)^{-\frac{1+\alpha'}{2}} \|\partial_z (v(s)Uu(s))\|_\infty ds \\
&\leq C \int_0^t (t-s)^{-\frac{1+\alpha'}{2}} \left(\|\partial_z v(s)\|_\infty \|u(s)\|_\infty + \|v(s)\|_\infty \|\partial_z u(s)\|_\infty\right) ds \\
&\leq CT^{\frac{1-\alpha'}{2}} \|v\|_{L^\infty([0,T],W^{1,\infty}(\mathbb{R}))} \|u\|_{L^\infty([0,T],W^{1,\infty}(\mathbb{R}))} \\
&\leq CT^{\frac{1-\alpha'}{2}} \left\|\frac{\rho^0}{U}\right\|_{C^{1,\alpha}(\mathbb{R})} \|u\|_{L^\infty([0,T],W^{1,\infty}(\mathbb{R}))},
\end{aligned}$$

where we used the fact that $U \in W^{1,\infty}(\mathbb{R})$. And:

$$\begin{aligned}
\left[\partial_{zz} \int_0^t e^{D(t-s)\partial_{zz}} \dot{\bar{x}}(s)\partial_z u(s)ds\right]_{\alpha'} &\leq C \int_0^t (t-s)^{-\frac{1+\alpha'}{2}} |\dot{\bar{x}}(s)| \|\partial_{zz} u(s)\|_\infty ds \\
&\leq CA \|u\|_{L^\infty([0,T],W^{2,\infty}(\mathbb{R})} T^{\frac{1-\alpha'}{2}}.
\end{aligned}$$

By using Bound (19), we can choose T such that the Lipschitz constant for G becomes $\frac{1}{2}$. Therefore G is a contraction and this yields existence and uniqueness of the solution $\tilde{N}$ to Equation (16b) and $\tilde{N}$ satisfies the bound:

$$\left\|\tilde{N}\right\|_{L^\infty([0,T],C^{2,\alpha'}(\mathbb{R}))} \leq 2 \left\|N^0\right\|_{C^{2,\alpha'}(\mathbb{R})} \leq 2 \left\|N^0\right\|_{W^{3,\infty}}. \tag{22}$$

As before, we show that $\bar{x} \in B(A) \mapsto N \in L^\infty([0,T], C^{2,\alpha'}(\mathbb{R}))$ is Lipschitz continuous. Given $\bar{x}_1, \bar{x}_2 \in B(A)$, consider the corresponding $\tilde{N}_1, \tilde{N}_2$, as well as the two corresponding maps G_1, G_2. We recall that $w = v_1 - v_2$ and set $P := \tilde{N}_1 - \tilde{N}_2$:

$$P = G_1[P] - G_1[0] + \int_0^t e^{(t-s)D\partial_{zz}} \left((\dot{\bar{x}}_1(s) - \dot{\bar{x}}_2(s))\partial_z \tilde{N}_2(s)ds - w(s)U\tilde{N}_2(s)\right) ds.$$

By the same arguments as before, we have the following bounds:

$$\begin{aligned}
\|G_1[P] - G_1[0]\|_{L^\infty([0,T],C^{2,\alpha'}(\mathbb{R}))} &\leq \frac{1}{2} \|P\|_{L^\infty([0,T],C^{2,\alpha'}(\mathbb{R}))} \\
\left\|\int_0^t e^{(t-s)D\partial_{zz}} (\dot{\bar{x}}_1(s) - \dot{\bar{x}}_2(s))\partial_z \tilde{N}_1(s)ds\right\|_{L^\infty([0,T],C^{2,\alpha'}(\mathbb{R}))} &\leq CT^{\frac{p(1-\alpha')-2}{2p}} \left\|N^0\right\|_{C^{2,\alpha'}(\mathbb{R})} \|\dot{\bar{x}}_2 - \dot{\bar{x}}_1\|_p \\
\left\|\int_0^t e^{(t-s)\partial_{zz}} w(s)U\tilde{N}_1(s)ds\right\|_{L^\infty([0,T],C^{2,\alpha'}(\mathbb{R}))} &\leq CT^{\frac{1-\alpha'}{2}} \|w\|_{L^\infty([0,T],C^{1,\alpha}(\mathbb{R}))} \left\|N^0\right\|_{C^{2,\alpha'}(\mathbb{R})}.
\end{aligned}$$

By recalling Inequality (20) on $\|w\|_{L^\infty([0,T],C^{1,\alpha}(\mathbb{R}))}$, we obtain that the map $\bar{x} \mapsto N$ is Lipschitz continuous, with:

$$\|P\|_{L^\infty([0,T],C^{2,\alpha'}(\mathbb{R}))} \leq C\left(T^{\frac{p(1-\alpha')-2}{2p}} + T^{\frac{(1-\alpha')(p(1-\alpha)-2)}{4p}}\right)\left\|\frac{\rho^0}{U}\right\|_{C^{1,\alpha}(\mathbb{R})}\left\|N^0\right\|_{C^{2,\alpha'}(\mathbb{R})}\|\dot{\bar{x}}_2 - \dot{\bar{x}}_1\|_p. \quad (23)$$

Step 2: Enhanced regularity on $\tilde{N} \in L^\infty([0,T], W^{3,\infty}(\mathbb{R}))$.
First let us point out that $\tilde{N}$ satisfies Equation (16b):

$$\partial_t \tilde{N} - \partial_{zz}\tilde{N} - \dot{\bar{x}}(t)\partial_z \tilde{N} = -vU\tilde{N}.$$

Suppose that $vU\tilde{N}$ had $C^{1,\alpha''}$ regularity in space for some $\alpha'' \in (0,1)$. Then by standard Parabolic Schauder Estimates (see Chapter 8 in [32]), $\tilde{N}$ would have $C^{3,\alpha''}$ regularity in space. But because of the C^1-discontinuity of U at $z=0$, this fails and $vU\tilde{N}$ is merely Lipschitz continuous. This constitutes the endpoint case for the Parabolic Schauder Estimates and it cannot generally be deduced that $\tilde{N}$ has $W^{3,\infty}$ regularity in space. However, in our case this result remains true, as we can single out the C^1-discontinuity of U at $z=0$, then prove that this explicit contribution enjoys the endpoint $W^{3,\infty}$ regularity. Finally, we refer the reader to Remark 2 at the end of this proof, where we give an argument why any higher regularity is not to be expected.

From the preceding point, we have the following representation for $\tilde{N}$:

$$\tilde{N}(t) = e^{tD\partial_{zz}}N^0 + \int_0^t e^{(t-s)D\partial_{zz}}\left(\dot{\bar{x}}(s)\partial_z\tilde{N}(s) - v(s)U\tilde{N}(s)\right)ds. \quad (24)$$

The term $e^{tD\partial_{zz}}N^0 \in L^\infty([0,T], W^{3,\infty}(\mathbb{R}))$, as by assumption $N^0 \in W^{3,\infty}(\mathbb{R})$.
In addition, by using Bound (15):

$$\begin{aligned}\left\|\partial_{zzz}\int_0^t e^{(t-s)D\partial_{zz}}\dot{\bar{x}}(s)\partial_z\tilde{N}(s)ds\right\|_\infty &\leq \int_0^t \left\|\partial_{zz}e^{(t-s)\partial_{zz}}\dot{\bar{x}}(s)\partial_{zz}\tilde{N}(s)\right\|_\infty ds \\ &\leq CT^{\frac{p\alpha'-2}{2p}}\|\dot{\bar{x}}\|_p\left\|\tilde{N}\right\|_{L^\infty([0,T],C^{2,\alpha'}(\mathbb{R}))},\end{aligned}$$

where the integrability of $\left\|s \mapsto (t-s)^{\frac{\alpha'}{2}-1}\right\|_{\frac{p}{p-1}}$ is due to the condition $\alpha' > \frac{2}{p}$.
It remains to be shown that $\int_0^t e^{(t-s)D\partial_{zz}}\left(v(s)U\tilde{N}(s)\right)ds \in L^\infty([0,T], W^{3,\infty}(\mathbb{R}))$. In order to do so, we will decompose the term $\partial_z\left(v(s)U\tilde{N}(s)\right)$ as the sum of a $C^{0,\alpha}$ function and a discontinous function. In fact for $s \in [0,T], z \in \mathbb{R}$:

$$\partial_z\left(vU\tilde{N}\right)(s,z) = \underbrace{\partial_z\left(vU\tilde{N}\right)(s,z) - [\![\partial_z\left(vU\tilde{N}\right)(s)]\!]_{z=0}\mathbb{1}_{z\geq 0}}_{g(s,z):=} + \underbrace{[\![\partial_z\left(vU\tilde{N}\right)(s)]\!]_{z=0}}_{h(s):=}\mathbb{1}_{z\geq 0}, \quad (25)$$

where $[\![f]\!]_{z_0} = \lim_{z\to z_0^+} f(z) - \lim_{z\to z_0^-} f(z)$. Here $[\![\partial_z\left(vU\tilde{N}\right)(s)]\!]_{z=0}$ is well-defined, since $v(s), \tilde{N}(s) \in C^{1,\alpha}(\mathbb{R})$ and $U \in C^1(\mathbb{R}_+) \cap C^1(\mathbb{R}_-)$. We will conclude by treating both terms separately and using the following bound:

$$\left\|\partial_{zzz}\int_0^t e^{(t-s)\partial_{zz}}\left(v(s)U\tilde{N}(s)\right)ds\right\|_\infty \leq \left\|\int_0^t \partial_{zz}e^{(t-s)\partial_{zz}}g(s)ds\right\|_\infty + \left\|\int_0^t \partial_{zz}e^{(t-s)\partial_{zz}}h(s)\mathbb{1}_{z\geq 0}ds\right\|_\infty. \quad (26)$$

Lemma 2.1.1. *Let $f \in L^\infty([0,T], C^{0,\alpha}(\mathbb{R}_+)) \cap L^\infty([0,T], C^{0,\alpha}(\mathbb{R}_-))$, where we understand $C^{0,\alpha}(\mathbb{R}_\pm)$ as a normed space, equipped with the norm $\|\cdot\|_\infty + [\,\cdot\,]_{\alpha;\mathbb{R}_\pm}$.*

Then we have that $g := f - [\![f]\!]_{z=0}\mathbb{1}_{z\geq 0} \in L^\infty([0,T], C^{0,\alpha}(\mathbb{R}))$ and:

$$\|g\|_{L^\infty([0,T],C^{0,\alpha}(\mathbb{R}))} \leq 2^{1-\alpha} \max\left\{ \sup_{t\in[0,T]} [f(t)]_{\alpha;\mathbb{R}_+}, \sup_{t\in[0,T]} [f(t)]_{\alpha;\mathbb{R}_-} \right\} + 3\|f\|_\infty .$$

Proof. By construction we have that for $t \in [0,T]$, $g(t) \in C^0(\mathbb{R})$ and $\|g\|_\infty \leq 3\|f\|_\infty$, since $|[\![f]\!]_{z=0}| \leq 2\|f\|_\infty$. Let $t \in [0,T]$ and $x, y \in \mathbb{R}$ and suppose that $x < 0 < y$:

$$\begin{aligned}
\frac{|f(t,y) - f(t,x)|}{|y-x|^\alpha} &\leq \frac{|f(t,y) - f(t,0)|}{|y-x|^\alpha} + \frac{|f(t,0) - f(t,x)|}{|y-x|^\alpha} \\
&\leq 2\left(\frac{1}{2} \cdot \frac{|y|^\alpha}{|y-x|^\alpha} [f(t)]_{\alpha;\mathbb{R}_-} + \frac{1}{2} \cdot \frac{|x|^\alpha}{|y-x|^\alpha} [f(t)]_{\alpha;\mathbb{R}_+} \right) \\
&\leq 2\left(\frac{|y|+|x|}{2|y-x|} \right)^\alpha \max\left\{ \sup_{t\in[0,T]} [f(t)]_{\alpha;\mathbb{R}_+}, \sup_{t\in[0,T]} [f(t)]_{\alpha;\mathbb{R}_-} \right\} \\
&= 2^{1-\alpha} \max\left\{ \sup_{t\in[0,T]} [f(t)]_{\alpha;\mathbb{R}_+}, \sup_{t\in[0,T]} [f(t)]_{\alpha;\mathbb{R}_-} \right\},
\end{aligned} \tag{27}$$

where we have used the concavity of $z \mapsto z^\alpha$ and the fact that $x < 0 < y$.

If $x, y < 0$ (resp. $x, y > 0$), then the left handside of (27) is simply bounded by $[f(t)]_{\alpha;\mathbb{R}_-}$ (resp. $[f(t)]_{\alpha;\mathbb{R}_+}$). □

Applying Lemma 2.1.1 with $f = \partial_z \left(vU\tilde{N} \right) \in L^\infty([0,T], C^{0,\tilde{\alpha}}(\mathbb{R}_+)) \cap L^\infty([0,T], C^{0,\tilde{\alpha}}(\mathbb{R}_-))$, where $\tilde{\alpha} := \min(\alpha, \alpha')$, we obtain that $g \in L^\infty([0,T], C^{0,\tilde{\alpha}}(\mathbb{R}))$. Through Bound (15), this leads to:

$$\begin{aligned}
\left\| \int_0^t \partial_{zz} e^{(t-s)\partial_{zz}} g(s) ds \right\|_\infty &\leq CT^{\frac{\tilde{\alpha}}{2}} \|g\|_{L^\infty([0,T],C^{0,\tilde{\alpha}}(\mathbb{R}))} \\
&\leq CT^{\frac{\tilde{\alpha}}{2}} \left(\max\left\{ \sup_{t\in[0,T]} [\partial_z \left(vU\tilde{N} \right)(t)]_{\tilde{\alpha};\mathbb{R}_+}, \sup_{t\in[0,T]} [\partial_z \left(vU\tilde{N} \right)(t)]_{\tilde{\alpha};\mathbb{R}_-} \right\} \right. \\
&\quad \left. + \left\| \partial_z \left(vU\tilde{N} \right) \right\|_{L^\infty([0,T],C^{0,\tilde{\alpha}}(\mathbb{R}))} \right) \\
&\leq CT^{\frac{\tilde{\alpha}}{2}} \|v\|_{L^\infty([0,T],C^{1,\tilde{\alpha}}(\mathbb{R}))} \|N\|_{L^\infty([0,T],C^{1,\tilde{\alpha}}(\mathbb{R}))} \\
&\leq CT^{\frac{\tilde{\alpha}}{2}} \left\| \frac{\rho^0}{U} \right\|_{C^{1,\alpha}} \|N^0\|_{W^{3,\infty}} ,,
\end{aligned} \tag{28}$$

where we have used that the product of $C^{1,\tilde{\alpha}}$ functions are $C^{1,\tilde{\alpha}}$.
The last term is treated differently by using explicit computations:

$$\begin{aligned}
\int_0^t \partial_{zz} e^{(t-s)D\partial_{zz}} h(s) \mathbb{1}_{z\geq 0} ds &= \int_0^t \partial_z e^{(t-s)D\partial_{zz}} h(s) \delta_0 ds \\
&= -\int_0^t \frac{zh(s)}{4\pi^{\frac{1}{2}} D^{\frac{3}{2}} (t-s)^{\frac{3}{2}}} e^{-\frac{z^2}{4(t-s)}} ds \\
&= -\int_{\frac{|z|}{\sqrt{t}}}^\infty \frac{h\left(t - \frac{z^2}{u^2}\right)}{2\pi^{\frac{1}{2}} D^{\frac{3}{2}}} e^{-u^2} du \qquad \text{where } u = \frac{|z|}{\sqrt{(t-s)}}.
\end{aligned}$$

Hence, by the integrability of e^{-u^2}:

$$\left\| \int_0^t \partial_{zz} e^{(t-s)D\partial_{zz}} h(s) \mathbb{1}_{z\geq 0} ds \right\|_\infty \leq C \|h\|_\infty$$

But we have the following identity:

$$\begin{aligned}
&h(s)\\
&= \lim_{z\to 0^+} \partial_z(vU\tilde{N})(s,z) - \lim_{z\to 0^-} \partial_z(vU\tilde{N})(s,z)\\
&= \lim_{z\to 0^+} U(z)\partial_z(v\tilde{N})(s,z) + \lim_{z\to 0^+} (v\tilde{N})(s,z)\partial_z U(z) - \lim_{z\to 0^-} U(z)\partial_z(v\tilde{N})(s,z) - \lim_{z\to 0^-} (v\tilde{N})(s,z)\partial_z U(z)\\
&= v(s,0)\tilde{N}(s,0)[\![\partial_z U]\!]_{z=0}\\
&= -\chi v(s,0)\tilde{N}(s,0).
\end{aligned}$$

Therefore:

$$\left\| \int_0^t \partial_{zz} e^{(t-s)D\partial_{zz}} h(s) \mathbb{1}_{z\geq 0} ds \right\|_\infty \leq C \|v\|_\infty \left\|\tilde{N}\right\|_\infty . \tag{29}$$

Bringing Bounds (26, 28, 29) together, we conclude that $\int_0^t e^{(t-s)D\partial_{zz}} \left(v(s)U\tilde{N}(s)\right) ds \in L^\infty([0,T], W^{3,\infty}(\mathbb{R}))$. Thus $\tilde{N} \in L^\infty([0,T], W^{3,\infty}(\mathbb{R}))$.

Step 3: Definition of the map $\bar{x} \mapsto \bar{y}$: Existence and Uniqueness of the solution $N(t,\,\cdot\,) = N_{\text{th}}$.

We consider ρ, N again in the initial frame (t,x), where they satisfy Equations (8). Note that $\|N\|_\infty = \left\|\tilde{N}\right\|_\infty$ and $\|\rho\|_\infty \leq \|v\|_\infty$. By assumption, we have that $\partial_x N^0 > \underline{m}$ on the interval $[-\zeta, +\zeta]$. Therefore by setting $\varepsilon = \zeta\underline{m}$, we have that for $x < -\zeta, N^0(x) < N_{\text{th}} - \varepsilon$ and $x > \zeta, N^0(x) > N_{\text{th}} + \varepsilon$.

1. We start by showing that there exists $T > 0$, such that for $t \in [0,T]$ and $x \leq -\zeta$, we have that $N(t,x) < N_{\text{th}}$. Note that:

$$\begin{aligned}
\left\| -\int_0^t e^{(t-s)\partial_{zz}} \rho(s)N(s)ds \right\|_\infty &\leq T \|N\|_\infty \|\rho\|_\infty\\
&\leq 4T \left\|\rho^0\right\|_{C^{1,\alpha}} \left\|N^0\right\|_{W^{3,\infty}} .
\end{aligned}$$

So for $T > 0$ small enough, the right handside is smaller than $\frac{\varepsilon}{4}$.
Choose $T > 0$ small engouh, so that for $t \in (0,T]$ we have that:

$$\frac{1}{\sqrt{4\pi Dt}} \int_\zeta^{+\infty} e^{-\frac{x^2}{4Dt}} dx < \frac{\varepsilon}{4}.$$

From this we can deduce that for $t \in (0,T], x < -\zeta$, by recalling that $\left\|N^0\right\|_\infty = 1$, we have:

$$\begin{aligned}
N(t,x) &= e^{t\partial_{xx}} N^0|_x - \int_0^t e^{(t-s)\partial_{xx}} \rho(s)N(s)ds|_x\\
&\leq \frac{1}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4Dt}} N^0(y)dy + \left\| -\int_0^t e^{(t-s)\partial_{xx}} \rho(s)N(s)ds \right\|_\infty\\
&< \frac{N_{\text{th}} - \varepsilon}{2} + \frac{N_{\text{th}}}{2} + \frac{\left\|N^0\right\|_\infty}{\sqrt{4\pi Dt}} \int_\zeta^{+\infty} e^{-\frac{x^2}{4t}} dx + \frac{\varepsilon}{4}\\
&< N_{\text{th}}.
\end{aligned}$$

2. By a similar reasoning, there exists $T > 0$, such that for $t \in [0, T]$ and $x \geq \zeta$, we have that $N(t, x) > N_{\text{th}}$.

3. We now show that there exists $T > 0$, such that for $t \in [0, T], x \in [-\zeta, +\zeta]$, we have that $\partial_x N(t, x) \geq \frac{m}{8}$.
The reasoning is again similar. On the one hand:
$$\left\| \partial_x \left(- \int_0^t e^{(t-s)D\partial_{xx}} \rho(s) N(s) ds \right) \right\|_\infty \leq \int_0^t \frac{C}{\sqrt{t-s}} \|\rho(s)\|_\infty \|N(s)\|_\infty ds.$$
As before we can choose $T > 0$ such that the right handside becomes smaller than $\frac{m}{8}$ (here the constant C does not depend on $\underline{m}$).
On the other hand, by choosing $T > 0$ small enough such that for every $t \in [0, T]$:
$$\frac{1}{\sqrt{4\pi D t}} \int_0^\zeta e^{-\frac{y^2}{4Dt}} dy \geq \frac{1}{4}.$$
In that fashion for $x \in [-\zeta, +\zeta]$:
$$\begin{aligned} e^{tD\partial_{xx}} \partial_x N^0|_x &= \frac{1}{\sqrt{4\pi D t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4Dt}} \partial_x N^0(y) dy \\ &\geq \frac{1}{\sqrt{4\pi D t}} \int_{-\zeta}^{\zeta} e^{-\frac{(x-y)^2}{4Dt}} \partial_x N^0(y) dy \\ &\geq \frac{\underline{m}}{\sqrt{4\pi D t}} \int_{-\zeta-x}^{\zeta-x} e^{-\frac{y^2}{4Dt}} dy \\ &\geq \frac{\underline{m}}{4}, \end{aligned}$$
by noticing that either $[0, \zeta] \subset [-\zeta - x, \zeta - x]$, when $x \leq 0$, or alternatively that $[-\zeta, 0] \subset [-\zeta - x, \zeta - x]$, when $x \geq 0$. From this, we conclude that:
$$\partial_x N(t, x) = e^{tD\partial_{xx}} \partial_x N^0|_x - \partial_x \left(\int_0^t e^{(t-s)D\partial_{xx}} \rho(s) N(s) ds \right) \Big|_x \geq \frac{\underline{m}}{4} - \frac{\underline{m}}{8} = \frac{\underline{m}}{8}.$$

4. From the considerations above, we see that there exists $T > 0$, such that for $t \in [0, T]$, the equation $N(t) = N_{\text{th}}$ has a unique solution, which we denote by $\bar{y}(t)$. We know that $\bar{y}(t) \in [-\zeta, +\zeta]$ and $\bar{y}(0) = 0$. Furthermore from the preceding analysis we know that N is differentiable and by differentiating the relation $N(t, \bar{y}(t)) = N_{\text{th}}$, $\bar{y}(t)$ satisfies an ODE:
$$\dot{\bar{y}}(t) = -\frac{\partial_t N(t, \bar{y}(t))}{\partial_x N(t, \bar{y}(t))} =: \mathcal{F}(t, \bar{y}(t)).$$
Since $\|N\|_{L^\infty([0,T], W^{3,\infty}(\mathbb{R}))} = \left\| \tilde{N} \right\|_{L^\infty([0,T], W^{3,\infty}(\mathbb{R}))}$, $\|\rho\|_{L^\infty([0,T], W^{1,\infty}(\mathbb{R}))} \leq C \|v\|_{L^\infty([0,T], W^{1,\infty}(\mathbb{R}))}$ and $\partial_t N = D\partial_{xx} N - \rho N$, we have that $\partial_t N \in L^\infty([0, T], W^{1,\infty}(\mathbb{R}))$. Additionally, since $\partial_x N(t, \bar{y}(t)) \geq \frac{\underline{m}}{4}$ and $\partial_x N \in L^\infty([0, T], W^{1,\infty}(\mathbb{R}))$, we have that $(\partial_x N)^{-1} \in L^\infty([0, T], W^{1,\infty}(\mathbb{R}))$. Therefore $\mathcal{F} \in L^\infty([0, T] \times [-\zeta, zeta])$ is uniformly in time Lipschitz continuous in the second variable. Hence the ODE is well-posed and it admits a unique solution $\bar{y} \in W^{1,\infty}([0, T])$.
We have the bound:
$$\begin{aligned} |\dot{\bar{y}}(t)| &\leq |\partial_x N(t, \bar{y}(t))|^{-1} \|D\partial_{xx} N - \rho N\|_\infty \\ &\leq \frac{8}{\underline{m}} \left(D \|\partial_{xx} N\|_\infty + \|\rho\|_\infty \|N\|_\infty \right) \\ &\leq C \|N^0\|_{W^{3,\infty}} \left\| \frac{\rho^0}{U} \right\|_{C^{1,\alpha'}}, \end{aligned}$$

where at the end, we have used Bounds (19,22). Hence, by the convention that $T < 1$, we have $\|\bar{y}\|_{W^{1,p}} \leq C \left\|N^0\right\|_{W^{3,\infty}} \left\|\frac{\rho^0}{U}\right\|_{C^{1,\alpha'}}$. It therefore suffices that A is bigger than the right handside and in that case the map $\bar{x} \mapsto \bar{y}$ maps $B(A)$ into itself.

Step 4: Unique Fixed Point of the map $\bar{x} \in B(A) \mapsto \bar{y} \in B(A)$.

Given $\bar{x}_1, \bar{x}_2 \in B(A)$, consider $\bar{y}_1, \bar{y}_2 \in B(A)$. Set $\mathcal{F}_i = -\frac{\partial_t N_i}{\partial_x N_i}$, such that $\dot{\bar{y}}_i(t) = \mathcal{F}_i(t, \bar{y}_i(t)$. Note that $\bar{y}_i(t) \in [-\zeta, +\zeta]$ and that therefore $\partial_x N_i(t, \bar{y}_i(t)) > \frac{m}{8}$.

$$\begin{aligned}|\dot{\bar{y}}_1(t) - \dot{\bar{y}}_2(t)| &= |\mathcal{F}_1(t, \bar{y}_1(t)) - \mathcal{F}_2(t, \bar{y}_2(t))| \\ &\leq |\mathcal{F}_1(t, \bar{y}_1(t)) - \mathcal{F}_1(t, \bar{y}_2(t))| + |\mathcal{F}_1(t, \bar{y}_2(t)) - \mathcal{F}_2(t, \bar{y}_2(t))|\,.\end{aligned}$$

For $x \in [-\zeta, +\zeta]$, $t \in [0, T]$ and by using the lower bounded of $\partial_x N(t, x)$ from Step 3, we have that:

$$\begin{aligned}|\partial_x \mathcal{F}_1(t, x)| &\leq \left(\sup_{x \in [-\zeta, +\zeta]} \frac{1}{\partial_x N(t, x)}\right)^2 \|\partial_{tx} N \partial_x N - \partial_t N \partial_{xxx} N\|_\infty \\ &\leq C \left\|(\partial_{xxx} N - \partial_x(\rho N))\, \partial_x N - (\partial_{xx} N - \rho N) \partial_{xxx} N\right\|_\infty .\end{aligned}$$

This latter term is bounded, in particular because of Step 2. Hence:

$$|\mathcal{F}_1(t, \bar{y}_1(t)) - \mathcal{F}_1(t, \bar{y}_2(t))| \leq C \left\|N^0\right\|^2_{W^{3,\infty}(\mathbb{R})} \left\|\frac{\rho^0}{U}\right\|_{C^{1,\alpha}(\mathbb{R})} |\bar{y}_1(t) - \bar{y}_2(t)|. \tag{30}$$

For the second term:

$$\begin{aligned}&|\mathcal{F}_1(t, \bar{y}_2(t)) - \mathcal{F}_2(t, \bar{y}_2(t))| \\ \leq &\frac{|\partial_t N_1(t, \bar{y}_2(t))|}{|\partial_x N_1(t, \bar{y}_2(t))||\partial_x N_2(t, \bar{y}_2(t))|} |\partial_x N_1(t, \bar{y}_2(t)) - \partial_x N_2(t, \bar{y}_2(t))| \\ &+ \frac{1}{|\partial_x N_2(t, \bar{y}_2(t))|} |\partial_t N_1(t, \bar{y}_2(t)) - \partial_t N_2(t, \bar{y}_2(t))| \\ \leq &C \left(\|\partial_{xx} N_1 - \rho_1 N_1\|_\infty \|\partial_x N_1 - \partial_x N_2\|_\infty + \|\partial_{xx} N_1 - \partial_{xx} N_2\|_\infty \right. \\ &\left. + \|\rho_1 - \rho_2\|_\infty \|N_1\|_\infty + \|\rho_2\|_\infty \|N_1 - N_2\|\right).\end{aligned}$$

Now take $(t, y) \in [0, T] \times \mathbb{R}$, we have:

$$\begin{aligned}&|\partial_{xx} N_1(t, y) - \partial_{xx} N_2(t, y)| \\ =&|\partial_{zz} \tilde{N}_1(t, y - \bar{x}_1(t)) - \partial_{zz} \tilde{N}_2(t, y - \bar{x}_2(t))| \\ \leq&|\partial_{zz} \tilde{N}_1(t, y - \bar{x}_1(t)) - \partial_{zz} \tilde{N}_1(t, y - \bar{x}_2(t))| + |\partial_{zz} \tilde{N}_1(t, y - \bar{x}_2(t)) - \partial_{zz} \tilde{N}_2(t, y - \bar{x}_2(t))| \\ \leq &\left\|\partial_{zzz} \tilde{N}_1\right\|_\infty |\bar{x}_1(t) - \bar{x}_2(t)| + \left\|\partial_{zz} \tilde{N}_1 - \partial_{zz} \tilde{N}_2\right\|_\infty .\end{aligned}$$

We have similar (and slightly easier) bounds for $\|\rho_1 - \rho_2\|_\infty$, $\|N_1 - N_2\|_\infty$ and $\|\partial_x N_1 - \partial_x N_2\|_\infty$. Hence, by using $\|\bar{x}_1 - \bar{x}_2\|_\infty \leq T^{1-\frac{1}{p}} \|\dot{\bar{x}}_1 - \dot{\bar{x}}_2\|_p$ and Bounds (20,23) this leads to:

$$\begin{aligned}&|\mathcal{F}_1(t, \bar{y}_1(t)) - \mathcal{F}_2(t, \bar{y}_2(t))| \\ \leq &C \left\|N^0\right\|^2_{W^{3,\infty}} \left\|\frac{\rho^0}{U}\right\|^2_{C^{1,\alpha}} \left(T^{1-\frac{1}{p}} + T^{\frac{p(1-\alpha')-2}{2p}} + T^{\frac{(1-\alpha')(p(1-\alpha)-2)}{4p}}\right) \|\dot{\bar{x}}_1 - \dot{\bar{x}}_2\|_p .\end{aligned} \tag{31}$$

Combining Inequalities (30,31), using the conventions and setting $K := C \left\|N^0\right\|^2_{W^{3,\infty}} \left\|\frac{\rho^0}{U}\right\|^2_{C^{1,\alpha}}$, we find that:

$$|\dot{\bar{y}}_1(t) - \dot{\bar{y}}_2(t)| \leq K \left(|\bar{y}_1(t) - \bar{y}_2(t)| + \|\dot{\bar{x}}_1 - \dot{\bar{x}}_2\|_p\right).$$

By Grönwall's lemma, we obtain:

$$\|\bar{y}_1 - \bar{y}_2\|_\infty \leq \left(e^{KT} - 1\right) \|\dot{\bar{x}}_1 - \dot{\bar{x}}_2\|_p .$$

Bootstrapping the penultimate estimate, we can prove that:

$$\|\dot{\bar{y}}_1 - \dot{\bar{y}}_2\|_p \leq KT^{\frac{1}{p}} e^{KT} \|\dot{\bar{x}}_1 - \dot{\bar{x}}_2\|_p .$$

By noticing that $\|\bar{y}_1 - \bar{y}_2\|_p \leq T^{\frac{1}{p}} \|\bar{y}_1 - \bar{y}_2\|_\infty$ and using the two last inequalities, we find that the map $\bar{x} \in B(A) \mapsto \bar{y} \in B(A)$ is a contraction in the $W^{1,p}$-norm for $T > 0$ small enough. Thus, we have a unique fixed point, which concludes the proof of Theorem 2.1. □

Remark. *The Step 2 of the preceding proof naturally leads to the question whether $\tilde{N} \in L^\infty([0,T], C^{3,\alpha''}(\mathbb{R}))$ for $\alpha'' \in (0,1)$. In fact, by the reasoning in Step 2, this is equivalent to wondering, whether $\int_0^t \partial_{zz} e^{(t-s)D\partial_{zz}} h(s) \mathbb{1}_{z\geq 0} d$ $L^\infty([0,T], C^{0,\alpha''}(\mathbb{R}))$. But we see that if we take $h \equiv 1$ and denote $\theta > 0$ the constant such that $\frac{1}{\pi^{\frac{1}{2}}} \int_0^\theta e^{-u^2} du = \frac{1}{4}$. Then, by applying the preceding computations between $(t,z) = (\min(y^2\theta^{-2}, T), y)$ and $(\min(y^2\theta^{-2}, T), 0)$, we find:*

$$\begin{aligned}
& y^{-\alpha''} \left| \left(\int_0^{\min(y^2\theta^{-2},T)} \partial_{zz} e^{(t-s)D\partial_{zz}} \mathbb{1}_{z\geq 0} ds \Big|_{z=0} \right) - \left(\int_0^{\min(y^2\theta^{-2},T)} \partial_{zz} e^{(t-s)D\partial_{zz}} \mathbb{1}_{z\geq 0} ds \Big|_{z=y} \right) \right| \\
= & \frac{y^{-\alpha''}}{2\pi^{\frac{1}{2}} D^{\frac{3}{2}}} \int_0^{\max\left(\theta, \frac{y}{\sqrt{T}}\right)} e^{-u^2} du \\
\geq & \frac{y^{-\alpha''}}{8D^{\frac{3}{2}}}.
\end{aligned}$$

Hence the expression is unbounded and we have that $\int_0^t \partial_{zz} e^{(t-s)D\partial_{zz}} \mathbb{1}_{z\geq 0} ds \notin L^\infty([0,T], C^{0,\alpha''}(\mathbb{R}))$. This establishes that the regularity $\tilde{N} \in L^\infty([0,T], W^{3,\infty}(\mathbb{R}))$ is in fact critical.

Corollary 2.1.2. *Suppose that in addition to the assumptions of Theorem 2.1, the initial conditions (ρ^0, N^0) satisfy the following conditions:*

1. $\frac{\partial_x N^0}{N^0}, \frac{\partial_x \rho^0}{\rho^0} \in L^\infty(\mathbb{R})$,
2. $\liminf_{x\to-\infty} \frac{\partial_x N^0}{N^0} \geq \nu > 0$ *and* $\limsup_{x\to+\infty} \frac{\partial_x \rho^0}{\rho^0} \leq -\eta < 0$,
3. $1 - N^0$ *is square-integrable at* $x = +\infty$.

Then (ρ, N), the solution given by Theorem 2.1, satisfies the condition $\partial_x N \geq 0$ locally in time and hence (ρ, N) is in fact a solution to System (1).

Let us briefly comment on the assumptions of Corollary 2.1.2. The assumption on N^0 implies that N^0 increases at least exponentially at $x = -\infty$. The assumption, that $1 - N^0$ is square-integrable at $x = +\infty$, is of course more restrictive than the condition $\lim_{x\to+\infty} N^0(x) = 1$. We will see in the next Section that the traveling wave solution satisfies the property $\tilde{N}(z) = 1 + p(z)$ with p a function that is dominated by an exponentially decreasing function at $z = +\infty$, which is then square-integrable at $z = +\infty$. Concerning the assumption on ρ^0, we already know that $\tilde{\rho} = vU$ with U an exponentially decreasing function at $z = +\infty$ and v a bounded function. Hence the additional condition translates the fact that $\limsup_{x\to+\infty} \left(\frac{\partial_x v^0}{v^0}\right) \leq \chi - \eta$. Considering that we must have $\liminf_{x\to+\infty} \frac{\partial_x v^0}{v^0} \leq 0$, otherwise v^0 is not bounded, this assumption translates a restriction on the oscillations of $\frac{\partial_x v^0}{v^0}$.

Proof. 1. We start by showing that $w := \frac{\partial_z \tilde{N}}{\tilde{N}}$ is well-defined. In fact by dividing Equation (16b) by $\tilde{N}$, we obtain:

$$\frac{\partial_t \tilde{N}}{\tilde{N}} = D(\partial_z w + w^2) + \dot{\bar{x}} w - vU.$$

Then by observing that $\partial_z \left(\frac{\partial_t \tilde{N}}{\tilde{N}}\right) = \partial_{zt} \log \tilde{N} = \partial_t \left(\frac{\partial_z \tilde{N}}{\tilde{N}}\right)$ and by differentiating the preceding Equation, w satisfies the following equation:

$$\partial_t w = D(\partial_{zz} w + \partial_z \left(w^2\right)) + \dot{\bar{x}} \partial_z w - \partial_z \left(vU\right).$$

This leads to the following representation formula for w:

$$w(t) = e^{tD\partial_{zz}} w^0 + \int_0^t e^{(t-s)D\partial_{zz}} \left(D\partial_z \left(w^2\right)(s) + \dot{\bar{x}}(s)\partial_z w(s) - \partial_z \left(v(s)U\right)\right) ds.$$

By arguments similar to the ones exposed in the proof of Theorem 2.1, we simply must show that for $R > 0$ big enough there exists $T > 0$ such that the right handside defines a contraction from $L^\infty([0,T], B(R))$ into itself, where $B(R) = \{f \in L^\infty(\mathbb{R}), \|f\|_\infty \leq R\}$. We merely treat the term $w \in L^\infty([0,T], B(R)) \mapsto \int_0^t e^{(t-s)D\partial_{zz}} D\partial_z \left(w^2(s)\right) ds \in L^\infty([0,T], B(R))$. Let $w_1, w_2 \in L^\infty([0,T] \times \mathbb{R})$, we have:

$$\begin{aligned}
&\left\| \int_0^t e^{(t-s)D\partial_{zz}} \left(\partial_z \left(w_1^2\right)(s) - \partial_z \left(w_2^2\right)(s)\right) ds \right\|_\infty \\
\leq & C \int_0^t \left\| w_1^2(s) - w_2^2(s) \right\|_\infty \frac{ds}{\sqrt{t-s}} \\
\leq & C \int_0^t \|w_1(s) - w_2(s)\|_\infty \|w_1(s) + w_2(s)\|_\infty \frac{ds}{\sqrt{t-s}} \\
\leq & CR\sqrt{T} \|w_1 - w_2\|_\infty .
\end{aligned}$$

Hence $w = \frac{\partial_x \tilde{N}}{\tilde{N}} \in L^\infty([0,T] \times \mathbb{R})$.

Furthermore, we show that $w \in C([0,T], L^\infty(\mathbb{R}))$. The map $t \mapsto e^{tD\partial_{xx}} w_0$ is continuous and in addition by noticing, for instance, that:

$$\begin{aligned}
&\left\| \int_0^{t+h} e^{(t+h-s)D\partial_{zz}} D\partial_z \left(w^2\right)(s) ds - \int_0^t e^{(t-s)D\partial_{zz}} D\partial_z \left(w^2\right)(s) ds \right\|_\infty \\
= & \left\| \int_t^{t+h} e^{(t+h-s)D\partial_{zz}} D\partial_z \left(w^2\right)(s) ds + \left(1 - e^{hD\partial_{zz}}\right) \int_0^t e^{(t-s)D\partial_{zz}} D\partial_z \left(w^2\right)(s) ds \right\|_\infty \\
\leq & C \|w\|_\infty^2 \int_t^{t+h} \frac{ds}{\sqrt{t+h-s}} + \left\| \left(1 - e^{hD\partial_{zz}}\right) \int_0^t e^{(t-s)D\partial_{zz}} D\partial_z \left(w^2\right)(s) ds \right\|_\infty .
\end{aligned}$$

When $h \to 0$, the first term clearly tends to 0 and so does the second term by strong continuity of the heat semi-group.

2. Next we show that $q = \frac{\partial_z v}{v}$ is well-defined. q satisfies:

$$\partial_t q = \partial_{zz} q + \partial_z (q^2) + \partial_z (\beta q) + \partial_z \gamma.$$

The proof is similar to the preceding point. However, one should notice the following fact, in order to prove that the map: $\int_0^t e^{(t-s)\partial_{zz}} \partial_z \gamma ds \in L^\infty([0,T] \times \mathbb{R})$. Indeed, by observing that $\partial_z \gamma =$

$\chi\left(\chi+\frac{1}{\chi}-\dot{\tilde{x}}(t)\right)\delta_0$ and applying Hölder's inequality, we have that:

$$\begin{aligned}\left\|\int_0^t e^{(t-s)\partial_{zz}}\partial_z\gamma ds\right\|_\infty &= \left\|\chi\int_0^t \frac{\chi+\frac{1}{\chi}-\dot{\tilde{x}}(s)}{\sqrt{4\pi(t-s)}}e^{-\frac{z^2}{4(t-s)}}ds\right\|_\infty \\ &\leq C\left\|\chi+\frac{1}{\chi}-\dot{\tilde{x}}\right\|_p.\end{aligned}$$

3. We establish that $\tilde{N}$ is nondecreasing. To do so we start by noticing that since $\limsup_{z\to+\infty}\frac{\partial_z\tilde{\rho}^0}{\tilde{\rho}^0}<-\eta<0$. There exists an $A>0$ such that for every $z>A$, $\frac{\partial_z v^0(z)}{v^0(z)}+\frac{U'(z)}{U}<-\eta$. But by continuity of $t\mapsto\frac{\partial_z v(t,\cdot)}{v(t,\cdot)}$ in $L^\infty(\mathbb{R})$, there exists $T>0$, such that for every $t\in[0,T]$ and $z>A$, $\frac{\partial_z\tilde{\rho}(t,z)}{\tilde{\rho}(t,z)}=\frac{\partial_z v(t,z)}{v(t,z)}+\frac{U'(z)}{U}<-\frac{\eta}{2}$. In particular $\partial_z\tilde{\rho}(t,z)<0$ for $(t,z)\in[0,T]\times[A,+\infty)$.
Now because of Condition 1 and 3, we have in fact that for $z\in(-\infty,A], w^0(z)\geq\frac{\nu}{2}>0$ and by the same argument, we must have for $T>0$ small enough that for $t\in[0,T], z\in(-\infty,A], w(t,z)\geq\frac{\nu}{4}>0$. Therefore it remains to show that on the interval $[A,+\infty)$, we also have that $\partial_z\tilde{N}\geq 0$. For $h>0$, let $f(t,z):=\tilde{N}(t,z+h)-\tilde{N}(t,z)$ and $g(t,z):=-\tilde{\rho}(t,z+h)\tilde{N}(t,z+h)+\tilde{\rho}(t,z)\tilde{N}(t,z)$. Then we have that:

$$\partial_t f-\dot{\tilde{x}}\partial_z f-D\partial_{zz}f=g. \tag{32}$$

Furthermore for $(t,z)\in[0,T]\times(-\infty,A)$, we have that $f\geq 0$ by the preceding and for $z>A$, if $f(t,z)=\tilde{N}(t,z+h)-\tilde{N}(t,z)<0$, then we must have that $g(t,z)=-\tilde{\rho}(t,z+h)\tilde{N}(t,z+h)+\tilde{\rho}(t,z)\tilde{N}(t,z)\geq 0$, since $\tilde{\rho}(t,z+h)\leq\tilde{\rho}(t,z)$. Therefore we have $f_-g\geq 0$, where $(\,\cdot\,)_-=-\min(0,\,\cdot\,)$.

We have that $g(t,\,\cdot\,)\in L^2(\mathbb{R})$, since $\tilde{N}$ is dominated at $z=-\infty$ by $e^{\frac{\nu z}{4}}$ and $\tilde{\rho}$ is dominated at $z=+\infty$ by $e^{-\frac{\eta z}{2}}$. Notice that $f(0,\,\cdot\,)\in L^2(\mathbb{R})$ by the assumption that $1-N^0$ is square-integrable at $x=+\infty$. Therefore $f(t\,\cdot\,)\in L^2(\mathbb{R})$ as solution to Equation (32). Hence, the following computations are justified:

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}}(f_-)^2\right) &= \int_{\mathbb{R}} f_-\partial_t f_- \\ &= -\int_{\mathbb{R}} f_-\partial_t f, \text{ since } f_-\partial_t(f_-)=-f_-\partial_t f \\ &= -\dot{\tilde{x}}(t)\int_{\mathbb{R}} f_-\partial_z f-D\int_{\mathbb{R}} f_-\partial_{zz}f-\int f_-g \\ &= \dot{\tilde{x}}(t)\int_{\mathbb{R}} f_-\partial_z(f_-)-D\int_{\mathbb{R}} f_-\partial_{zz}f-\int f_-g, \text{ since } f_-\partial_z(f_-)=-f_-\partial_z f \\ &\leq \frac{\dot{\tilde{x}}(t)}{2}\int_{\mathbb{R}}\partial_z((f_-)^2)+D\int_{\mathbb{R}}(\partial_z f_-)\partial_z f, \text{ since } f_-g\geq 0 \\ &= -D\int_{\mathbb{R}}(\partial_z f_-)^2, \text{ since } \partial_z f_-\partial_z f=-(\partial_z f_-)^2 \\ &\leq 0.\end{aligned}$$

But, by assumption $f_-(0,\,\cdot\,)\equiv 0$. Hence $f_-(t,\,\cdot\,)\equiv 0$ and $\tilde{N}(t,\,\cdot\,)$ is nondecreasing. □

3 Traveling Waves for the Parabolic System

In this Section, we will investigate the existence of waves for the parabolic System (1), *i.e.* solutions of the form $(\rho(t,x),N(t,x))=(\tilde{\rho}(x-\sigma t),\tilde{N}(x-\sigma t))$, for a velocity σ to be determined. Set $z=x-\sigma t$,

any traveling wave solution must satisfy the following equations:

$$-\sigma\tilde{\rho}' - \tilde{\rho}' + (\chi \mathrm{sign}(\partial_z N)\mathbb{1}_{\tilde{N}\leq N_{\mathrm{th}}}\tilde{\rho})' = \mathbb{1}_{\tilde{N}>N_{\mathrm{th}}}\tilde{\rho} \tag{33a}$$

$$-\sigma\tilde{N}' - D\tilde{N}'' = -\tilde{\rho}\tilde{N}. \tag{33b}$$

For the sake of concision, we will drop the diacritical $\tilde{}$. Applying the assumption that N is increasing, Equation (33a) reduces to a second-order linear ordinary differential equation with piecewise-constant coefficients. By translation invariance of the traveling waves, we suppose that $N(0) = N_{\mathrm{th}}$. Adding the C^1-jump relation (3), that comes from the continuity of the flux, we obtain the following problem:

$$\begin{cases} -\sigma\rho' - \rho'' + \chi\rho' = 0 & \text{for } z < 0 \\ -\sigma\rho' - \rho'' = \rho & \text{for } z > 0 \end{cases} \quad \text{and} \quad \rho'(0^+) - \rho'(0^-) = -\chi\rho(0). \tag{34}$$

We solve this problem explicitly and thus deduce all bounded and nonnegative traveling wave profiles for ρ. Moreover, there exists a minimal speed σ^*, such that for every $\sigma \in [\sigma^*, +\infty)$, there exists a unique (up to a multiplicative factor) traveling wave profile ρ^σ. In a second step, given the profile ρ^σ, we construct a corresponding traveling wave profile N^σ and the condition $N(0) = N_{\mathrm{th}}$ will fix the multiplicative factor of ρ, thus leading to a unique traveling wave profile (ρ^σ, N^σ) for each $\sigma \geq \sigma^*$.

Let us introduce some notations, before moving on to the statement of Theorem 3.1. Define the Fisher/Kolmogorov–Petrovsky–Piskunov speed $\sigma_{\mathrm{F/KPP}} := 2$. Note that $\chi + \frac{1}{\chi} \geq \sigma_{\mathrm{F/KPP}}$, with equality if and only if $\chi = 1$. Furthermore set for $\sigma \geq 2$, $\mu_\pm(\sigma) := \frac{\sigma \pm \sqrt{\sigma^2 - 4}}{2}$. We then have the following inequality for $\sigma > \sigma_{\mathrm{F/KPP}} = 2$:

$$0 < \mu_-(\sigma) < \mu_-\left(\sigma_{\mathrm{F/KPP}}\right) = 1 = \mu_+\left(\sigma_{\mathrm{F/KPP}}\right) < \mu_+(\sigma)$$

In addition, the function $\sigma \mapsto \mu_+(\sigma)$ (resp. $\sigma \mapsto \mu_-(\sigma)$) is increasing (resp. decreasing).

Theorem 3.1. *Under the assumption that N is increasing, there exists a minimal speed σ^*, such that there exists a bounded and nonnegative traveling wave profile $(\rho^\sigma(z), N^\sigma(z))$ if and only if $\sigma \geq \sigma^*$. Given $\sigma \geq \sigma^*$, the traveling wave profile $(\rho^\sigma(z), N^\sigma(z))$ is unique. Moreover, the exact value of σ^* is given by Formula (4) and depends on the value of χ:*

- *if $\chi > 1$, then $\sigma^* = \chi + \frac{1}{\chi}$,*
- *if $\chi \leq 1$, then $\sigma^* = \sigma_{F/KPP} = 2$*

Furthermore, the functions ρ^σ satisfy the following properties for $z \geq 0$ with $C^\sigma, D^\sigma > 0$:

- *for $\sigma > \sigma^*, \rho^\sigma(z) = A^\sigma e^{-\mu_-(\sigma)z} + B^\sigma e^{-\mu_+(\sigma)z}$*
- *for $\chi > 1$, $\sigma = \sigma^* = \chi + \frac{1}{\chi}, \rho^{\sigma^*}(z) = A^{\sigma^*} e^{-\mu_+(\sigma^*)z}$ and $\mu_+(\sigma^*) = \chi$*
- *for $\chi \leq 1, \sigma = \sigma_{F/KPP}, \rho^{\sigma_{F/KPP}}(z) = A^{\sigma_{F/KPP}}((1-\chi)z + 1)e^{-z}$*

In addition, let $\mu > 0$, with $\mu \neq \frac{\sigma}{D}$, such that $\rho(z) \leq Ce^{-\mu z}$ for a constant $C > 0$, then there exists another constant $C > 0$, such that for $z \in \mathbb{R}$:

$$|N(z) - 1| \leq C\left(e^{-\frac{\sigma}{D}z} + e^{-\mu z}\right). \tag{35}$$

Proof. Integrating Equation (34) over the whole line yields $(\sigma - \chi)\rho^\sigma(-\infty) = \int_{\mathbb{R}_+} \rho^\sigma(z)dz$ (as we will see just below, ρ^σ is integrable at $z = +\infty$). Therefore by nonnegativity of the left handside, we find that $\sigma > \chi$. Consider Equation (34) for $z < 0$. Its characteristic polynomial is $X^2 + (\sigma - \chi)X$ and has roots 0 and $\chi - \sigma$. There exist two constants $A^-, B^- \in \mathbb{R}$, such that $\rho^\sigma(z) = A^- + B^- e^{(\chi-\sigma)z}$, for

$z < 0$. Since $\sigma > \chi$, the term $e^{(\chi-\sigma)z}$ is unbounded on $\mathbb{R}_-$, which leads to $B^- = 0$.
Consider Equation (34) for $z > 0$. Its characteristic polynomial is $X^2 + \sigma X + 1$ and its discriminant is $\sigma^2 - 4$. If the discriminant is negative, the roots are complex and ρ^σ would be a linear combination of two oscillating functions, which is prohibited by the nonnegativity condition. Hence $\sigma^2 \geq 4$, or, by positivity of σ, $\sigma \geq 2$.
Suppose $\sigma > 2 = \sigma_{\mathrm{F/KPP}}$, the roots of the characteristic polynomial are then $-\mu_\pm(\sigma)$ and there exist two constants $A^+, B^+ \in \mathbb{R}$ such that $\rho^\sigma(z) = A^+ e^{-\mu_+(\sigma)z} + B^+ e^{-\mu_-(\sigma)z}$. By the continuity at $z = 0$ of ρ^σ, we obtain equality $A^- = A^+ + B^+$ and by the C^1-jump relation (3), we obtain equality $-\mu_+ A^+ - \mu_- B^+ = -\chi A^-$. Thus we find that $\rho^\sigma(z) = \frac{A^-}{\sqrt{\sigma^2-4}} \left((\mu_+(\sigma) - \chi)e^{-\mu_-(\sigma)z} + (\chi - \mu_-(\sigma))e^{-\mu_+(\sigma)z}\right)$. One checks that this expression is nonegative for all z, if and only if, $\mu_+(\sigma) \geq \chi$. In the case of small bias $\chi \leq 1$, this inequality is always verified. In the case of large bias $\chi > 1$, this inequality is verified, if and only if $\sigma \geq \chi + \frac{1}{\chi}$. This proves all cases of Theorem for $\sigma > \sigma_{\mathrm{F/KPP}}$.
Suppose $\sigma = 2 = \sigma_{\mathrm{F/KPP}}$, then there exist two constants $A^+, B^+ \in \mathbb{R}$ such that $\rho^\sigma(z) = (A^+ z + B^+)e^{-z}$. By the same arguments as above, we have $B^+ = A^-$, $A^+ = (1-\chi)A^-$ and this leads to $\rho^\sigma(z) = A^- \left((1-\chi)z + 1\right) e^{-z}$. To satisfy the nonnegativity condition, we must have $\chi \leq 1$, which shows that $\sigma = 2 = \sigma_{\mathrm{F/KPP}}$, is the speed of a traveling wave, if and only if $\chi \leq 1$.

The behavior of ρ^σ for $z \geq 0$ is simply a reformulation of the considerations above. In particular, note that in the case of large bias and $\sigma = \sigma^*$, we have that $\mu_+(\sigma^*) = \chi$, which leads to $\rho^{\sigma^*}(z) = A^- e^{-\chi z}$, for $z \geq 0$.

It remains to show that for every wave profile ρ^σ, we have a corresponding wave profile N^σ, which satisfies the condition $N^\sigma(0) = N_{\mathrm{th}}$. To do so, notice that by linearity of the equation on ρ, the profile ρ^σ is defined up to the multiplicative constant A^-, that is yet to be determined. Denote for a given constant $A^- > 0$, $\rho^\sigma_{A^-}$ its corresponding profile. Equation (33b) with boundary condition $N(+\infty) = 1$, then has a unique solution, which we denote $N^\sigma_{A^-}$. But it is clear that the map $A^- \mapsto N^\sigma_{A^-}(0)$ is continuous and decreasing and its range is $(0, 1)$. Therefore there exists a unique A^-, such that $N^\sigma_{A^-}(0) = N_{\mathrm{th}}$ and we obtain for a fixed $\sigma \geq \sigma^*$ a unique traveling wave profile $(\rho^\sigma_{A^-}, N^\sigma_{A^-})$.

Let us finally prove Estimate (35). By integrating Equation (33b) on the interval $(z, +\infty)$ and noticing that $N(+\infty) = 1, N'(+\infty) = 0$:

$$\sigma\left(N(z) - 1\right) + D\left(N(z) - 1\right)' = -\int_z^{+\infty} \rho(y)N(y)dy.$$

Hence for a constant $C > 0$:

$$\left|\left(e^{\frac{\sigma}{D}z}(N(z) - 1)\right)'\right| \leq Ce^{\left(\frac{\sigma}{D} - \mu\right)z}.$$

By integrating this inequality on the interval $(0, z)$ for another $C > 0$, we find:

$$|N(z) - 1| \leq C\left(e^{-\frac{\sigma}{D}z} + e^{-\mu z}\right).$$

□

Theorem 3.1 shows that there exist a large number of traveling waves. However, in Section 5 we will show that in the circumstances of a biologically relevant initial conditions, the interesting traveling wave will be that of minimal speed σ^*.

4 Inside Dynamics of Traveling Waves

We elaborate on the properties of the traveling waves, following the lines of [23], as well as [42] to a small extent. We establish that in the large bias case, the wave is *pushed*, and in the small bias case

(or when $\sigma > \sigma^*$), the wave is *pulled*, according to the definition proposed in [23] (see the discussion in the Introduction on the ambiguity of *pushed* and *pulled* waves). To do so, we are using the formalism of neutral fractions. The aim is to study the behaviour of partitions of the traveling wave profile (see [23] and [15] for the biological relevance of this decomposition).

Definition. *Define $L := -\partial_{zz} - \beta\partial_z$, where $\beta(z) = \sigma - \chi \mathbb{1}_{z\leq 0} + 2\frac{\partial_z \rho^\sigma}{\rho^\sigma}$. A neutral fraction ν (of the traveling wave ρ^σ) is a solution to the following equation:*

$$\begin{cases} \partial_t \nu + L\nu = 0 \\ \nu(0,\,\cdot\,) = \nu^0 \end{cases}. \tag{36}$$

It is clear that any constant is a neutral fraction, as stationary solutions to Equation (36). The interest in Equation (36) stems from the following observation. Suppose we have neutral fractions $(\nu_i)_{i=1}^k \geq 0$ that satisfy $\sum_{i=1}^k \nu_i^0 = 1$. It amounts to marking each part $\nu_i^0 \rho^\sigma$ of the population with neutral labels, *i.e.* that do not interfer with the dynamics. The neutral fractions $(\nu_i(t)\rho^\sigma)_{i=1}^k$ then describe the evolution over time of the distribution of these labels. Because of this interpretation, it is natural to suppose that ν^0 takes its values in $[0,1]$, but such a restriction is of no relevance for the subsequent analysis. Of note, describing ν_i or $\nu_i\rho^\sigma$ is equivalent, only the expression of the operator L changes.

4.1 Pushed Front Dynamics in the Large Bias Case

In this Section, we develop arguments very similar to the ones developped in the works [20] and [23] (see also [42]). The evolution of neutral fractions is characterized in the regime of large bias ($\chi > 1$) and minimal velocity ($\sigma = \sigma^*$) by Theorem 4.1.

Consider the operator L in the space $L^2(e^V dz)$, where $V' = \beta$. On the appropriate domain, L is self-adjoint, has 0 as eigenvalue and an exact spectral gap $\gamma := \frac{1}{4}\min\left(\sigma^2 - 4, \frac{1}{\chi^2}\right) > 0$. This leads to the following convergence result:

Theorem 4.1. *Suppose that $\chi > 1$ and that $\sigma = \sigma^*$. Let ν be a neutral fraction (36) that satisfies $\nu^0 \in L^2(e^V dz)$. Then we have the following convergence result:*

$$\left\|\nu(t) - \langle \nu^0 \rangle\right\|_{L^2(e^V dz)} \leq \left\|\nu^0\right\|_{L^2(e^V dz)} e^{-\gamma t}, \tag{37}$$

with $\langle \nu^0 \rangle = \frac{\int \nu^0 e^V dz}{\int e^V dz}$ and $\gamma := \frac{1}{4}\min\left(\sigma^2 - 4, \frac{1}{\chi^2}\right) > 0$.

Furthermore for a constant $C > 0$, we have:

$$\left\|(\nu(t) - \langle \nu^0 \rangle)e^{\frac{V}{2}}\right\|_\infty \leq C\left(1 + t^{-\frac{1}{2}}\right)e^{-\gamma t}. \tag{38}$$

And as a consequence ν converges exponentially fast to a constant on compact sets $K \subset \mathbb{R}$, i.e.:

$$\sup_{z\in K} |\nu(t,z) - \langle \nu^0 \rangle| \leq \frac{C\left(1 + t^{-\frac{1}{2}}\right)e^{-\gamma t}}{\inf_K e^{\frac{V}{2}}}. \tag{39}$$

Theorem 4.1 states that every neutral fraction converges to a constant. In other words, independently of the initial datum for the neutral fraction, the neutral fraction will after some time, uniformly on compact sets in space represent a multiple (which does depend on the initial datum) of the total population. This is in stark contrast with the case when $\sigma = \sigma_{F/KPP}$ (or when $\sigma > \sigma^*$) in Corollary 4.2.1, where neutral fractions will in general go extinct (unless they are part of the leading edge). In fact, in the large bias case when $\sigma = \sigma^*$, for $z < 0, \beta(z) > 0$ and for $z > 0, \beta(z) < 0$, which shows that

$1 \in L^2(e^V dz)$. But this property is not true in the small bias case when $\sigma = \sigma_{\mathrm{F/KPP}}$ (or when $\sigma > \sigma^*$), as we will see in Subsection 4.2.

We define L on $L^2(e^V dz)$ with domain $D(L) = H^2(e^V dz)$. Nevertheless to simplify the spectral study of L, we introduce the pullback $\mathcal{L}$ of L to the space $L^2(dz)$, that is $\mathcal{L} = e^{\frac{V}{2}} L\left(e^{-\frac{V}{2}} \cdot\right)$, with domain $D(\mathcal{L}) = \left\{ f \in H^1(dz) \middle| f' - \frac{\beta}{2} f \in H^1(dz) \right\}$. $\mathcal{L}$ is symmetric and monotone, as for $f, g \in D(\mathcal{L})$,

$$\begin{aligned}
&\langle f, \mathcal{L} g \rangle \\
=& \int_{\mathbb{R}} e^{\frac{V}{2}} f \left(-\left(e^{-\frac{V}{2}} g\right)'' - \beta \left(e^{-\frac{V}{2}} g\right)' \right) dz \\
=& \int_{\mathbb{R}} \left(\left(e^{\frac{V}{2}} f\right)' \left(e^{-\frac{V}{2}} g\right)' - \beta e^{\frac{V}{2}} f \left(e^{-\frac{V}{2}} g\right)' \right) dz \\
=& \int_{\mathbb{R}} \left(\left(f' + \frac{\beta}{2} f\right) \left(g' - \frac{\beta}{2} g\right) - \beta f \left(g' - \frac{\beta}{2} g\right) \right) dz \\
=& \int_{\mathbb{R}} \left(f' - \frac{\beta}{2} f\right) \left(g' - \frac{\beta}{2} g\right) dz \qquad (40) \\
=& \langle \mathcal{L} f, g \rangle,
\end{aligned}$$

where $\langle f | g \rangle := \int_{\mathbb{R}} f g dz$. In fact Equality (40) holds also true if $f \in D(\mathcal{L})$ and $g \in H^1(dz)$. By observing that $D(\mathcal{L}) \subset H^1(dz)$ and density of $D(\mathcal{L})$ in $H^1(dz)$, we obtain that for every $f \in H^1(dz)$:

$$\langle f, \mathcal{L} f \rangle = \int_{\mathbb{R}} \left(f' - \frac{\beta}{2} f\right)^2 dz \geq 0. \qquad (41)$$

Finally $\mathcal{L}$ has the following expressions:

$$\begin{aligned}
&\mathcal{L} f \\
=& -f'' + \frac{\beta^2}{4} f + \frac{\beta'}{2} f \qquad (42) \\
=& -\left(e^{\frac{V}{2}} \left(e^{-\frac{V}{2}} f\right)'\right)'. \qquad (43)
\end{aligned}$$

Proposition 4.1.1. *The operator $\mathcal{L} : D(\mathcal{L}) \to L^2(dz)$ is closed.*

Proof. Let $(f_n, \mathcal{L} f_n) \in \Gamma(\mathcal{L})$ such that $(f_n, \mathcal{L} f_n) \xrightarrow[n \to +\infty]{L^2(dz)} (f, g) \in L^2(dz) \times L^2(dz)$. We will show that $f \in D(\mathcal{L})$ and $\mathcal{L} f = g$.

1. First, we prove boundedness of (f_n) in $H^1(dz)$:

$$\int_{\mathbb{R}} \left(f_n' - \frac{\beta}{2} f_n\right)^2 dz = \langle f_n | \mathcal{L} f_n \rangle \leq \|f_n\|_2 \|\mathcal{L} f_n\|_2 .$$

The right handside is bounded and thus $f_n' - \frac{\beta}{2} f_n$ is bounded in $L^2(dz)$. But $\frac{\beta}{2} f_n$ is also bounded in $L^2(dz)$ (since $\beta \in L^\infty(\mathbb{R})$). Hence (f_n) is bounded in $H^1(dz)$

2. The boundedness of (f_n) in $H^1(dz)$ implies weak compactness of the sequences. Hence up to extraction of a subsequence, we can assume that $f_n \xrightarrow[n \to +\infty]{H^1(dz)} \ell \in H^1(dz)$. But uniqueness of the limits in $L^2(dz)$ implies $\ell = f$, which leads to $f \in H^1(dz)$.

3. We now show strong convergence in $H^1(dz)$. Denote $r_n := f - f_n$

$$\begin{aligned}
&\int_{\mathbb{R}} \left(r_n' - \frac{\beta}{2} r_n\right)^2 dz \\
&= \left\langle f' - \frac{\beta}{2} f \middle| r_n' - \frac{\beta}{2} r_n \right\rangle - \left\langle f_n' - \frac{\beta}{2} f_n \middle| r_n' - \frac{\beta}{2} r_n \right\rangle \\
&= \underbrace{\left\langle f' - \frac{\beta}{2} f \middle| r_n' - \frac{\beta}{2} r_n \right\rangle}_{\to 0 \text{ by weak convergence in } H^1(dz)} - \underbrace{\langle \mathcal{L} f_n | r_n \rangle}_{\to 0 \text{ since } r_n \to 0 \text{ in } L^2(dz) \text{ and } \mathcal{L} f_n \text{ is bounded}},
\end{aligned}$$

where we have applied (40) with $f_n \in D(\mathcal{L})$ and $r_n \in H^1(dz)$. Thus $f_n \xrightarrow[n\to+\infty]{H^1(dz)} f$.

4. We show that $f \in D(\mathcal{L})$:

$$\left(f_n' - \frac{\beta}{2} f_n\right)' = -\mathcal{L} f_n - \frac{\beta}{2} f_n' - \frac{\beta^2}{4} f_n.$$

The right handside converges in $L^2(dz)$, which establishes that $f_n' - \frac{\beta}{2} f_n \xrightarrow[n\to+\infty]{H^1(dz)} f' - \frac{\beta}{2} f \in H^1(dz)$. Therefore $f \in D(\mathcal{L})$.

5. Finally we show that $\mathcal{L} f = g$. Let $h \in D(\mathcal{L}^*)$. By definition of $D(\mathcal{L}^*)$, we have that $\langle h | \mathcal{L} f_n \rangle \xrightarrow[n\to+\infty]{} \langle h | \mathcal{L} f \rangle$. But we also know that $\langle h | \mathcal{L} f_n \rangle \xrightarrow[n\to+\infty]{} \langle h | g \rangle$. Therefore $\mathcal{L} f - g \in D(\mathcal{L}^*)^\perp$. But since $\mathcal{L}$ is a symmetric operator, we have the inclusion $D(\mathcal{L}) \subset D(\mathcal{L}^*)$ and thus $D(\mathcal{L}^*)$ is dense. Hence $\mathcal{L} f = g$.

Thus, the operator $\mathcal{L}$ is closed. □

Proposition 4.1.2. *The operator $\mathcal{L} : D(\mathcal{L}) \to L^2(dz)$ is self-adjoint.*

Proof. We already know that the operator $\mathcal{L}$ is symmetric. It remains to show that it shares the same domain as its adjoint.

Let $g \in D(\mathcal{L}^*)$ and $f \in C_0^\infty(\mathbb{R}) \subset D(\mathcal{L})$. By definition of $D(\mathcal{L}^*)$, we have that:

$$|\langle g | \mathcal{L} f \rangle| \leq C(g) \|f\|_{L^2(dz)}. \tag{44}$$

But since f is a test function, we can view $\mathcal{L} g$ as a distribution. Let us take $g_n \in D(\mathcal{L})$ such that $g_n \xrightarrow[n\to+\infty]{L^2(dz)} g$. Then $\mathcal{L} g_n \xrightarrow[n\to+\infty]{D'(\mathbb{R})} \mathcal{L} g$, so that $\langle \mathcal{L} g_n | f \rangle \xrightarrow[n\to+\infty]{} \langle \mathcal{L} g | u f \rangle_{L^2(dz)}$. But from the symmetry of $\mathcal{L}$, we get $\langle \mathcal{L} g_n | f \rangle = \langle g_n | \mathcal{L} f \rangle \xrightarrow[n\to+\infty]{} \langle g | \mathcal{L} f \rangle$. Hence $\langle \mathcal{L} g | f \rangle = \langle g | \mathcal{L} f \rangle$

Therefore:

$$|\langle \mathcal{L} g | f \rangle| = |\langle g | \mathcal{L} f \rangle| \leq C(g) \|f\|_2. \tag{45}$$

By Bound (45), the linear form $f \mapsto \langle \mathcal{L} g | f \rangle$ is bounded on $C_0^\infty(\mathbb{R})$ in the L^2-norm and we can extend it to the whole space $L^2(dz)$ by uniform continuity and density of $C_0^\infty(\mathbb{R})$ in $L^2(dz)$. This shows that $\mathcal{L} g \in L^2(dz)$.

Then:

$$\begin{aligned}
&\mathcal{L} g = -\left(e^{\frac{V}{2}} \left(e^{-\frac{V}{2}} g\right)'\right)' \in L^2(dz) \\
&\implies e^{\frac{V}{2}} \left(e^{-\frac{V}{2}} g\right)' \in H^1(dz) \\
&\implies g' - \frac{\beta}{2} g \in H^1(dz) \\
&\implies g \in D(\mathcal{L}).
\end{aligned}$$

Thus $D(\mathcal{L}^*) \subset D(\mathcal{L})$, which concludes the proof. □

We are now ready to show a lower bound on the essential spectrum of the operator $\mathcal{L}$.

Proposition 4.1.3.

$$\sigma_{ess}(\mathcal{L}) = [\gamma, +\infty),$$

with $\gamma := \frac{1}{4}\min\left(\sigma^2 - 4, \frac{1}{\chi^2}\right) > 0$

In order to prove this proposition, we require two standard lemmata, whose proof we give for the sake of completeness.

Lemma 4.1.4. *Let $f \in H^1(dz)$, then $f \in C^0(\mathbb{R})$ and:*

$$\|f\|_\infty^2 \leq \frac{1}{2\pi}\left\|f'\right\|_2 \|f\|_2 .$$

Proof. Let $z \in \mathbb{R}, \eta > 0$, then:

$$\begin{aligned} f(z)^2 = \left(\frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)e^{iz\xi}d\xi\right)^2 &\leq \left(\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\sqrt{1+\eta|\xi|^2}\left|\hat{f}(\xi)\right|}{\sqrt{1+\eta|\xi|^2}}d\xi\right)^2 \\ &\leq \frac{1}{4\pi^2}\frac{\pi}{\sqrt{\eta}}\left(\|f\|_2^2 + \eta\left\|f'\right\|_2^2\right), \end{aligned}$$

where we have used Cauchy-Schwartz Inequality and the fact that $\int_{\mathbb{R}} \frac{d\xi}{1+\eta|\xi|^2} = \frac{\pi}{\sqrt{\eta}}$. By taking $\eta = \frac{\|f\|_2^2}{\|f'\|_2^2}$, we obtain the desired bound.

By similar computations, we also have that:

$$(f(z+h) - f(z))^2 \leq \frac{1}{4\pi}\int_{\mathbb{R}} \left(1 + |\xi|^2\right)\left|\hat{f}(\xi)\right|^2 \left|e^{i(z+h)\xi} - e^{iz\xi}\right|^2 d\xi.$$

By Dominated Convergence, we have that $\lim_{h\to 0} f(z+h) = f(z)$ and hence $f \in C^0(\mathbb{R})$. □

Lemma 4.1.5 (Weyl's criterion)**.** *Let T be a self-adjoint operator in the Hilbert space H. The following properties are equivalent:*
(i) $\lambda \in \sigma_{ess}(T)$.
(ii) There exists a sequence $(u_n) \subset D(T)$ such that $\|u_n\|_H = 1$, (u_n) is not relatively compact and $(T - \lambda I)u_n \xrightarrow[n\to\infty]{H} 0$.

Proof. i) $\implies$ (ii):
Let $\lambda \in \sigma_{\text{ess}}(T)$, $T - \lambda I$ is not a Fredholm operator, which means that $N(T - \lambda I)$ infinite dimensional. Therefore there exists an orthonormal family $(u_n) \in N(T - \lambda I)$, which satisfies proposition (ii).

(ii) $\implies$ (i):
Suppose $\lambda \notin \sigma_{\text{ess}}(T)$, then $T - \lambda I$ is a Fredholm operator. Consider a sequence $(u_n) \subset D(T)$ such that $\|u_n\|_H = 1$ and $(T - \lambda I)u_n \xrightarrow[n\to\infty]{H} 0$. Define $v_n \in N(T - \lambda I), w_n \in N(T - \lambda I)^\perp$ such that $u_n = v_n + w_n$. We have that $w_n = A^{-1}(T - \lambda I)$ where $A = (T - \lambda I)_{|N(T-\lambda I)^\perp}$. A^{-1} is bounded and therefore $w_n \xrightarrow[n\to\infty]{H} 0$. Furthermore (v_n) is bounded, since $\|v_n\|_H \leq \|u_n\|_H = 1$ and as a bounded sequence in a finite-dimensional subspace it admits a converging subsequence. Thus (u_n) admits a converging subsequence. □

We now prove Proposition 4.1.3.

Proof. 1. Let $\lambda < \gamma$, we show that $\lambda \notin \sigma_{\text{ess}}(\mathcal{L})$.

By a straightforward computation, we have that $\frac{\beta^2}{4} > \lambda$. Suppose there exists a sequence $(f_n) \subset D(\mathcal{L})$ that satisfies $(\mathcal{L} - \lambda I) f_n \xrightarrow[n\to\infty]{L^2(dz)} 0$ and $\|f_n\|_2 = 1$. We will show that (f_n) has a converging subsequence.

(f_n) is bounded in $H^1(dz)$ and we may apply Equality (41):

$$\begin{aligned}\langle (\mathcal{L} - \lambda I) f_n | f_n \rangle &= \int_{\mathbb{R}} \left((f_n')^2 + \left(\frac{\beta^2}{4} - \lambda \right) f_n^2 \right) dz - \frac{\chi f_n(0)^2}{2} \\ &\geq \|f_n'\|_2^2 - \frac{\chi f_n(0)^2}{2} \\ &\geq \|f_n'\|_2^2 - \frac{\chi}{4\pi} \|f_n'\|_2 .\end{aligned}$$

We used the fact that by Lemma 4.1.4 the domain $D(\mathcal{L}) \subset C^0(\mathbb{R})$ and thus the distribution $\frac{\beta'}{2} f_n = -\frac{\chi f_n(0)}{2} \delta_0$ is well-defined as a linear function on $C^0(\mathbb{R})$. The left handside is bounded above as a converging sequence and thus the second-order polynomial in $\|f_n'\|_2$ on the right handside is also bounded above, which in turn shows that $\|f_n'\|_2$ is uniformly bounded.

By boundedness of $\|f_n'\|_2$ and Lemma 4.1.4, we have that $(f_n(0))$ is a bounded sequence and thus admits a converging subsequence. Up to extraction, we can suppose that $(f_n(0))$ converges.

$$\langle (\mathcal{L} - \lambda I)(f_n - f_m) | (f_n - f_m) \rangle + \frac{\chi}{2} (f_n(0) - f_m(0))^2 \geq \|f_n' - f_m'\|_2^2 + (\inf \gamma - \lambda) \|f_n - f_m\|_2^2 ,$$

since $\gamma(z) \geq \lambda$ for $z \in \mathbb{R}$. Therefore (f_n) is a Cauchy sequence in $L^2(dz)$ (in fact even in $H^1(z)$) and converges. By Lemma 4.1.5, we have that $\lambda \notin \sigma_{\text{ess}}(\mathcal{L})$.

2. Let $\lambda \geq \gamma$, we show that $\lambda \in \sigma_{\text{ess}}(\mathcal{L})$.

If $\gamma = \frac{1}{4} \min\left(\sigma^2 - 4, \frac{1}{\chi^2}\right) = \frac{\sigma^2 - 4}{4}$, or equivalently when $\frac{\beta^2(z)}{4} - \gamma = 0$ for $z \geq 0$, we have that for a function $f \in D(\mathcal{L})$, such that $\text{supp}(f) \subset \mathbb{R}_+ \setminus \{0\}$, $(\mathcal{L} - \lambda) f = -f'' + (\gamma - \lambda) f$.
Take a smooth increasing nonnegative function $\phi : [0,1] \to [0,1]$ such that $\phi(0) = \phi'(0) = \phi'(1) = 0$ and $\phi(1) = 0$. Define the function $f_k : \mathbb{R} \to \mathbb{R}$ like so:

$$f_k : z \mapsto c_k \cdot \begin{cases} \phi(z) & \text{if } z \in [0,1] \\ \cos\left(\sqrt{\lambda - \gamma}(z-1)\right) & \text{if } z \in [1, a_k + 1] \\ \phi(a_k + 2 - z) & \text{if } z \in [a_k + 1, a_k + 2] \\ 0 & \text{else} \end{cases},$$

where $a_k = \frac{2k\pi}{\sqrt{\lambda - \gamma}}$, $c_k = \left(\frac{a_k}{2} + 2 \int_0^1 \phi^2(z) dz \right)^{-\frac{1}{2}}$, if $\lambda > \gamma$, or $a_k = 2^k$, $c_k = \left(a_k + 2 \int_0^1 \phi^2(z) dz \right)^{-\frac{1}{2}}$, if $\lambda = \gamma$. In all cases $f_k \in D(\mathcal{L}), \|f_k\|_2 = 1$ and $(\mathcal{L} - \lambda) f_k(z) = 0$ for $z \in [1, a_k + 1]$. Hence $\|(\mathcal{L} - \lambda) f_k\|_2^2 = 2c_k^2 \int_0^1 (\phi''(z) + (\lambda - \gamma)\phi(z))^2 dz \to 0$, because clearly $\lim_{k \to +\infty} c_k = 0$. Finally, it can be easily shown that (f_k) is not relatively compact (its only possible accumulation point would be 0, but (f_k) cannot converge to 0, since $\|f_k\|_2 = 1$). Thus by Lemma 4.1.5, $\lambda \in \sigma_{\text{ess}}(\mathcal{L})$.

If $\gamma = \frac{1}{4} \min\left(\sigma^2 - 4, \frac{1}{\chi^2}\right) = \frac{1}{4\chi^2}$, or equivalently when $\frac{\beta^2(z)}{4} - \gamma = 0$ for $z \leq 0$, the same reasoning applies *mutatis mutandis.* □

Proposition 4.1.6.

$$\sigma(\mathcal{L}) \cap (-\infty, \gamma) = \{0\}$$

Furthermore, the eigenvalue 0 *is simple.*

Proof. Let $\lambda < \gamma$ be an eigenvalue for $\mathcal{L}$ in the space $L^2(dz)$.

For $z > 0$, the characteristic polynomial of the ordinary differential equation $\mathcal{L} - \lambda = 0$ is $P(\mu) = -\mu^2 + \frac{\sigma^2}{4} - 1 - \lambda$, whose roots are $\mu_\pm = \pm\frac{1}{2}\sqrt{\sigma^2 - 4(1+\lambda)}$, which is well-defined, since $\lambda < \frac{1}{4}(\sigma^2 - 4)$. This gives rise to two eigenvectors $e^{\mu_\pm z}$, but it is clear that only $e^{\mu_- z} \in L^2(\mathbb{R}_+, dz)$.

For $z < 0$, by a similar reasoning we obtain that $e^{\nu_+ z}$ is an eigenvector for $\mathcal{L}$ in the space $L^2(\mathbb{R}_-, dz)$, with $\nu_+ = \frac{1}{2}\sqrt{\frac{1}{\chi^2} - 4\lambda}$, which is well-defined, since $\lambda < \frac{1}{4\chi^2}$, .

The eigenvector associated with the eigenvalue λ is therefore of the shape:

$$f_\lambda(z) = \begin{cases} e^{\nu_+ z} & \text{if } z < 0 \\ e^{\mu_- z} & \text{if } z \geq 0 \end{cases}.$$

However we must have that $f_\lambda \in D(\mathcal{L})$ which implies that $f'_\lambda - \frac{\beta}{2} f_\lambda$ is continuous.

$$\begin{aligned} \left(f'_\lambda - \frac{\beta}{2} f_\lambda\right)(0^+) = \left(f'_\lambda - \frac{\beta}{2} f_\lambda\right)(0^-) &\iff \mu_- + \frac{\sigma}{2} = \nu_+ + \frac{\sigma - \chi}{2} \\ &\iff \chi = \sqrt{\sigma^2 - 4(1+\lambda)} + \sqrt{\frac{1}{\chi^2} - 4\lambda} \end{aligned}$$

The right handside is a strictly decreasing function in λ and thus the equation admits at most one root. One checks that this root is $\lambda = 0$ and we already know that $\lambda = 0$ is indeed an eigenvalue with the eigenvector $e^{\frac{V}{2}}$. □

We move on with the proof of Theorem 4.1.

Proof. $\mathcal{L}$ is monotone and self-adjoint, therefore by Semigroup theory (see Section 7.4 in [11]), it generates the semi-group $e^{-t\mathcal{L}}$. We consider the spectral projection $\mathcal{P}$ onto the eigenspace of $\mathcal{L}$ associated with the eigenvalue 0, which is $\mathcal{P}u = \frac{1}{\int e^V dz} \int u e^{\frac{V}{2}} dz$. Let $\mathcal{A}$ be the restriction of $\mathcal{L}$ to $N(\mathcal{P})^\perp$. We have that $\inf \sigma(\mathcal{A}) = \gamma$ and for $\lambda < \gamma$, $(\mathcal{A} - \lambda I)^{-1}$ is a bounded operator. Since $\mathcal{A}$ is a closed self-adjoint operator, $\inf \sigma(\mathcal{A}) = \inf_{\|u\|_2 = 1} \langle \mathcal{A}u | u \rangle$ and therefore $\langle (A - \lambda I)u | u \rangle \geq (\gamma - \lambda) \|u\|_2^2$, which shows that $\left\|(\mathcal{A} - \lambda I)^{-1}\right\| \leq \frac{1}{\gamma - \lambda}$. From Hille-Yosida theorem in the self-adjoint case (see Section 7.4 in [11]), we then have for every $u \in N(\mathcal{P})^\perp$ (and not just $u \in D(\mathcal{A})$) that $\left\|e^{-t\mathcal{A}} u\right\|_2 \leq e^{-\gamma t} \|u\|_2$ and $\left\|\mathcal{A} e^{-t\mathcal{A}} u\right\|_2 \leq \frac{e^{-\gamma t}}{t} \|u\|_2$. By setting $w = e^{\frac{V}{2}} \nu$, we have that:

$$w(t) = \mathcal{P} w_0 + e^{-t\mathcal{A}} (I - \mathcal{P}) w_0.$$

This leads to the bound:

$$\|w(t) - \mathcal{P} w_0\|_{L^2(dz)} \leq \|w_0\|_{L^2(dz)} e^{-\gamma t}.$$

Bound (37) is simply a rewritten form of this bound.

Finally, we prove Bound (38). Set $r(t) := e^{-t\mathcal{A}}(I - \mathcal{P}) w_0$, then we have:

$$\int_{\mathbb{R}} \left(\partial_z r(t) - \frac{\beta}{2} r(t)\right)^2 = \langle r(t), \mathcal{A} r(t) \rangle \leq \|r(t)\|_2 \|\mathcal{A} r(t)\|_2 \leq \frac{e^{-2\gamma t}}{t} \|r(0)\|_2^2,$$

where we have used the bounds obtained from Hille-Yosida Theorem in the self-adjoint case. Therefore $\|\partial_z r(t)\|_2 \leq \left(\frac{\|\beta\|_\infty \|r(0)\|_2}{2} + \frac{\|r(0)\|_2}{\sqrt{t}}\right) e^{-\gamma t}$. By Lemma 4.1.4, we then have that $\|r(t)\|_\infty \leq C\left(1 + t^{-\frac{1}{2}}\right) e^{-\gamma t}$. Or equivalently:

$$\left\|(\nu(t) - \langle \nu^0 \rangle) e^{\frac{V}{2}}\right\|_\infty \leq C\left(1 + t^{-\frac{1}{2}}\right) e^{-\gamma t}.$$

□

4.2 Pulled Front Dynamics in the Small Bias Case

In the small bias case $\chi \leq 1$ (or when $\sigma > \sigma^*$), the inside dynamics of the wave is drastically different. In fact, we can start by observing that Theorem 4.1 doesn't apply to these cases. For $\sigma = \sigma_{\mathrm{F/KPP}}$, $\frac{\partial_z \rho^\sigma}{\rho^\sigma} = \frac{1-\chi}{(1-\chi)z+1} - 1$, for $z \geq 0$. Hence $1 \cdot e^{V(z)} = C((1-\chi)z+1)^2$ for a multiplicative constant $C > 0$ and $1 \notin L^2(e^V)$. Similarily, in the case $\sigma > \sigma^*$, $1 \cdot e^V$ behaves like $e^{(\sigma - 2\mu_-(\sigma))z}$, but $\sigma - 2\mu_-(\sigma) = \sqrt{\sigma^2 - 4} > 0$ and thus $1 \notin L^2(e^V dz)$. To describe the behavior in these cases, we start by stating first a general Theorem concerning Equation (36) under a condition on β:

Theorem 4.2. *Consider Equation:*

$$\begin{cases} \partial_t \nu - \partial_{zz}\nu - \beta(z)\partial_z \nu = 0 \\ \nu(0,\,\cdot\,) = \nu^0 \end{cases}. \tag{46}$$

Suppose that β verifies the following conditions:

- *There exists $K > 0$, such that $\beta' \leq K$ in the sense of distributions.*
- *$\int_{\mathbb{R}} \beta_-(z)dz < +\infty$, where $(\,\cdot\,)_- = -\min(0,\,\cdot\,)$.*

We recall that $V' = \beta$. Suppose that the initial datum ν^0 satisfies the following condition:

$$\int_{\mathbb{R}} \left(\nu^0(z)\right)^2 e^{V(z)} dz < +\infty. \tag{47}$$

Then the solution ν to Equation (46) satisfies the following decay property:

$$\lim_{t \to +\infty} \left\| \nu(t) e^{\frac{V}{2}} \right\|_\infty = 0. \tag{48}$$

And as a consequence ν converges uniformly to 0 on intervals of the form $[a, +\infty)$, for $a \in \mathbb{R}$, i.e.:

$$\lim_{t \to +\infty} \sup_{z \in [a, +\infty)} |\nu(t,z)| = 0. \tag{49}$$

Next, we apply Theorem 4.2 to establish the pulled nature of the waves in the two mentioned cases (small bias case $\chi \leq 1$, or $\sigma > \sigma^*$).

Corollary 4.2.1. *Suppose that $\chi \leq 1$ and that $\sigma = \sigma_{F/KPP} = 2$ and consider a neutral fraction ν of the associated wave profile that satisfies Condition (47). In particular, any function $\nu^0 \in L^\infty(\mathbb{R})$ such that $z \mapsto \left(\nu^0(z)z\right)^2$ is integrable at $z = +\infty$ satisfies Condition (47). Then $\nu(t)$ converges to 0 uniformly on intervals of the form $[a, +\infty)$, with $a \in \mathbb{R}$.*
The same results holds true if we suppose that $\sigma > \sigma^*$. In particular, any function $\nu^0 \in L^\infty(\mathbb{R})$ such that $z \mapsto \left(\nu^0(z)\right)^2 e^{(\sigma - 2\mu_-(\sigma))z}$ is integrable at $z = +\infty$ satisfies Condition (47).

Proof. We recall that $\beta(z) = \sigma - \chi \mathbb{1}_{z \leq 0} + 2\frac{\partial_z \rho^\sigma}{\rho^\sigma}$.
In the case where $\chi \leq 1$ and $\sigma = 2$, we have from Theorem 3.1 that $\rho^\sigma(z) = \begin{cases} 1 & \text{if } z \leq 0 \\ ((1-\chi)z+1)e^{-z} & \text{if } z > 0 \end{cases}$, which leads to $\beta(z) = \begin{cases} 2-\chi & \text{if } z \leq 0 \\ \frac{2(1-\chi)}{(1-\chi)z+1} & \text{if } z > 0 \end{cases}$. β' is clearly bounded everywhere except at $z = 0$, but $\beta(0^-) = 2-\chi$ and $\beta(0^+) = 2(1-\chi)$. This leads to β' being bounded above in the sense of distributions and $\beta \geq 0$, hence Theorem 4.2 applies. Furthermore, by an easy computation up to a mulitplicative constant we have that $e^{V(z)} = \begin{cases} e^{(2-\chi)z} & \text{if } z \leq 0 \\ ((1-\chi)z+1)^2 & \text{if } z > 0 \end{cases}$, hence a bounded function $\nu^0 \in L^\infty(\mathbb{R})$ satisfies Condition (47), if $\left(\nu^0(z)z\right)^2$ is integrable at $z = +\infty$.

In the case where $\sigma > \sigma^*$, we have $\rho^\sigma(z) = \begin{cases} 1 & \text{if } z \leq 0 \\ \frac{\mu_+ - \chi}{\sigma - \chi} e^{-\mu_- z} - \frac{\mu_- - \chi}{\sigma - \chi} e^{-\mu_+ z} & \text{if } z > 0 \end{cases}$, where we recall that $\mu_\pm = \frac{\sigma \pm \sqrt{\sigma^2 - 4}}{2}$. This leads to $\beta(z) = \begin{cases} \sigma - \chi & \text{if } z \leq 0 \\ \sigma - 2\frac{(\mu_+ - \chi)\mu_- e^{-\mu_- z} - (\mu_- - \chi)\mu_+ e^{-\mu_+ z}}{(\mu_+ - \chi)e^{-\mu_- z} - (\mu_- - \chi)e^{-\mu_+ z}} & \text{if } z > 0 \end{cases}$. By the same reasoning β' is bounded above. For $z \leq 0, \beta(z) \geq 0$ and $\lim_{z \to +\infty} \beta(z) = \sigma - 2\mu_- > 0$, which establishes that $\int_{\mathbb{R}} \beta_-(z) dz < +\infty$ and hence Theorem 4.2 applies. Finally, we have up to a multiplicative constant that $e^{V(z)} = \begin{cases} e^{(\sigma - \chi)z} & \text{if } z \leq 0 \\ e^{\sigma z} \left(\rho^\sigma(z)\right)^2 & \text{if } z > 0 \end{cases}$, hence a bounded function $\nu^0 \in L^\infty(\mathbb{R})$ satisfies Condition (47), if $\left(\nu^0(z)\right)^2 e^{(\sigma - 2\mu_-)z}$ is integrable at $z = +\infty$. □

Before moving on to the proof of Theorem 4.2, let us make some comments. Notice that Corollary 4.2.1 does not apply to the neutral fraction $\nu^0 \equiv 1$, which is consistent with the fact that the neutral fraction $\nu \equiv 1$ stays constant and does not converge to 0. In fact, Condition (47) or its counterparts in Corollary 4.2.1 may be seen as a characterization of an initial datum ν^0, that constitutes a negligible part of the leading edge of the traveling wave. If, for the sake of the argument, we were to accept this property as a definition, then Theorem 4.2 or Corollary 4.2.1 tell us that neutral fractions, which constitute a negligible part of the leading edge of the traveling wave, go extinct in the traveling wave, *i.e* they converge to 0.
Nevertheless, just like in the work [23], Theorem 4.2 does not give any rate of convergence, contrary to Theorem 4.1, and this remains an open question. Finally let us note, that unsurprisingly in the case of large bias with $\sigma = \sigma^*$, for $\beta(z)$ is bounded above by a negative constant for $z > 0$, which establishes that $\int \beta_-(z) = +\infty$ and thus Theorem 4.2 conversely does not apply to that case.

We now move on to the proof of Theorem 4.2.

Proof. Consider $L := -\partial_{zz} - \beta(z)\partial_z$ in the weighted space $L^2(e^V dz)$ with domain $D(L) = H^2(e^V dz)$. From the arguments of the proof of Theorem 4.1, we have that L is self-adjoint and it furthermore satisfies the following property for $f \in D(L)$:

$$\begin{aligned} \int_{\mathbb{R}} f(Lf)e^V dz &= \int_{\mathbb{R}} f(-f'' - \beta f')e^V \partial_z \\ &= \int_{\mathbb{R}} \left(f'^2 + \beta f f' - \beta f f'\right) e^V dz \\ &= \left\|f'\right\|^2_{L^2(e^V dz)}. \end{aligned}$$

This leads to the following dissipation rate:

$$\frac{d}{dt}\left(\frac{1}{2}\left\|\nu\right\|^2_{L^2(e^V dz)}\right) = -\left\|\partial_z \nu\right\|^2_{L^2(e^V dz)}. \tag{50}$$

Furthermore:

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\left\|\partial_z\nu\right\|^2_{L^2(e^V dz)}\right)\\
=&\int_{\mathbb{R}}\partial_z(\partial_t\nu)\partial_z\nu e^V dz\\
=&-\int_{\mathbb{R}}\partial_t\nu\partial_z(e^V\partial_z\nu)dz\\
=&-\int_{\mathbb{R}}\partial_{zz}\nu\partial_z(e^V\partial_z\nu)dz-\int_{\mathbb{R}}\beta\partial_z\nu\partial_z(e^V\partial_z\nu)dz\\
=&-\left\|\partial_{zz}\nu\right\|^2_{L^2(e^V dz)}-\frac{1}{2}\int_{\mathbb{R}}\partial_z\left((\partial_z\nu)^2\right)\beta e^V dz-\int_{\mathbb{R}}(\partial_z\nu)^2\beta^2 e^V dz-\frac{1}{2}\int_{\mathbb{R}}\beta\partial_z\left((\partial_z\nu)^2\right)e^V dz\\
=&-\left\|\partial_{zz}\nu\right\|^2_{L^2(e^V dz)}-\int_{\mathbb{R}}\left(\partial_z((\partial_z\nu)^2)\beta e^V+(\partial_z\nu)^2\beta^2 e^V\right)dz\\
=&-\left\|\partial_{zz}\nu\right\|^2_{L^2(e^V dz)}-\int_{\mathbb{R}}\left(\partial_z\left((\partial_z\nu)^2\beta e^V\right)-(\partial_z\nu)^2\beta' e^V\right)dz\\
=&-\left\|\partial_{zz}\nu\right\|^2_{L^2(e^V dz)}+\int_{\mathbb{R}}(\partial_z\nu)^2\beta' e^V dz.
\end{aligned}
$$

For $K>0$ such that $\beta'\leq K$, we have that:

$$
\frac{d}{dt}\left(\frac{K}{2}\left\|\nu\right\|^2_{L^2(e^V dz)}+\frac{1}{2}\left\|\partial_z\nu\right\|^2_{L^2(e^V dz)}\right)=-\left\|\partial_{zz}\nu\right\|^2_{L^2(e^V dz)}+\int_{\mathbb{R}}(\partial_z\nu)^2\left(\beta'-K\right)e^V dz\leq 0.
$$

Hence $\|\nu(t)\|_{L^2(e^V dz)}$ converges to a limit and so does $\|\partial_z\nu(t)\|_{L^2(e^V dz)}$. But by Equation (50), the limit of $\|\partial_z\nu(t)\|_{L^2(e^V dz)}$ can only be 0, otherwise $\|\nu(t)\|_{L^2(e^V dz)}$ could not converge.
Finally, set $c:=\int_{-\infty}^{+\infty}\beta_-$ and $W(z)=\int_0^z\beta_+-c$. If we fix the constant of integration of V such that $V(z)=\int_0^z\beta$. Then we have the following inequality:

$$
V-2c\leq W\leq V.
$$

We can conclude by the following argument:

$$
\begin{aligned}
e^{-2c}\left\|\nu(t)^2e^V\right\|_\infty&\leq\left\|\nu(t)^2e^W\right\|_\infty\\
&\leq\int_{\mathbb{R}}\left|\partial_z\left(\nu^2e^W\right)\right|dz\\
&\leq 2\int_{\mathbb{R}}|\nu||\partial_z\nu|e^W dz+\int_{\mathbb{R}}|\beta_+e^W|\nu^2 dz\\
&\leq 2\int_{\mathbb{R}}|\nu||\partial_z\nu|e^W dz+\int_{\mathbb{R}}\left(e^W\right)'\nu^2 dz\\
&\leq 2\int_{\mathbb{R}}|\nu||\partial_z\nu|e^W dz-2\int_{\mathbb{R}}\nu\partial_z\nu e^W dz\\
&\leq 4\int_{\mathbb{R}}|\nu||\partial_z\nu|e^W dz\\
&\leq 4\left\|\nu(t)\right\|_{L^2(e^W dz)}\left\|\partial_z\nu(t)\right\|_{L^2(e^W dz)}\\
&\leq 4\left\|\nu(t)\right\|_{L^2(e^V dz)}\left\|\partial_z\nu(t)\right\|_{L^2(e^V dz)},
\end{aligned}
$$

where at the end we have used Cauchy-Schwarz inequality, followed by the equivalence of norms. Hence $\lim_{t\to\infty}\sup_{z\in\mathbb{R}}|\nu(t,z)^2e^V(z)|=0$. Finally, since $\int_{\mathbb{R}}\beta_-(z)dz<+\infty$, for every $a\in\mathbb{R}$, we have that $\inf_{z\in[a,+\infty)}e^{V(z)}>0$ and therefore $\lim_{t\to+\infty}\sup_{z\in[a,+\infty)}|\nu(t,z)|=0$. □

5 Asymptotic Spreading Properties

In this Section, we work under the hypothesis that the solution (ρ, N) of System (1) is well-defined for all time, that $x \mapsto N(t,x)$ is an increasing function and we recall that $\bar{x}(t)$ is defined as the unique solution $N(t, \bar{x}(t)) = N_{\text{th}}$. As before, we consider the system in the moving frame of reference and study $(\rho(t,z), N(t,z))$ solution of System (16), droping the diacritical ˜ for the sake of concision.

The aim of the Section is to describe the asymptotic behavior of $t \mapsto \bar{x}(t)$. In fact, we will show that under some assumptions on the initial datum, we have $\liminf_{t\to+\infty} \dot{\bar{x}}(t) \leq \sigma^*$ and $\limsup_{t\to+\infty} \dot{\bar{x}}(t) \geq \sigma^*$. The strategy of proof for both results is similar and is based on an argument by contradiction: if those properties were not satisfied, then this would first lead to an abnormal behavior of ρ, which in turn contradicts the equation $N(t,0) = N_{\text{th}}$. We have not been able to prove a stronger result, such as for instance $\lim_{t\to+\infty} \dot{\bar{x}}(t) = \sigma^*$ and we believe that in order to achieve such a result, one needs to study the behavior of ρ and N simultaneously, which is much more involved than our present study. One major difficulty comes from the fact that System (1) does not have a comparison principle.

5.1 The Spreading may not be too fast

Theorem 5.1. *Suppose that* $\dot{\bar{x}} \in L^\infty(\mathbb{R}_+)$ *and that:*

$$\frac{\rho^0}{\rho^{\sigma^*}} \in L^\infty.$$

Then $\liminf_{t\to+\infty} \dot{\bar{x}}(t) \leq \sigma^*$.

Proof. 1. We argue by contradiction and start by showing that ρ converges to 0 uniformly on intervals of the form $[a, +\infty)$ for $a \in \mathbb{R}$.
Let $t_0 \geq 0, \delta > 0$, such that for $t \geq t_0, \dot{\bar{x}}(t) \geq \sigma^* + \delta$. Set $v := \frac{\rho}{\rho^{\sigma^*+\delta}}$, where $\rho^{\sigma^*+\delta}$ is the traveling wave profile for speed $\sigma^* + \delta$ (see Theorem 3.1). We have that v satisfies the following Equation:

$$\partial_t v - \partial_{zz} v - \beta(t,z)\partial_z v - \gamma(t,z) v = 0,$$

with $\beta(t,z) := \dot{\bar{x}}(t) - \chi \mathbb{1}_{z<0} + 2\frac{\partial_z \rho^{\sigma^*+\delta}}{\rho^{\sigma^*+\delta}}$ and $\gamma(t,z) := (\dot{\bar{x}}(t) - (\sigma^*+\delta))\frac{\partial_z \rho^{\sigma^*+\delta}}{\rho^{\sigma^*+\delta}}$. By noticing that $(t,z) \mapsto \left\|\frac{\rho^0}{\rho^{\sigma^*}}\right\|_\infty \exp\left(\left\|\frac{\partial_z \rho^{\sigma^*+\delta}}{\rho^{\sigma^*+\delta}}\right\|_\infty \int_0^{t_0} (\sigma^* + \delta - \dot{\bar{x}}(s))_+ ds\right)$ is a super-solution to $\partial_t - \partial_{zz} - \beta(t,z)\partial_z - \gamma(t,z)$, we observe that $\frac{\rho(t_0)}{\rho^{\sigma^*}} \in L^\infty$. Hence we can suppose without loss of generality that for $t \geq 0, \dot{\bar{x}}(t) \geq \sigma^* + \delta$. Therefore, we have that $\beta(t,z) \geq \sigma^* + \delta - \chi \mathbb{1}_{z<0} + 2\frac{\partial_z \rho^{\sigma^*+\delta}}{\rho^{\sigma^*+\delta}} =: \beta^{\sigma^*+\delta}(z)$ and $\gamma(t,z) \leq 0$. Moreover, by linearity we can suppose that $\left\|\frac{\rho^0}{\rho^{\sigma^*}}\right\|_\infty \leq 1$.

Since $\gamma(t,z) \leq 0$ and $v \geq 0$, we have that v is a subsolution of $\partial_t - \partial_{zz} - \beta(t,z)\partial_z$. Hence we consider the solution $\bar{v}$ of $\partial_t - \partial_{zz} - \beta(t,z)\partial_z$, such that $v \leq \bar{v}$, with initial datum $\bar{v}^0$, which we define now. Set $\eta := \frac{\mu_+(\sigma^*) - \mu_-(\sigma^*+\delta)}{2}$ and define $g(z) = \frac{e^{-\eta z}}{1+e^{-\eta z}}$. Let $K > 0$ be big enough such that:

$$\frac{\rho^0(z)}{\rho^{\sigma^*+\delta}(z)} \leq \frac{\rho^{\sigma^*}(z)}{\rho^{\sigma^*+\delta}(z)} \leq K g(z).$$

Such a constant exists, as a consequence of Theorem 3.1, since for $z < 0$, $\frac{\rho^{\sigma^*}(z)}{\rho^{\sigma^*+\delta}(z)} = 1$ and for $z \to +\infty$, $\frac{\rho^{\sigma^*}(z)}{\rho^{\sigma^*+\delta}(z)} = O\left(z e^{-(\mu_+(\sigma^*) - \mu_-(\sigma^*+\delta))z}\right) = O(g(z))$. Thus, we set the initial datum $\bar{v}^0(z) = K g(z)$.

Let us show that $\partial_z \bar{v} \leq 0$. By construction, $\partial_z \bar{v}^0 \leq 0$. Furthermore $\bar{v}^0 \in H^2_{\text{loc}}(\mathbb{R}, dz)$ and hence by standard regularity theory, we have that $\bar{v} \in C(\mathbb{R}_+, H^2_{\text{loc}}(\mathbb{R}, dz))$. Setting $w := \partial_z \bar{v}$, combining the facts that $w(0, \cdot) \in H^1(\mathbb{R}, dz)$, $w \in C(\mathbb{R}_+, H^1_{\text{loc}}(\mathbb{R}))$ and that w is solution to $\partial_t w - \partial_{zz} w - \partial_z(\beta w) = 0$,

shows in fact that $w \in C(\mathbb{R}_+, H^1(\mathbb{R}, dz))$. Now we can procede as in the proof of Corollay 2.1.2, to show that $w_+ \equiv 0$, where $(\,\cdot\,)_+ = \max(\,\cdot\,, 0)$:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}} w_+^2(t,z)\right) &= \int_{\mathbb{R}} w_+ \partial_t w_+ dz \\
&= \int_{\mathbb{R}} w_+ \partial_t w dz \\
&= \int_{\mathbb{R}} w_+ (\partial_{zz} w + \partial_z(\beta w)) dz \\
&= -\int_{\mathbb{R}} \left(\partial_z w_+ \partial_z w + \beta w \partial_z w_+\right) dz \\
&= -\int_{\mathbb{R}} (\partial_z w_+)^2 dz - \int_{\mathbb{R}} \beta w_+ \partial_z w_+ dz \\
&\leq -\int_{\mathbb{R}} (\partial_z w_+)^2 dz + \frac{1}{2}\int_{\mathbb{R}} (\partial_z w_+)^2 dz + \frac{\|\beta\|_\infty^2}{2}\int_{\mathbb{R}} (w_+)^2 dz \\
&\leq \frac{\|\beta\|_\infty^2}{2}\int_{\mathbb{R}} (w_+)^2 dz,
\end{aligned}$$

where we have applied arithmetic and geometric means inequality $|\beta w_+ \partial_z w_+| \leq \frac{(\partial_z w_+)^2}{2} + \frac{\|\beta\|_\infty^2 (w_+)^2}{2}$. Since $w_+(0,\,\cdot\,) \equiv 0$, by Grönwall's lemma, we have that $w_+(t,\,\cdot\,) \equiv 0$. Thus $\partial_z \bar{v} \leq 0$.

By the bound $\beta^{\sigma^*+\delta} \leq \beta$ and the inequality $\partial_z \bar{v} \leq 0$, we have that $\bar{v}$ is a sub-solution (and not a super-solution!) of the parabolic operator $\partial_t - \partial_{zz} - \beta^{\sigma^*+\delta}\partial_z$. But from Corollary 4.2.1, we know that the corresponding solution with initial datum $\bar{v}^0(z) = Kg(z)$ converges to 0 uniformly on intervals of the form $[a, +\infty)$. Hence so does $\bar{v}$ and v by the comparison principle and thus ρ converges to 0 uniformly on compact sets.

2. Let us now show that the condition $N(t,0) = N_{\text{th}}$ cannot be satisfied for time $t > 0$ sufficiently large. Let $\mu > 0$ such that $\mu < \frac{\sigma^*}{D}$ and $\mu < \mu_-(\sigma^* + \delta)$. Let $\varepsilon > 0$ be sufficiently small, such that $\frac{1+N_{\text{th}}}{2} - \frac{\varepsilon}{D\mu\left(\frac{\sigma^*}{D}-\mu\right)} > N_{\text{th}}$ and denote $B := \frac{\varepsilon}{D\mu\left(\frac{\sigma^*}{D}-\mu\right)}$. Choose $h > 0$ big enough such that $\frac{1+N_{\text{th}}}{2}\left(1 - e^{-\frac{\sigma^* h}{D}}\right) - B\left(1 - e^{-\left(\frac{\sigma^*}{D}-\mu\right)h}\right) > N_{\text{th}}$, which exists by noticing that in the limit $h \to +\infty$ the inequality is satisfied by the above.

As v converges to 0 uniformly on the set $[-h, +\infty)$, and by noticing that $\rho^{\sigma^*+\delta}(z) = o\left(e^{-\mu z}\right)$, there exists t_1, such that for $t \geq t_1$ and $z \geq -h$, we have $\rho(t,z)N(t,z) \leq \rho(t,z) \leq \varepsilon e^{-\mu z}$. Finally let $Z > 0$ be such that $N(t_1, Z) > \frac{1+N_{\text{th}}}{2}$.
We consider the parabolic equation in the domain $\Omega = (t_1, +\infty) \times (-h, +\infty)$:

$$\begin{aligned}
\partial_t n - \sigma^* \partial_z n - D\partial_{zz} n + \varepsilon e^{-\mu z} &= 0 \\
n(t,-h) = 0, n(t_1,\,\cdot\,) &= n^{t_1}(\,\cdot\,).
\end{aligned} \tag{51}$$

A stationary solution to Equation (51) is:

$$n_\infty(z) = \frac{1+N_{\text{th}}}{2} - Be^{-\mu z} + Ce^{-\frac{\sigma^*}{D}z},$$

with $C := -Be^{-\left(\frac{\sigma^*}{D}-\mu\right)h} - \frac{1+N_{\text{th}}}{2}e^{-\frac{\sigma^*}{D}h}$. Notice that $n_\infty(0) > N_{\text{th}}$, by the choices made above.
Consider now the solution n to Equation (51) with initial datum $n(t_1,\,\cdot\,) = n_\infty \mathbb{1}_{z \geq Z}$. We will show that $n(t,\,\cdot\,) \to n_\infty$ uniformly and in particular for $z = 0$, $n(t,0) \to n_\infty(0) > N_{\text{th}}$.

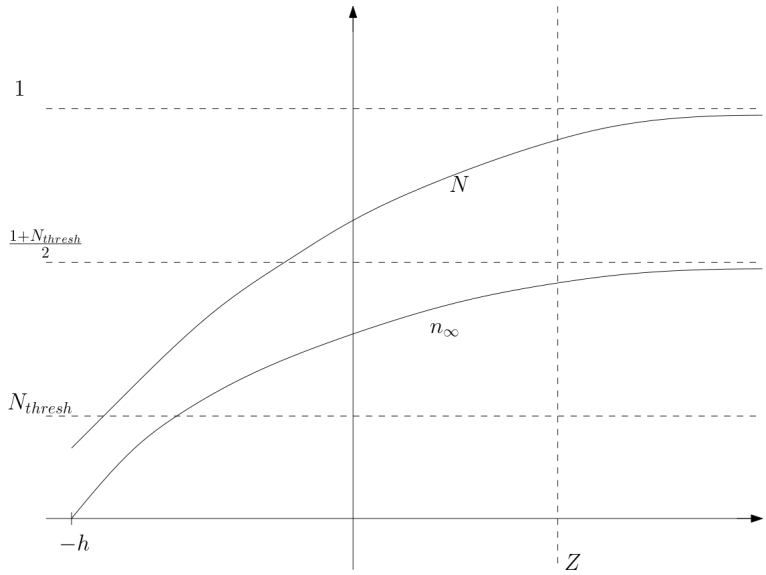


Figure 2: In order to prove that condition $N(t,0) = N_{\text{th}}$ cannot be satisfied, we consider solution $n(t,z)$ to Equation (51) with inital datum $n(t_1,z) = n_\infty(z)$ for $z \geq Z$ and $n(t_1,z) = 0$ for $z \in [-h, Z)$. Then $n(t) \to n_\infty$ uniformly. Furthermore N is a super-solution of Equation (51) and $N(t_1,\,\cdot\,) > n(t_1,\,\cdot\,)$, which leads to $N(t,\,\cdot\,) \geq n(t,\,\cdot\,)$ for all $t \geq t_1$. But $n(t,0) \to n_\infty(0) > N_{\text{th}}$ and thus the condition $N(t,0) = N_{\text{th}}$ is not satisfied after some time.

Indeed set $w := n_\infty - n$, then w satisfies for $(t,z) \in \Omega$:

$$\begin{aligned} \partial_t w - \sigma^* \partial_z w - D\partial_{zz} w &= 0 \\ w(t,-h) = 0, w(t_1,z) &= n_\infty(z)\mathbb{1}_{z<Z}. \end{aligned}$$

It can be verified that for $(t,z) \in \Omega$, we have the following explicit expression:

$$\begin{aligned} |w(t,z)| &= \left| \frac{e^{-\frac{\sigma^{*2}}{4}t - \frac{\sigma^*}{2}z}}{\sqrt{4\pi D(t-t_1)}} \int_{-h}^{+\infty} \left(e^{-\frac{(z+h-y)^2}{4D(t-t_1)}} - e^{-\frac{(z+h+y)^2}{4D(t-t_1)}} \right) e^{\frac{\sigma^*}{2}y} n_\infty(y) \mathbb{1}_{y \leq Z} dy \right| \\ &\leq \frac{C e^{\frac{\sigma^*}{2}h} e^{-\frac{\sigma^{*2}}{4}t}}{\sqrt{(t-t_1)}} \left\| e^{\frac{\sigma^*}{2}y} n_\infty(y) \right\|_{L^1([-h,Z])}. \end{aligned}$$

Hence, $\lim_{t\to+\infty} w(t,z) = 0$ uniformly, which is equivalent to establishing that $\lim_{t\to+\infty} n(t,z) = n_\infty(z)$ uniformly. In particular there exist $t_2 > t_1$, such that $n(t_2,0) > N_{\text{th}}$.

In the final step, it remains to show that N is a supersolution of Equation (51). Indeed for $(t,z) \in \Omega$:

$$\begin{aligned} &\partial_t N - \sigma^* \partial_z N - D\partial_{zz} N + \varepsilon e^{-\mu z} \\ \geq &\partial_t N - \dot{\bar{x}}(t)\partial_z N - D\partial_{zz} N + \rho N \\ = &0. \end{aligned}$$

By construction $N(t_1,\,\cdot\,) \geq n(t_1,\,\cdot\,)$ and $N(\,\cdot\,,-h) > 0 = n(\,\cdot\,,-h)$. Hence we have for all $(t,z) \in \Omega, N(t,z) \geq n(t,z)$ and in particular $N(t_2,0) \geq n(t_2,0) > N_{\text{th}}$, which leads to a contradiction.

□

5.2 The Spreading may not be too slow

In this Section, we will show that $\limsup_{t\to+\infty}\dot{\bar{x}}(t)\geq\sigma^*$. The proof is based on the observation that the norm of ρ increases exponentially in a certain weighted L^1-spaces, when $\dot{\bar{x}}<\sigma^*$. In fact, as a starter, in the large bias case $\chi>1$, take for example the weight $e^{\frac{z}{\chi}}$. By noticing that it is an eigenvalue of the dual of the elliptic operator, one sees that:

$$\frac{d}{dt}\int_{\mathbb{R}}e^{\frac{z}{\chi}}\rho(t,z)dz=\frac{\chi+\frac{1}{\chi}-\dot{\bar{x}}(t)}{\chi}\int_{\mathbb{R}}e^{\frac{z}{\chi}}\rho(t,z)dz.$$

Then if $\limsup_{t\to+\infty}\dot{\bar{x}}(t)<\sigma^*$, we have that $\int_{\mathbb{R}}e^{\frac{z}{\chi}}\rho(t,z)dz$ grows exponentially. Nevertheless, since $e^{\frac{z}{\chi}}$ is unbounded, this observation is not sufficiently instructive. Hence, instead of $e^{\frac{z}{\chi}}$ an exact eigenvalue of the dual of the elliptic operator, we consider a supersolution $e^u\in L^\infty(\mathbb{R})$ of the dual of the elliptic operator, which enables us to show that $\rho(t)$ diverges to $+\infty$ in $L^1(\mathbb{R},e^udz)$. Then we show that a similar statement remains true for ρ in $L^1(\mathbb{R}_+,e^udz)$ and finally this accumulation of mass on the half-line $\mathbb{R}_+$ leads to a contradiction on the condition $N(t,0)=N_{\text{th}}$.

Theorem 5.2. *Suppose that $\dot{\bar{x}}\in L^\infty(\mathbb{R}_+)$ and that $\rho^0\in L^\infty$. Then, we have that $\limsup_{t\to+\infty}\dot{\bar{x}}(t)\geq\sigma^*$.*

Proof. Suppose by contradiction that $\limsup_{t\to+\infty}\dot{\bar{x}}(t)<\sigma^*$ ($=2$ in the small bias case $\chi\leq1$, $=\chi+\frac{1}{\chi}$ in the large bias case $\chi>1$). Then there exists $\delta>0,t_0\geq0$, such that for $t\geq t_0,\dot{\bar{x}}(t)\leq\sigma^*-\delta$.

1. We start by constructing a function $e^u\in L^\infty(\mathbb{R})$, such that the quantity $\int_{\mathbb{R}}e^{u(z)}\rho(t,z)dz$ tends to infinity. Given a function $u\in C^2(\mathbb{R})$, set $w(t,z):=e^{u(z)}\rho(t,z)$, then:

$$\partial_tw-\partial_{zz}w-\partial_z\left(\underbrace{\left(\dot{\bar{x}}(t)-2u'-\chi\mathbb{1}_{z\leq0}\right)}_{=:\beta(t,z)}w\right)-\underbrace{\left(u'^2+u''-\dot{\bar{x}}(t)u'+\mathbb{1}_{z>0}+\chi\mathbb{1}_{z\leq0}u'\right)}_{=:\gamma(t,z)}w. \tag{52}$$

We now show that we can construct a function u such that γ is bounded below by a positive constant $\eta>0$. u will be of the shape $u(z)=\mu(z)z$ with $\mu(z)=\frac{1}{\chi}$ for $z\leq0$, $\mu(z)=-\frac{1}{\|\dot{\bar{x}}\|_\infty}$ for $z>B$, with $B>0$, and μ will decrease slowly on the interval $[0,B]$. The key ingredients for the construction of the function u are Bounds (53,54) and Lemma 5.2.1.

First notice that for $t\geq t_0$ and by assuming that $2\eta<\frac{\delta}{\chi}$, we have the following Bound:

$$\left(\frac{1}{\chi}\right)^2-\dot{\bar{x}}(t)\left(\frac{1}{\chi}\right)+\chi\left(\frac{1}{\chi}\right)=\frac{\chi+\frac{1}{\chi}-\dot{\bar{x}}(t)}{\chi}\geq\frac{\delta}{\chi}. \tag{53}$$

For the second bound we introduce the functions $g_t:\mu\in I:=\left[-\frac{1}{\|\dot{\bar{x}}\|_\infty},\frac{1}{\chi}\right]\mapsto\mu^2-\dot{\bar{x}}(t)\mu+1$ and show that it is uniformly in time bounded below by 2η for each $\dot{\bar{x}}(t)$. Without loss of generality we suppose that $\|\dot{\bar{x}}\|_\infty>1$. The minimum of g_t on $\mathbb{R}$ is reached for $\mu=\frac{\dot{\bar{x}}(t)}{2}$ and its minimal value is $1-\frac{\dot{\bar{x}}(t)^2}{4}$.

- If $\frac{\dot{\bar{x}}(t)}{2}<-\frac{1}{\|\dot{\bar{x}}\|_\infty}$ then the minimum of g_t on its domain I is $g_t\left(-\frac{1}{\|\dot{\bar{x}}\|_\infty}\right)=\frac{1}{\|\dot{\bar{x}}\|_\infty^2}-\frac{\dot{\bar{x}}}{\|\dot{\bar{x}}\|_\infty}+1\geq\frac{1}{\|\dot{\bar{x}}\|_\infty^2}$.
- If $\frac{\dot{\bar{x}}(t)}{2}>\frac{1}{\chi}$ then the minimum of g_t is $g_t\left(\frac{1}{\chi}\right)\geq\frac{\delta}{\chi}$.

- Else if $\frac{\dot{\bar{x}}(t)}{2} \in I$, then: (i) in the large bias case $\chi > 1$, we have that $g_t(\mu) \geq 1 - \min\left(\frac{1}{\|\dot{\bar{x}}\|_\infty^2}, \frac{1}{\chi^2}\right) > 0$. (ii) In the small bias case $\chi \leq 1$, we know that $\dot{\bar{x}}(t) \leq 2 - \delta$ and $\dot{\bar{x}}(t) \geq -\frac{2}{\|\dot{\bar{x}}\|_\infty} > -2$. Hence the minimum of g_t, which is $1 - \frac{\dot{\bar{x}}(t)^2}{2}$, is in that case positive.

Finally this leads for $\eta > 0$ small enough and for every $t \geq t_0$ to the Bound:

$$\min_{\mu \in I} g_t(\mu) \geq 2\eta. \tag{54}$$

Next, we introduce the following technical Lemma 5.2.1, which will be an essential ingredient for the construction of $u(z) = \mu(z)z$ with the function $z \mapsto \mu(z)$ varying slowly in the interval I, such that γ will stay bounded below by η.

Lemma 5.2.1. *For every $\varepsilon > 0$, there exists a nondecreasing function $\theta_\varepsilon \in W^{2,\infty}([0,1],[0,1])$ such that $\theta_\varepsilon(0) = 0$, $\theta'_\varepsilon(0) = 0$, $\theta_\varepsilon(1) = 1$ and $\theta'_\varepsilon(1) = 0$, and that:*

$$\sup_{z \in [0,1]} \theta'_\varepsilon(z)z \leq \varepsilon.$$

Let us give a quick proof of Lemma 5.2.1.

Proof. Consider the function $f(z) = z^\varepsilon$. Then $f'(z)z = \varepsilon z^{\varepsilon - 1}$. Hence $\sup_{z \in [0,1]} f'(z)z \leq \varepsilon$. Set:

$$\theta_\varepsilon : z \mapsto \begin{cases} \varepsilon^{(\varepsilon-3)\left(1+\frac{1}{\varepsilon}\right)}\left(\varepsilon^{1+\frac{1}{\varepsilon}}(3-\varepsilon)z^2 - (2-\varepsilon)z^3\right) & \text{if } z \in \left[0, \varepsilon^{1+\frac{1}{\varepsilon}}\right) \\ f(z) & \text{if } z \in [\varepsilon^{1+\frac{1}{\varepsilon}}, 1-\varepsilon) \\ g(z) & \text{if } z \in [1-\varepsilon, 1] \end{cases},$$

where g is any concave C^1 function such that $g(1-\varepsilon) = f(1-\varepsilon)$, $g'(1-\varepsilon) = f'(1-\varepsilon)$, $g(1) = 1$ and $g'(1) = 0$. Notice that $g'(1-\varepsilon) \geq g'(1)$, hence such a concave function g exists.

First notice that by construction $\theta_\varepsilon(0) = \theta'_\varepsilon(0) = 0$, $\theta_\varepsilon\left(\left(\varepsilon^{1+\frac{1}{\varepsilon}}\right)^-\right) = f\left(\varepsilon^{1+\frac{1}{\varepsilon}}\right)$ and $\theta'_\varepsilon\left(\left(\varepsilon^{1+\frac{1}{\varepsilon}}\right)^-\right) = f'\left(\varepsilon^{1+\frac{1}{\varepsilon}}\right)$. By straightforward computations, we show that θ_ε is increasing on the interval $\left[0, \varepsilon^{1+\frac{1}{\varepsilon}}\right)$ and that $\theta'_\varepsilon(z)z$ reaches its maximum on the interval $\left[0, \varepsilon^{1+\frac{1}{\varepsilon}}\right)$ at point $z_\varepsilon := \left(\frac{2}{3}\right)^2 \frac{3-\varepsilon}{2-\varepsilon}\varepsilon^{1+\frac{1}{\varepsilon}}$, with value:

$$\begin{aligned} \theta'_\varepsilon(z_\varepsilon)z_\varepsilon &= \left(\frac{2}{3}\right)^5 \frac{(3-\varepsilon)^3}{(2-\varepsilon)^2}\varepsilon^{\varepsilon+1} \\ &\leq \left(\frac{2}{3}\right)^5 \frac{3^3}{1^2}\varepsilon^\varepsilon\varepsilon \\ &\leq \frac{32}{9}\varepsilon. \end{aligned}$$

For $z \in [1-\varepsilon, 1]$, θ_ε is increasing, as by concavity of g, we have that $g'(z) \geq g'(0) = 0$. In addition:

$$\theta'_\varepsilon(z)z = g'(z)z \leq g'(1-\varepsilon)z \leq g'(1-\varepsilon) = \frac{(1-\varepsilon)f'(1-\varepsilon)}{1-\varepsilon} \leq \frac{\varepsilon}{1-\varepsilon}.$$

Finally, by construction, we have that $\theta_\varepsilon \in W^2([0,1],[0,1])$ □

Using the function θ_ε from Lemma 5.2.1, for $\varepsilon > 0, B > 0$ to be determined later, we choose $u(z) = \mu(z)z$ with:

$$\mu : z \mapsto \begin{cases} \frac{1}{\chi} & \text{if } z \leq 0 \\ \frac{1}{\chi} - \theta_\varepsilon\left(\frac{z}{B}\right)\left(\frac{1}{\chi} + \frac{1}{\|\dot{\bar{x}}\|_\infty}\right) & \text{if } z \in (0, B] \\ -\frac{1}{\|\dot{\bar{x}}\|_\infty} & \text{if } z > B \end{cases}.$$

- Notice that by Bound (53), we have that for $t \geq t_0, z < 0$, $\gamma(t,z) = u'^2 + u'' - \dot{\bar{x}}(t)u' + \chi u' = \left(\frac{1}{\chi}\right)^2 - \dot{\bar{x}}(t)\left(\frac{1}{\chi}\right) + \chi\left(\frac{1}{\chi}\right) \geq \frac{\delta}{\chi} \geq 2\eta$.
- Additionally by Bound (54), we have that for $t \geq t_0, z > B$, $\gamma(t,z) = u'^2 + u'' - \dot{\bar{x}}(t)u' + 1 = g_t\left(-\frac{1}{\|\dot{\bar{x}}\|_\infty}\right) \geq 2\eta$.
- Then for $t \geq t_0, z \in (0,B)$, first notice that $\mu(z) \in I$ and we will show that for $\varepsilon > 0$ small enough and $B > 0$ big enough, we have $|\gamma(t,z) - g_t(\mu(z))| \leq \eta$.
  We have that:

$$u'(z) = \mu(z) - \frac{z}{B}\theta'_\varepsilon\left(\frac{z}{B}\right)\left(\frac{1}{\chi} + \frac{1}{\|\dot{\bar{x}}\|_\infty}\right).$$

  But according to Lemma 5.2.1, $\sup_{z\in[0,B]} \theta'_\varepsilon\left(\frac{z}{B}\right)\frac{z}{B} \leq \sup_{z\in[0,1]} \theta'_\varepsilon(z)z \leq \varepsilon$. Hence for $\varepsilon > 0$ small enough the quantity $|u'(z) - \mu(z)|$ can be bounded uniformly in time by any arbitrary positive constant. Similarily:

$$u''(z) = -\frac{1}{B}\underbrace{\left(\frac{z}{B}\theta''_\varepsilon\left(\frac{z}{B}\right)\left(\frac{1}{\chi} + \frac{1}{\|\dot{\bar{x}}\|_\infty}\right) + 2\theta'_\varepsilon\left(\frac{z}{B}\right)\left(\frac{1}{\chi} + \frac{1}{\|\dot{\bar{x}}\|_\infty}\right)\right)}_{\text{bounded by } C\left(\|\theta'_\varepsilon\|_\infty + \|\theta''_\varepsilon\|_\infty\right)}.$$

  For $B > 0$ big enough, $|u''(z)|$ can be bounded by any arbitrary positive constant. Therefore, we can pick $\varepsilon > 0$ and $B > 0$ such that $|\gamma(t,z) - g_t(\mu(z))| \leq \eta$. Then by Bound (54), we have that for $t \geq t_0, z \in (0,B)$:

$$\gamma(t,z) \geq \eta.$$

As an intermediary conclusion, on each interval $(-\infty, 0], [0,B], [B,+\infty)$, γ is lower bounded by the positive constant $\eta > 0$.

Therefore, if we consider ω the solution to Equation for $t \geq t_0$:

$$\begin{aligned} \partial_t\omega - \partial_{zz}\omega - \partial_z(\beta(t,z)\omega) &= 0 \\ \omega(t_0,\,\cdot\,) &= e^{u(\,\cdot\,)}\rho(t_0,\,\cdot\,). \end{aligned} \tag{55}$$

Then $\omega(t,z)e^{\eta(t-t_0)}$ is a subsolution of Equation (52). Of note by the asymptotic properties of e^u, we have that $\omega(t_0,\,\cdot\,) \in L^1(\mathbb{R})$. Moreover, Equation (55) is under conservative form and hence mass is conserved. Without loss of generality, we suppose that $\int_\mathbb{R}\omega(t_0,z)dz = 1$ and for every $t \geq t_0, \int_\mathbb{R}\omega(t,z)dz = 1$. Hence for $t \geq t_0$:

$$\int_\mathbb{R} e^{u(z)}\rho(t,z)dz \geq e^{\eta(t-t_0)}$$

2. In the next step, we show that the mass of ρ in $L^1(e^u dz)$ is not exclusively concentrated on $\mathbb{R}_-$. More precisely, we show that $\liminf_{t\to\infty}\int_{t-4}^t\int_{\mathbb{R}_+}\omega(s,z)dzds > 0$. We start by considering the quantity:

$$I(t) = \int_{\mathbb{R}_-}\left(e^{-\frac{z}{2\chi}} - 1 + \frac{z}{2\chi}\right)\omega(t,z)dz.$$

Notice that for $z<0$, we have $\beta(t,z)=\dot{\bar{x}}(t)-\frac{2}{\chi}-\chi=\dot{\bar{x}}(t)-\left(\chi+\frac{1}{\chi}\right)-\frac{1}{\chi}\leq\dot{\bar{x}}(t)-\sigma^*-\frac{1}{\chi}\leq-\delta-\frac{1}{\chi}$ and that this quantity is finite, since ω will be dominated by $e^{\frac{z}{\chi}}$ for $z<0$ and so the following series of integration by parts is justified:

$$\begin{aligned}
\dot{I}(t) &= \int_{\mathbb{R}_-}\left(e^{-\frac{z}{2\chi}}-1+\frac{z}{2\chi}\right)(\partial_{zz}\omega+\partial_z(\beta\omega))dz\\
&= \left[\left(e^{-\frac{z}{2\chi}}-1+\frac{z}{2\chi}\right)(\partial_z\omega(t,z)+\beta(t,z)\omega(t,z))\right]_{z=-\infty}^{z=0}+\frac{1}{2\chi}\int_{\mathbb{R}_-}\left(e^{-\frac{z}{2\chi}}-1\right)(\partial_z\omega+\beta\omega)dz\\
&= \frac{1}{4\chi^2}\int_{\mathbb{R}_-}e^{-\frac{z}{2\chi}}\omega dz+\frac{1}{2\chi}\int_{\mathbb{R}_-}\left(e^{-\frac{z}{2\chi}}-1\right)\beta\omega dz\\
&= \frac{1}{2\chi}\int_{\mathbb{R}_-}\left(\frac{1}{2\chi}+\beta\right)\left(e^{-\frac{z}{2\chi}}-1\right)\omega dz+\frac{1}{4\chi^2}\int_{\mathbb{R}_-}\omega dz\\
&= \frac{1}{2\chi}\int_{\mathbb{R}_-}\underbrace{\left(\frac{1}{2\chi}+\beta\right)}_{\leq-\delta-\frac{1}{2\chi}}\left(e^{-\frac{z}{2\chi}}-1+\frac{z}{2\chi}\right)\omega dz+\frac{1}{4\chi^2}\underbrace{\int_{\mathbb{R}_-}\omega dz}_{\leq\int_{\mathbb{R}}\omega dz}\underbrace{-\frac{1}{2\chi}\int_{\mathbb{R}_-}\left(\frac{1}{2\chi}+\beta\right)\frac{z}{2\chi}\omega dz}_{\leq 0}\\
&\leq -\frac{1}{2\chi}\left(\delta+\frac{1}{2\chi}\right)I(t)+\frac{1}{2\chi}.
\end{aligned}$$

Thus by Grönwall's Lemma, we obtain:

$$I(t)\leq I(t_0)e^{-\frac{1}{2\chi}\left(\delta+\frac{1}{2\chi}\right)(t-t_0)}+\frac{1}{4\chi^2}\leq I(t_0)+\frac{1}{4\chi^2}.$$

Furthermore $z\mapsto e^{-\frac{z}{2\chi}}-1+\frac{z}{2\chi}$ is decreasing on $\mathbb{R}_-$, hence by a Markov inequality, we obtain that:

$$\int_{-\infty}^{-h}\omega(t,z)dz\leq\frac{I(t)}{e^{\frac{h}{2\chi}}-1-\frac{h}{2\chi}}\leq\frac{8\chi^2}{h^2}\left(I(t_0)+\frac{1}{4\chi^2}\right),$$

where we haved used that $e^{\frac{h}{2\chi}}-1-\frac{h}{2\chi}\geq\frac{h^2}{8\chi^2}$. Hence, if we choose h sufficiently large then for $t\geq t_0$, we have:

$$\int_{-\infty}^{-h}\omega(t,z)dz\leq\frac{1}{2}.$$

Next, we use a parabolic Harnack inequality, such as it is stated in Theorem 1.1 in [49], to obtain the following Lemma:

Lemma 5.2.2. *Let $Q_1:=(-1,0)\times(-h,h)$ and $Q_2:=(-4,-2)\times(-h,h)$, then there exists a constant $C>0$, such that for every $t_2>t_0+4$, we have the following inequality:*

$$\sup_{(t_2,0)+Q_2}\omega\leq C\inf_{(t_2,0)+Q_1}\omega. \tag{56}$$

We will use Inequality (56) to establish that for every $t_2>t_1+4$:

$$\int_{t_2-4}^{t_2}\int_{\mathbb{R}_+}\omega(s,z)dzds\geq\frac{1}{1+2C}. \tag{57}$$

Either, for every $s \in (0,1)$, we have that $\int_{\mathbb{R}} \omega(t_2 - s, z)dz \geq \frac{1}{1+2C}$ and then Inequality (57) follows. Or, there exists $s \in (0,1)$, such that $\int_{\mathbb{R}} \omega(t_2 - s, z)dz < \frac{1}{1+2C}$. Then:

$$\frac{1}{1+2C} > \int_{\mathbb{R}_+} \omega(t_2 - s, z)dz \geq \int_0^h \omega(t_2 - s, z)dz \geq h \inf_{(t_2,0)+Q_1} \omega.$$

By using Inequality (56), we then have that for every $s \in (-4,-2)$:

$$\int_{-h}^0 \omega(t - s, z)dz \leq h \sup_{(t_2,0)+Q_2} \omega \leq Ch \inf_{(t_2,0)+Q_1} \omega \leq \frac{C}{1+2C}.$$

But:

$$\int_{\mathbb{R}_+} \omega(t - s, z)dz = 1 - \int_{-\infty}^{-h} \omega(t - s, z)dz - \int_{-h}^0 \omega(t - s, z)dz \geq \frac{1}{2} - \frac{C}{1+2C}.$$

Hence:

$$\int_{t_2-4}^{t_2-2} \int_{\mathbb{R}_+} \omega(s, z)dzds \geq \frac{1}{1+2C}.$$

Thus, we have established Inequality (57).

3. In the final step we show that the last result contradicts with the condition that $N(t,0) = N_{\text{th}}$. We multiply Equation (16b) by e^u and integrate over $\mathbb{R}_+$:

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}_+} e^u N dz =& \dot{\bar{x}} \int_{\mathbb{R}_+} e^u \partial_z N dz + D \int_{\mathbb{R}_+} e^u \partial_{zz} N - \int_{\mathbb{R}_+} \rho e^u N \\
=& - \dot{\bar{x}} \left(N(t,0) + \int_{\mathbb{R}_+} N u' e^u dz \right) \\
&+ D \left(\underbrace{-\partial_z N(t,0)}_{\leq 0} + \frac{N(t,0)}{\chi} + \int_{\mathbb{R}_+} \left(u'' + u'^2\right) e^u N dz \right) - \int_{\mathbb{R}_+} \rho e^u N dz \\
\leq& C(1 + \|\dot{\bar{x}}\|_\infty) \|N\|_\infty - N_{\text{th}} \int_{\mathbb{R}_+} \rho e^u dz \\
\leq& C(1 + \|\dot{\bar{x}}\|_\infty) - N_{\text{th}} \int_{\mathbb{R}_+} \omega e^{\eta(t-t_0)} dz,
\end{aligned}$$

where we have used the fact that e^u and its derivatives are integrable, N is bounded above by 1 and below by N_{th} on $\mathbb{R}_+$ and ρe^u is bounded below by $\omega e^{\eta(t-t_0)}$ (the latter being a subsolution). Finally, we integrate between $[t-4, t]$ and obtain:

$$\begin{aligned}
\int_{\mathbb{R}_+} e^u N(t,z)dz \leq& \int_{\mathbb{R}_+} e^u N(t-4,z)dz + 4C(1 + \|\dot{\bar{x}}\|_\infty) - N_{\text{th}} \int_{t-4}^t \int_{\mathbb{R}_+} e^{\eta(s-t_0)} \omega dz ds \\
\leq& \int_{\mathbb{R}_+} e^u dz + C(1 + \|\dot{\bar{x}}\|_\infty) - N_{\text{th}} e^{\eta(t-4-t_0)} \int_{t-4}^t \int_{\mathbb{R}_+} \omega dz \\
\leq& C(1 + \|\dot{\bar{x}}\|_\infty) - \frac{N_{\text{th}} e^{\eta(t-4-t_0)}}{1+2C}.
\end{aligned}$$

By letting $t \to +\infty$, we have that $\int_{\mathbb{R}_+} e^u N(t,z)dz < 0$, which is a contradiction. □

6 Traveling Waves for a Two-Velocity System with Persistence

In this Section, we exhibit all subsonic ($\sigma < \varepsilon^{-1}$) traveling wave solutions to System (6). It is known [9] that in hyperbolic models with growth supersonic traveling wave solutions can exist, but for the sake of concision we discard them in this discussion. Furthermore by following the terminology in [9], there exist a parabolic regime $\varepsilon^{-2} > 1$ and a hyperbolic regime $\varepsilon^{-2} < 1$. Of note, the relevant quantities to compare are the tumbling rate, normalized to ε^{-2} here, and the growth rate, normalized to 1 here. Therefore we write the parabolic regime as $\varepsilon^{-2} > 1$ and not as $\varepsilon^{-1} > 1$, which is equivalent in our case. Theorem 6.1 notably states that subsonic traveling waves only exist in the parabolic regime, which was already observed in another model [9].

We will procede similarily than in the parabolic case (Section 3) and consider that $\partial_z N > 0$. In that case Equation (6) reduces in the moving frame of reference to:

$$\text{for } z < 0, \quad \begin{cases} (-\sigma + \varepsilon^{-1}) f^{+\prime} = \frac{\varepsilon^{-2}}{2} \left((1 + \varepsilon\chi) f^- - (1 - \varepsilon\chi) f^+ \right) \\ (-\sigma - \varepsilon^{-1}) f^{-\prime} = \frac{\varepsilon^{-2}}{2} \left((1 - \varepsilon\chi) f^+ - (1 + \varepsilon\chi) f^- \right) \end{cases} \tag{58a}$$

$$\text{for } z > 0, \quad \begin{cases} (-\sigma + \varepsilon^{-1}) f^{+\prime} = \frac{\varepsilon^{-2}+1}{2} f^- - \frac{\varepsilon^{-2}-1}{2} f^+ \\ (-\sigma - \varepsilon^{-1}) f^{-\prime} = \frac{\varepsilon^{-2}+1}{2} f^+ - \frac{\varepsilon^{-2}-1}{2} f^- \end{cases} \tag{58b}$$

Theorem 6.1. *Assume that $\partial_z N > 0$. In the parabolic regime $\varepsilon^{-2} > 1$, there exists a minimal speed $\sigma^* \in (1, \varepsilon^{-1})$, such that for any $\sigma \in [\sigma^*, \varepsilon^{-1})$, there exists a corresponding bounded and nonnegative traveling wave profile $(f^{+,\sigma}, f^{-,\sigma}, N^\sigma)$. In addition, for $\sigma \in [\sigma^*, \varepsilon^{-1})$ fixed, the traveling wave profile $(f^{+,\sigma}(z), f^{-,\sigma}(z), N^\sigma(z))$ is unique. For $\sigma \in [0, \sigma^*)$, there does not exist a traveling wave profile. The expression of σ^* is given by Formula (7) and depends on the value of χ:*

- *If $\chi \in (1, \varepsilon^{-1})$, then $\sigma^* = \frac{\chi + \frac{1}{\chi}}{1+\varepsilon^2}$. Note that in that case $\sigma^* < \varepsilon^{-1}$.*
- *If $\chi \leq 1$, then $\sigma^* = \sigma_{F/KPP} := \frac{2}{1+\varepsilon^2}$.*

In the hyperbolic regime $\varepsilon^{-2} < 1$, there doesn't exist any subsonic traveling wave profile, i.e a wave travaling with speed $\sigma < \varepsilon^{-1}$.

Furthermore, in the parabolic regime $\varepsilon^{-2} > 1$, for $\sigma \in [\frac{2}{1+\varepsilon^2}, \varepsilon^{-1})$, define $\mu_\pm(\sigma) := \frac{\sigma(1-\varepsilon^2) \pm \sqrt{\sigma^2(1+\varepsilon^2)^2 - 4}}{2(1-\varepsilon^2\sigma^2)}$. We then have the inequality, for $\sigma > \sigma_{F/KPP}$:

$$0 < \mu_-(\sigma) < \mu_- \left(\sigma_{F/KPP}\right) = 1 = \mu + \left(\sigma_{F/KPP}\right) < \mu + (\sigma)$$

The functions $f^{\pm,\sigma}$ have the following behavior for $z > 0$:

- *for $\sigma \in (\sigma^*, \varepsilon^{-1}), z > 0, f^{\pm,\sigma}(z) = A^\pm e^{-\mu_-(\sigma)z} + B^\pm e^{-\mu_+(\sigma^*)z}$*
- *for $\chi > 1, \sigma = \sigma^* = \frac{\chi + \frac{1}{\chi}}{1+\varepsilon^2}, z > 0, f^{\pm,\sigma^*}(z) = B^\pm e^{-\mu_+(\sigma^*)z}$ and $\mu_+(\sigma^*) = \frac{\chi(1+\varepsilon^2)}{1-\varepsilon^2\chi^2}$*
- *for $\chi < 1, z > 0, \sigma = \sigma_{F/KPP}, f^{\pm\sigma_{F/KPP}}(z) = (A^\pm z + B^\pm) e^{-z}$*
- *for $\chi = 1, z > 0, \sigma = \sigma_{F/KPP}, f^{\pm\sigma_{F/KPP}}(z) = B^\pm e^{-z}$*

As before note that $\frac{\chi + \frac{1}{\chi}}{1+\varepsilon^2} \geq \frac{2}{1+\varepsilon^2} = \sigma_{\mathrm{F/KPP}}$, with equality if and only if $\chi = 1$.

Proof. The proof relies on similar arguments than the proof in the parabolic case (Section 3). Since we are looking for subsonic solution, we suppose througout the proof that $\sigma^2 < \varepsilon^{-2}$.

Set $F(z) = \begin{pmatrix} f^+(z) \\ f^-(z) \end{pmatrix}$, $A_- = \frac{\varepsilon^{-2}}{2} \begin{pmatrix} -\frac{1-\varepsilon\chi}{\varepsilon^{-1}-\sigma} & \frac{1+\varepsilon\chi}{\varepsilon^{-1}-\sigma} \\ -\frac{1-\varepsilon\chi}{\varepsilon^{-1}+\sigma} & \frac{1+\varepsilon\chi}{\varepsilon^{-1}+\sigma} \end{pmatrix}$ and $A_+ = \frac{1}{2} \begin{pmatrix} -\frac{\varepsilon^{-2}-1}{\varepsilon^{-1}-\sigma} & \frac{\varepsilon^{-2}+1}{\varepsilon^{-1}-\sigma} \\ -\frac{\varepsilon^{-2}+1}{\varepsilon^{-1}+\sigma} & \frac{\varepsilon^{-2}-1}{\varepsilon^{-1}+\sigma} \end{pmatrix}$. Then for $z < 0, F'(z) = A_- F(z)$ and for $z > 0, F'(z) = A_+ F(z)$.

As in the parabolic case (Section 3), the characteristic polynomial of A_- has two roots 0 and $\frac{\varepsilon^{-2}}{2}\left(\frac{1+\varepsilon\chi}{\varepsilon^{-1}+\sigma} - \frac{1-\varepsilon\chi}{\varepsilon^{-1}-\sigma}\right)$, the later being negative, by an argument similar to the proof of Theorem 3.1. Therefore there exist a constant $a \in \mathbb{R}$, such that $F(z) = a \begin{pmatrix} 1+\varepsilon\chi \\ 1-\varepsilon\chi \end{pmatrix}$. The negativity of the second root also shows that there cannot exist a traveling wave profile with velocity $\sigma < \chi$.

The characteristic polynomial of A_+ is (up to a multiplicative constant) $P(X) = (\varepsilon^{-2} - \sigma^2)X^2 + \sigma(\varepsilon^{-2} - 1)X + \varepsilon^{-2}$. In the hyperbolic regime $\varepsilon^{-2} < 1$, we have that $P(0) > 0$ and $P'(0) < 0$. But the leading coefficient of P is positive, hence the roots of P have positive real part, which is in contradiction with the fact that we are looking for a bounded solution. Hence in the hyperbolic regime, there do not exist any (subsonic) traveling wave solutions.

For the rest of the proof, we suppose that we are in the parabolic regime $\varepsilon^{-2} > 1$. The discriminant of the characteristic polynomial P is $\sigma^2(\varepsilon^{-2} - 1)^2 - 4\varepsilon^{-2}(\varepsilon^{-2} - \sigma^2) = \sigma^2(\varepsilon^{-2} + 1)^2 - 4\varepsilon^{-4}$. As in the parabolic case (Section 3) $\sigma = \sigma_{F/KPP}$ cancels the discriminant and we yield the condition that $\sigma \geq \sigma_{F/KPP}$, since otherwise we would have complex roots and oscillating functions.

Suppose $\sigma > \sigma_{\mathrm{F/KPP}}$, the roots of the characteristic polynomial are then $-\mu_\pm(\sigma)$. By continuity of F at $z = 0$ and elementary computations, we find that:

$$F(z) = a\frac{\mu_+(\sigma-\chi)-1}{\varepsilon^{-2}(\mu^+ - \mu^-)} \begin{pmatrix} \mu_-(\sigma)(\varepsilon^{-1}+\sigma)-1 \\ \mu_-(\sigma)(\varepsilon^{-1}-\sigma)+1 \end{pmatrix} e^{-\mu_-(\sigma)z} - a\frac{\mu_-(\sigma-\chi)-1}{\varepsilon^{-2}(\mu^+ - \mu^-)} \begin{pmatrix} \mu_+(\sigma)(\varepsilon^{-1}+\sigma)-1 \\ \mu_+(\sigma)(\varepsilon^{-1}-\sigma)+1 \end{pmatrix} e^{-\mu_+(\sigma)z}.$$

First we show that the two components of the vectors are of the same sign. Indeed $(\mu_\pm(\varepsilon^{-1} + \sigma) - 1)(\mu_\pm(\varepsilon^{-1} - \sigma) + 1) \geq 0 \iff -\mu_\pm \leq -\frac{1}{\sigma}$. Or equivalently $-\frac{1}{\sigma}$ is bigger than the two roots of the characteristic polynomial P. But $P\left(-\frac{1}{\sigma}\right) = \frac{1}{\varepsilon^2\sigma^2} > 0$ and $P'\left(-\frac{1}{\sigma}\right) = \frac{1}{\sigma(\varepsilon^{-2}+1)}\left(\sigma^2 - \frac{2}{1+\varepsilon^2}\right) \geq \frac{1}{\sigma(\varepsilon^{-2}+1)}\left(\sigma^2 - \frac{4\varepsilon^{-4}}{(\varepsilon^{-2}+1)^2}\right) \geq 0$, where we used the fact that $\frac{4\varepsilon^{-4}}{(\varepsilon^{-2}+1)^2} \geq \frac{2}{1+\varepsilon^2}$, which is equivalent to $\varepsilon^{-2} \geq 1$. Hence the components are of the same sign, that is the positive sign, since the second component of each vector is positive.

Therefore we observe that F is positive if and only if $g(\sigma) := \mu_+(\sigma)(\sigma - \chi) - 1 \geq 0$. g is an increasing function, as $g'(\sigma) = \mu_+(\sigma) + \mu_+'(\sigma)(\sigma - \chi) > 0$ (one easily checks that $\mu_+'(\sigma) > 0$). One checks that in the case $\chi \leq 1$, $g(\sigma_{F/KPP}) \geq 0$, which establishes existence of waves for $\sigma > \sigma_{F/KPP}$, and that in the case $\chi > 1$, $g(\sigma^*) = 0$, which establishes the existence of waves for $\sigma \geq \sigma^* = \frac{\chi+\frac{1}{\chi}}{1+\varepsilon^2}$ and the nonexistence of waves for $\sigma \in (\sigma_{F/KPP}, \sigma^*)$.

Suppose $\sigma = \sigma_{\mathrm{F/KPP}}$, then one shows that for $z \geq 0$:

$$F(z) = A \begin{pmatrix} 1+\varepsilon\chi + z\frac{(1+\varepsilon^2)^2(1-\chi)}{(1-\varepsilon^2)(1-\varepsilon)} \\ 1-\varepsilon\chi + z\frac{(1+\varepsilon^2)^2(1-\chi)}{(1-\varepsilon^2)(1+\varepsilon)} \end{pmatrix} e^{-z}. \tag{59}$$

Hence F is positive if and only if $\chi \leq 1$, which establishes the criterion for existence in the last remaining case.

The decay properties follow immediately and the existence of the profile for N is treated exactly as in the parabolic case. □

Let us make some comments on Theorem 6.1 to show how it is linked to Theorem 3.1 in the limit $\varepsilon \to 0$. First of all, considering Formula (7) for σ^* given by Theorem 6.1, we observe that in the limit $\varepsilon \to 0$, Formula (4) given by Theorem 3.1 is recovered. Furthermore, the limit of all the values $\mu_\pm(\sigma^*)$

coincide and so do the shapes of the wave. For example, consider Expression (59) in the limit $\varepsilon \to 0$, we obtain for $z > 0$:

$$F(z) = A \begin{pmatrix} 1 + (1-\chi)z \\ 1 + (1-\chi)z \end{pmatrix} e^{-z}.$$

As a consequence $\rho(t,z) = A(1+(1-\chi)z)e^{-z}$, which is exactly the form that we obtained in the proof of Theorem 3.1.

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