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Abstract

The problem of fault detection and isolation has been introduced into automatic control as a paradigm for the design of algorithms capable of detecting the occurrence of faults and isolating their causes. Many methods have been developed for the class of integer systems. In contrast, few methods have been developed for the class of fractional order systems. In this paper, a new fault detection and isolation method, based on the flatness property, is introduced for the class of fractional order linear systems to detect and isolate faults on sensors and actuators. In this method, the measurement of the so-called fractional flat output vector is used to construct the redundant components of the system and to generate fault indicators called residual signals. In addition, new definitions of fault detection and isolation are introduced. The effectiveness of this method is validated by simulations on a fractional order system, the thermal bi-dimensional system.

Keywords: fault detection and isolation, fractional system, flat system, flat output, thermal system.

1 Introduction

Over the past three decades, fault detection and isolation (FDI) has attracted increasing interest among researchers in the field of automatic control. It consists in designing algorithms capable of detecting the occurrence of faults on a system and identifying their causes. Examples of survey papers on FDI may be found in [37] and [33]. The FDI process consists of two steps:

- *Fault detection*: consists in detecting the abrupt change in the system behavior;
- *Fault isolation*: consists in determining the exact location of the fault.

The first proposed FDI technique is called the *hardware redundancy* in which multiple redundant components are used in parallel to the process components [10]. In this technique, a fault is detected and isolated if the behavior of a process component is different from those of the redundant components. However, this method incurs additional equipment and maintenance costs, and additional space is required to accommodate the equipments. Later on, this approach was replaced by the *analytical redundancy* where the redundant components are calculated using algorithms and then avoiding extra costs [37]. Examples of analytical redundancy methods are the observer-based methods [11, 13], parity-space methods [12, 17] and artificial neural network (ANN) approaches [16].

The analytical redundancy methods are in general based on the notion of generating residual signals, usually denoted by $r(t)$, which are the difference between the measured output and the estimated process output. In the ideal case where there is no uncertainties and disturbances on the system, the residual signal takes two values:

$$\begin{aligned} r(t) &= 0 \qquad \text{in the fault-free case;} \\ r(t) &\neq 0 \qquad \text{in the faulty case.} \end{aligned}$$

However, in practice it will always involve model uncertainties and device noise. Then, the residue will be compared next to a threshold. If the residue exceeds its threshold then a fault is detected, otherwise there is no fault on the system. There exist several research studies on fixing thresholds and can be found in [13, 18].

Among the analytical redundancy techniques there is the method, introduced in [23] and further developed in [29], based on the flatness property of nonlinear systems. Roughly speaking, a nonlinear system is said to be flat if, and only if, all the system states, inputs and outputs can be expressed in function of a vector, called flat output, and its successive time derivatives [19]. Therefore, in this method, the measurement of the flat output vector is used to compute the redundant sensors and actuators and then generate the residual signals. Definitions of detectability and isolability of this method are presented in [29].

Despite the advanced research in the field of FDI for nonlinear systems, few works have been done for fractional order systems, which are systems modeled by fractional order differential equations, such as thermal systems [8], nuclear magnetic resonance systems [22] and viscoelastic systems [25]. A first development of fractional models in the FDI field was initiated in [3]. This method is an extension of the dynamic parity space FDI method. See also [4, 5]. Later on in [6], a scheme of FDI, based on a bank of fractional unknown input observers, is extended to diagnose fractional order systems.

Recently, the flatness property of nonlinear systems has been extended to include the class of linear fractional order systems, see [35]. Methods of computation of flat output vectors were developed in [36], based on the *Smith decomposition*, and in [30], based on the *unimodular completion algorithm*. In this paper, a new method of FDI for fractional order linear systems is introduced, based on the extension of the flatness-based method developed in [23] and [29] for the class of nonlinear systems.

In this context, this paper is organized as follows: Section 2 presents recalls on fractional calculus and fractional order linear flat systems. The Flatness-based FDI method is presented in Section 3 and the application of this method on a thermal bi-dimensional system is presented in Section 4. Finally, Section 5 concludes the paper.

2 fractional order Linear Flat System

The fractional derivative has been introduced by [20] and [31] in the $19th$ century as a generalization of the traditionally used derivative. For more recent references, see [24, 32, 14]. However, it was considered only as a theoretical notion until the discovery of physical systems that could be modeled by fractional differential equations [34], such as thermal systems. This notion of fractional calculation has also shown its usefulness in robust control [27] and system identification [1, 21].

2.1 Fractional Calculus

Fractional integral:

Let $n \in \mathbb{N}^*$ be a non zero integer,
$a \in \mathbb{R}$ and $f(t) \in \mathcal{C}^\infty([a, +\infty[)$ the set of infinitely continuously differentiable functions. The integration of order n of the function $f(t)$ is defined, according to Cauchy's formula, by:

$$\mathbf{I}_a^n f(t) = \frac{1}{(n-1)!} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-n}} d\tau. \tag{1}$$

The generalization of the Cauchy formula to the fractional integration of order $\gamma \in \mathbb{R}_+$, introduced by Riemann and Liouville, is given by [24]:

$$\mathbf{I}_a^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\gamma}} d\tau, \tag{2}$$

where

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt, \quad \forall x \in \mathbb{R}^* \setminus \mathbb{N}^- \tag{3}$$

is the Euler's function or the generalized factorial. Note that for all $n \in \mathbb{N}$

$$\Gamma(n+1) = n!, \tag{4}$$

and for all $\gamma \in \mathbb{R}_+$

$$\gamma\Gamma(\gamma) = \Gamma(\gamma+1). \tag{5}$$

Fractional derivative:

Let $\gamma \in \mathbb{R}_+$ be a positive real number, $n = \min\{k \in \mathbb{N} \mid k > \gamma\}$ the smallest integer greater than γ and $\nu = n - \gamma \in [0, 1[$. The fractional derivative of order $\gamma = n - \nu$ of a function $f \in \mathcal{C}^{\infty}([a, +\infty[)$ at time t, denoted by $\mathbf{D}_a^{\gamma} f(t)$, is defined by the n$^{\text{th}}$ order derivative of the fractional integral of order ν:

$$\begin{aligned} \mathbf{D}_a^{\gamma} f(t) &= \mathbf{D}^n \big(\mathbf{I}_a^{\nu} f(t)\big) \\ &\triangleq \Big(\frac{d}{dt}\Big)^n \Big(\frac{1}{\Gamma(\nu)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\nu}} d\tau\Big). \end{aligned} \tag{6}$$

If $\gamma = n \in \mathbb{N}$, the fractional derivative coincides with the ordinary derivative:

$$\mathbf{D}_a^{\gamma} f(t) = \mathbf{D}_a^n f(t).$$

If $\gamma < 0$, the fractional derivative is in fact the fractional integral:

$$\mathbf{D}_a^{\gamma} f(t) = \mathbf{I}_a^{-\gamma} f(t).$$

Properties of differentiability, integrability and commutativity of the fractional operator can be found in [28].

In systems theory, the signal space is defined as the space of causal functions $\mathfrak{H}_a$ given by:

$$\begin{aligned} \mathfrak{H}_a \triangleq \{f : \mathbb{R} \mapsto \mathbb{R} \mid f \in \mathcal{C}^{\infty}([a, +\infty[), \\ f(t) = 0, \forall t \leq a\}. \end{aligned} \tag{7}$$

The operator $\mathbf{D}_a^{\gamma}$ is an endomorphism from $\mathfrak{H}_a$ to $\mathfrak{H}_a$ [28].

Let $\mathbb{R}[\mathbf{D}_a^\gamma]$ be the set of $\mathbf{D}_a^\gamma$-polynomials with real coefficients of the form $\sum_{k=0}^{K} c_k \mathbf{D}_a^{k\gamma}$. One can easily verify that such set, endowed with the usual addition and multiplication of polynomials $(\mathbb{R}[\mathbf{D}_a^\gamma], +, \times)$, is a commutative principal ideal domain.

Let p and $q \in \mathbb{N}$, we denote by $\mathbb{R}[\mathbf{D}_a^\gamma]^{p\times q}$ the set of $\mathbf{D}_a^\gamma$-polynomial matrices of size $p \times q$. An invertible square matrix of $\mathbb{R}[\mathbf{D}_a^\gamma]^{p\times p}$ whose inverse is also in $\mathbb{R}[\mathbf{D}_a^\gamma]^{p\times p}$ is called unimodular matrix. The set of unimodular matrices is denoted by $GL_p(\mathbb{R}[\mathbf{D}_a^\gamma])$. $\mathbf{D}_a^\gamma$-polynomial matrices admit the following property ([2]):

Theorem 1 *A matrix $M \in \mathbb{R}[\mathbf{D}_a^\gamma]^{p\times q}$ with $p \leq q$ (resp. $p \geq q$) is said to be hyper-regular if, and only if, there exists a matrix $U \in GL_p(\mathbb{R}[\mathbf{D}_a^\gamma])$ (resp. $V \in GL_q(\mathbb{R}[\mathbf{D}_a^\gamma])$) such that:*

$$MU = \begin{pmatrix} I_p & 0_{p\times(q-p)} \end{pmatrix} \quad \left(\text{resp. } VM = \begin{pmatrix} I_q \\ 0_{(p-q)\times q} \end{pmatrix} \right). \tag{8}$$

The decomposition (8) of the matrix M is called Smith diagonal decomposition.

2.2 Flatness of fractional order Linear System

The state representation of a fractional order linear system, called pseudo-state representation[1] where $x \in (\mathfrak{H}_a)^n$ represents the n-dimensional pseudo-state vector, $u \in (\mathfrak{H}_a)^m$ is the m-dimensional control vector, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are real matrices. The matrix B is supposed to be of rank m and $m \leq n$. The controllability and observability properties of such systems can be found in ([15]).

The system (9) can be written into the form:

$$\mathcal{A}x = \mathcal{B}u \tag{10}$$

where $\mathcal{A} \in \mathbb{R}[\mathbf{D}_a^\gamma]^{n\times n}$ and $\mathcal{B} \in \mathbb{R}[\mathbf{D}_a^\gamma]^{n\times m}$ are $\mathbf{D}_a^\gamma$-polynomial matrices and $\mathcal{B}$ is supposed to be of rank m. In turn, the system (10) can be transformed into the form:

$$F \begin{pmatrix} x \\ u \end{pmatrix} = 0 \tag{11}$$

where $F \triangleq \begin{pmatrix} \mathcal{A} & -\mathcal{B} \end{pmatrix} \in \mathbb{R}[\mathbf{D}_a^\gamma]^{n\times(n+m)}$ is assumed to be of full row rank.

Inspired by the work of [2] on the flatness of integer order linear systems, the definition of fractional order linear flatness is then introduced by [36] as follows:

[1] This notation is specified for the class of fractional order system and it refers to [27], is given by:

$$x^{(\nu)} = Ax + Bu \tag{9}$$

Definition 1 The fractional order linear system (11) is said to be fractionally flat if, and only if, there exist two matrices $P \in \mathbb{R}[\mathbf{D}_a^\gamma]^{m\times(n+m)}$ and $Q \in \mathbb{R}[\mathbf{D}_a^\gamma]^{(n+m)\times m}$ and a vector $z \in (\mathfrak{H}_a)^m$ such that

1. $PQ = I_m$;
2. For all $(x, u)^T$ satisfying (11), we have $z = P\begin{pmatrix} x \\ u \end{pmatrix}$ and conversely $\begin{pmatrix} x \\ u \end{pmatrix} = Qz$.

The vector z is called fractional flat output vector and the matrices P and Q are called defining matrices.

The main property of fractional order linear flatness is given by the following theorem [36]:

Theorem 2 *The system (11) is fractionally flat if, and only if, the matrix F is hyper-regular over $\mathbb{R}[\mathbf{D}_a^\gamma]$.*

The flatness-based FDI method, introduced in [23] for the class of nonlinear systems, uses the measurement of the flat output to construct the redundant sensors and actuators and then to generate residual signals. In the next section, the flatness-based FDI for the class of fractional order systems is introduced.

3 Flatness-based FDI

3.1 Residual Generation

Consider the following linear system of fractional order ν:

$$\begin{cases} x^{(\nu)}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{12}$$

where $x \in (\mathfrak{H}_a)^n$ is the pseudo-state vector, $u \in (\mathfrak{H}_a)^m$ the input vector, $y \in (\mathfrak{H}_a)^p$ the output vector of dimension $p \geq m$, and A, B, C and D are real matrices with proper dimensions.

We suppose that the system (12) is flat with $z = (z_1, \ldots, z_m) \in (\mathfrak{H}_a)^m$ as a fractional flat output. Then, according to Definition 1, the expressions of the state, input and output vectors are linear combinations of the fractional flat output z and its successive fractional derivatives:

$$x_i = \sum_{j=1}^{m} \sum_{k=1}^{\alpha_j} a_{i,j,k} z_j^{(k\nu)} \qquad i = 1, \ldots, n \tag{13}$$

$$u_l = \sum_{j=1}^{m} \sum_{k=1}^{\alpha_j+1} b_{l,j,k} z_j^{(k\nu)} \qquad l = 1, \ldots, m \tag{14}$$

$$y_q = \sum_{j=1}^{m} \sum_{k=1}^{\beta_j} c_{q,j,k} z_j^{(k\nu)} \qquad q = 1, \ldots, p. \tag{15}$$

In turn, the fractional flat output z is a linear combination of x, u and successive fractional derivatives of u:

$$z = h(x, u, u^{(\nu)}, \ldots, u^{(\kappa\nu)}). \tag{16}$$

In the following, we suppose that the components $y_1, \ldots, y_p$ of the output y are measured by sensors $\mathsf{S}_1, \ldots, \mathsf{S}_p$, respectively, and we denote their measurements by:

$$y^s = (y_1^s, \ldots, y_p^s). \tag{17}$$

Moreover, the values of the input components $u_1, \ldots, u_m$, corresponding to the actuators $\mathsf{A}_1, \ldots, \mathsf{A}_m$, are assumed to be available at every time. Therefore, we denote by

$$\zeta = (y_1^s, \ldots, y_p^s, u_1, \ldots, u_m) \tag{18}$$

the vector of dimension $p + m$ of the available measurements.

The redundant input vector, denoted by u^z, and the redundant output vector, denoted by y^z, are calculated by flatness using expressions (14) and (15), respectively. For this reason, a necessary condition for this method to be applicable is that the measurement of the components z_i, for $i = 1, \ldots, m$, of the fractional flat output must be available at every time. Then, residual signals are defined by the difference between the available measurements and their redundancies:

Definition 2 The k^{th}-sensor residue R_{S_k} and l^{th}-input residue R_{A_l}, for $k = 1, \ldots, p$ and $l = 1, \ldots, m$, are given by:

$$R_{\mathsf{S}_k} = y_k^s - y_k^z, \quad \text{and} \quad R_{\mathsf{A}_l} = u_l - u_l^z. \tag{19}$$

respectively.

In total, we have $p + m$ residues and we denote by r the full vector of residues:

$$r = (r_{\mathsf{S}_1}, \ldots, r_{\mathsf{S}_p}, r_{\mathsf{A}_1}, \ldots, r_{\mathsf{A}_m}). \tag{20}$$

3.2 Fault Detection and Isolation

For the aim of detecting and isolating faults on sensors and actuators of a fractional order linear system, the following definition of fractional signature matrix is introduced:

Definition 3 (Fractional signature matrix) Given the vector of residues r defined in (20) and ζ the vector of available measurements, defined in (18), the fractional signature matrix, denoted by $\mathbf{S}$ and associated to the fractional flat output z, is given by:

$$\mathbf{S} = \begin{array}{c} \\ r_1 \\ \vdots \\ r_{p+m} \end{array} \begin{array}{c} \begin{array}{cccc} \zeta_1 & \zeta_2 & \cdots & \zeta_{p+m} \end{array} \\ \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,p+m} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{p+m,1} & \sigma_{p+m,2} & \cdots & \sigma_{p+m,p+m} \end{pmatrix} \end{array} \tag{21}$$

with

$$\sigma_{i,j} \triangleq \begin{cases} 0 \text{ if } \frac{\partial r_i}{\partial \zeta_j^{(\varrho\nu)}} = 0 \quad \forall \varrho \in \{0,1,\ldots\} \\ 1 \text{ if } \exists\, \varrho \in \{0,1,\ldots\} \text{ s.t. } \frac{\partial r_i}{\partial \zeta_j^{(\varrho\nu)}} \neq 0 \end{cases}. \tag{22}$$

A column Σ_j, for $j = 1, \ldots, p+m$, of the signature matrix $\mathbf{S}$ indicates whether a residue r_i is or is not functionally affected by a fault on the measurement ζ_j. Therefore, in (22), $\sigma_{i,j} = 0$ means that the residue r_i is not affected by a fault on the measurement ζ_j and $\sigma_{i,j} = 1$ otherwise.

Definition 4 (Fault alarm signature) A column Σ_j of the signature matrix $\mathbf{S}$ is called fault alarm signature, associated to the sensor/actuator ζ_j.

In the following, definitions of the *detectability* and the *isolability* in the fractional flatness context, are introduced:

Definition 5 (Detectability) A fault on a sensor/actuator ζ_j is detectable if, and only if there exists at least one $i \in \{1, \ldots, p\}$ such that $\sigma_{i,j} = 1$.

Definition 6 (Isolability) A fault on a sensor S_k, $k = 1, \ldots, p$, is said isolable if, and only if, its corresponding fault alarm signature Σ_k in the signature matrix $\mathbf{S}$ is distinct from the others, *i.e.*

$$\Sigma_k \neq \Sigma_j, \quad \forall j = 1, \ldots, p+m,\ j \neq k. \tag{23}$$

An isolable fault on the actuator A_l, for $l = 1, \ldots, m$, is defined analogously:

$$\Sigma_{p+l} \neq \Sigma_j, \quad \forall j = 1, \ldots, p+m,\ j \neq p+l. \tag{24}$$

In the next section, the proposed fractional flatness-based FDI method is applied on the thermal bi-dimensional system. Moreover, the effectiveness of this method is proved by simulations.

4 Application to a Thermal Bi-dimensional System

Thermal systems are those that involve the storage and transfer of heat. Examples of thermal systems are solar thermal systems, radiators, electric stove, among many others. The thermal system paradigm is a popular example in automatic control. It has been used as a benchmark for designing controllers, trajectory planning and system diagnostic. In this section, the fractional flatness-based FDI method is applied to the thermal bi-dimensional system.

4.1 System description

The proposed FDI method will be evaluated on a thermal bi-dimensional system which is constituted by a two-dimensional isolated (without heat losses) metallic sheet (see Figure 1). The variable $T(x, y, t)$ represents the temperature at a point (x, y) at time t. The temperature is controlled by the heat flux $\varphi(t)$, applied at the point $(0, 0)$.

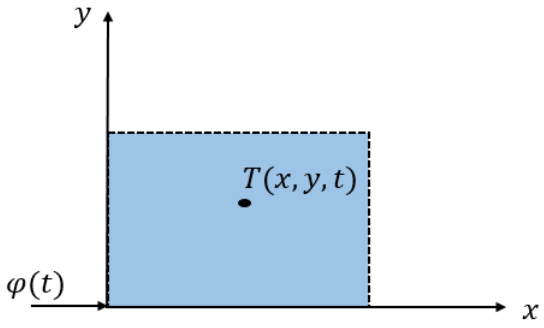

Fig. 1 *Thermal bi-dimensional system*

The heated metallic model is represented by the following heat equation:

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{\alpha}\frac{\partial}{\partial t}\Big)(T(x, y, t) + T_{amb}) = 0 \tag{25}$$

where T_{amb} represents the ambient temperature. Then, this equation can be written as follows:

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{\alpha}\frac{\partial}{\partial t}\Big)T(x, y, t) = 0 \tag{26}$$

where α is the coefficient of diffusivity. In polar coordinates:

$$x = \rho\cos(\theta), \qquad y = \rho\sin(\theta), \tag{27}$$

equation (26) becomes:

$$\frac{1}{\alpha}\frac{\partial T(\rho,\theta,t)}{\partial t} = \frac{\partial^2 T(\rho,\theta,t)}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial T(\rho,\theta,t)}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 T(\rho,\theta,t)}{\partial^2 \theta}. \tag{28}$$

On the thermal bi-dimensional system, we consider the following conditions:

– the boundary conditions:

$$-\lambda \lim_{\rho\to 0} \frac{\pi}{2}\rho\frac{\partial T(\rho,\theta,t)}{\partial \rho} = \varphi(t) \qquad \forall t > 0 \tag{29}$$

and

$$\frac{\partial T(\rho,0,t)}{\partial \rho} = 0 \quad \text{and} \quad \frac{\partial T(\rho,\frac{\pi}{2},t)}{\partial \rho} = 0, \qquad \forall \rho > 0 \tag{30}$$

where λ is the coefficient of conductivity

– the limit condition:

$$\lim_{\rho\to\infty} T(\rho,\theta,t) = 0 \qquad \forall\, \theta \in \left[0,\frac{\pi}{2}\right], \quad \forall t > 0 \tag{31}$$

– the initial condition known as Cauchy condition:

$$T(\rho,\theta,0) = 0 \qquad \forall \rho > 0, \quad \forall \theta \in \left[0,\frac{\pi}{2}\right]. \tag{32}$$

A method for resolving the bi-dimensional thermal system has been developed in [9]. In this method, a Fourier transformation is applied to a space variable, which transforms the bi-dimensional system into a mono-dimensional one. Then, the problem of mono-dimensional control has been solved using the Laplace transformation of the variable t and the differential flatness. In this paper, a solution for the system is determined using the Laplace transformation of the equation (28) and the method of separation of variables. The solution is a modified Bessel function of second kind of order 0. This method leads to polynomials in s^ν, where $s \in \mathbb{C}$ is the Laplace variable, in explicit form, which are well adapted for the pseudo-state representation (12).

The Laplace transformation of the equation (28) is given by:

$$\frac{s}{\alpha}\widehat{T}(\rho,\theta,s) = \frac{\partial^2 \widehat{T}(\rho,\theta,s)}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \widehat{T}(\rho,\theta,s)}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \widehat{T}(\rho,\theta,s)}{\partial^2 \theta} \tag{33}$$

where

$$\widehat{T}(\rho,\theta,s)=\int_0^{+\infty}T(\rho,\theta,t)e^{-st}dt \tag{34}$$

is the Laplace transformation of $T(\rho,\theta,t)$. Using the separation of variables method, the temperature $\widehat{T}(\rho,\theta,s)$ can be written into the form

$$\widehat{T}(\rho,\theta,s)=\widehat{T}_\rho(\rho,s)\widehat{T}_\theta(\theta,s) \tag{35}$$

where $\widehat{T}_\rho(\rho,s)$ is function of ρ and s and $\widehat{T}_\theta(\theta,s)$ is function of θ and s. By injecting (35) in (33), we get

$$\frac{\rho^2}{\widehat{T}_\rho(\rho,s)}\frac{\partial^2\widehat{T}_\rho(\rho,s)}{\partial\rho^2}+\frac{\rho}{\widehat{T}_\rho(\rho,s)}\frac{\partial\widehat{T}_\rho(\rho,s)}{\partial\rho} - \frac{s\rho^2}{\alpha}+\frac{1}{\widehat{T}_\theta(\theta,s)}\frac{\partial^2\widehat{T}_\theta(\theta,s)}{\partial\theta^2}=0. \tag{36}$$

Because of the symmetry of the metallic sheet, the temperature $\widehat{T}_\theta(\theta,s)$ is constant with respect to θ, *i.e.* $\widehat{T}_\theta(\theta,s)=A_1(s)$, then equation (36) becomes:

$$\rho^2\frac{\partial^2\widehat{T}_\rho(\rho,s)}{\partial\rho^2}+\rho\frac{\partial\widehat{T}_\rho(\rho,s)}{\partial\rho}-\frac{s\rho^2}{\alpha}\widehat{T}_\rho(\rho,s)=0 \tag{37}$$

Equation (37) is a modified Bessel equation. Its solution is a modified Bessel function of the form:

$$\widehat{T}_\rho(\rho,s)=B_1(s)I_0\Big(\rho\sqrt{\frac{s}{\alpha}}\Big)+B_2(s)K_0\Big(\rho\sqrt{\frac{s}{\alpha}}\Big). \tag{38}$$

The modified Bessel function of the first kind I_0 is an exponentially growing function, then, according to the limit condition (31), $B_1(s)=0$, and hence:

$$\widehat{T}_\rho(\rho,s)=B_2(s)K_0\Big(\rho\sqrt{\frac{s}{\alpha}}\Big). \tag{39}$$

Finally, the solution of the thermal bi-dimensional system is given by:

$$\widehat{T}(\rho,\theta,s)=A_1(s)\,B_2(s)K_0\Big(\rho\sqrt{\frac{s}{\alpha}}\Big). \tag{40}$$

According to the boundary condition, the heat flux is given by:

$$\widehat{\varphi}(s)=\lambda\lim_{\rho\to0}\frac{\pi}{2}A_1(s)B_2(s)\,\rho\sqrt{\frac{s}{\alpha}}K_1\Big(\rho\sqrt{\frac{s}{\alpha}}\Big)$$

$$= \lim_{\rho \to 0} \lambda \frac{\pi}{2} A_1(s) B_2(s)\, \rho \sqrt{\frac{s}{\alpha}} \frac{1}{\rho \sqrt{\frac{s}{\alpha}}} \tag{41}$$
$$= \lambda \frac{\pi}{2} A_1(s) B_2(s)$$

since $\dfrac{\partial K_0\left(\rho\sqrt{\frac{s}{\alpha}}\right)}{\partial \rho} = -\sqrt{\frac{s}{\alpha}} K_1\left(\rho\sqrt{\frac{s}{\alpha}}\right)$ and $K_1\left(\rho\sqrt{\frac{s}{\alpha}}\right) \sim \dfrac{1}{\rho\sqrt{\frac{s}{\alpha}}}$ in the neighbourhood of 0.

The transfer function of the system, called the thermal impedance, is defined by

$$\widehat{H}(\rho, \theta, s) = \frac{\widehat{T}(\rho, \theta, s)}{\widehat{\varphi}(s)} = \frac{A_1(s) B_2(s) K_0\left(\rho\sqrt{\frac{s}{\alpha}}\right)}{\lambda \frac{\pi}{2} A_1(s) B_2(s)} \tag{42}$$

which, for ρ sufficiently large, is equivalent to:

$$\widehat{H}(\rho, \theta, s) = \frac{2\sqrt{2\pi}}{\lambda \pi} \frac{1}{\sqrt{\rho\sqrt{\frac{s}{\alpha}}}} e^{-\rho\sqrt{\frac{s}{\alpha}}}. \tag{43}$$

Applying the Padé approximation at the order $\mathbf{K}$ of the pure delay [7] at a point (x_0, y_0), which corresponds to (ρ_0, θ_0) with $\rho_0 = \sqrt{x_0^2 + y_0^2}$, gives:

$$\widehat{H}_{\mathbf{K}}(\rho_0, \theta_0, s) \approx \frac{\sqrt{2\pi\sqrt{\alpha}}}{\lambda \pi \sqrt{\rho_0}} \frac{\sum_{k=0}^{\mathbf{K}} (-1)^k C'_k s^{\frac{k}{2}}}{\sum_{k=0}^{\mathbf{K}} C'_k s^{\frac{2k+1}{4}}} \tag{44}$$

with $C'_k = \frac{C_k}{|C_{\mathbf{K}}|}$ and $C_k = \dfrac{(2\mathbf{K}-k)!\mathbf{K}!}{(2\mathbf{K}!)k!(\mathbf{K}-k)!} \left(\dfrac{\rho_0}{\sqrt{\alpha}}\right)^k$.

4.2 Pseudo-state Representation

The transfer function $H_{\mathbf{K}}$ of the thermal bi-dimensional system is of fractional order multiple of $\nu = \frac{1}{4}$ and can be written in the form of a pseudo-state representation:

$$\begin{cases} X^{(\nu)} = AX + BU, \\ T_{\mathbf{K}}(x_0, y_0, t) = CX \end{cases} \tag{45}$$

with

$$A = \begin{pmatrix} 0 & -C'_{\mathbf{K}-1} & 0 & -C'_{\mathbf{K}-2} & \cdots & 0 & -C'_0 & 0 \\ & & & I_{2\mathbf{K}} & & & & 0_{2\mathbf{K}\times 1} \end{pmatrix} \tag{46}$$

is a real square matrix of dimension $(2\mathbf{K}+1)$,

$$B = \begin{pmatrix} 1 \\ 0_{2\mathbf{K}\times 1} \end{pmatrix} \in \mathbb{R}^{(2\mathbf{K}+1)\times 1}, \tag{47}$$

and

$$C = \frac{\sqrt{2\pi\sqrt{\alpha}}}{\lambda\pi\sqrt{\rho_0}} \begin{pmatrix} (-1)^{\mathbf{K}} C'_{\mathbf{K}} & 0 & \cdots & 0 & -C'_1 & 0 & C'_0 \end{pmatrix} \in \mathbb{R}^{1\times(2\mathbf{K}+1)}. \tag{48}$$

The pseudo-state vector X is of size $n = 2\mathbf{K}+1$ and denoted by

$$X = \begin{pmatrix} X_0 \\ \vdots \\ X_{2\mathbf{K}} \end{pmatrix} \tag{49}$$

and the input U is the heat flux, *i.e.* $U = \varphi(t)$. The dimension of the input U and the output $T_{\mathbf{K}}$ are respectively $m = 1$ and $p = 1$. The system (45) can also be written into the form

$$\begin{cases} \mathcal{A}X = BU \\ T_{\mathbf{K}}(x_0, y_0, t) = CX \end{cases} \tag{50}$$

where $\mathcal{A}$ is a $\mathbb{R}[\mathbf{D}^{\frac{1}{4}}]$-polynomial matrix of size $(2\mathbf{K}+1)\times(2\mathbf{K}+1)$ and given by

$$\mathcal{A} = \begin{pmatrix} \mathbf{D}^{\frac{1}{4}} & C'_{\mathbf{K}-1} & 0 & \cdots & 0 & C'_0 & 0 \\ -1 & \mathbf{D}^{\frac{1}{4}} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 0 \\ 0 & & \cdots & & 0 & -1 & \mathbf{D}^{\frac{1}{4}} \end{pmatrix}. \tag{51}$$

Using the Smith decomposition or the unimodular completion algorithm for the computation of fractional flat outputs, we can prove that the system (50) is a flat system and a fractional flat output of the thermal system (50) is

given by:

$$Z = X_{2\mathbf{K}}. \tag{52}$$

Therefore, the state X can be expressed using Z and its successive fractional derivatives:

$$X = \begin{pmatrix} \mathbf{D}^{\frac{\mathbf{K}}{2}} \\ \mathbf{D}^{\frac{\mathbf{K}}{2}-\frac{1}{4}} \\ \vdots \\ \mathbf{D}^{\frac{1}{4}} \\ 1 \end{pmatrix} Z. \tag{53}$$

Moreover, the input U is computed by

$$U = \sum_{k=0}^{\mathbf{K}} C'_k \mathbf{D}^{\frac{2k+1}{4}} Z \tag{54}$$

and the output $T_{\mathbf{K}}$ is given by

$$T_{\mathbf{K}} = CX = \frac{\sqrt{2\pi\sqrt{\alpha}}}{\lambda\pi\sqrt{\rho_0}} \sum_{k=0}^{\mathbf{K}} (-1)^k C'_k \mathbf{D}^{\frac{k}{2}} Z. \tag{55}$$

4.3 Fault detection and isolation

In the thermal bi-dimensional system, the temperature at the point (x_0, y_0) of the metallic sheet is measured. Thus, there is only one sensor at this point, and the measured temperature is denoted by $T^s_{\mathbf{K}}(x_0, y_0, t)$. Moreover, there is a single actuator that produces the heat flux $\varphi(t)$.

In order to compute the redundant output and redundant input using the fractional flatness-based method, the measurement of the flat output Z must be available at every time. For the thermal bi-dimensional system, the only available measurements are the output $T^s_{\mathbf{K}}(x_0, y_0, t)$ and the input $\varphi(t)$. However, the measurement of the flat output Z, denoted by Z^s, can be computed from the measurement of the temperature $T^s_{\mathbf{K}}(x_0, y_0, t)$. In fact, from (55) we have

$$T_{\mathbf{K}}(x_0, y_0, t) = W(\mathbf{D}^{\frac{1}{4}}) Z(t) \tag{56}$$

where $W(\mathbf{D}^{\frac{1}{4}})$ is given by

$$W(\mathbf{D}^{\frac{1}{4}}) = \frac{\sqrt{2\pi\sqrt{\alpha}}}{\lambda\pi\sqrt{\rho_0}} \sum_{k=0}^{\mathbf{K}} (-1)^k C'_k \mathbf{D}^{\frac{k}{2}}. \tag{57}$$

$W(\mathbf{D}^{\frac{1}{4}})$ is a $(\mathbf{D}^{\frac{1}{4}})$-polynomial, its inverse is given by

$$W_{inv}(\mathbf{D}^{\frac{1}{4}}) = \frac{\lambda\pi\sqrt{\rho_0}}{\sqrt{2\pi\sqrt{\alpha}}\sum_{k=0}^{\mathbf{K}}(-1)^k C'_k \mathbf{D}^{\frac{k}{2}}}. \tag{58}$$

Then, the measurement of the fractional flat output at every time is computed by

$$Z^s(t) = W_{inv}(\mathbf{D}^{\frac{1}{4}}) T^s_{\mathbf{K}}(x_0, y_0, t). \tag{59}$$

The redundant output, denoted by $T^z_{\mathbf{K}}(x_0, y_0, t)$, is computed using (15), and the redundant input, denoted by $\varphi^z(t)$, is computed using (14). The fractional derivatives of the flat output measurements are computed using the function **dn** of the toolbox CRONE [26]. Then, the vector of residues, associated to Z^s, is given by

$$r = \begin{pmatrix} R_{\mathsf{S}} \\ R_{\mathsf{A}} \end{pmatrix} = \begin{pmatrix} T^s_{\mathbf{K}}(x_0, y_0, t) - T^z_{\mathbf{K}}(x_0, y_0, t) \\ \varphi(t) - \varphi^z(t) \end{pmatrix}. \tag{60}$$

It is important to note that we work with an open-loop system. Then, only the residue that depends on the measurement of the temperature $T^s_{\mathbf{K}}$ is affected if a fault occurs on the sensor S. Similarly, only the residue that depends on the measurement of the heat flux φ is affected, if a fault occurs on the actuator A. Then, the signature matrix $\mathbf{S}$, associated to the fractional flat output Z^s, is given by:

$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{61}$$

According to Definition 5, all faults on the system sensor and actuator are detectable, and according to Definition 6, all faults are isolable.

4.4 Simulation Results

The fractional flatness-based FDI method, applied on the thermal bi-dimensional system, is proved by simulations.

4.4.1 Trajectory planning

In these simulations, the temperature goes from a resting state of temperature $T_0 = 0°\text{C}$ to a resting state of temperature $T_f = 30°\text{C}$ in a period of 2500s, at the point $x_0 = 0.005\text{m}$ and $y_0 = 0.002\text{m}$. Then, the initial and final conditions of the temperature are the following:

$$T(x_0, y_0, 0) \triangleq T_0 = 0,$$

$$\begin{aligned} &T(x_0, y_0, t_f) \triangleq T_f = 30, \quad t_f = 2500\text{s}, \\ &T^{(l)}(x_0, y_0, 0) = 0, \quad l = 1, 2, \\ &T^{(l)}(x_0, y_0, t_f) = 0, \quad l = 1, 2. \end{aligned} \tag{62}$$

Then, the desired reference trajectory is calculated by polynomial interpolation of order 5:

$$T_{ref}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5. \tag{63}$$

The reference trajectory of the temperature $T_{ref}(t)$ and its derivatives are illustrated in Figure 2. The reference trajectory of the fractional flat output

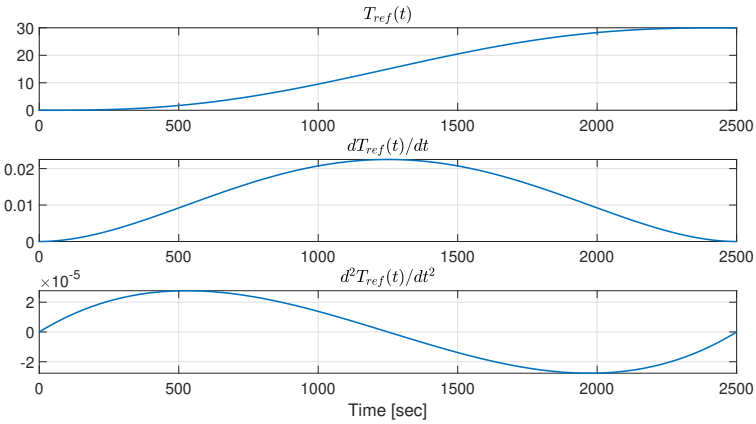

Fig. 2 *Reference trajectory of the temperature $T(x_0, y_0, t)$ and its derivatives*

$Z_{ref}(t)$ is deduced using (59) with $\mathbf{K} = 20$[2], and it is illustrated in Figure 3.

Finally, a reference trajectory of the heat flux $\varphi_{ref}(t)$ is deduced by flatness using (54). See Figure 4.

Figure 5 shows that, by applying the heat flux $\varphi_{ref}(t)$ on the system (45), in the absence of perturbations, the output of the system follows its reference trajectory.

In order to make the simulations close to the reality, white Gaussian noise is added to the sensor and the actuator with a level corresponding to the actual process level, see Table 1.

	Mean	Variance	Power
Sensor	0	0.13	1×10^{-2}
Actuator	0	0.13	1×10^{-2}

Table 1 *Parameter values of the added white Gaussian noise*

[2] $\mathbf{K} = 20$ is sufficient for the convergence of $\widehat{H}_{\mathbf{K}}(\rho_0, \theta_0, s)$ to $\widehat{H}(\rho_0, \theta_0, s)$ [36].

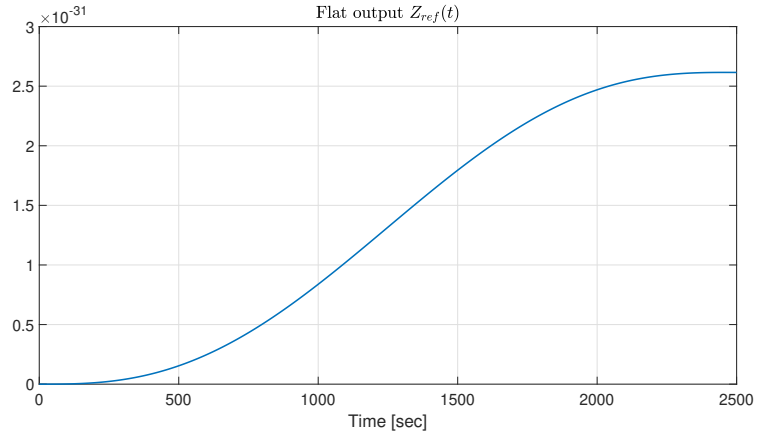

Fig. 3 *Reference trajectory of the flat output* $Z_{ref}(t)$

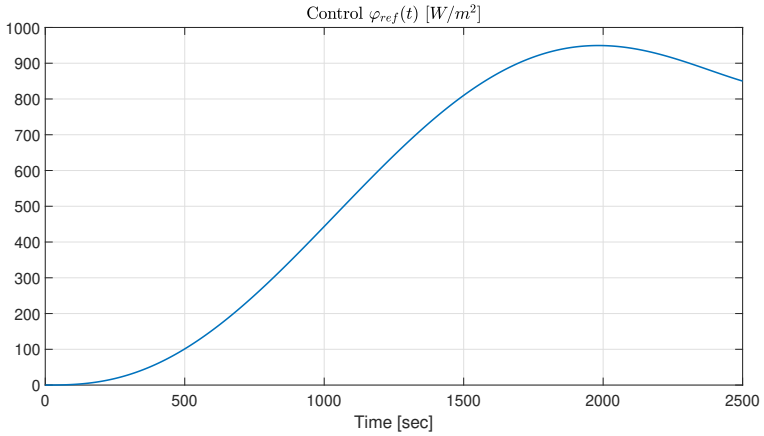

Fig. 4 *Reference trajectory of the heat flux* $\varphi(t)$

Since there is noise on the system sensor and actuator, a threshold is fixed for each residue. For this purpose, several nominal simulations were realized with different initial and final conditions. The amplitude of the detection threshold is calculated by selecting the worst case among all simulations results, plus a 5% safety margin to avoid false alarms. The values of the maximum and minimum threshold for each residue are given in Table 2. In the figures below, the thresholds are normalized between −1 and 1.

	Max	Min
R_S [°C]	0.0089	−0.0078
R_A[W/m^2]	0.0404	−0.1122

Table 2 *Values of the maximum and minimum threshold for each residue*

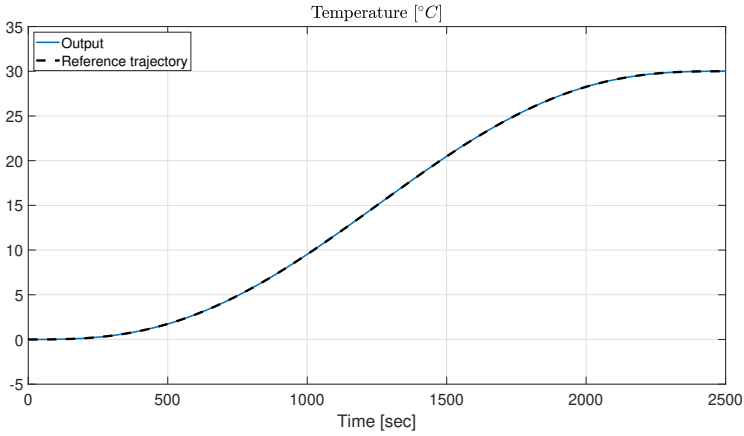

Fig. 5 *Reference trajectory vs. exact solution*

Moreover, because of the presence of noise, a low-pass filter of order 4 and cutoff frequency $f_c = 0.2$Hz is used in order to filter the noise.

In the following, multiplicative and additive faults are applied on the sensor and the actuator, separately.

4.4.2 Multiplicative sensor fault

For multiplicative sensor fault, at time $t = 1000$s, the sensor measures only 80% of the temperature, so the actual temperature is 20% above the measurement indicated by the sensor. On the other hand, since there is no controller on the system, the input $\varphi(t)$ and therefore the system states including the fractional flat output $Z(t)$ are not affected by this fault. The redundant output and the redundant input are then calculated analytically, using the flat output $Z(t)$. The residue vector associated to the flat output $Z(t)$ is calculated by the difference between the measurements and their redundancies. In the case of multiplicative fault on the sensor, only the residue R_{S} is affected by the fault on the sensor, see Figure 6, which corresponds to the fault alarm signature

$$\Sigma_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{64}$$

of the signature matrix $\mathbf{S}$ given in (61).

4.4.3 Multiplicative actuator fault

At time $t = 1000$s, a multiplicative fault is applied to the actuator, that is the actuator can provide only 80% of the reference trajectory of the heat flux $\varphi_{ref}(t)$. Moreover, the fault on the actuator affects the system states, including the flat output, and therefore affects the system output. In order to construct the residue vector, first the value of the flat output $Z^s(t)$ is computed

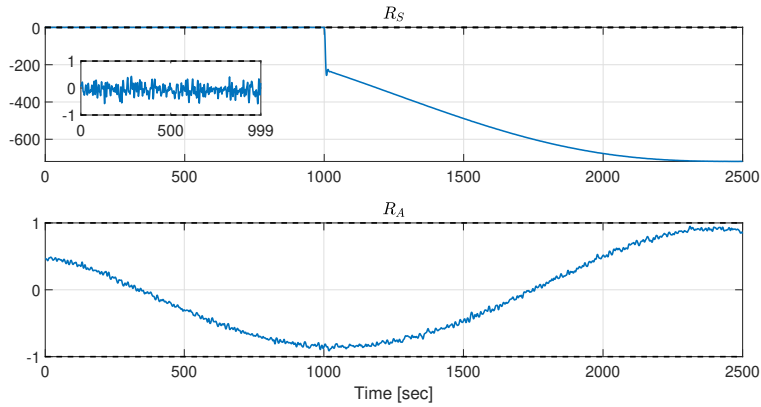

Fig. 6 *Multiplicative fault on the sensor at time* $t = 1000$s

using relation (59). Then, the redundant input $\varphi^z(t)$ and the redundant output $T^z(x_0, y_0, t)$ are computed using (14) and (15).

Finally, the vector of residues can be computed, see Figure 7. The fault that affects the actuator, affects only the residue R_{A}, which corresponds to the fault alarm signature

$$\Sigma_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{65}$$

of the signature matrix $\mathbf{S}$ given in (61). Then, all multiplicative faults on the thermal bi-dimensional system can be detected and isolated using the fractional flatness-based FDI method.

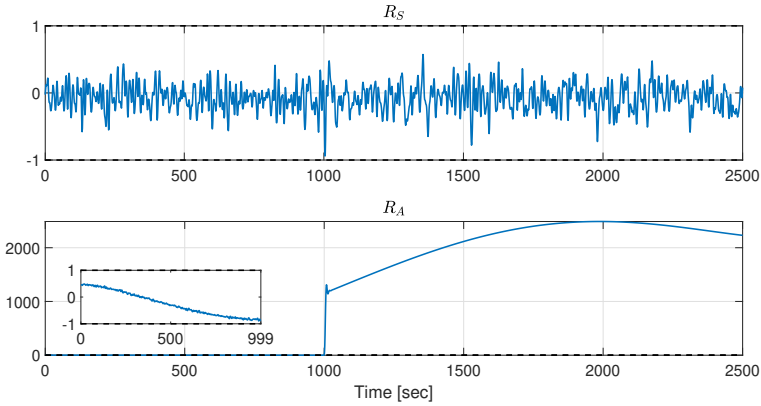

Fig. 7 *Multiplicative fault on the actuator at time* $t = 1000$s

4.4.4 Additive sensor and actuator faults

Additive faults on the sensor and the actuator are represented by biases. For the sensor fault, a $+2°C$ is added to the temperature at time $t = 1000$s, and for the actuator fault an extra heat flux of 10^2 W/m^2 is added to the heat flux. The residue values are illustrated in Figures 8 and 9.

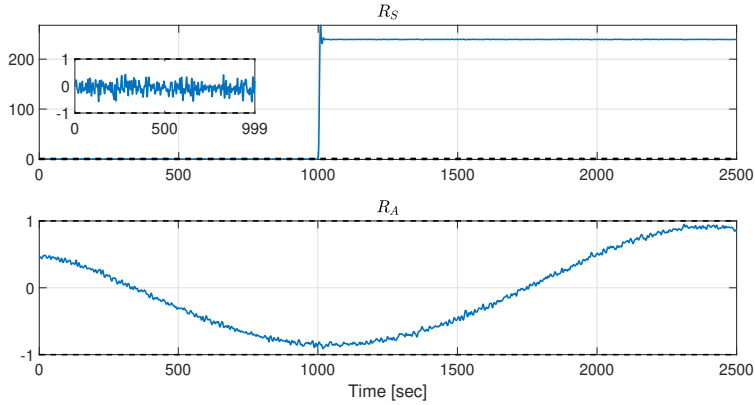

Fig. 8 *Additive fault on the sensor at time* $t = 1000$s

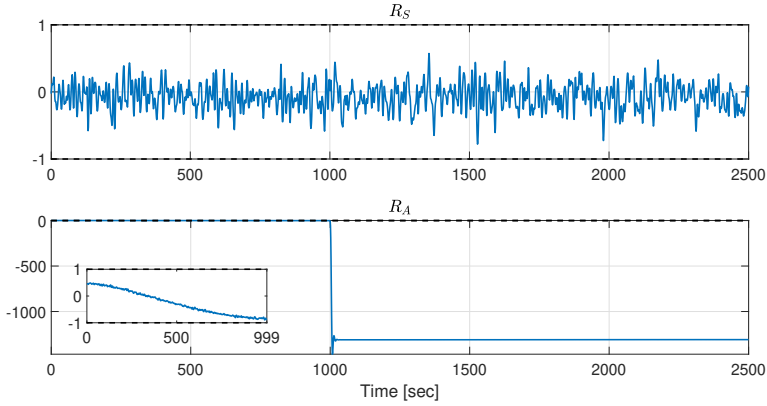

Fig. 9 *Additive fault on the actuator at time* $t = 1000$s

The same signature matrix obtained theoretically:

$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{66}$$

Then, additive faults are also detectable and isolable using the fractional flatness-based FDI method.

5 Conclusion

In this paper, the flatness-based FDI method, introduced in [23], and generalized in [29], has been extended to the class of fractional order linear flat systems, in order to detect and isolate faults on sensors and actuators. In particular, definitions of residual generation, signature matrix, detectability and isolability were extended. This new method of FDI has been applied on the thermal bi-dimensional system and its efficiency has been proved by simulations on an open-loop system.

This paper presents a first work for the generalization of the flatness-based FDI method to the class of fractional order linear systems. Future work must take into account the impact of a controller on the system, and apply it to a real thermal system such as the thermal bar. Moreover, since the FDI process is a stage of the FTC process, a study can be made to show the effectiveness of the fractional flatness-based method on the control reconfiguration and the fault tolerant control.

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