



**HAL**  
open science

# Pseudoconvex non-Stein domains in primary Hopf surfaces

Christian Miebach

► **To cite this version:**

Christian Miebach. Pseudoconvex non-Stein domains in primary Hopf surfaces. *Izvestiya: Mathematics*, 2014, 78 (5), pp.1028-1035. 10.1070/IM2014v078n05ABEH002717 . hal-03600354

**HAL Id: hal-03600354**

**<https://hal.science/hal-03600354>**

Submitted on 7 Mar 2022

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# PSEUDOCONVEX NON-STEIN DOMAINS IN PRIMARY HOPF SURFACES

CHRISTIAN MIEBACH

ABSTRACT. We describe pseudoconvex non-Stein domains in primary Hopf surfaces using techniques developed by Hirschowitz.

## 1. INTRODUCTION

Let  $H$  be a primary Hopf surface. In [LY12] Levenberg and Yamaguchi characterize locally pseudoconvex domains  $D \subset H$  having smooth real-analytic boundary that are not Stein. In this note we generalize their result to arbitrary pseudoconvex domains using ideas developed by Hirschowitz in [Hir74] and [Hir75]. For the readers' convenience these ideas are reviewed in a slightly generalized form in Section 2. In Section 3 we review the structure of primary Hopf surfaces in order to describe a certain (singular) holomorphic foliation  $\mathcal{F}$  of  $H$ . This allows us to formulate the following Main Theorem, which is proven in Sections 4 and 5.

**Main Theorem.** *Let  $D \subset H$  be a pseudoconvex domain. If  $D$  is not Stein, then  $D$  contains with every point  $p \in D$  the topological closure  $\bar{F}_p$  of the leaf  $F \in \mathcal{F}$  passing through  $p$ .*

I would like to thank Karl Oeljeklaus for helpful discussions on the subject of this paper and Stefan Nemirovski for a suggestion on how to prove Lemma 5.3. I am also grateful to Peter Heinzner and the SFB/TR 12 for an invitation to the Ruhr-Universität Bochum where a part of this paper has been written.

## 2. A REVIEW OF HIRSCHOWITZ' METHODS

In this section we present the methods developed by Hirschowitz in [Hir75] in a slightly more general setup.

Let  $X$  be a complex manifold with holomorphic tangent bundle  $TX \rightarrow X$ , and let  $\pi: \mathbb{P}TX \rightarrow X$  be the projectivized holomorphic tangent bundle. A continuous function on  $X$  is called *strictly plurisubharmonic* on  $X$  if it is everywhere locally the sum of a continuous plurisubharmonic and a smooth strictly plurisubharmonic function.

---

2010 *Mathematics Subject Classification.* 32M05 (primary); 32E40 (secondary).

**Definition 2.1.** Let  $\varphi \in \mathcal{C}(X)$  be plurisubharmonic. Then we define  $S(\varphi)$  to be the set of  $[v] \in \mathbb{P}TX$  such that  $\varphi$  is in a neighborhood of  $\pi[v]$  the sum of a plurisubharmonic function and a smooth function that is strictly plurisubharmonic on any germ of a holomorphic curve defining  $[v]$ .

**Lemma 2.2.** Let  $\varphi \in \mathcal{C}(X)$  be plurisubharmonic.

- (1) The set  $S(\varphi)$  is open in  $\mathbb{P}TX$ .
- (2) If  $S(\varphi) = \mathbb{P}TX$ , then  $\varphi$  is strictly plurisubharmonic on  $X$ .
- (3) If  $\sum_k \varphi_k$  converges uniformly on compact subsets of  $X$  where  $\varphi_k \in \mathcal{C}(X)$ , then we have  $S(\sum_k \varphi_k) \supset \bigcup_k S(\varphi_k)$ .

*Proof.* This is [Hir75, Proposition 1.3].  $\square$

For any plurisubharmonic function  $\varphi \in \mathcal{C}(X)$  we define  $C(\varphi)$  to be

$$\mathbb{P}TX \setminus \{[v] \in \mathbb{P}TX; \varphi \text{ is smooth around } \pi[v] \text{ and } \partial\varphi(v) \neq 0\}$$

and then set

$$(2.1) \quad C(X) := \bigcap_{\substack{\varphi \in \mathcal{C}(X) \\ \text{plurisubharmonic}}} C(\varphi).$$

Every set  $C(\varphi)$  (and thus  $C(X)$ ) is closed in  $\mathbb{P}TX$ . The next lemma is a slight generalization of [Hir75, Proposition 1.5].

**Lemma 2.3.** Let  $X$  be a complex manifold and let  $\Omega := X \setminus \pi(C(X))$ . Then there exists a plurisubharmonic function  $\psi \in \mathcal{C}(X)$  which is strictly plurisubharmonic on  $\Omega$ .

*Proof.* Since  $\pi: \mathbb{P}TX \rightarrow X$  is proper, the set  $\Omega$  is open in  $X$ . If  $\Omega$  is empty, there is nothing to prove. Therefore let us suppose that  $\Omega$  is a non-empty open subset of  $X$ . Consequently,  $\mathbb{P}TX \setminus C(X)$  is non-empty.

For every  $[v] \in \mathbb{P}TX \setminus C(X)$  we find a plurisubharmonic function  $\varphi_{[v]} \in \mathcal{C}^\infty(X)$  with  $\partial\varphi_{[v]}v \neq 0$ . We claim that the function  $\psi_{[v]} := \exp \circ \varphi_{[v]}$  is strictly plurisubharmonic in the direction of  $[v]$ . To see this, we calculate

$$\partial\bar{\partial}\psi_{[v]} = e^{\varphi_{[v]}} (\partial\varphi_{[v]} \wedge \bar{\partial}\varphi_{[v]} + \partial\bar{\partial}\varphi_{[v]}).$$

In other words, we obtain  $[v] \in S(\psi_{[v]})$ . Since  $X$  has countable topology, we get an open covering

$$\mathbb{P}TX \setminus C(X) \subseteq \bigcup_{k=1}^{\infty} S(\psi_k).$$

It is possible to find  $\lambda_k > 0$  such that  $\sum_{k=1}^{\infty} \lambda_k \psi_k$  converges uniformly on compact subsets of  $X$ . To prove this, choose a countable exhaustion  $X = \bigcup_j K_j$  by compact sets with  $K_j \subset \overset{\circ}{K}_{j+1}$ . For every  $j$  there are

$\lambda_{k,j} > 0$  such that

$$\sum_{k=1}^{\infty} \lambda_{k,j} \|\psi_k\|_{K_j}$$

converges. Since  $\|\psi_k\|_{K_j} \leq \|\psi_k\|_{K_{j+1}}$  for every  $j$ , we may suppose that  $\lambda_{k,j'} \leq \lambda_{k,j}$  for all  $j \leq j'$ . Defining  $\lambda_k := \lambda_{k,k}$  and noting that every compact subset  $K \subset X$  is contained in  $K_{j_0}$  for some  $j_0$ , we conclude

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda_k \|\psi_k\|_K &\leq \sum_{k=1}^{\infty} \lambda_k \|\psi_k\|_{K_{j_0}} \\ &\leq \sum_{k=1}^{j_0} \lambda_k \|\psi_k\|_{K_{j_0}} + \sum_{k=j_0+1}^{\infty} \lambda_{k,j_0} \|\psi_k\|_{K_{j_0}} < \infty \end{aligned}$$

which proves the claim. It follows that the limit function  $\psi := \sum_k \psi_k$  is continuous and satisfies  $S(\psi) \supset \bigcup_k S(\psi_k) \supset \mathbb{P}TX \setminus C(X)$ , hence it is strictly plurisubharmonic on  $\Omega$ .  $\square$

In the following we say that a complex manifold  $X$  is *pseudoconvex* if there is a continuous plurisubharmonic exhaustion function  $\rho: X \rightarrow \mathbb{R}^{>0}$ .

**Lemma 2.4.** *Let  $X$  be a pseudoconvex complex manifold and let  $\gamma: U \rightarrow X$  be the integral curve of a holomorphic vector field on  $X$  where  $U$  is a domain in  $\mathbb{C}$ . If  $\gamma'(U)$  meets  $C(X)$ , then  $\gamma'(U)$  is contained in  $C(X)$ . If  $X$  admits a smooth plurisubharmonic exhaustion function, then  $\gamma'(U) \subset C(X)$  implies that  $\gamma(U)$  is relatively compact in  $X$ . In particular, in this case we have  $U = \mathbb{C}$ .*

*Proof.* Let  $\zeta$  be the holomorphic vector field on  $X$  with integral curve  $\gamma$  and suppose that  $\gamma'(0) = \zeta(x_0) \in C(X)$ . It is enough to show that 0 is an inner point of the set of  $t \in U$  with  $\gamma'(t) \in C(X)$ , for then the closed set  $(\gamma')^{-1}(C(X))$  is also open, hence equal to  $U$ . In other words, we must prove that for every plurisubharmonic function  $\varphi \in \mathcal{C}(X)$  smooth in a neighborhood of  $x_t := \gamma(t)$  we have  $\zeta(\varphi)(x_t) = 0$  whenever  $|t|$  is sufficiently small.

To do this, choose  $\alpha \in \mathbb{R}^{>0}$  such that  $x_0 = \gamma(t_0) \in X_\alpha := \{x \in X; \rho(x) < \alpha\}$  where  $\rho$  is a continuous plurisubharmonic exhaustion function of  $X$ . Let  $\Phi_t^\zeta$  be the holomorphic local flow of  $\zeta$ . For  $|t|$  sufficiently small we have

$$\Phi_t^\zeta(X_{\alpha+1}) \supset X_\alpha \ni x_t := \Phi_t^\zeta(x_0).$$

Since  $\Phi_t^\zeta: X_\alpha \rightarrow X_{\alpha+1}$  is holomorphic,  $\varphi_t := \varphi \circ \Phi_t^\zeta$  is continuous plurisubharmonic on  $X_\alpha$  and smooth in a neighborhood of  $x_0$  for each plurisubharmonic function  $\varphi \in \mathcal{C}(X_{\alpha+1})$  that is smooth in a neighborhood of  $x_t$ . Following the proof of [Hir75, Proposition 1.6] we construct a continuous plurisubharmonic function  $\psi_t$  on  $X$  which

coincides with  $\varphi_t$  in a neighborhood of  $x_0$ . Choose  $\beta \in \mathbb{R}$  such that  $\varphi_t(x_0) < \beta < \alpha$  and note that  $K := \rho^{-1}(\beta) \subset X_\alpha$  is compact. Then choose a convex increasing function  $\chi$  on  $\mathbb{R}$  fulfilling

$$\begin{aligned} \chi(\rho(x_0)) &< \varphi_t(x_0) \quad \text{and} \\ \chi(\beta) &> \|\varphi_t\|_K. \end{aligned}$$

Finally, define  $\psi_t: X \rightarrow \mathbb{R}$  by

$$\psi_t(x) := \begin{cases} \max(\varphi_t(x), \chi \circ \rho(x)) & : \rho(x) \leq \beta \\ \chi \circ \rho(x) & : \rho(x) \geq \beta. \end{cases}$$

One checks directly that  $\psi_t$  is continuous plurisubharmonic and coincides with  $\varphi_t$  in some neighborhood of  $x_0$ . Consequently, we may calculate

$$\zeta \varphi(x_t) = \frac{d}{ds} \Big|_t \varphi(\Phi_s^{\zeta}(x_0)) = \zeta \varphi_t(x_0) = \zeta \psi_t(x_0) = 0$$

since  $\zeta_{x_0} = \gamma'(0) \in C(X)$ . Therefore we see that  $\gamma'(t) \in C(X)$  for every  $t \in U$  sufficiently close to 0, which proves the first part of the lemma.

If  $\rho$  is smooth, then choosing  $\varphi = \rho$  in the argument given above, we see that  $\gamma(U)$  lies in a fiber of  $\rho$ , hence is relatively compact.  $\square$

### 3. STATEMENT OF THE MAIN THEOREM

Let us fix  $a_1, a_2 \in \mathbb{C}$  such that  $0 < |a_1| \leq |a_2| < 1$ . The automorphism  $\varphi: \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{C}^2 \setminus \{0\}$ ,  $(z_1, z_2) \mapsto (a_1 z_1, a_2 z_2)$ , generates a free proper  $\mathbb{Z}$ -action on  $\mathbb{C}^2 \setminus \{0\}$ . By definition, the compact complex surface  $H_a := (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$  for  $a = (a_1, a_2)$  is a *primary Hopf surface*. We will write  $[z_1, z_2] := \pi(z_1, z_2)$  where  $\pi: \mathbb{C}^2 \setminus \{0\} \rightarrow H_a$  is the quotient map.

The torus  $T = \mathbb{C}^* \times \mathbb{C}^*$  acts holomorphically on  $H_a$  with three orbits. More precisely, we have  $H_a = E_1 \cup H_a^* \cup E_2$  where  $H_a^* := (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}$  is the open  $T$ -orbit, and where  $E_1 := (\mathbb{C}^* \times \{0\})/\mathbb{Z} = T \cdot [1, 0]$  and  $E_2 := (\{0\} \times \mathbb{C}^*)/\mathbb{Z} = T \cdot [0, 1]$  are elliptic curves.

Note that  $H_a^*$  is a connected Abelian complex Lie group which thus can be represented as  $\mathbb{C}^2/\Gamma_3$  where  $\Gamma_3$  is a discrete subgroup of rank 3 of  $\mathbb{C}^2$ . The map  $p: \mathbb{C}^2 \rightarrow \mathbb{C}^2/\Gamma_3 \cong H_a^*$  is the universal covering of  $H_a^*$ . Let  $V$  be the real span of  $\Gamma_3$  and set  $W := V \cap iV$ . There are two possibilities. Either  $p(W)$  is dense in  $V/\Gamma_3 \cong (S^1)^3$ , or  $p(W)$  is closed, hence compact, hence an elliptic curve  $E$ . In the first case, we have  $\mathcal{O}(H_a^*) = \mathbb{C}$ , i.e.,  $H_a^*$  is a *Cousin group*, while in the second case  $H_a^* \cong \mathbb{C}^* \times E$ .

For the following result we refer the reader to [BHPV04, Chapter V.18].

**Proposition 3.1.** *The open orbit  $H_a^*$  is not Cousin if and only if  $a_1^{k_1} = a_2^{k_2}$  for some relatively prime  $k_1, k_2 \in \mathbb{Z}$ .*

*Remark.* If there exist relatively prime integers  $k_1, k_2$  with  $a_1^{k_1} = a_2^{k_2}$ , then we have the elliptic fibration  $H_a \rightarrow \mathbb{P}^1, [z_1, z_2] \mapsto [z_1^{k_1} : z_2^{k_2}]$ . The generic fiber is the elliptic curve  $E = \mathbb{C}^*/(z \sim cz)$  where  $c := a_1^{k_1} = a_2^{k_2}$ . Note that for a generic choice of  $a = (a_1, a_2)$  the open subset  $H_a^*$  is a Cousin group.

Suppose that  $H_a^*$  is Cousin and let  $\xi \in \mathfrak{t}$  be the generator of the relatively compact one parameter subgroup  $p(W)$ . Let  $\xi_{H_a}$  be the holomorphic vector field induced by the  $T$ -action on  $H_a$ . One checks directly that  $\xi_{H_a}$  has no zeros in  $H_a$ , hence defines a holomorphic foliation of  $H_a$ . Note that the open subset  $H_a^*$  is saturated with respect to  $\mathcal{F}$  and that the leaves of  $\mathcal{F}|_{H_a^*}$  are relatively compact in  $H_a^*$ . The closure of a leaf  $F \subset H_a^*$  in  $H_a^*$  is a Levi-flat compact smooth hypersurface. In fact, these Levi-flat hypersurfaces are the fibers of the pluriharmonic function  $[z_1, z_2] \mapsto \frac{\log|z_1|}{\log|a_1|} - \frac{\log|z_2|}{\log|a_2|}$  defined on  $H_a^*$ , see [LY12]. If  $H_a$  is elliptic, then it is foliated by elliptic curves. Again,  $H_a^*$  is saturated with respect to this foliation and the leaves are compact in  $H_a^*$ . This shows that in both cases we obtain a (singular) holomorphic foliation  $\mathcal{F}$  of  $H_a$  such that the leaves of  $\mathcal{F}|_{H_a^*}$  are relatively compact in  $H_a^*$ .

We now state the main result of this note.

**Theorem 3.2.** *Let  $H_a$  be a primary Hopf surface and let  $D \subset H_a$  be a pseudoconvex domain. If  $D$  is not Stein, then  $D$  contains with every point  $p \in D$  the topological closure  $\overline{F}_p$  of the leaf  $F \in \mathcal{F}$  passing through  $p$ .*

*Remark.* For locally pseudoconvex domains having smooth real-analytic boundary this result has been obtained by Levenberg and Yamaguchi using the theory of  $c$ -Robin functions, see [LY12].

#### 4. EXISTENCE OF PLURISUBHARMONIC EXHAUSTIONS

In this section we will show that every smoothly bounded locally pseudoconvex domain  $D \subset H_a$  admits a continuous plurisubharmonic exhaustion function. For this we will modify Hirschowitz' proof of [Hir74, Théorème 2.1].

**Proposition 4.1.** *Let  $D \subset H_a$  be locally pseudoconvex and suppose that neither  $E_1$  nor  $E_2$  is a component of  $\partial D$ . Then  $D$  admits a continuous plurisubharmonic exhaustion function.*

*Remark.* The hypothesis of Proposition 4.1 is fulfilled if  $D$  is locally pseudoconvex and smoothly bounded. Hence, Theorem 3.2 indeed generalizes the main result of [LY12].

*Proof.* We define  $\Omega := \{(x, \xi) \in D \times \mathfrak{t} ; \exp(\xi) \cdot x \in D\}$ . By definition,  $\Omega$  is an open subset of  $D \times \mathfrak{t}$  containing  $D \times \{0\}$ . Since  $D$  is locally pseudoconvex in  $H_a$ , it follows that  $\Omega$  is locally pseudoconvex in  $D \times \mathfrak{t}$ .

We define the boundary distance  $d: D \rightarrow \mathbb{R}^{>0}$  by

$$d(x) := \sup\{r > 0; \{x\} \times B_r(0) \subset \Omega\}.$$

It is elementary to check that  $d$  is lower semicontinuous. Since  $\Omega$  is locally pseudoconvex in  $D \times T$ , every point  $x \in \overline{D}$  has an open Stein neighborhood  $U$  such that  $\Omega \cap (U \times \mathfrak{t})$  is pseudoconvex. Due to a result of Lelong, see [Lel68, Theorem 2.4.2], the function  $-\log d$  is plurisubharmonic on  $U \cap D$  and therefore everywhere on  $D$ .

Note that  $-\log d \equiv -\infty$  if and only if  $D$  contains  $H_a^*$ . Since neither  $E_1$  nor  $E_2$  is a component of  $\partial D$ , this implies  $D = H_a$  so that we may exclude this case in the following.

For every  $x \in H_a$  the orbit map  $\mathfrak{t} \rightarrow T \cdot x, \xi \mapsto \exp(\xi) \cdot x$ , is open into its image  $T \cdot x$ . Thus we see that  $-\log d(x)$  goes to infinity as  $x$  approaches a point in  $\partial D$ . Since  $\partial D$  is compact, this implies that  $-\log d$  is an exhaustion.

To end this proof, one verifies directly that  $d$  is upper semicontinuous in any point  $x \in D$  such that  $d(x) \neq \infty$ . Therefore, for every constant  $C > 0$  the map  $\sup(C, -\log d)$  is a continuous plurisubharmonic exhaustion of  $D$ .  $\square$

*Remark.* The proof of Proposition 4.1 shows that the polar set given by  $\{-\log d = -\infty\}$  is non-empty if and only if  $D$  contains  $E_1$  or  $E_2$ .

*Remark.* In [DF82], Diederich and Fornæss give an example of a relatively compact pseudoconvex domain in a  $\mathbb{P}_1$ -bundle over a Hopf surface that has smooth real-analytic boundary but does *not* admit an exhaustion by pseudoconvex subdomains.

## 5. PROOF OF THEOREM 3.2

Let us start by noting the following simple but important observation.

**Lemma 5.1.** *The domain  $H_a^*$  admits a smooth plurisubharmonic exhaustion function. Consequently, if  $D \subset H_a$  is pseudoconvex, then  $D^* := D \cap H_a^*$  is likewise pseudoconvex.*

If  $H_a^*$  is Cousin, this lemma follows from [Hu10, Proposition 2.4]. If  $H_a^*$  is not Cousin, then  $H_a^* \cong \mathbb{C}^* \times E$  clearly has a smooth plurisubharmonic exhaustion.

Let  $D \subset H_a$  be a pseudoconvex domain which is not Stein. If the pseudoconvex domain  $D^* \subset H_a^*$  is not Stein, then  $D^*$  is saturated with respect to the foliation  $\mathcal{F}|_{H_a^*}$ , see [GMO13, Theorem 3.1]. Since

the leaves of  $\mathcal{F}$  are relatively compact orbits of a one parameter subgroup of  $T$ , continuity of the action map  $T \times H_a \rightarrow H_a$  implies that  $D$  is saturated with respect to  $\mathcal{F}$  in this case. Therefore, let us assume that  $D \neq D^*$  is Stein. We will complete the proof of Theorem 3.2 by showing that then  $D$  is Stein as well.

We note first that  $D$  cannot contain  $E_1$  or  $E_2$  if  $D^*$  is Stein. Indeed, due to the continuity of the leaves of  $\mathcal{F}$  remarked above, if  $E_1$  was contained in  $D$ , then some of the relatively compact leaves of  $\mathcal{F}$  would lie in  $D^*$ , contradicting the assumption that  $D^*$  is Stein.

Let us consider the subset  $C(D) \subset \mathbb{P}TD$  defined in (2.1) where  $\pi: \mathbb{P}TD \rightarrow D$  is the projectivized tangent bundle. The proof of the following lemma relies essentially on the explicit knowledge of the structure of primary Hopf surfaces.

**Lemma 5.2.** *Let  $D \subset H_a$  be a pseudoconvex domain. If  $D^* = D \cap H_a^*$  is Stein, then we have  $\pi(C(D)) \subset D^* \setminus D$ .*

*Proof.* Suppose that  $\pi(C(D))$  meets  $D^*$ . Since  $H_a^*$  is an Abelian complex Lie group, it has a biinvariant Haar measure. Therefore, we can apply the usual convolution technique in order to approximate the continuous plurisubharmonic exhaustion of  $D$  uniformly on compact subsets by smooth ones. This allows us to apply Lemma 2.4 to prove existence of a complex one parameter subgroup  $A$  of  $T$  and a point  $x \in D^*$  such that  $A \cdot x$  is relatively compact in  $D$ . Note that  $A \cdot x$  is not relatively compact in  $D^*$  since the latter is assumed to be Stein.

We claim that that the closure of such a curve  $A \cdot x$  in  $D$  (and hence  $D$  itself) would have to contain  $E_1$  or  $E_2$ , which then, as noted above, will contradict our assumption that  $D^*$  is Stein. In order to prove this claim, consider  $A = \{(e^{tz_1}, e^{tz_2}); t \in \mathbb{C}\}$  where  $z = (z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$ . If  $z_1 = 0$  or  $z_2 = 0$ , we see directly that the closure of  $A \cdot x$  contains  $E_2$  or  $E_1$ . Hence, suppose that  $z_1, z_2 \neq 0$ . If  $\frac{z_1}{z_2} \notin \mathbb{R}$ , already the closure of  $A$  in  $\mathbb{C}^2 \setminus \{0\}$  contains  $\{z_1 = 0\} \cup \{z_2 = 0\}$ , thus the closure of  $A \cdot x$  in  $D$  contains  $E_1 \cup E_2$  as well in this case. Therefore, we are left to deal with the case  $A = \{(e^t, e^{\lambda t}); t \in \mathbb{C}\}$  where  $\lambda = \frac{z_2}{z_1} \in \mathbb{R}$ . Consider the smooth map  $\pi_A: \mathbb{C}^* \times \mathbb{C}^* \rightarrow \mathbb{R}^{>0}$  defined by  $\pi_A(w_1, w_2) := \frac{|w_1|^\lambda}{|w_2|}$ . The closure  $\overline{A}$  of  $A$  in  $T = \mathbb{C}^* \times \mathbb{C}^*$  is contained in the kernel of  $\pi_A$ . Since  $\overline{A} \cdot x$  is closed and non-compact in  $D^*$ , we conclude that  $\pi_A(a_1, a_2) = \frac{|a_1|^\lambda}{|a_2|}$  is closed and non-compact in  $\mathbb{R}^{>0}$ ; in particular we have  $|a_1|^\lambda \neq |a_2|$ . Now choose  $t_m \in \mathbb{C}$  such that  $a_1^m e^{\lambda t_m} = c$  for all  $m \in \mathbb{Z}$ . It follows that

$$\left| a_2^m e^{\lambda t_m} \right| = \left( \frac{|a_2|}{|a_1|^\lambda} \right)^m |c|^\lambda.$$

Since  $\frac{|a_2|}{|a_1|^\lambda} \neq 1$ , we see that  $(a_2^m e^{\lambda t_m})$  converges to 0 for  $m \rightarrow \infty$  or  $m \rightarrow -\infty$ . This proves again that the closure of  $A \cdot x$  in  $D$  contains  $E_1$  or  $E_2$ .  $\square$

Combining the Lemmas 5.2 and 2.3, we see that there exists a plurisubharmonic function  $\varphi$  on  $D$  which is strictly plurisubharmonic on  $D^*$ .

*Remark.* Due to [Ri68] (see the formulation given in [Dem12, Chapter I.5.E]), we may assume without loss of generality that  $\varphi$  is smooth on  $D^*$ .

As we have noted above,  $D$  cannot contain  $E_1$  or  $E_2$ , so that  $D \cap E_1$  and  $D \cap E_2$  are closed Stein submanifolds of  $D$ . Therefore, the following lemma implies that  $D$  is Stein, which then completes the proof of Theorem 3.2.

**Lemma 5.3.** *Let  $X$  be a connected complex manifold endowed with a plurisubharmonic exhaustion  $\varphi$ . Suppose that there exists a closed Stein submanifold  $A$  of  $X$  with at most finitely many connected components such that  $\varphi$  is strictly plurisubharmonic on  $X \setminus A$ . Then  $X$  is Stein.*

*Proof.* For  $\alpha \in \mathbb{R}$  write  $X_\alpha := \{\varphi < \alpha\}$ . In the first step we will show that  $X_n$  is Stein for all  $n \geq 1$ . Since  $A$  is Stein, every  $A_n := A \cap X_n$  is a closed Stein submanifold of  $X_n$ .

Due to [Siu77], we find an open Stein neighborhood  $U_{n+1}$  of  $A_{n+1}$  in  $X_{n+1}$ . Consequently, there exists a strictly plurisubharmonic exhaustion function  $\psi_{n+1}$  on  $U_{n+1}$ . Let us choose a relatively compact open neighborhood  $V_n$  of  $A_n$  in  $U_{n+1}$  as well as a cutoff function  $\chi_{n+1}$  which is identically 1 on  $V_n$  and which vanishes near  $\partial U_{n+1}$ . Then  $\chi_{n+1}\psi_{n+1}: X \rightarrow \mathbb{R}$  is strictly plurisubharmonic in a neighborhood of  $A_n$  and its Levi form is uniformly bounded from below on  $X_n$ . Moreover, let  $\rho_n$  be a plurisubharmonic exhaustion of  $X_n$  which is strictly plurisubharmonic and smooth on  $X_n \setminus A_n$ . Then, for  $k$  sufficiently large the function  $\chi_{n+1}\psi_{n+1}|_{X_n} + ke^{\rho_n}$  is a smooth strictly plurisubharmonic exhaustion function of  $X_n$ , proving that  $X_n$  is Stein.

Since we know now that  $X_n$  is Stein, we can apply [Nar62, Corollary 1] which implies that  $X_{n-1}$  is Runge in  $X_n$  for every  $n \geq 2$ . Therefore  $X = \bigcup_{n \geq 1} X_n$  is a Runge exhaustion of  $X$  by relatively compact Stein open subsets, hence  $X$  is Stein, see [GuRo65, Theorem VII.A.10].  $\square$

## REFERENCES

- [BHPV04] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven, *Compact complex surfaces*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 4, Springer-Verlag, Berlin, 2004.

- [Dem12] Jean-Pierre Demailly, *Complex Analytic and Differential Geometry*, at [www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf](http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf), version of June 21, 2012.
- [DF82] Klas Diederich and John Erik Fornæss, *A smooth pseudoconvex domain without pseudoconvex exhaustion*, *Manuscripta Math.* **39** (1982), no. 1, 119–123.
- [GMO13] Bruce Gilligan, Christian Miebach, and Karl Oeljeklaus, *Pseudoconvex domains spread over complex homogeneous manifolds*, *Manuscripta Math.* **142** (2013), no. 1-2, 35–59.
- [GuRo65] Robert C. Gunning and Hugo Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965.
- [Hir74] André Hirschowitz, *Pseudoconvexité au-dessus d'espaces plus ou moins homogènes*, *Invent. Math.* **26** (1974), 303–322.
- [Hir75] ———, *Le problème de Lévi pour les espaces homogènes*, *Bull. Soc. Math. France* **103** (1975), no. 2, 191–201.
- [Hu10] Alan T. Huckleberry, *Remarks on homogeneous complex manifolds satisfying Levi conditions*, *Boll. Unione Mat. Ital. (9)* **3** (2010), no. 1, 1–23.
- [Lel68] Pierre Lelong, *Fonctionnelles analytiques et fonctions entières (n variables)*, Les Presses de l'Université de Montréal, Montréal, Que., 1968, Séminaire de Mathématiques Supérieures, No. 13 (Été, 1967).
- [LY12] Norman Levenberg and Hiroshi Yamaguchi, *Pseudoconvex domains in the Hopf surface*, arXiv:1205.3346 (2012).
- [Nar62] Raghavan Narasimhan, *The Levi problem for complex spaces. II*, *Math. Ann.* **146** (1962), 195–216.
- [Ri68] Rolf Richberg, *Stetige streng pseudokonvexe Funktionen*, *Math. Ann.* **175** (1968), 257–286.
- [Siu77] Yum Tong Siu, *Every Stein subvariety admits a Stein neighborhood*, *Invent. Math.* **38** (1976/77), no. 1, 89–100.

LABORATOIRE DE MATHÉMATIQUES PURES ET APPLIQUÉES, CNRS-FR 2956,  
UNIVERSITÉ DU LITTORAL CÔTE D'OPALE, 50, RUE F. BUISSON, 62228 CALAIS  
CEDEX, FRANCE

*Email address:* miebach@lmpa.univ-littoral.fr