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► **To cite this version:**

Christian Miebach. Pseudoconvex non-Stein domains in primary Hopf surfaces. *Izvestiya: Mathematics*, 2014, 78 (5), pp.1028-1035. 10.1070/IM2014v078n05ABEH002717 . hal-03600354

HAL Id: hal-03600354

<https://hal.science/hal-03600354>

Submitted on 7 Mar 2022

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PSEUDOCONVEX NON-STEIN DOMAINS IN PRIMARY HOPF SURFACES

CHRISTIAN MIEBACH

ABSTRACT. We describe pseudoconvex non-Stein domains in primary Hopf surfaces using techniques developed by Hirschowitz.

1. INTRODUCTION

Let H be a primary Hopf surface. In [LY12] Levenberg and Yamaguchi characterize locally pseudoconvex domains $D \subset H$ having smooth real-analytic boundary that are not Stein. In this note we generalize their result to arbitrary pseudoconvex domains using ideas developed by Hirschowitz in [Hir74] and [Hir75]. For the readers' convenience these ideas are reviewed in a slightly generalized form in Section 2. In Section 3 we review the structure of primary Hopf surfaces in order to describe a certain (singular) holomorphic foliation \mathcal{F} of H . This allows us to formulate the following Main Theorem, which is proven in Sections 4 and 5.

Main Theorem. *Let $D \subset H$ be a pseudoconvex domain. If D is not Stein, then D contains with every point $p \in D$ the topological closure \bar{F}_p of the leaf $F \in \mathcal{F}$ passing through p .*

I would like to thank Karl Oeljeklaus for helpful discussions on the subject of this paper and Stefan Nemirovski for a suggestion on how to prove Lemma 5.3. I am also grateful to Peter Heinzner and the SFB/TR 12 for an invitation to the Ruhr-Universität Bochum where a part of this paper has been written.

2. A REVIEW OF HIRSCHOWITZ' METHODS

In this section we present the methods developed by Hirschowitz in [Hir75] in a slightly more general setup.

Let X be a complex manifold with holomorphic tangent bundle $TX \rightarrow X$, and let $\pi: \mathbb{P}TX \rightarrow X$ be the projectivized holomorphic tangent bundle. A continuous function on X is called *strictly plurisubharmonic* on X if it is everywhere locally the sum of a continuous plurisubharmonic and a smooth strictly plurisubharmonic function.

2010 *Mathematics Subject Classification.* 32M05 (primary); 32E40 (secondary).

Definition 2.1. Let $\varphi \in \mathcal{C}(X)$ be plurisubharmonic. Then we define $S(\varphi)$ to be the set of $[v] \in \mathbb{P}TX$ such that φ is in a neighborhood of $\pi[v]$ the sum of a plurisubharmonic function and a smooth function that is strictly plurisubharmonic on any germ of a holomorphic curve defining $[v]$.

Lemma 2.2. Let $\varphi \in \mathcal{C}(X)$ be plurisubharmonic.

- (1) The set $S(\varphi)$ is open in $\mathbb{P}TX$.
- (2) If $S(\varphi) = \mathbb{P}TX$, then φ is strictly plurisubharmonic on X .
- (3) If $\sum_k \varphi_k$ converges uniformly on compact subsets of X where $\varphi_k \in \mathcal{C}(X)$, then we have $S(\sum_k \varphi_k) \supset \bigcup_k S(\varphi_k)$.

Proof. This is [Hir75, Proposition 1.3]. □

For any plurisubharmonic function $\varphi \in \mathcal{C}(X)$ we define $C(\varphi)$ to be

$$\mathbb{P}TX \setminus \{[v] \in \mathbb{P}TX; \varphi \text{ is smooth around } \pi[v] \text{ and } \partial\varphi(v) \neq 0\}$$

and then set

$$(2.1) \quad C(X) := \bigcap_{\substack{\varphi \in \mathcal{C}(X) \\ \text{plurisubharmonic}}} C(\varphi).$$

Every set $C(\varphi)$ (and thus $C(X)$) is closed in $\mathbb{P}TX$. The next lemma is a slight generalization of [Hir75, Proposition 1.5].

Lemma 2.3. Let X be a complex manifold and let $\Omega := X \setminus \pi(C(X))$. Then there exists a plurisubharmonic function $\psi \in \mathcal{C}(X)$ which is strictly plurisubharmonic on Ω .

Proof. Since $\pi: \mathbb{P}TX \rightarrow X$ is proper, the set Ω is open in X . If Ω is empty, there is nothing to prove. Therefore let us suppose that Ω is a non-empty open subset of X . Consequently, $\mathbb{P}TX \setminus C(X)$ is non-empty.

For every $[v] \in \mathbb{P}TX \setminus C(X)$ we find a plurisubharmonic function $\varphi_{[v]} \in \mathcal{C}^\infty(X)$ with $\partial\varphi_{[v]}v \neq 0$. We claim that the function $\psi_{[v]} := \exp \circ \varphi_{[v]}$ is strictly plurisubharmonic in the direction of $[v]$. To see this, we calculate

$$\partial\bar{\partial}\psi_{[v]} = e^{\varphi_{[v]}} (\partial\varphi_{[v]} \wedge \bar{\partial}\varphi_{[v]} + \partial\bar{\partial}\varphi_{[v]}).$$

In other words, we obtain $[v] \in S(\psi_{[v]})$. Since X has countable topology, we get an open covering

$$\mathbb{P}TX \setminus C(X) \subseteq \bigcup_{k=1}^{\infty} S(\psi_k).$$

It is possible to find $\lambda_k > 0$ such that $\sum_{k=1}^{\infty} \lambda_k \psi_k$ converges uniformly on compact subsets of X . To prove this, choose a countable exhaustion $X = \bigcup_j K_j$ by compact sets with $K_j \subset \overset{\circ}{K}_{j+1}$. For every j there are

$\lambda_{k,j} > 0$ such that

$$\sum_{k=1}^{\infty} \lambda_{k,j} \|\psi_k\|_{K_j}$$

converges. Since $\|\psi_k\|_{K_j} \leq \|\psi_k\|_{K_{j+1}}$ for every j , we may suppose that $\lambda_{k,j'} \leq \lambda_{k,j}$ for all $j \leq j'$. Defining $\lambda_k := \lambda_{k,k}$ and noting that every compact subset $K \subset X$ is contained in K_{j_0} for some j_0 , we conclude

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda_k \|\psi_k\|_K &\leq \sum_{k=1}^{\infty} \lambda_k \|\psi_k\|_{K_{j_0}} \\ &\leq \sum_{k=1}^{j_0} \lambda_k \|\psi_k\|_{K_{j_0}} + \sum_{k=j_0+1}^{\infty} \lambda_{k,j_0} \|\psi_k\|_{K_{j_0}} < \infty \end{aligned}$$

which proves the claim. It follows that the limit function $\psi := \sum_k \psi_k$ is continuous and satisfies $S(\psi) \supset \bigcup_k S(\psi_k) \supset \mathbb{P}TX \setminus C(X)$, hence it is strictly plurisubharmonic on Ω . \square

In the following we say that a complex manifold X is *pseudoconvex* if there is a continuous plurisubharmonic exhaustion function $\rho: X \rightarrow \mathbb{R}^{>0}$.

Lemma 2.4. *Let X be a pseudoconvex complex manifold and let $\gamma: U \rightarrow X$ be the integral curve of a holomorphic vector field on X where U is a domain in \mathbb{C} . If $\gamma'(U)$ meets $C(X)$, then $\gamma'(U)$ is contained in $C(X)$. If X admits a smooth plurisubharmonic exhaustion function, then $\gamma'(U) \subset C(X)$ implies that $\gamma(U)$ is relatively compact in X . In particular, in this case we have $U = \mathbb{C}$.*

Proof. Let ζ be the holomorphic vector field on X with integral curve γ and suppose that $\gamma'(0) = \zeta(x_0) \in C(X)$. It is enough to show that 0 is an inner point of the set of $t \in U$ with $\gamma'(t) \in C(X)$, for then the closed set $(\gamma')^{-1}(C(X))$ is also open, hence equal to U . In other words, we must prove that for every plurisubharmonic function $\varphi \in \mathcal{C}(X)$ smooth in a neighborhood of $x_t := \gamma(t)$ we have $\zeta(\varphi)(x_t) = 0$ whenever $|t|$ is sufficiently small.

To do this, choose $\alpha \in \mathbb{R}^{>0}$ such that $x_0 = \gamma(t_0) \in X_\alpha := \{x \in X; \rho(x) < \alpha\}$ where ρ is a continuous plurisubharmonic exhaustion function of X . Let Φ_t^ζ be the holomorphic local flow of ζ . For $|t|$ sufficiently small we have

$$\Phi_t^\zeta(X_{\alpha+1}) \supset X_\alpha \ni x_t := \Phi_t^\zeta(x_0).$$

Since $\Phi_t^\zeta: X_\alpha \rightarrow X_{\alpha+1}$ is holomorphic, $\varphi_t := \varphi \circ \Phi_t^\zeta$ is continuous plurisubharmonic on X_α and smooth in a neighborhood of x_0 for each plurisubharmonic function $\varphi \in \mathcal{C}(X_{\alpha+1})$ that is smooth in a neighborhood of x_t . Following the proof of [Hir75, Proposition 1.6] we construct a continuous plurisubharmonic function ψ_t on X which

coincides with φ_t in a neighborhood of x_0 . Choose $\beta \in \mathbb{R}$ such that $\varphi_t(x_0) < \beta < \alpha$ and note that $K := \rho^{-1}(\beta) \subset X_\alpha$ is compact. Then choose a convex increasing function χ on \mathbb{R} fulfilling

$$\begin{aligned} \chi(\rho(x_0)) &< \varphi_t(x_0) \quad \text{and} \\ \chi(\beta) &> \|\varphi_t\|_K. \end{aligned}$$

Finally, define $\psi_t: X \rightarrow \mathbb{R}$ by

$$\psi_t(x) := \begin{cases} \max(\varphi_t(x), \chi \circ \rho(x)) & : \rho(x) \leq \beta \\ \chi \circ \rho(x) & : \rho(x) \geq \beta. \end{cases}$$

One checks directly that ψ_t is continuous plurisubharmonic and coincides with φ_t in some neighborhood of x_0 . Consequently, we may calculate

$$\zeta \varphi(x_t) = \frac{d}{ds} \Big|_t \varphi(\Phi_s^{\zeta}(x_0)) = \zeta \varphi_t(x_0) = \zeta \psi_t(x_0) = 0$$

since $\zeta_{x_0} = \gamma'(0) \in C(X)$. Therefore we see that $\gamma'(t) \in C(X)$ for every $t \in U$ sufficiently close to 0, which proves the first part of the lemma.

If ρ is smooth, then choosing $\varphi = \rho$ in the argument given above, we see that $\gamma(U)$ lies in a fiber of ρ , hence is relatively compact. \square

3. STATEMENT OF THE MAIN THEOREM

Let us fix $a_1, a_2 \in \mathbb{C}$ such that $0 < |a_1| \leq |a_2| < 1$. The automorphism $\varphi: \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{C}^2 \setminus \{0\}$, $(z_1, z_2) \mapsto (a_1 z_1, a_2 z_2)$, generates a free proper \mathbb{Z} -action on $\mathbb{C}^2 \setminus \{0\}$. By definition, the compact complex surface $H_a := (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$ for $a = (a_1, a_2)$ is a *primary Hopf surface*. We will write $[z_1, z_2] := \pi(z_1, z_2)$ where $\pi: \mathbb{C}^2 \setminus \{0\} \rightarrow H_a$ is the quotient map.

The torus $T = \mathbb{C}^* \times \mathbb{C}^*$ acts holomorphically on H_a with three orbits. More precisely, we have $H_a = E_1 \cup H_a^* \cup E_2$ where $H_a^* := (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}$ is the open T -orbit, and where $E_1 := (\mathbb{C}^* \times \{0\})/\mathbb{Z} = T \cdot [1, 0]$ and $E_2 := (\{0\} \times \mathbb{C}^*)/\mathbb{Z} = T \cdot [0, 1]$ are elliptic curves.

Note that H_a^* is a connected Abelian complex Lie group which thus can be represented as \mathbb{C}^2/Γ_3 where Γ_3 is a discrete subgroup of rank 3 of \mathbb{C}^2 . The map $p: \mathbb{C}^2 \rightarrow \mathbb{C}^2/\Gamma_3 \cong H_a^*$ is the universal covering of H_a^* . Let V be the real span of Γ_3 and set $W := V \cap iV$. There are two possibilities. Either $p(W)$ is dense in $V/\Gamma_3 \cong (S^1)^3$, or $p(W)$ is closed, hence compact, hence an elliptic curve E . In the first case, we have $\mathcal{O}(H_a^*) = \mathbb{C}$, i.e., H_a^* is a *Cousin group*, while in the second case $H_a^* \cong \mathbb{C}^* \times E$.

For the following result we refer the reader to [BHPV04, Chapter V.18].

Proposition 3.1. *The open orbit H_a^* is not Cousin if and only if $a_1^{k_1} = a_2^{k_2}$ for some relatively prime $k_1, k_2 \in \mathbb{Z}$.*

Remark. If there exist relatively prime integers k_1, k_2 with $a_1^{k_1} = a_2^{k_2}$, then we have the elliptic fibration $H_a \rightarrow \mathbb{P}_1, [z_1, z_2] \mapsto [z_1^{k_1} : z_2^{k_2}]$. The generic fiber is the elliptic curve $E = \mathbb{C}^*/(z \sim cz)$ where $c := a_1^{k_1} = a_2^{k_2}$. Note that for a generic choice of $a = (a_1, a_2)$ the open subset H_a^* is a Cousin group.

Suppose that H_a^* is Cousin and let $\xi \in \mathfrak{t}$ be the generator of the relatively compact one parameter subgroup $p(W)$. Let ξ_{H_a} be the holomorphic vector field induced by the T -action on H_a . One checks directly that ξ_{H_a} has no zeros in H_a , hence defines a holomorphic foliation of H_a . Note that the open subset H_a^* is saturated with respect to \mathcal{F} and that the leaves of $\mathcal{F}|_{H_a^*}$ are relatively compact in H_a^* . The closure of a leaf $F \subset H_a^*$ in H_a^* is a Levi-flat compact smooth hypersurface. In fact, these Levi-flat hypersurfaces are the fibers of the pluriharmonic function $[z_1, z_2] \mapsto \frac{\log|z_1|}{\log|a_1|} - \frac{\log|z_2|}{\log|a_2|}$ defined on H_a^* , see [LY12]. If H_a is elliptic, then it is foliated by elliptic curves. Again, H_a^* is saturated with respect to this foliation and the leaves are compact in H_a^* . This shows that in both cases we obtain a (singular) holomorphic foliation \mathcal{F} of H_a such that the leaves of $\mathcal{F}|_{H_a^*}$ are relatively compact in H_a^* .

We now state the main result of this note.

Theorem 3.2. *Let H_a be a primary Hopf surface and let $D \subset H_a$ be a pseudoconvex domain. If D is not Stein, then D contains with every point $p \in D$ the topological closure \overline{F}_p of the leaf $F \in \mathcal{F}$ passing through p .*

Remark. For locally pseudoconvex domains having smooth real-analytic boundary this result has been obtained by Levenberg and Yamaguchi using the theory of c -Robin functions, see [LY12].

4. EXISTENCE OF PLURISUBHARMONIC EXHAUSTIONS

In this section we will show that every smoothly bounded locally pseudoconvex domain $D \subset H_a$ admits a continuous plurisubharmonic exhaustion function. For this we will modify Hirschowitz' proof of [Hir74, Théorème 2.1].

Proposition 4.1. *Let $D \subset H_a$ be locally pseudoconvex and suppose that neither E_1 nor E_2 is a component of ∂D . Then D admits a continuous plurisubharmonic exhaustion function.*

Remark. The hypothesis of Proposition 4.1 is fulfilled if D is locally pseudoconvex and smoothly bounded. Hence, Theorem 3.2 indeed generalizes the main result of [LY12].

Proof. We define $\Omega := \{(x, \xi) \in D \times \mathfrak{t} ; \exp(\xi) \cdot x \in D\}$. By definition, Ω is an open subset of $D \times \mathfrak{t}$ containing $D \times \{0\}$. Since D is locally pseudoconvex in H_a , it follows that Ω is locally pseudoconvex in $D \times \mathfrak{t}$.

We define the boundary distance $d: D \rightarrow \mathbb{R}^{>0}$ by

$$d(x) := \sup\{r > 0; \{x\} \times B_r(0) \subset \Omega\}.$$

It is elementary to check that d is lower semicontinuous. Since Ω is locally pseudoconvex in $D \times T$, every point $x \in \overline{D}$ has an open Stein neighborhood U such that $\Omega \cap (U \times \mathfrak{t})$ is pseudoconvex. Due to a result of Lelong, see [Lel68, Theorem 2.4.2], the function $-\log d$ is plurisubharmonic on $U \cap D$ and therefore everywhere on D .

Note that $-\log d \equiv -\infty$ if and only if D contains H_a^* . Since neither E_1 nor E_2 is a component of ∂D , this implies $D = H_a$ so that we may exclude this case in the following.

For every $x \in H_a$ the orbit map $\mathfrak{t} \rightarrow T \cdot x, \xi \mapsto \exp(\xi) \cdot x$, is open into its image $T \cdot x$. Thus we see that $-\log d(x)$ goes to infinity as x approaches a point in ∂D . Since ∂D is compact, this implies that $-\log d$ is an exhaustion.

To end this proof, one verifies directly that d is upper semicontinuous in any point $x \in D$ such that $d(x) \neq \infty$. Therefore, for every constant $C > 0$ the map $\sup(C, -\log d)$ is a continuous plurisubharmonic exhaustion of D . \square

Remark. The proof of Proposition 4.1 shows that the polar set given by $\{-\log d = -\infty\}$ is non-empty if and only if D contains E_1 or E_2 .

Remark. In [DF82], Diederich and Fornæss give an example of a relatively compact pseudoconvex domain in a \mathbb{P}_1 -bundle over a Hopf surface that has smooth real-analytic boundary but does *not* admit an exhaustion by pseudoconvex subdomains.

5. PROOF OF THEOREM 3.2

Let us start by noting the following simple but important observation.

Lemma 5.1. *The domain H_a^* admits a smooth plurisubharmonic exhaustion function. Consequently, if $D \subset H_a$ is pseudoconvex, then $D^* := D \cap H_a^*$ is likewise pseudoconvex.*

If H_a^* is Cousin, this lemma follows from [Hu10, Proposition 2.4]. If H_a^* is not Cousin, then $H_a^* \cong \mathbb{C}^* \times E$ clearly has a smooth plurisubharmonic exhaustion.

Let $D \subset H_a$ be a pseudoconvex domain which is not Stein. If the pseudoconvex domain $D^* \subset H_a^*$ is not Stein, then D^* is saturated with respect to the foliation $\mathcal{F}|_{H_a^*}$, see [GMO13, Theorem 3.1]. Since

the leaves of \mathcal{F} are relatively compact orbits of a one parameter subgroup of T , continuity of the action map $T \times H_a \rightarrow H_a$ implies that D is saturated with respect to \mathcal{F} in this case. Therefore, let us assume that $D \neq D^*$ is Stein. We will complete the proof of Theorem 3.2 by showing that then D is Stein as well.

We note first that D cannot contain E_1 or E_2 if D^* is Stein. Indeed, due to the continuity of the leaves of \mathcal{F} remarked above, if E_1 was contained in D , then some of the relatively compact leaves of \mathcal{F} would lie in D^* , contradicting the assumption that D^* is Stein.

Let us consider the subset $C(D) \subset \mathbb{P}TD$ defined in (2.1) where $\pi: \mathbb{P}TD \rightarrow D$ is the projectivized tangent bundle. The proof of the following lemma relies essentially on the explicit knowledge of the structure of primary Hopf surfaces.

Lemma 5.2. *Let $D \subset H_a$ be a pseudoconvex domain. If $D^* = D \cap H_a^*$ is Stein, then we have $\pi(C(D)) \subset D^* \setminus D$.*

Proof. Suppose that $\pi(C(D))$ meets D^* . Since H_a^* is an Abelian complex Lie group, it has a biinvariant Haar measure. Therefore, we can apply the usual convolution technique in order to approximate the continuous plurisubharmonic exhaustion of D uniformly on compact subsets by smooth ones. This allows us to apply Lemma 2.4 to prove existence of a complex one parameter subgroup A of T and a point $x \in D^*$ such that $A \cdot x$ is relatively compact in D . Note that $A \cdot x$ is not relatively compact in D^* since the latter is assumed to be Stein.

We claim that that the closure of such a curve $A \cdot x$ in D (and hence D itself) would have to contain E_1 or E_2 , which then, as noted above, will contradict our assumption that D^* is Stein. In order to prove this claim, consider $A = \{(e^{tz_1}, e^{tz_2}); t \in \mathbb{C}\}$ where $z = (z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$. If $z_1 = 0$ or $z_2 = 0$, we see directly that the closure of $A \cdot x$ contains E_2 or E_1 . Hence, suppose that $z_1, z_2 \neq 0$. If $\frac{z_1}{z_2} \notin \mathbb{R}$, already the closure of A in $\mathbb{C}^2 \setminus \{0\}$ contains $\{z_1 = 0\} \cup \{z_2 = 0\}$, thus the closure of $A \cdot x$ in D contains $E_1 \cup E_2$ as well in this case. Therefore, we are left to deal with the case $A = \{(e^t, e^{\lambda t}); t \in \mathbb{C}\}$ where $\lambda = \frac{z_2}{z_1} \in \mathbb{R}$. Consider the smooth map $\pi_A: \mathbb{C}^* \times \mathbb{C}^* \rightarrow \mathbb{R}^{>0}$ defined by $\pi_A(w_1, w_2) := \frac{|w_1|^\lambda}{|w_2|}$. The closure \overline{A} of A in $T = \mathbb{C}^* \times \mathbb{C}^*$ is contained in the kernel of π_A . Since $\overline{A} \cdot x$ is closed and non-compact in D^* , we conclude that $\pi_A(a_1, a_2) = \frac{|a_1|^\lambda}{|a_2|}$ is closed and non-compact in $\mathbb{R}^{>0}$; in particular we have $|a_1|^\lambda \neq |a_2|$. Now choose $t_m \in \mathbb{C}$ such that $a_1^m e^{\lambda t_m} = c$ for all $m \in \mathbb{Z}$. It follows that

$$|a_2^m e^{\lambda t_m}| = \left(\frac{|a_2|}{|a_1|^\lambda} \right)^m |c|^\lambda.$$

Since $\frac{|a_2|}{|a_1|^\lambda} \neq 1$, we see that $(a_2^m e^{\lambda t_m})$ converges to 0 for $m \rightarrow \infty$ or $m \rightarrow -\infty$. This proves again that the closure of $A \cdot x$ in D contains E_1 or E_2 . \square

Combining the Lemmas 5.2 and 2.3, we see that there exists a plurisubharmonic function φ on D which is strictly plurisubharmonic on D^* .

Remark. Due to [Ri68] (see the formulation given in [Dem12, Chapter I.5.E]), we may assume without loss of generality that φ is smooth on D^* .

As we have noted above, D cannot contain E_1 or E_2 , so that $D \cap E_1$ and $D \cap E_2$ are closed Stein submanifolds of D . Therefore, the following lemma implies that D is Stein, which then completes the proof of Theorem 3.2.

Lemma 5.3. *Let X be a connected complex manifold endowed with a plurisubharmonic exhaustion φ . Suppose that there exists a closed Stein submanifold A of X with at most finitely many connected components such that φ is strictly plurisubharmonic on $X \setminus A$. Then X is Stein.*

Proof. For $\alpha \in \mathbb{R}$ write $X_\alpha := \{\varphi < \alpha\}$. In the first step we will show that X_n is Stein for all $n \geq 1$. Since A is Stein, every $A_n := A \cap X_n$ is a closed Stein submanifold of X_n .

Due to [Siu77], we find an open Stein neighborhood U_{n+1} of A_{n+1} in X_{n+1} . Consequently, there exists a strictly plurisubharmonic exhaustion function ψ_{n+1} on U_{n+1} . Let us choose a relatively compact open neighborhood V_n of A_n in U_{n+1} as well as a cutoff function χ_{n+1} which is identically 1 on V_n and which vanishes near ∂U_{n+1} . Then $\chi_{n+1}\psi_{n+1}: X \rightarrow \mathbb{R}$ is strictly plurisubharmonic in a neighborhood of A_n and its Levi form is uniformly bounded from below on X_n . Moreover, let ρ_n be a plurisubharmonic exhaustion of X_n which is strictly plurisubharmonic and smooth on $X_n \setminus A_n$. Then, for k sufficiently large the function $\chi_{n+1}\psi_{n+1}|_{X_n} + ke^{\rho_n}$ is a smooth strictly plurisubharmonic exhaustion function of X_n , proving that X_n is Stein.

Since we know now that X_n is Stein, we can apply [Nar62, Corollary 1] which implies that X_{n-1} is Runge in X_n for every $n \geq 2$. Therefore $X = \bigcup_{n \geq 1} X_n$ is a Runge exhaustion of X by relatively compact Stein open subsets, hence X is Stein, see [GuRo65, Theorem VII.A.10]. \square

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