Classification of spherical algebraic subalgebras of real simple Lie algebras of rank 1

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LISA KNAUSS AND CHRISTIAN MIEBACH

Abstract. We determine all spherical algebraic subalgebras in any simple Lie algebra of real rank 1.

1. Introduction

Let $U^{\mathbb{C}}$ be a complex reductive group with maximal compact subgroup U. It has been proved in [Bri87a] (see also [HW90]) that smooth compact complex spherical $U^{\mathbb{C}}$-varieties Z may be characterized by the fact that a moment map $\mu\colon Z \to \mathfrak{u}^*$ separates the U-orbits in Z.

In [HS07] it has been shown that the so-called gradient maps are the right analogue for moment maps when one is interested in actions of a real reductive group $G = K\exp(\mathfrak{p})$. Spherical gradient manifolds have been introduced in [MS10] in order to carry over Brion's theorem to the real reductive case. To be more precise, we call a G-gradient manifold $X \subset Z$ with gradient map $\mu_{\mathfrak{p}}\colon X \to \mathfrak{p}$ spherical if a minimal parabolic subgroup of G has an open orbit in X. If G is connected complex reductive, then a minimal parabolic subgroup is the same as a Borel subgroup of G, so that there is no ambiguity in this definition. The main result of [MS10] states that X is spherical if and only if $\mu_{\mathfrak{p}}$ almost separates the K-orbits in X.

Recently, real spherical manifolds have attracted attention from the representation theoretical view point (see [KO13] and [KS16b]) as well as from a geometric one (see [KS16a] and [KKS15]). In [Bie93] and [KS16a] the authors have shown that, given a homogeneous real spherical manifold $X = G/H$, any minimal parabolic subgroup of G has only finitely many orbits in X. Moreover, the paper [KS16a] contains the list of all reductive spherical subalgebras of $\mathfrak{g} = \mathfrak{so}(n, 1)$. In [Mat79] the author has found *all* decompositions of $\mathfrak{so}(n, 1)$ as the sum of two subalgebras. In [KKPS16] the authors classify the reductive spherical subalgebras of arbitrary simple real Lie algebras.

As the main result of this paper we describe the *non-reductive* spherical algebraic subalgebras of $\mathfrak{g}$ where $\mathfrak{g}$ is a simple Lie algebra of real rank 1 by methods in the spirit of [MS10]. We then apply this result to classify the reductive spherical subalgebras of $\mathfrak{g}$, thus obtaining a second proof of the rank one case in [KKPS16]. Although a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is spherical whenever $\mathfrak{h}^{\mathbb{C}}$ is a complex spherical subalgebra of $\mathfrak{g}^{\mathbb{C}}$, we would like to stress the fact that the converse is not true (see the example in Section 3). In particular one cannot reduce the question to the complex classification. In contrast to the complex case, there are continuous families of spherical subalgebras of $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{su}(n, 1)$ the geometry of such a family was studied in detail by means of an explicit slice model in [Kna16].

Let us outline the main steps of the proof as well as the organization of this paper. In Section 2 we show that the homogeneous manifold $X = G/H$ admits a G-gradient map if and only if H is an algebraic subgroup of G. This is the reason why we classify spherical *algebraic* subalgebras. In Section 3 we characterize reductive and non-reductive spherical algebraic subgroups H of G by the fact that a maximal compact subgroup of H acts transitively on the spheres in a certain representation related to the inclusion $H \hookrightarrow G$. More precisely, our starting point is the following, see Propositions 3.1 and 3.2.

Proposition 1.1. *Let $G = K\exp(\mathfrak{p})$ be a connected simple Lie group of real rank 1 with Iwasawa decomposition $G = KAN$. Let $M := \mathcal{N}_K(\mathfrak{a})$.*

(1) Let $H = K_H\exp(\mathfrak{p}_H)$ be a reductive algebraic subgroup of G. Then H is spherical if and only if K_H acts transitively on the connected components of the spheres in $\mathfrak{p}_H^\perp$.
(2) Let $H = M_HA_HN_H$ be a non-reductive algebraic subgroup of G. If $N_H = N$, then H is spherical. If $\dim N/N_H \geq 1$, then H is spherical if and only if $A_H = A$ and M_H acts transitively on the connected components of the spheres in $\mathfrak{n}_H^\perp$.

Suppose that $H = M_HA_HN_H$ is a non-reductive spherical algebraic subgroup of G with $\dim N/N_H \geq 1$. Let us write $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ where $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$ are the restricted root spaces with respect to the maximal Abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Let us fix an M-invariant scalar product on $\mathfrak{g}_\alpha$. Then the Lie algebra of $H = M_HA_HN_H$ is of the form $\mathfrak{h} = \mathfrak{m}_H \oplus \mathfrak{a} \oplus W^\perp \oplus \mathfrak{g}_{2\alpha}$ where $W \subset \mathfrak{g}_\alpha$ is an M_H-stable subspace such that M_H acts transitively on the connected components of the spheres in W, see Corollary 3.3. Thus, in a second step we will determine the subspaces W of $\mathfrak{g}_\alpha$ such that $\mathcal{N}_M(W)$, which contains M_H, acts irreducibly on W. It turns out that in many cases this already implies that $\mathcal{N}_M(W)$ acts transitively on the connected components of the spheres in W. In the final step, we make use of Onishchik's classification of transitive actions on spheres in order to find all subgroups of $\mathcal{N}_M(W)$ that still act transitively on the connected components of the spheres in W. More details of this general scheme are given in Section 4. In the remaining Sections 5 to 8 we carry out our program case by case for $\mathfrak{so}(n,1)$, $\mathfrak{su}(n,1)$, $\mathfrak{sp}(n,1)$ and $\mathfrak{f}_4$. The tables containing all spherical algebraic subalgebras are given in Theorems 5.2 and 5.4 for $\mathfrak{g} = \mathfrak{so}(n,1)$, in Theorems 6.7 and 6.8 for $\mathfrak{g} = \mathfrak{su}(n,1)$, in Theorems 7.14 and 7.15 for $\mathfrak{g} = \mathfrak{sp}(n,1)$, and in Theorems 8.2 and 8.6 for the exceptional Lie algebra $\mathfrak{g} = \mathfrak{f}_4 = \mathfrak{f}_{4(-20)}$.

After this paper was finished, we learned that Kimelfeld has considered the classification problem of algebraic spherical subgroups in real simple Lie groups $G = K\exp(\mathfrak{p})$ of rank 1, too. In [Kim87] he obtains Proposition 1.1 by differential geometric arguments based on the Karpelevich compactification of the hyperbolic space G/K and then gives a general description of all spherical algebraic subgroups of G. However, he does not provide an explicit list of all spherical algebraic subalgebras of $\mathfrak{g}$. Bien obtains an explicit but not complete list in [Bie93].

Acknowledgements. We would like to thank Peter Heinzner and Valdemar Tsanov for helpful discussions on the subject presented here. We are much obliged to Friedrich Knop for informing us about Kimelfeld's paper [Kim87] as well as about an inaccuracy in the statement of Theorem 5.2 in an earlier version of this manuscript. The first author gratefully acknowledges the financial support by SPP 1388 "Darstellungstheorie" and SFB/TR 12 "Symmetries and Universality in Mesoscopic Systems" of the DFG.

2. (Homogeneous) gradient manifolds

Let G be a connected semisimple Lie group that embeds as a closed subgroup into its universal complexification $G^{\mathbb{C}}$. Let U be a compact real form of $G^{\mathbb{C}}$ such that G is stable under the corresponding Cartan involution of $G^{\mathbb{C}} = U^{\mathbb{C}}$. Then we obtain the Cartan decomposition $K \times \mathfrak{p} \to G$, $(k, \xi) \mapsto k\exp(\xi)$, where $K := G \cap U$ is a maximal compact subgroup of G and where $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$.

By a *G-gradient manifold* we mean the following. Let Z be a Kähler manifold endowed with a holomorphic $G^{\mathbb{C}}$-action and U-invariant Kähler form ω such that there exists a U-equivariant moment map $\mu\colon Z \to \mathfrak{u}^*$ for the U-action on (Z, ω). We call such a Z a *Hamiltonian $G^{\mathbb{C}}$-manifold.* Basic examples are given by $G^{\mathbb{C}}$-stable complex submanifolds of some projective space $\mathbb{P}(V)$ where V is a finite-dimensional complex $G^{\mathbb{C}}$-representation space. In particular, homogeneous algebraic $G^{\mathbb{C}}$-varieties are Hamiltonian. Identifying $\mathfrak{u}^*$ with $i\mathfrak{u}$ and composing μ with the orthogonal projection to $\mathfrak{p} \subset i\mathfrak{u}$ with respect to a U-invariant inner product, we obtain the K-equivariant *gradient map* $\mu_{\mathfrak{p}}\colon Z \to \mathfrak{p}$. Any G-stable closed real submanifold X of Z is called a G-gradient manifold with gradient map $\mu_{\mathfrak{p}}|_X$. For more details and the basic properties of gradient maps we refer the reader to [HS07] and [HSS08].

Proposition 2.1. *Let G be a connected semisimple Lie group and let H be a closed subgroup of G. Then $X = G/H$ is a G-gradient manifold if and only if H is algebraic, i.e., if $\overline{H} \cap G = H$ holds for the Zariski closure $\overline{H}$ of H in $G^{\mathbb{C}}$.*

Proof. Suppose first that $\overline{H} \cap G = H$ holds. It follows that $X = G/H$ is a real submanifold of the homogeneous space $G^{\mathbb{C}}/\overline{H}$. Since the latter is quasi-projective, it is in particular Kähler, and since $G^{\mathbb{C}}$ is semisimple, there exists a unique U-equivariant moment map on $G^{\mathbb{C}}/\overline{H}$. Then the construction described above yields a G-gradient map on $X = G/H$.

If $X = G/H$ is a gradient manifold, then by definition there exists a G-equivariant diffeomorphism $G/H \cong G \cdot z \subset Z$ where Z is a Hamiltonian $G^{\mathbb{C}}$-manifold. From this we obtain $G/H \hookrightarrow G^{\mathbb{C}}/(G^{\mathbb{C}})_z$, and $G^{\mathbb{C}}/(G^{\mathbb{C}})_z$ is again Kähler. Therefore $(G^{\mathbb{C}})_z$ is an algebraic subgroup of $G^{\mathbb{C}}$, see [GMO11, Corollary 4.12], hence contains $\overline{H}$. This implies $\overline{H} \cap G \subset (G^{\mathbb{C}})_z \cap G = G_z = H$, as was to be shown. □

3. Characterization of spherical homogeneous gradient manifolds

As in the previous section let $G = K \exp(\mathfrak{p})$ be a connected semisimple Lie group which embeds into its complexification $G^{\mathbb{C}}$. Let $H \subset G$ be a closed subgroup such that $X = G/H$ is a G-gradient manifold with gradient map $\mu_{\mathfrak{p}}\colon X \to \mathfrak{p}$. We say that $X = G/H$ is *spherical* if a minimal parabolic subgroup of G has an open orbit in X. In this case we call H a spherical subgroup of G and $\mathfrak{h}$ a spherical subalgebra of $\mathfrak{g}$. As shown in [MS10] sphericity of X is equivalent to the fact that $\mu_{\mathfrak{p}}$ almost separates the K-orbits in X, i.e., that the map $X/K \to \mathfrak{p}/K$ induced by $\mu_{\mathfrak{p}}\colon X \to \mathfrak{p}$ has discrete fibers.

Example. Any symmetric subalgebra of $\mathfrak{g}$ is spherical in $\mathfrak{g}$, see [MS10, §6.1].

It is not hard to see that, if $\mathfrak{h}^{\mathbb{C}}$ is spherical in $\mathfrak{g}^{\mathbb{C}}$ in the usual sense, then $\mathfrak{h}$ is spherical in $\mathfrak{g}$. However, as the following examples show, the converse does not hold, i.e., there are more spherical subalgebras of $\mathfrak{g}$ than just real forms of complex spherical subalgebras of $\mathfrak{g}^{\mathbb{C}}$.

Example. The unipotent radical N of a minimal parabolic subgroup of G is always spherical in G. However, in general $N^{\mathbb{C}}$ is not a spherical subgroup of $G^{\mathbb{C}}$. As a concrete example one may take $G = \mathrm{SO}^{\circ}(5,1)$.

A semisimple example is given by the spherical subgroup $H = \mathrm{Sp}(1,1)$ of $G = \mathrm{Sp}(2,1)$ (see Theorem 7.15) since $H^{\mathbb{C}} = \mathrm{Sp}(2,\mathbb{C})$ is not spherical in $G^{\mathbb{C}} = \mathrm{Sp}(3,\mathbb{C})$, see [Krä79].

For the rest of this paper we assume that *G is simple and has real rank* 1. Then $\mathfrak{g}$ is isomorphic to either $\mathfrak{so}(n,1)$ $(n \geq 3)$ or $\mathfrak{su}(n,1)$ $(n \geq 1)$ or $\mathfrak{sp}(n,1)$ $(n \geq 2)$ or the exceptional Lie algebra $\mathfrak{f}_4 = \mathfrak{f}_{4(-20)}$, see [Kna02, Chapter VI.11].

As our main result we will describe the algebraic spherical subalgebras of $\mathfrak{g}$ up to conjugation by an element of G, i.e., those subalgebras $\mathfrak{h}$ for which there exists a closed subgroup $H \subset G$ having $\mathfrak{h}$ as Lie algebra such that $X = G/H$ is a spherical gradient manifold.

For the classification we distinguish the cases that H is reductive or not. If H is reductive, then according to [Vin94, Theorem 6.3.6], after conjugation by an element of G, we have a Cartan decomposition $H = K_H \exp(\mathfrak{p}_H)$ where $K_H := H \cap K$ is a maximal compact subgroup of H and where $\mathfrak{p}_H := \mathfrak{h} \cap \mathfrak{p}$ is a K_H-invariant subspace of $\mathfrak{p}$ with $[\mathfrak{p}_H, \mathfrak{p}_H] \subset \mathfrak{k}_H$. We write $\mathfrak{p} = \mathfrak{p}_H \oplus \mathfrak{p}_H^\perp$ with respect to the K-invariant inner product on $\mathfrak{p}$ that comes from the U-invariant inner product on $i\mathfrak{u}$. In this situation, sphericity of $X = G/H$ has been characterized in [MS10, Proposition 6.1]. Under the additional assumption $\mathrm{rk}_{\mathbb{R}} G = 1$ we can make the following more precise statement.

Proposition 3.1. *Let $G = K\exp(\mathfrak{p})$ be a connected simple Lie group of real rank 1 and let $H = K_H \exp(\mathfrak{p}_H) \subset G$ be a closed reductive subgroup. Then $X = G/H$ is spherical if and only if K_H has an open orbit in every sphere in $\mathfrak{p}_H^\perp \subset \mathfrak{p}$.*

Remark. Except for $\operatorname{codim} \mathfrak{p}_H = 1$ Proposition 3.1 says that $X = G/H$ is spherical if and only if K_H acts transitively on the spheres in $\mathfrak{p}_H^\perp$. In particular, X can only be spherical if the K_H-representation on $\mathfrak{p}_H^\perp$ is irreducible.

Proof of Proposition 3.1. By [MS10, Theorem 1.1] the homogeneous gradient manifold $X = G/H$ is spherical if and only if any gradient map on it almost separates the K-orbits. The Mostow decomposition (see [HS07] for a proof using gradient maps) exhibits X as K-equivariantly isomorphic to the twisted product $K \times_{K_H} \mathfrak{p}_H^\perp$ [1]. A particular gradient map is given by $\mu_{\mathfrak{p}}[k, \xi] = -\operatorname{Ad}(k)\xi$ for $k \in K$ and $\xi \in \mathfrak{p}_H^\perp$. Since the K-orbits in $K \times_{K_H} \mathfrak{p}_H^\perp$ correspond to the K_H-orbits in $\mathfrak{p}_H^\perp$, this gradient map $\mu_{\mathfrak{p}}$ separates the K-orbits if and only if the map $\mathfrak{p}_H^\perp/K_H \to \mathfrak{p}/K$, induced by the inclusion $\mathfrak{p}_H^\perp \hookrightarrow \mathfrak{p}$, has discrete fibers. Since these fibers are precisely the K_H-orbits in the compact sets $(K \cdot \xi) \cap \mathfrak{p}_H^\perp$ with $\xi \in \mathfrak{p}_H^\perp$, we note in particular that the fibers have to be finite.

In the case $\mathrm{rk}_{\mathbb{R}} G = 1$, the K-orbits in $\mathfrak{p}$ are spheres, so their intersections with any subspace of $\mathfrak{p}$ are again spheres and in particular connected (unless the subspace is a line). Thus, X is spherical if and only if K_H has an open orbit in every sphere $(K \cdot \xi) \cap \mathfrak{p}_H^\perp$, $\xi \in \mathfrak{p}_H^\perp$, hence in any sphere in $\mathfrak{p}_H^\perp$. $\square$

In the rest of this section we give a similar criterion for non-reductive H. For this we fix a minimal parabolic subalgebra $\mathfrak{q}_0 = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{g}$ such that $\mathfrak{a}$ is a maximal Abelian subspace (i.e., a line) of $\mathfrak{p}$. The corresponding group is $Q_0 = MAN$ where $M := \mathcal{Z}_K(\mathfrak{a})$. Let $\mathfrak{h}$ be a non-reductive algebraic subalgebra of $\mathfrak{g}$ and let H be a corresponding subgroup of G. As is shown in [BT71] (compare [KS16a, Lemma 3.1]), after conjugation by an element of G we may assume that $\mathfrak{h} = \mathfrak{l}_H \oplus \mathfrak{n}_H$ where $\mathfrak{l}_H \subset \mathfrak{m} \oplus \mathfrak{a}$ is reductive in $\mathfrak{g}$ and $\{0\} \neq \mathfrak{n}_H \subset \mathfrak{n}$ is a nilpotent ideal of $\mathfrak{h}$. On the group level we have $H \cong L_H \ltimes N_H$ with a reductive group $L_H = M_H A_H \subset MA$ and $N_H \subset N$. Note that L_H acts by conjugation on N and stabilizes N_H, hence acts on N/N_H. On the Lie algebra level we have the decomposition $\mathfrak{n} = \mathfrak{n}_H \oplus \mathfrak{n}_H^\perp$ of $\mathfrak{n}$ as an M_H-module.

The following result is proven in [KS16a, Lemma 3.2], compare also [Bri87b, Proposition 1.1]. For the reader's convenience we repeat the argument here.

Proposition 3.2. *Let G be a connected simple Lie group of real rank 1 and let $X = G/H$ be a G-gradient manifold such that $H = M_H A_H N_H$ is non-reductive. Then X is spherical if and only if either*

(1) $N_H = N$ and $L_H = M_H A_H$ is arbitrary, or
(2) $\dim N/N_H \geq 1$ and $A_H = A$ and $M_H \subset \mathcal{N}_M(\mathfrak{n}_H)$ acts transitively on the connected components of the spheres in $\mathfrak{n}_H^\perp$.

[1] If H is a subgroup of G and Y is a set on which H acts, then the twisted product $G \times_H Y$ is defined as the quotient of $G \times Y$ with respect to the diagonal H-action $h \cdot (g, y) := (gh^{-1}, h \cdot y)$.

Remark. Every algebraic subgroup $H \subset G$ that contains N is spherical. Therefore, we will concentrate on the case $\dim \mathfrak{n}_H^{\perp} \geq 1$.

Proof of Proposition 3.2. Let H be a non-reductive algebraic subgroup of G of the form $H = M_H A_H N_H \subset Q_0 = MAN$. Since we have the G-equivariant fiber bundle $X = G/H \to G/Q_0$, we see that X is G-equivariantly diffeomorphic to the twisted product $G \times_{Q_0} (Q_0/H)$. By [Hel01, Corollary IX.1.8] the unique open orbit of the opposite minimal parabolic subgroup $Q_0^- = MAN^-$ in G/Q_0 is $Q_0^- \cdot eQ_0$. This implies that $X = G/H$ is spherical if and only if there is some $xH \in Q_0/H$ such that $Q_0^- \cdot [e, xH]$ is open in $G \times_{Q_0} (Q_0/H)$. The latter is the case if and only if $L := MA = Q_0^- \cap Q_0$ has an open orbit in Q_0/H. Using the fact that N_H is normal in H, we see that Q_0/H is L-equivariantly diffeomorphic to $L \times_{L_H} (N/N_H)$ where L_H acts by conjugation on N/N_H. This proves that $X = G/H$ is spherical if and only if L_H has an open orbit in N/N_H.

If $A_H \neq A$ then $L_H = M_H$ can only have an open orbit in N/N_H if the latter is compact forcing $\mathfrak{n} = \mathfrak{n}_H$. In particular if $\dim N/N_H \geq 1$ then $X = G/H$ can only be spherical if $A_H = A$. The L_H-equivariant diffeomorphism $\exp : \mathfrak{n} \to N$ induces via $\xi \mapsto \exp(\xi)N_H$ an L_H-equivariant map from $\mathfrak{n}_H^{\perp}$ to N/N_H. It follows from [Hel00, Lemma IV.6.8] that this map is a diffeomorphism. Therefore, if $A_H = A$ holds, sphericity of X is equivalent to the fact that L_H has an open orbit in $\mathfrak{n}_H^{\perp}$.

If $\dim \mathfrak{n}_H^{\perp} = 1$, then $L_H = M_H A$ has an open orbit for any $M_H \subset \mathcal{N}_M(\mathfrak{n}_H)$. Since every A-orbit in $\mathfrak{n}_H^{\perp}$ intersects any sphere in $\mathfrak{n}_H^{\perp}$ precisely once, the claim follows. $\square$

As we have seen, in order to classify non-reductive spherical algebraic subalgebras $\mathfrak{h} \subset \mathfrak{g}$ we may assume without loss of generality $\dim \mathfrak{n}_H^{\perp} \geq 1$. In particular, M_H must act irreducibly on $\mathfrak{n}_H^{\perp}$ if $X = G/H$ is spherical. In closing this section, we state the following corollary of Proposition 3.2 which exploits this observation a little further.

Corollary 3.3. *Let $H = L_H N_H$ be a spherical non-reductive algebraic subgroup of G. Then $\mathfrak{n}_H = (\mathfrak{n}_H \cap \mathfrak{g}_\alpha) \oplus \mathfrak{g}_{2\alpha}$.*

Conversely, for every subspace $W \subset \mathfrak{g}_\alpha$ and for every algebraic subalgebra $\mathfrak{m}_H \subset \mathcal{N}_{\mathfrak{m}}(W)$ the direct sum $\mathfrak{h}_W := \mathfrak{m}_H \oplus \mathfrak{a}_H \oplus W^{\perp} \oplus \mathfrak{g}_{2\alpha}$ is an algebraic subalgebra of $\mathfrak{g}$. Moreover, $\mathfrak{h}_W$ is spherical if and only if $M_H A_H$ has an open orbit in W.

Proof. We have $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ according to the restricted root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$ where α is a simple restricted root. If 2α is not a restricted root, we set $\mathfrak{g}_{2\alpha} = 0$. (Note that this is only the case for $\mathfrak{g} = \mathfrak{so}(n, 1)$.) Suppose that $X = G/H$ is spherical with $\dim \mathfrak{n}_H^{\perp} \geq 1$. Since $A \subset L_H$ acts with two different weights on $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$, we obtain

$$\mathfrak{n}_H^{\perp} = (\mathfrak{n}_H^{\perp} \cap \mathfrak{g}_\alpha) \oplus (\mathfrak{n}_H^{\perp} \cap \mathfrak{g}_{2\alpha}).$$

Consequently, M_H acts irreducibly on $\mathfrak{n}_H^{\perp}$ only if $\mathfrak{n}_H$ contains $\mathfrak{g}_\alpha$ or $\mathfrak{g}_{2\alpha}$. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = \mathfrak{g}_{2\alpha}$, see [Hel01, p. 408], and since $\mathfrak{n}_H$ is a Lie algebra, $\mathfrak{g}_{2\alpha} \subset \mathfrak{n}_H$ must hold. $\square$

4. Strategy of the classification

Let us outline the principal steps that will lead to the classification result. Recall that $G = K \exp(\mathfrak{p})$ is a connected simple Lie group of real rank 1 that embeds into $G^{\mathbb{C}}$.

Let $\mathfrak{h} \subset \mathfrak{g}$ be an algebraic subalgebra which we assume first to be non-reductive. Motivated by Proposition 3.2 and Corollary 3.3 we will first determine the real subspaces $W \subset \mathfrak{g}_\alpha$ such that $\mathcal{N}_M(W)$ acts irreducibly on W. Then we will bring W (by an element in M) into a suitable normal form and calculate $\mathcal{N}_M(W)$. This step will be carried out case-by-case for every simple Lie algebra of real rank one. Let us therefore identify the relevant representations in each case.

Remark 4.1. Suppose first that $G = \mathrm{SO}^\circ(n,1)$. The K-action on $\mathfrak{p}$ is isomorphic to the defining representation of $K \cong \mathrm{SO}(n)$ on $\mathfrak{p} \cong \mathbb{R}^n$. Moreover, the Lie algebra $\mathfrak{n} = \mathfrak{g}_\alpha$ is Abelian and the action of $M \cong \mathrm{SO}(n-1)$ on $\mathfrak{n} \cong \mathbb{R}^{n-1}$ is again isomorphic to the defining representation of $\mathrm{SO}(n-1)$.

If $G = \mathrm{SU}(n,1)$, then we can choose

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} ;\ A \in \mathrm{U}(n), a = \det(A)^{-1} \right\} \cong S(\mathrm{U}(n) \times \mathrm{U}(1)).$$

The group $K \cong S(\mathrm{U}(n) \times \mathrm{U}(1))$ acts on $\mathfrak{p} \cong \mathbb{C}^n$ by $(A,a) \cdot v = Ava^{-1}$. The group M is a 2-to-1-covering of $S(\mathrm{U}(n-1) \times \mathrm{U}(1))$. Its action on $\mathfrak{g}_\alpha \cong \mathbb{C}^{n-1}$ factorizes through the covering map $M \to S(\mathrm{U}(n-1) \times \mathrm{U}(1))$, and the action of $S(\mathrm{U}(n-1) \times \mathrm{U}(1))$ on $\mathfrak{g}_\alpha$ is given by the analogous formula.

Suppose now that $G = \mathrm{Sp}(n,1)$. The group $K \cong \mathrm{Sp}(n) \times \mathrm{Sp}(1)$ acts on $\mathfrak{p} \cong \mathbb{H}^n$ by $(A,a) \cdot v := Ava^{-1}$. The group $M \cong \mathrm{Sp}(1) \times \mathrm{Sp}(n-1)$ acts in the same way on $\mathfrak{g}_\alpha \cong \mathbb{H}^{n-1}$.

If $G = F_4$, then the K-action on $\mathfrak{p}$ is isomorphic to the unique irreducible representation of $\mathrm{Spin}(9)$ on $\mathbb{R}^{16}$. The M-action on $\mathfrak{g}_\alpha$ is equivalent to the unique irreducible representation of $\mathrm{Spin}(7)$ on $\mathbb{R}^8$, see Lemma 8.3.

In the second step we single out those subgroups of $\mathcal{N}_M(W)$ that do indeed act transitively on the spheres in W. In order to do so, we use results of Montgomery and Samelson as well as of Onishchik which we recall here for the reader's convenience.

Montgomery and Samelson considered the case of a connected compact Lie group L which acts transitively and effectively on S^n and obtained the following result, see [MS43, Theorem I].

Theorem 4.2. *Let L be a connected compact Lie group acting transitively and effectively on S^n. If n is even, then L is simple, while for n odd L is either simple or finitely covered by $L_1 \times L_2$, where L_2 is either $\mathrm{SO}(2)$ or $\mathrm{Sp}(1)$ and L_1 is simple and acts already transitively on S^n.*

In the same paper Montgomery and Samelson found all simple, compact, connected groups acting transitively on S^n for almost all n (see [MS43, Theorem II-IV]). In the case that n is even, their result was sharpened in [Bor49]. Given any compact group G acting transitively and effectively on some homogeneous space, Onishchik found all subgroups that also act transitively in [Oni94, Theorem 4.1]. This enabled him to find all transitive effective actions of connected compact Lie groups on S^n for all n. (see [Oni94, Theorem 3 in §18.3]).

We recall his result in the following theorem where we consider the defining representations of $\mathrm{O}(n)$ on $\mathbb{R}^n$, of $\mathrm{U}(n)$ on $\mathbb{C}^n$ and of $\mathrm{Sp}(n)$ on $\mathbb{H}^n$.

Theorem 4.3. *Let K be either $\mathrm{O}(n)$ or $\mathrm{U}(n)$ or $\mathrm{Sp}(n)$ and let V denote the respective defining representation of K. The following table lists all connected proper subgroups L of K that act transitively on the spheres in V, up to conjugation in K, where $p\colon \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$ is the universal covering and $L_2 \subset \mathrm{Sp}(1)$ is an arbitrary connected subgroup.*

K	L
$\mathrm{U}(2k+1), k \geq 1$	$\mathrm{SU}(2k+1)$
$\mathrm{U}(2k), k \geq 2$	$\mathrm{SU}(2k),\ \mathrm{Sp}(k) \times \mathrm{U}(1),\ \mathrm{Sp}(k)$
$\mathrm{O}(2k+1), k \geq 0$	$\mathrm{SO}(2k+1)$
$\mathrm{O}(4k+2), k \geq 1$	$\mathrm{SO}(4k+2),\ \mathrm{U}(2k+1),\ \mathrm{SU}(2k+1)$
$\mathrm{O}(4k), k = 3, k \geq 5$	$\mathrm{SO}(4k),\ \mathrm{U}(2k),\ \mathrm{SU}(2k),\ \mathrm{Sp}(k) \times \mathrm{Sp}(1),\ \mathrm{Sp}(k) \times \mathrm{U}(1),\ \mathrm{Sp}(k)$

O(16)	SO(16), U(8), SU(8), Sp(4) × Sp(1), Sp(4) × U(1), Sp(4), Spin(9)
O(8)	SO(8), U(4), SU(4), Sp(2) × Sp(1), Sp(2) × U(1), Sp(2), Spin(7)
O(7)	SO(7), G_2
O(4)	SO(4), $p\big(\mathrm{Sp}(1) \times L_2\big)$, $p\big(L_2 \times \mathrm{Sp}(1)\big)$
O(2)	SO(2)

Note that $K = \mathrm{Sp}(n)$ does not contain such a subgroup.

Proof. Since a connected subgroup of $\mathrm{O}(n)$ lies in $\mathrm{SO}(n)$ and since $\mathrm{SO}(n)$ ($n \geq 3$, $n \neq 4$) and $\mathrm{Sp}(n)$ are simple, the result follows directly from [Oni94, Table 8, p. 227] for these groups.

Let us discuss the case $K = \mathrm{O}(4)$. Identifying $\mathbb{R}^4$ with $\mathbb{H}$ one sees that $\mathfrak{so}(4) \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. The isotropy algebra $\mathfrak{so}(4)_{e_1}$ of $e_1 \in S^3 \subset \mathbb{R}^4$ corresponds to the diagonal $\Delta_{\mathfrak{sp}(1)}$ in $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. Therefore we have to find all subalgebras $\mathfrak{l}$ of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ (up to conjugation) that verify

$$\mathfrak{l} + \Delta_{\mathfrak{sp}(1)} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1). \tag{4.1}$$

Consider the projections

$$\begin{array}{ccc} \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) & \xrightarrow{\pi_2} & \mathfrak{sp}(1) \\ {\scriptstyle \pi_1}\big\downarrow & & \\ \mathfrak{sp}(1). & & \end{array}$$

Let $\mathfrak{l}$ be a subalgebra of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ verifying (4.1). This implies

$$3 \leq \dim \mathfrak{l} = \dim \pi_1(\mathfrak{l}) + \dim \ker(\pi_1|_{\mathfrak{l}})$$

where $\ker(\pi_1|_{\mathfrak{l}}) = \mathfrak{l} \cap \big(\{0\} \oplus \mathfrak{sp}(1)\big)$ is an ideal in $\mathfrak{l}$.

If $\dim \pi_1(\mathfrak{l}) = 0$, then $\mathfrak{l} = \{0\} \oplus \mathfrak{sp}(1)$. If $\dim \pi_1(\mathfrak{l}) = 1$, then $\pi_1(\mathfrak{l}) =: \mathfrak{t}$ is a maximal torus in $\mathfrak{sp}(1)$ and $\dim \mathfrak{l} \cap \big(\{0\} \oplus \mathfrak{sp}(1)\big) \geq 2$, hence $\mathfrak{l} = \mathfrak{t} \oplus \mathfrak{sp}(1)$.

Finally, suppose that $\dim \pi_1(\mathfrak{l}) = 3$ and note that $\dim \mathfrak{l} \cap \big(\{0\} \oplus \mathfrak{sp}(1)\big) \in \{0, 1, 3\}$. If $\dim \mathfrak{l} \cap \big(\{0\} \oplus \mathfrak{sp}(1)\big) = 3$, then $\mathfrak{l} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. If $\dim \mathfrak{l} \cap \big(\{0\} \oplus \mathfrak{sp}(1)\big) = 0$, then $\dim \mathfrak{l} = 3$ and thus $\mathfrak{l}$ is simple. Therefore $\mathfrak{l} \cap \big(\mathfrak{sp}(1) \oplus \{0\}\big)$ is either $\mathfrak{sp}(1) \oplus \{0\}$ (and then $\mathfrak{l} = \mathfrak{sp}(1) \oplus \{0\}$) or $\{0\}$. In the latter case $\pi_1|_{\mathfrak{l}}$ and $\pi_2|_{\mathfrak{l}}$ are isomorphisms onto $\mathfrak{sp}(1)$ and $\mathfrak{l}$ coincides with the graph of $\varphi := (\pi_2|_{\mathfrak{l}}) \circ (\pi_1|_{\mathfrak{l}})^{-1} \in \mathrm{Aut}\big(\mathfrak{sp}(1)\big)$. Since we can identify $\mathfrak{l} \cap \Delta_{\mathfrak{sp}(1)}$ with the space of fixed points $\mathfrak{sp}(1)^{\varphi}$ in this case and since $\dim \mathfrak{sp}(1)^{\varphi} \geq 1$, we obtain $\dim \mathfrak{l} + \Delta_{\mathfrak{sp}(1)} \leq 5$, hence $\mathfrak{l}$ cannot verify (4.1). Finally consider the case $\dim \mathfrak{l} \cap \big(\{0\} \oplus \mathfrak{sp}(1)\big) = 1$. Then $\dim \mathfrak{l} = 4$ and $\mathfrak{l} \cap \big(\{0\} \oplus \mathfrak{sp}(1)\big)$ is a one-dimensional ideal in $\mathfrak{l}$ which implies $\mathfrak{l} \cap \big(\{0\} \oplus \mathfrak{sp}(1)\big) = \mathfrak{z}(\mathfrak{l})$. Thus $\mathfrak{l} = \mathfrak{sp}(1) \oplus \mathfrak{z}(\mathfrak{l})$.

In the unitary case we use [Oni94, Theorem 1.5.1] in order to reduce the classification to $\mathrm{SU}(n)$. We denote the isotropy algebra $\mathfrak{u}(n)_{e_1}$ of $e_1 \in S^{2n-1} \subset \mathbb{C}^n$ by $\mathfrak{u}(n-1)$. As above we look for subalgebras $\mathfrak{l}$ of $\mathfrak{u}(n)$ that verify $\mathfrak{l} + \mathfrak{u}(n-1) = \mathfrak{u}(n)$. According to [Oni94, Theorem 1.5.1] this holds if and only if $\mathfrak{l}' + \mathfrak{su}(n-1) = \mathfrak{su}(n)$ (where $\mathfrak{l}'$ is the derived algebra of $\mathfrak{l}$) and $\mathfrak{z}\big(\mathfrak{u}(n)\big) = \pi\big(\mathfrak{z}(\mathfrak{l}) + \mathfrak{z}\big(\mathfrak{u}(n-1)\big)\big)$ where π is the projection of $\mathfrak{u}(n)$ onto its center with kernel $\mathfrak{su}(n)$. Note that the second condition is automatically verified since already $\pi\big(\mathfrak{z}(\mathfrak{u}(n-1))\big) = \mathfrak{z}\big(\mathfrak{u}(n)\big)$. Consequently, the result follows from the classification of subgroups of $\mathrm{SU}(n)$ that act transitively on $S^{2n-1} \subset \mathbb{C}^n$ given in [Oni94, Table 8]. $\square$

Remark 4.4. The subalgebras $\mathfrak{sp}(1) \oplus \mathfrak{l}_2$ and $\mathfrak{l}_2 \oplus \mathfrak{sp}(1)$ of $\mathfrak{so}(4)$ are O(4)-conjugate (see [Kna16, Remark 8.5]) but not SO(4)-conjugate to each other, since any inner automorphism of SO(4) leaves the ideals $\mathfrak{sp}(1) \oplus \{0\}$ and $\{0\} \oplus \mathfrak{sp}(1)$ in $\mathfrak{so}(4)$ invariant.

Since for classical G the K-action on $\mathfrak{p}$ and the M-action on $\mathfrak{g}_\alpha$ are essentially the same, we can use the results of the non-reductive case also in the reductive one. Moreover, suppose that $\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H$ is a reductive spherical subalgebra of $\mathfrak{g}$. Then we have

$$[\mathfrak{p}_H, \mathfrak{p}_H] \subset \mathfrak{k}_H \subset \mathcal{N}_{\mathfrak{k}}(\mathfrak{p}_H). \tag{4.2}$$

Using the normal forms obtained in the first step, it is not hard to single out those $\mathfrak{k}_H$ and $\mathfrak{p}_H$ from the lists obtained in the non-reductive case that verify (4.2).

5. $G = \mathrm{SO}^\circ(n,1)$

5.1. **Non-reductive spherical subalgebras.** In this subsection we classify the spherical non-reductive algebraic subalgebras $\mathfrak{h}$ of $\mathfrak{g} = \mathfrak{so}(n,1)$, where $n \geq 2$. As we have seen we may assume $H = M_H A_H N_H$, where $1 \leq \dim \mathfrak{n}_H^\perp < n-1$ and $M_H \subset \mathcal{N}_M(\mathfrak{n}_H)$ (see Proposition 3.2). The M-action on $\mathfrak{n}$ is isomorphic to the defining representation of $\mathrm{SO}(n-1)$ on $\mathbb{R}^{n-1}$ (see Remark 4.1). Therefore any real subspace $\mathfrak{n}_H$ of $\mathfrak{n}$ can, and will, be identified with a real subspace W of $\mathbb{R}^{n-1}$.

Recall that $\mathbb{R}^{n-1}$ comes equipped with an $\mathrm{SO}(n-1)$-invariant real inner product s. Note that such an inner product is unique up to a positive factor. Let W be a real subspace of $\mathbb{R}^{n-1}$. Then $s|_{W\times W}$ is a real inner product on W and $\mathrm{O}(W)$ denotes the group of invertible endomorphisms of W that respect $s|_{W\times W}$. We write $\mathrm{O}(W)\times\mathrm{O}(W^\perp)$ for the subgroup of $\mathrm{O}(\mathbb{R}^{n-1})$ that stabilizes the decomposition $\mathbb{R}^{n-1} = W \oplus W^{\perp_s}$.

Lemma 5.1. *Let $W \subset \mathbb{R}^{n-1}$ be a real subspace. Then $\mathcal{N}_{\mathrm{SO}(n-1)}(W) = S(\mathrm{O}(W)\times\mathrm{O}(W^{\perp_s}))$ and its action on W and $W^{\perp_s}$ is induced by the standard $\mathrm{O}(W)$-action on W and $\mathrm{O}(W^{\perp_s})$-action on $W^{\perp_s}$ respectively. In particular $\mathcal{N}_{\mathrm{SO}(n-1)}(W)$ acts transitively on the connected components of the spheres in W. Moreover there exists a basis $(w_1,\dots,w_{n-1})$ of $\mathbb{R}^{n-1}$ such that $W = \mathbb{R}w_1\oplus\cdots\oplus\mathbb{R}w_l$ where $l = \dim_{\mathbb{R}} W$.*

Proof. Any element in $\mathcal{N}_{\mathrm{SO}(n-1)}(W)$ respects s, W and $W^{\perp_s}$ and has determinant 1, which implies that it lies in $S(\mathrm{O}(W)\times\mathrm{O}(W)^{\perp_s})$. On the other hand any element in $S(\mathrm{O}(W)\times\mathrm{O}(W)^{\perp_s})$ leaves W invariant and lies in $\mathrm{SO}(n-1)$. Thus $\mathcal{N}_{\mathrm{SO}(n-1)}(W) = S(\mathrm{O}(W)\times\mathrm{O}(W^{\perp_s}))$ holds. Choosing oriented s-orthonormal bases $(w_1,\dots,w_l)$ of W and $(w_{l+1},\dots,w_{n-1})$ of $W^{\perp_s}$ we obtain an element $(w_1\cdots w_{n-1}) \in \mathrm{SO}(n-1)$ that maps W_l to W. □

Combining this observation with Proposition 3.2, Theorem 4.3 and Remark 4.4, we obtain the following theorem.

Theorem 5.2. *Every spherical non-reductive algebraic subalgebra of $\mathfrak{g} = \mathfrak{so}(n,1)$, $n \geq 2$, is G-conjugate to exactly one in the following list where $\mathfrak{c}_l \subset \mathfrak{so}(n-l-1)$ is an arbitrary subalgebra (under the condition displayed in italic in Remark 5.3 below) and where*

$$\mathfrak{n}_l := \left\{ \begin{pmatrix} 0 & v & -v \\ -v^t & 0 & 0 \\ -v^t & 0 & 0 \end{pmatrix} : v \in \{0\}^l \times \mathbb{R}^{n-1-l} \right\} \cong W^{\perp_s}.$$

$\mathfrak{l}_H \oplus \mathfrak{n}$	$\mathfrak{l}_H \subset \mathfrak{m}\oplus\mathfrak{a}$ *arbitrary*
$\mathfrak{so}(l)\oplus\mathfrak{c}_l\oplus\mathfrak{a}\oplus\mathfrak{n}_l$	$1 \leq l \leq n-2,$
$\mathfrak{u}(m)\oplus\mathfrak{c}_l\oplus\mathfrak{a}\oplus\mathfrak{n}_l$	$1 \leq l \leq n-2, l = 2m, m \geq 3$
$\mathfrak{su}(m)\oplus\mathfrak{c}_l\oplus\mathfrak{a}\oplus\mathfrak{n}_l$	$1 \leq l \leq n-2, l = 2m, m \geq 3$
$\mathfrak{sp}(m)\oplus\mathfrak{sp}(1)\oplus\mathfrak{c}_l\oplus\mathfrak{a}\oplus\mathfrak{n}_l$	$1 \leq l \leq n-2, l = 4m, m \geq 2$

$\mathfrak{sp}(m) \oplus \mathfrak{u}(1) \oplus \mathfrak{c}_l \oplus \mathfrak{a} \oplus \mathfrak{n}_l$	$1 \leq l \leq n-2, l = 4m, m \geq 1$
$\mathfrak{sp}(m) \oplus \mathfrak{c}_l \oplus \mathfrak{a} \oplus \mathfrak{n}_l$	$1 \leq l \leq n-2, l = 4m, m \geq 1$
$\mathfrak{so}(9) \oplus \mathfrak{c}_{16} \oplus \mathfrak{a} \oplus \mathfrak{n}_{16}$	
$\mathfrak{so}(7) \oplus \mathfrak{c}_8 \oplus \mathfrak{a} \oplus \mathfrak{n}_8$	
$\mathfrak{g}_2 \oplus \mathfrak{c}_7 \oplus \mathfrak{a} \oplus \mathfrak{n}_7$	

Note that $\mathfrak{n}_1$ *is* 1*-codimensional and that* $\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_1) \cong \mathfrak{so}(n-2)$.

Remark 5.3. Let us explain the notation in Theorem 5.2. Let $\pi \colon \mathfrak{so}(l) \oplus \mathfrak{so}(n-l) \to \mathfrak{so}(l)$ denote the projection onto the first factor. Then π is a Lie algebra homomorphism and the kernel of its restriction to an arbitrary subalgebra $\mathfrak{k} \subset \mathfrak{so}(l) \oplus \mathfrak{so}(n-l)$ is an ideal in $\mathfrak{k}$. Since $\mathfrak{k}$ is compact the orthogonal complement to this ideal with respect to the Killing form κ is also an ideal in $\mathfrak{k}$ and we obtain $\mathfrak{k} = (\ker \pi|_{\mathfrak{k}})^{\perp_\kappa} \oplus \ker \pi|_{\mathfrak{k}}$. Note that $(\ker \pi|_{\mathfrak{k}})^{\perp_\kappa}$ is isomorphic to the subalgebra $\pi(\mathfrak{k})$ of $\mathfrak{so}(l)$. Let $\Phi : \pi(\mathfrak{k}) \to (\ker \pi|_{\mathfrak{k}})^{\perp_\kappa}$ be a Lie algebra isomorphism. Then $\varphi = \pi_{\mathfrak{so}(n-l)} \circ \Phi : \pi(\mathfrak{k}) \to \mathfrak{so}(n-l)$ is a Lie algebra homomorphism, where $\pi_{\mathfrak{so}(n-l)} : \mathfrak{so}(l) \oplus \mathfrak{so}(n-l) \to \mathfrak{so}(n-l)$ is the projection onto the second factor. Then

$$\mathfrak{k} = \big\{ \big(\xi, \varphi(\xi)\big) \colon\ \xi \in \pi(\mathfrak{k}) \big\} \oplus \ker \pi|_{\mathfrak{k}}.$$

In order to simplify the notation, we write here and in the rest of this work $\mathfrak{k} = \pi(\mathfrak{k}) \oplus \mathfrak{c}$ where $\mathfrak{c} = \ker \pi|_{\mathfrak{k}} = \mathfrak{k} \cap \big(\{0\} \oplus \mathfrak{so}(n-l)\big)$ is the ineffectivity of the $\mathcal{N}_{\mathfrak{m}}(W)$-action on W, i.e., we omit the representation φ from the notation.

> *In particular,* $\pi(\mathfrak{k}) \oplus \mathfrak{c}$ *is a subalgebra of* $\mathfrak{so}(n,1)$ *if and only if* $\mathfrak{c}$ *is contained in the centralizer of* $\varphi\big(\pi(\mathfrak{k})\big)$ *in* $\mathfrak{so}(n-l)$.

Note that, if $\pi(\mathfrak{k})$ is simple, then φ is either identically zero or injective.

5.2. **Reductive spherical subalgebras.** Let $\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H$ be a reductive algebraic subalgebra of $\mathfrak{g} = \mathfrak{so}(n,1)$ (where $n \geq 2$) such that $\mathfrak{k}_H \subset \mathfrak{k}$ and $\mathfrak{p}_H \subset \mathfrak{p}$. On the group level we have $H = K_H \exp(\mathfrak{p}_H)$.

Since the K-representation on $\mathfrak{p}$ is essentially the same as the M-representation on $\mathfrak{n}$ (see Remark 4.1), we may apply Lemma 5.1 to this situation and obtain, after conjugation in K, that $W = \mathbb{R}^l \times \{0\}^{n-l}$, where $l = \dim_{\mathbb{R}}(W)$. In particular

$$\mathfrak{p}_H = \mathfrak{p}_{H,l} := \left\{ \left(\begin{smallmatrix} 0 & x \\ x^t & 0 \end{smallmatrix}\right) : x \in \{0\}^l \times \mathbb{R}^{n-l} \right\},$$

for some $0 \leq l \leq n$. A direct calculation shows that

$$[\mathfrak{p}_{H,l}, \mathfrak{p}_{H,l}] = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ & & \vdots \\ 0 & B & \vdots \\ 0 & \dots & 0 \end{pmatrix} \in \mathfrak{k} : B \in \mathfrak{so}(n-l) \right\},$$

$$\mathcal{N}_{\mathfrak{k}}(\mathfrak{p}_{H,l}) = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{k} : A \in \mathfrak{so}(l), B \in \mathfrak{so}(n-l) \right\},$$

and thus $[\mathfrak{p}_{H,l}, \mathfrak{p}_{H,l}] \oplus \mathfrak{p}_{H,l} \cong \mathfrak{so}(n-l,1)$ (with $\mathfrak{so}(0,1) := \{0\}$). Therefore the condition $[\mathfrak{p}_H, \mathfrak{p}_H] \subset \mathfrak{k}_H \subset \mathcal{N}_{\mathfrak{k}}(\mathfrak{p}_H)$ (see (4.2)) implies that $\mathfrak{h}$ is of the form $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{so}(n-l,1)$, where $\mathfrak{b}$ is a subalgebra of $\mathfrak{so}(l)$ and $0 \leq l \leq n$. Together with Proposition 3.1 and Theorem 4.3 we obtain the following theorem.

Theorem 5.4. *All spherical reductive algebraic subalgebras $\mathfrak{h}$ of $\mathfrak{so}(n,1)$, $n \geq 2$, are (up to conjugation in G) one of the following, where $\mathfrak{so}(0,1) := \{0\}$ and $\mathfrak{l}_2 \subset \mathfrak{sp}(1)$ is arbitrary.*

$\mathfrak{so}(l) \oplus \mathfrak{so}(n-l,1)$	$0 \leq l \leq n$
$\mathfrak{u}(m) \oplus \mathfrak{so}(n-l,1)$	$0 \leq l \leq n, l = 2m, m \geq 3$
$\mathfrak{su}(m) \oplus \mathfrak{so}(n-l,1)$	$0 \leq l \leq n, l = 2m, m \geq 3$
$\mathfrak{sp}(m) \oplus \mathfrak{l}_2 \oplus \mathfrak{so}(n-l,1)$	$0 \leq l \leq n, l = 4m, m \geq 1$
$\mathfrak{so}(9) \oplus \mathfrak{so}(n-16,1)$	$n \geq 16$
$\mathfrak{so}(7) \oplus \mathfrak{so}(n-8,1)$	$n \geq 8$
$\mathfrak{g}_2 \oplus \mathfrak{so}(n-7,1)$	$n \geq 7$
$\mathfrak{l}_2 \oplus \mathfrak{sp}(1) \oplus \mathfrak{so}(n-4,1)$	$n \geq 4$

According to [Ber57, Table 2]*, the symmetric subalgebras are $\mathfrak{so}(n) \oplus \mathfrak{so}(1)$ and $\mathfrak{so}(l) \oplus \mathfrak{so}(n-l,1)$ for $0 \leq l < n$.*

Remark 5.5. Contrary to the non-reductive case, see Remark 5.3, the condition $[\mathfrak{p}_H, \mathfrak{p}_H] \subset \mathfrak{k}_H \subset \mathcal{N}_{\mathfrak{k}}(\mathfrak{p}_H)$ (see (4.2)) implies that there is no ambiguity in choosing the representation φ. More precisely, we have $\varphi = 0$.

6. $G = \mathrm{SU}(n,1)$

6.1. **Notation.** Let V be a complex vector space of complex dimension n and let h be a hermitian inner product on V. We denote the unitary and special unitary groups of V with respect to h by $\mathrm{U}(V)$ and $\mathrm{SU}(V)$, respectively. Similarly, if W is a real subspace of V, we write $\mathrm{O}(W)$ (and $\mathrm{SO}(W)$) for the group of $\mathbb{R}$-linear transformations of W that leave the scalar product $s|_{W\times W}$ invariant (and have determinant 1) where $s := \mathrm{Re}(h)$. If $W \subset V$ is a totally real subspace, then we extend the action of $\mathrm{O}(W)$ by $\mathbb{C}$-linearity to $W^{\mathbb{C}} = W \oplus iW$. By abuse of notation, we consider $\mathrm{O}(W)$ also as a group of $\mathbb{C}$-linear maps of $W^{\mathbb{C}}$.

Remark 6.1. If W is a real form of V, i.e., a maximal totally real subspace, then the complex $\mathrm{O}(W)$-representation on V is irreducible. Moreover, if $n \geq 3$, then the complex $\mathrm{SO}(W)$-representation on V is irreducible, too, see [Kna16, Lemma 6.3].

Suppose that $V = V_1 \oplus V_2$ is an orthogonal direct sum of two complex subspaces. As in Section 5 we write $\mathrm{U}(V_1) \times \mathrm{U}(V_2)$ for the subgroup of $\mathrm{U}(V)$ that stabilizes this decomposition.

6.2. **Preliminaries.** From now on we fix a real subspace W of V such that a compact subgroup K of $\mathcal{N}_{\mathrm{U}(V)}(W)$ acts irreducibly on W. Since K acts complex linearly on V, the maximal complex subspace $W \cap iW$ of W is K-invariant. Consequently, W is either complex or totally real. We first treat the complex case.

Lemma 6.2. *Let W be any complex subspace of V. Then $\mathcal{N}_{\mathrm{U}(V)}(W)$ coincides with $\mathrm{U}(W) \times \mathrm{U}(W^{\perp})$ and acts transitively on the spheres in W. Moreover, there is an orthonormal basis $(v_1, \ldots, v_n)$ of V such that $W = \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_l$ where $l = \dim_{\mathbb{C}} W$.*

Let us now suppose that W is totally real in V. Our goal is to understand how W lies in its complexification $W^{\mathbb{C}} = W \oplus iW \subset V$ with respect to the symplectic form $\omega := \mathrm{Im}(h)$. For this we define a linear map $\mathcal{I} \colon W \to W$ by the identity

$$s(\mathcal{I}w_1, w_2) = \omega(w_1, w_2)$$

for all $w_1, w_2 \in W$. One verifies directly that $\mathcal{I}$ is $\mathcal{N}_{\mathrm{U}(V)}(W)$-equivariant and skew-symmetric with respect to s. Moreover, the kernel of $\mathcal{I}$ is $W \cap W^{\perp_\omega}$, i.e., the maximal subspace of W on which ω is degenerate.

Let $\mathcal{I}^{\mathbb{C}} \in \mathrm{End}(W^{\mathbb{C}})$ be the $\mathbb{C}$-linear extension of $\mathcal{I}$. For the proof of the following proposition we refer the reader to [Kna16, Proposition 6.11].

Lemma 6.3. *The map $\mathcal{I}^{\mathbb{C}} \colon W^{\mathbb{C}} \to W^{\mathbb{C}}$ is $\mathcal{N}_{\mathrm{U}(V)}(W)$-equivariant and skew-hermitian with respect to h. Moreover, $\mathcal{I}^{\mathbb{C}}$ commutes with the complex conjugation of $W^{\mathbb{C}}$ with respect to W, denoted in the following by σ.*

As a consequence, $\mathcal{I}^{\mathbb{C}}$ is diagonalizable having purely imaginary eigenvalues. Let

$$W^{\mathbb{C}} = \ker(\mathcal{I}^{\mathbb{C}}) \oplus \bigoplus_{\xi \in i\mathbb{R}^*} W^{\mathbb{C}}_{\xi}$$

be the decomposition of $W^{\mathbb{C}}$ into the eigenspaces of $\mathcal{I}^{\mathbb{C}}$. This decomposition is $\mathcal{N}_{\mathrm{U}(V)}(W)$-invariant and h-orthogonal, and we have $\sigma(W^{\mathbb{C}}_{\xi}) = W^{\mathbb{C}}_{-\xi}$. Hence, we obtain the $\mathcal{N}_{\mathrm{U}(V)}(W)$-invariant decomposition

$$W = \ker(\mathcal{I}) \oplus \bigoplus_{\xi \in i\mathbb{R}^{>0}} (W^{\mathbb{C}}_{\xi} \oplus W^{\mathbb{C}}_{-\xi})^{\sigma}.$$

Since by assumption there exists a compact subgroup K of $\mathcal{N}_{\mathrm{U}(V)}(W)$ that acts irreducibly on W, we have either $W = \ker(\mathcal{I})$ or $W = (W^{\mathbb{C}}_{\xi} \oplus W^{\mathbb{C}}_{-\xi})^{\sigma}$ for some $\xi \in i\mathbb{R}^{>0}$. In the first case W is Lagrangian in $W^{\mathbb{C}}$ with respect to ω, while in the second case W is a symplectic subspace of $W^{\mathbb{C}}$ with respect to ω.

This discussion proves part of the following proposition.

Proposition 6.4. *Let W be a totally real subspace of (V, h) and $W^{\mathbb{C}} = W \oplus iW$ be its complexification. If a subgroup $K \subset \mathcal{N}_{\mathrm{U}(V)}(W)$ acts irreducibly on W, then*

- *(i) either $W \subset W^{\mathbb{C}}$ is Lagrangian with respect to ω, $W^{\mathbb{C}}$ is a complex irreducible $\mathcal{N}_{\mathrm{U}(V)}(W)$-representation and $\mathcal{N}_{\mathrm{U}(V)}(W) = \mathrm{O}(W) \times \mathrm{U}(W^{\perp_h})$,*
- *(ii) or $W \subset W^{\mathbb{C}}$ is symplectic with respect to ω, $W^{\mathbb{C}}$ decomposes h-orthogonally as $W^{\mathbb{C}} = W^{\mathbb{C}}_{\xi} \oplus W^{\mathbb{C}}_{-\xi}$ into two complex irreducible $\mathcal{N}_{\mathrm{U}(V)}(W)$-representations that are as real representations isomorphic to W and $\mathcal{N}_{\mathrm{U}(V)}(W) = \{(g, \varphi(g)) \in \mathrm{U}(W^{\mathbb{C}}_{\xi}) \times \mathrm{U}(W^{\mathbb{C}}_{-\xi})\} \times \mathrm{U}(W^{\perp_h})$, where $\varphi \colon \mathrm{U}(W^{\mathbb{C}}_{\xi}) \to \mathrm{U}(W^{\mathbb{C}}_{-\xi})$ is a Lie group isomorphism.*

In both cases $\mathcal{N}_{\mathrm{U}(V)}(W)$ acts transitively on the spheres in W.

Proof. If $W \subset V$ is Lagrangian, then W and iW are orthogonal with respect to s. Any element in $\mathcal{N}_{\mathrm{U}(V)}(W)$ leaves W and $W^{\perp_h} = (W^{\mathbb{C}})^{\perp_h}$ as well as $s|_{W \times W}$ invariant, and lies therefore in $\mathrm{O}(W) \times \mathrm{U}(W^{\perp_h})$. Conversely, any element in $\mathrm{O}(W)$ acts $\mathbb{C}$-linearly on $W^{\mathbb{C}}$. Again, due to the fact that W is Lagrangian in $W^{\mathbb{C}}$, the hermitian inner product $h|_{W^{\mathbb{C}} \times W^{\mathbb{C}}}$ is $\mathrm{O}(W)$-invariant, which implies $\mathrm{O}(W) \times \mathrm{U}(W^{\perp_h}) \subset \mathcal{N}_{\mathrm{U}(V)}(W)$. This shows $\mathcal{N}_{\mathrm{U}(V)}(W) = \mathrm{O}(W) \times \mathrm{U}(W^{\perp_q})$. Consequently, Remark 6.1 implies that $\mathcal{N}_{\mathrm{U}(V)}(W)$ acts transitively on the spheres in W and complex irreducibly on $W^{\mathbb{C}}$.

Let us now consider the case that $W^{\mathbb{C}} = W^{\mathbb{C}}_{\xi} \oplus W^{\mathbb{C}}_{-\xi}$ is the decomposition of $W^{\mathbb{C}}$ into $\mathcal{I}^{\mathbb{C}}$-eigenspaces corresponding to the eigenvalues $\pm\xi \in i\mathbb{R}\backslash\{0\}$. Recall that this decomposition is h-orthogonal and that the $\mathbb{C}$-anti-linear involution σ on $W^{\mathbb{C}}$ yields an $\mathbb{R}$-linear isomorphism between $W^{\mathbb{C}}_{\xi}$ and $W^{\mathbb{C}}_{-\xi}$. Hence, we obtain $\mathbb{R}$-linear, $\mathcal{N}_{\mathrm{U}(V)}(W)$-equivariant isomorphisms $f_{\pm}$ between $W^{\mathbb{C}}_{\pm\xi}$ and W given by $f_{\pm}(w) = \frac{1}{2}\big(w + \sigma(w)\big)$. In particular, the complex subspaces $W^{\mathbb{C}}_{\pm\xi}$ are real irreducible, hence complex irreducible, $\mathcal{N}_{\mathrm{U}(V)}(W)$-representations (isomorphic to W). We define another hermitian inner product on $W^{\mathbb{C}}_{-\xi}$ by $h_{\sigma}(v, u) = \overline{h(\sigma(v), \sigma(u))}$ for all $v, u \in W^{\mathbb{C}}_{-\xi}$. One verifies directly that h_σ and $h|_{W^{\mathbb{C}}_{-\xi}\times W^{\mathbb{C}}_{-\xi}}$ are both $\mathcal{N}_{\mathrm{U}(V)}(W)$-invariant. Thus there exists an $r \in \mathbb{R}^{>0}$ such that $h_\sigma = r^2 h|_{W^{\mathbb{C}}_{-\xi}\times W^{\mathbb{C}}_{-\xi}}$.

Suppose for a moment that $r = 1$, i.e., that σ is an isometry of s. Consequently, W is orthogonal to iW with respect to s, hence a Lagrangian subspace of $W^{\mathbb{C}}$ with respect to ω, contradicting our assumption. Thus we conclude $r \neq 1$.

We define the map $\varphi\colon \mathrm{U}(W^{\mathbb{C}}_{\xi}) \to \mathrm{GL}(W^{\mathbb{C}}_{-\xi})$ by $\varphi(g)v = \sigma(g \cdot \sigma^{-1}(v))$ for all $v \in W^{\mathbb{C}}_{-\xi}$. A direct calculation shows that φ is a Lie group isomorphism from $\mathrm{U}(W^{\mathbb{C}}_{\xi})$ to $\mathrm{U}(W^{\mathbb{C}}_{-\xi})$. This observation allows us to show that

$$\mathcal{N}_{\mathrm{U}(V)}(W) = \{(g, \varphi(g)) \in \mathrm{U}(W^{\mathbb{C}}_{\xi}) \times \mathrm{U}(W^{\mathbb{C}}_{-\xi})\} \times \mathrm{U}(W^{\perp_h}).$$

Indeed, any element in $\mathcal{N}_{\mathrm{U}(V)}(W)$ respects the decomposition $W^{\mathbb{C}}_{\xi} \oplus W^{\mathbb{C}}_{-\xi}$ and therefore lies in $\mathrm{U}(W^{\mathbb{C}}_{\xi}) \times \mathrm{U}(W^{\mathbb{C}}_{-\xi})$. Using the fact that any element in $\mathcal{N}_{\mathrm{U}(V)}(W)$ stabilizes $W = (W^{\mathbb{C}})^\sigma$ we obtain $(k_1, k_2) \cdot (v + \sigma(v)) = k_1 v + k_2\sigma(v) = k_1 v + \sigma(\varphi(k_2)v)$ has to lie in W for all $(k_1, k_2) \in \mathrm{U}(W^{\mathbb{C}}_{\xi}) \times \mathrm{U}(W^{\mathbb{C}}_{-\xi})$ and for all $v \in W^{\mathbb{C}}_{\xi}$. This implies $k_1 v = \varphi(k_2)v$ for all $v \in W^{\mathbb{C}}_{\xi}$ forcing $k_2 = \varphi(k_1)$ and thus

$$\mathcal{N}_{\mathrm{U}(V)}(W) \subset \{(g, \varphi(g)) \in \mathrm{U}(W^{\mathbb{C}}_{\xi}) \times \mathrm{U}(W^{\mathbb{C}}_{-\xi})\} \times \mathrm{U}(W^{\perp_h}).$$

The converse inclusion is elementary to check. □

Introducing coordinates we obtain the following normal form of real subspaces W of $\mathbb{C}^n$ equipped with the standard hermitian inner product, up to the action of $\mathrm{SU}(n)$.

Corollary 6.5. *Let W be a real subspace of $\mathbb{C}^n$. If some compact subgroup of $\mathcal{N}_{\mathrm{U}(n)}(W)$ acts irreducibly on W, then W lies in the same $\mathrm{SU}(n)$-orbit as one of the following*

$$W_{\mathbb{C},l} := \mathbb{C}^l \times \{0\}^{n-1-l}, \qquad 0 \le l \le n-1 \tag{1}$$

$$W_{\mathbb{R},l} := \mathbb{R}^l \times \{0\}^{n-1-l}, \qquad 0 \le l \le n-1 \tag{2}$$

$$W_{\mathbb{R},2l,r} := \left\{ \begin{pmatrix} z \\ r\bar{z} \\ 0 \end{pmatrix} : z \in \mathbb{C}^l \right\}, \qquad 0 \le l \le \lfloor \tfrac{n-1}{2} \rfloor \tag{3}$$

for some $r \in \mathbb{R}^{>0}\backslash\{1\}$. The corresponding normalizers (with respect to the action given in Remark 4.1) are given by

$$\mathcal{N}_{S(\mathrm{U}(n)\times\mathrm{U}(1))}(W_{\mathbb{C},l}) = S(\mathrm{U}(l) \times \mathrm{U}(n-1-l) \times \mathrm{U}(1)),$$
$$\mathcal{N}_{S(\mathrm{U}(n)\times\mathrm{U}(1))}(W_{\mathbb{R},l}) \cong S(\mathrm{O}(l) \times \mathrm{U}(n-1-l) \times \mathrm{U}(1)),$$
$$\mathcal{N}_{S(\mathrm{U}(n)\times\mathrm{U}(1))}(W_{\mathbb{R},2l,r}) \cong S(\mathrm{U}(l) \times \mathrm{U}(n-1-2l) \times \mathrm{U}(1)).$$

In all three cases $\mathcal{N}_{S(\mathrm{U}(n)\times\mathrm{U}(1))}(W)$ acts transitively on the connected components of the spheres in W.

Proof. If W is Lagrangian in $W^{\mathbb{C}}$ with respect to ω, then W is orthogonal to iW with respect to s. Therefore any s-orthonormal basis over $\mathbb{R}$ of W is an h-orthonormal basis over $\mathbb{C}$ of $W^{\mathbb{C}}$. In particular, if we choose an oriented s-orthonormal basis $(w_1, \ldots, w_l)$ of W and complete it to an h-orthonormal basis $(w_1, \ldots, w_n)$ of $\mathbb{C}^n$ we obtain an element $(w_1 \cdots w_n) \in \mathrm{SU}(n)$ that maps $W_{\mathbb{R},l}$ to W.

If W is symplectic in $W^{\mathbb{C}}$ with respect to ω, then we saw in the proof of Proposition 6.4 that there exists an $r \in \mathbb{R}^{>0}\backslash\{1\}$ such that $\overline{h(\sigma(v), \sigma(u))} = r^2 h(v, u)$ holds for all $v, u \in V_{-\xi}$. If we choose an h-orthonormal basis $(v_1, \ldots, v_{l/2})$ over $\mathbb{C}$ of V_ξ and set $v_{l/2+j} := \frac{1}{r}\sigma(v_j)$ for all $1 \le j \le l/2$ we obtain the h-orthonormal basis $(v_{l/2+1}, \ldots, v_l)$ of $V_{-\xi}$. Completing it to an oriented h-orthonormal basis $(v_1, \ldots, v_n)$ of $\mathbb{C}^n$ the element $(v_1 \cdots v_n) \in \mathrm{SU}(n)$ maps $W_{\mathbb{R},l,r} := \left\{ \begin{pmatrix} z \\ r\bar{z} \\ 0 \end{pmatrix} : z \in \mathbb{C}^{l/2} \right\}$ to W. Therefore W lies in the same $\mathrm{SU}(n)$-orbit as $W_{\mathbb{R},l,r}$ for some $r \in \mathbb{R}^{>0}\backslash\{1\}$ in this case. Moreover r can be chosen between 0 and 1, because the element $\left(\begin{smallmatrix} 0 & \mathrm{id} & 0 \\ \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} \end{smallmatrix}\right) \in \mathrm{U}(n)$ maps $W_{\mathbb{R},l,r}$ to $W_{\mathbb{R},l,1/r}$. □

Remark 6.6. In [Kna16, pp. 29–31] the following geometric strengthening of Corollary 6.5 is proven. Let us consider the set

$$\mathcal{M}_m := \left\{W \subset \mathbb{C}^n;\ \dim_{\mathbb{R}} W = m,\ \mathcal{N}_{\mathrm{U}(n)}(W) \text{ acts irreducibly on } W\right\}.$$

If m is odd, then $\mathcal{M}_m = \mathrm{U}(n) \cdot W_{\mathbb{R},m}$. If m is even with $\lfloor \frac{n}{2} \rfloor \leq \frac{m}{2} \leq n$, then $\mathcal{M}_m = \mathrm{U}(n) \cdot W_{\mathbb{C},m/2}$. If m is even with $0 \leq \frac{m}{2} \leq \lfloor \frac{n}{2} \rfloor$, then the set $\mathcal{S} := \left\{W_{\mathbb{R},m,r};\ r \in [0,1]\right\}$ is a geometric slice for the $\mathrm{U}(n)$-action on $\mathcal{M}_m$ in the following sense. The closed subset $\mathcal{S}$ is a real submanifold with boundary which meets each $\mathrm{U}(n)$-orbit in $\mathcal{M}_m$ exactly once, and $\left\{W_{\mathbb{R},m,r};\ r \in (0,1)\right\}$ intersects each $\mathrm{U}(n)$-orbit transversally.

6.3. **Non-reductive spherical subalgebras.** In this subsection we describe all non-reductive spherical algebraic subalgebras of $\mathfrak{su}(n,1)$, up to the action of M. Their unipotent radicals are of the form $W^\perp \oplus \mathfrak{g}_{2\alpha}$ where $W \subset \mathfrak{g}_\alpha$ is one of the subspaces described in Corollary 6.5. In order to find all possibilities for their maximal compact subalgebras we apply Onishchik's theorem to their normalizers and thus obtain the following.

Theorem 6.7. *Every spherical non-reductive algebraic subalgebra of $\mathfrak{g} = \mathfrak{su}(n,1)$, $n \geq 2$, is G-conjugate to one in the following list where $\mathfrak{b}_j \subset \mathfrak{u}(n-1-j)$ and $\mathfrak{c} \subset \mathfrak{u}(1)$ are arbitrary (under the condition displayed in italic in Remark 5.3 adapted to this situation).*

$\mathfrak{l}_H \oplus \mathfrak{n}$	$\mathfrak{l}_H \subset \mathfrak{m} \oplus \mathfrak{a}$ *arbitrary*
$\mathfrak{s}(\mathfrak{u}(l) \oplus \mathfrak{b}_l \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{C},l}$	$1 \leq l \leq n-1,$
$\mathfrak{s}(\mathfrak{su}(l) \oplus \mathfrak{b}_l \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{C},l}$	$2 \leq l \leq n-1$
$\mathfrak{s}(\mathfrak{sp}(m) \oplus \mathfrak{b}_l \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{C},l}$	$1 \leq l \leq n-1, l = 2m, m \geq 2$
$\mathfrak{s}(\mathfrak{sp}(m) \oplus \mathfrak{u}(1) \oplus \mathfrak{b}_l \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{C},l}$	$1 \leq l \leq n-1, l = 2m, m \geq 2$
$\mathfrak{s}(\mathfrak{c} \oplus \mathfrak{b}_1 \oplus \mathfrak{u}(1)) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{C},1}$	
$\mathfrak{s}(\mathfrak{so}(l) \oplus \mathfrak{b}_l \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},l}$	$1 \leq l \leq n-1,$
$\mathfrak{s}(\mathfrak{u}(m) \oplus \mathfrak{b}_l \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},l}$	$1 \leq l \leq n-1, l = 2m, m \geq 3$
$\mathfrak{s}(\mathfrak{su}(m) \oplus \mathfrak{b}_l \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},l}$	$1 \leq l \leq n-1, l = 2m, m \geq 3$
$\mathfrak{s}(\mathfrak{sp}(m) \oplus \mathfrak{sp}(1) \oplus \mathfrak{b}_l \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},l}$	$1 \leq l \leq n-1, l = 4m, m \geq 2$
$\mathfrak{s}(\mathfrak{sp}(m) \oplus \mathfrak{u}(1) \oplus \mathfrak{b}_l \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},l}$	$1 \leq l \leq n-1, l = 4m, m \geq 1$
$\mathfrak{s}(\mathfrak{sp}(m) \oplus \mathfrak{b}_l \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},l}$	$1 \leq l \leq n-1, l = 4m, m \geq 1$
$\mathfrak{s}(\mathfrak{so}(9) \oplus \mathfrak{b}_{16} \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},16}$	
$\mathfrak{s}(\mathfrak{so}(7) \oplus \mathfrak{b}_8 \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},8}$	
$\mathfrak{s}(\mathfrak{g}_2 \oplus \mathfrak{b}_7 \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},7}$	
$\mathfrak{s}(\mathfrak{u}(l) \oplus \mathfrak{b}_{2l} \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},2l,r}$	$1 \leq 2l \leq n-1, 0 < r \neq 1$
$\mathfrak{s}(\mathfrak{su}(l) \oplus \mathfrak{b}_{2l} \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},2l,r}$	$2 \leq 2l \leq n-1, 0 < r \neq 1$
$\mathfrak{s}(\mathfrak{sp}(m) \oplus \mathfrak{u}(1) \oplus \mathfrak{b}_{2l} \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},2l,r}$	$1 \leq 2l \leq n-1, l = 2m, m \geq 2, 0 < r \neq 1$
$\mathfrak{s}(\mathfrak{sp}(m) \oplus \mathfrak{b}_{2l} \oplus \mathfrak{c}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},2l,r}$	$2 \leq 2l \leq n-1, l = 2m, m \geq 2, 0 < r \neq 1$

Here we have written $\mathfrak{n}_{\mathbb{C},l}$ to denote $W_{\mathbb{C},l}^\perp \oplus \mathfrak{g}_{2\alpha}$ and so on. Note that $\mathfrak{n}_{\mathbb{R},1}$ is 1-codimensional and that $\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_{\mathbb{R},1}) = \mathfrak{s}(\mathfrak{u}(n-2) \oplus \mathfrak{u}(1))$.

6.4. **Reductive spherical subalgebras.** In this subsection we classify the reductive spherical subalgebras $\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H$ of $\mathfrak{su}(n,1)$. Since the K-action on $\mathfrak{p}$ is essentially the same as the M-action on $\mathfrak{g}_\alpha$, we may apply again Corollary 6.5 in order to obtain all the candidates for the subspace $\mathfrak{p}_H \subset \mathfrak{p}$. In the reductive case we have the additional restriction that $[\mathfrak{p}_H, \mathfrak{p}_H] \subset \mathfrak{k}_H \subset \mathcal{N}_{\mathfrak{k}}(\mathfrak{p}_H)$ must hold.

To be precise, let $\mathfrak{p}_H^\perp = W \subset \mathbb{C}^n$ be a real subspace. If $\mathcal{N}_K(W)$ acts irreducibly on W then $\mathfrak{p}_H$ fulfills $[\mathfrak{p}_H, \mathfrak{p}_H] \subset \mathcal{N}_{\mathfrak{k}}(\mathfrak{p}_H)$ only if W lies in the same $\mathrm{SU}(n)$-orbit as $W_{\mathbb{C},l}$ (in which case $[\mathfrak{p}_H, \mathfrak{p}_H] \oplus \mathfrak{p}_H \cong \mathfrak{su}(n-l,1)$) for some $0 \leq l \leq n$ or $W_{\mathbb{R},n}$ (in which case $[\mathfrak{p}_H, \mathfrak{p}_H] \oplus \mathfrak{p}_H \cong \mathfrak{so}(n,1)$), see [Kna16, Lemma 6.21].

Consequently, we obtain the following list of reductive spherical subalgebras of $\mathfrak{su}(n,1)$.

Theorem 6.8. *All spherical, reductive subalgebras $\mathfrak{h}$ of $\mathfrak{su}(n,1)$, where $n > 1$ are (up to conjugation in G) one of the following*

$\mathfrak{u}(n)$	
$\mathfrak{su}(n)$	$n > 1$
$\mathfrak{sp}(m)$	$n = 2m$ *and* $m \geq 2$
$\mathfrak{sp}(m) \oplus \mathfrak{u}(1)$	$n = 2m$ *and* $m \geq 2$
$\mathfrak{s}(\mathfrak{u}(l) \oplus \mathfrak{u}(n-l,1))$	$0 \leq l < n$
$\mathfrak{su}(l) \oplus \mathfrak{su}(n-l,1)$	$0 < l < n$
$\mathfrak{s}(\mathfrak{sp}(m) \oplus \mathfrak{u}(n-l,1))$	$0 \leq l < n, l = 2m, m \geq 2$
$\mathfrak{s}(\mathfrak{sp}(m) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(n-l,1))$	$0 \leq l < n, l = 2m, m \geq 2$
$\mathfrak{so}(n,1)$	

According to [Ber57, Table 2], *the symmetric subalgebras are $\mathfrak{s}(\mathfrak{u}(l) \oplus \mathfrak{u}(n-l,1))$ for $0 \leq l < n$, $\mathfrak{s}(\mathfrak{u}(n) \oplus \mathfrak{u}(1))$ and $\mathfrak{so}(n,1)$.*

Proof. Due to [Kna16, Lemma 6.21] we may assume that $\mathfrak{p}_H \cong W_{\mathbb{C},l}^\perp$ for some $0 \leq l \leq n$ or $\mathfrak{p}_H \cong W_{\mathbb{R},n}^\perp$. Note that $W_{\mathbb{C},n}^\perp = \{0\}$. Therefore $\mathfrak{h}$ is compact in that case. If $\mathfrak{p}_H \cong W_{\mathbb{C},l}^\perp$ for some $0 \leq l < n$ then $[\mathfrak{p}_H, \mathfrak{p}_H] \subset \mathfrak{k}_H \subset \mathcal{N}_{\mathfrak{k}}(\mathfrak{p}_H)$ together with $[\mathfrak{p}_H, \mathfrak{p}_H] \oplus \mathfrak{p}_H \cong \mathfrak{su}(n-l,1)$ implies that $\mathfrak{h} = \mathfrak{s}(\mathfrak{b} \oplus \mathfrak{u}(n-l,1))$ where $\mathfrak{b}$ is some subalgebra of $\mathfrak{u}(l)$ that acts transitively on the spheres in $\mathbb{C}^l$. If $\mathfrak{p}_H \cong W_{\mathbb{R},n}^\perp$ then $\mathfrak{so}(n) = [\mathfrak{p}_H, \mathfrak{p}_H] \subset \mathfrak{k}_H \subset \mathcal{N}_{\mathfrak{k}}(\mathfrak{p}_H) = \mathfrak{so}(n)$ implies that $\mathfrak{h} = \mathfrak{so}(n,1)$. $\square$

7. $G = \mathrm{Sp}(n,1)$

7.1. **Notation.** Let U be a quaternionic vector space (with scalar multiplication from the right) and let $q : U \times U \to \mathbb{H}$ be a quaternionic inner product on U (i.e., q is conjugate symmetric, $\mathbb{H}$-linear in the second argument and positive definite). We denote the group of invertible, $\mathbb{H}$-linear transformations on U by $\mathrm{GL}_{\mathbb{H}}(U)$ and define

$$\mathrm{Sp}(U) := \mathrm{Sp}(U,q) = \{A \in \mathrm{GL}_{\mathbb{H}}(U) : \ \forall\, v, w \in U, q(Av, Aw) = q(v,w)\}.$$

For $\mathbb{H}^n$ equipped with the standard quaternionic inner product (given by $q(v,w) = \overline{v}^t w$ for all $v, w \in \mathbb{H}^n$) we set $\mathrm{Sp}(n) := \mathrm{Sp}(\mathbb{H}^n)$. The standard quaternionic inner product on $\mathbb{H}$ is denoted by $q_{\mathbb{H}}$.

Let $\lambda \in \mathrm{Sp}(1) \cap \mathrm{Im}(\mathbb{H})$. Then $\lambda^2 = -1$ and $F_\lambda := \mathbb{R} \oplus \mathbb{R}\lambda$ is a subfield of $\mathbb{H}$ isomorphic to $\mathbb{C}$. If $\mu \in \mathrm{Sp}(1) \cap \mathrm{Im}(\mathbb{H})$ has the property $\mathrm{Re}(q_{\mathbb{H}}(\lambda,\mu)) = 0$, then $\lambda\mu$ lies in $\mathrm{Sp}(1) \cap \mathrm{Im}(\mathbb{H})$ and is $\mathrm{Re}\, q_{\mathbb{H}}$-orthogonal to λ and μ.

For the proof of the following result we refer the reader to [Kna16, Lemma 7.6].

Lemma 7.1. *Let $(1, \lambda, \mu, \lambda\mu)$ be a $\mathrm{Re}(q_{\mathbb{H}})$-orthonormal basis over $\mathbb{R}$ of $\mathbb{H}$ and let U be a quaternionic vector space with quaternionic inner product q. The projection $h_\lambda := \pi_{F_\lambda} \circ q$ of q onto F_λ is an Hermitian form on U and $q = h_\lambda + \mu\eta_\lambda$, where $\eta_\lambda : U \times U \to F_\lambda$ is a symplectic form on U. We define the 2-forms $s, \omega_\lambda, \omega_\mu, \omega_{\mu\lambda} : U \times U \to \mathbb{R}$ by $s := \mathrm{Re}\, h_\lambda$, $\omega_\lambda := \mathrm{Re}(-\lambda h_\lambda)$, $\omega_\mu := \mathrm{Re}\,\eta_\lambda$ and $\omega_{\mu\lambda} := \mathrm{Re}(-\lambda\eta_\lambda)$ and obtain that all of them are $\mathrm{Sp}(U, q)$-invariant and fulfill :*

$$\begin{aligned}
q = s + \lambda\omega_\lambda + \mu(\omega_\mu + \lambda\omega_{\mu\lambda}) &: U \times U \to \mathbb{H} \text{ is a quaternionic inner product,}\\
h_\lambda = s + \lambda\omega_\lambda &: U \times U \to F_\lambda \text{ is Hermitian,}\\
\eta_\lambda = \omega_\mu + \lambda\omega_{\mu\lambda} &: U \times U \to F_\lambda \text{ is symplectic,}\\
\omega_\lambda, \omega_\mu, \omega_{\mu\lambda} &: U \times U \to \mathbb{R} \text{ is symplectic,}\\
s &: U \times U \to \mathbb{R} \text{ is a real inner product.}
\end{aligned}$$

Since we choose q to be $\mathbb{H}$-linear in the second and $\mathbb{H}$-anti-linear in the first entry we have

$$\begin{aligned}
&s(v\alpha, w\beta) + \lambda\omega_\lambda(v\alpha, w\beta) + \mu\omega_\mu(v\alpha, w\beta) + \mu\lambda\omega_{\mu\lambda}(v\alpha, w\beta) = q(v\alpha, w\beta)\\
&= \overline{\alpha}q(v, w)\beta = \overline{\alpha}s(v, w)\beta + \overline{\alpha}\lambda\omega_\lambda(v, w)\beta + \overline{\alpha}\mu\omega_\mu(v, w)\beta + \overline{\alpha}\mu\lambda\omega_{\mu\lambda}(v, w)\beta
\end{aligned} \tag{7.1}$$

for all $v, w \in U$, $\alpha, \beta \in \mathbb{H}$. The same is true if one interchanges λ and μ.

Remark 7.2. If V is an F_λ-vector space whose quaternionification is $U = V \oplus V\mu$, then the quaternionic $\mathrm{U}(V)$-representation on U is irreducible. Furthermore, if W is a real form of U, i.e., if $U = W \oplus W\lambda \oplus W\mu \oplus W\lambda\mu$ holds, then the $\mathrm{O}(W)$-representation on U is irreducible, too, see [Kna16, Lemma 7.11].

Suppose that $U = U_1 \oplus U_2$ is a q-orthogonal direct sum of two quaternionic subspaces. As in Sections 5 and 6 we write $\mathrm{Sp}(U_1) \times \mathrm{Sp}(U_2)$ for the subgroup of $\mathrm{Sp}(U)$ that stabilizes this decomposition.

7.2. **Strategy of classification.** We consider the $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$-action on $\mathbb{H}^n$ given by $(A, a) \cdot v = Ava^{-1}$ for all $(A, a) \in \mathrm{Sp}(n) \times \mathrm{Sp}(1)$ and $v \in \mathbb{H}^n$. Recall that we want to find all real subspaces W of $\mathbb{H}^n$ and all connected subgroups K of $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W)$ which act transitively on the connected components of the spheres in W.

Given such a W, the group $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W)$ acts necessarily irreducibly on W. Since the $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$-action on $\mathbb{H}^n$ maps quaternionic subspaces to quaternionic subspaces, the maximal quaternionic subspace $W_{\mathbb{H}} = W \cap Wi \cap Wj \cap Wk$ of W is $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W)$-invariant. Consequently, W is either quaternionic or its maximal quaternionic subspace $W_{\mathbb{H}}$ is zero. Lemma 7.3 gives a normal form for quaternionic W which allows to complete the classification in this case.

Therefore we are left to deal with subspaces W with $W_{\mathbb{H}} = 0$ on which $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W)$ acts irreducibly. This case splits into two subcases: Either there exists a $\lambda \in \mathrm{Sp}(1) \cap \mathrm{Im}(\mathbb{H})$ such that the subfield $F_\lambda \cong \mathbb{C}$ acts by right multiplication on W or W is totally real in the sense that there exists no $\lambda \in \mathrm{Sp}(1) \setminus \{\pm 1\}$ with $W = W\lambda$. It turns out that the first of these subcases can be treated by the same arguments as the ones developed in Section 6, while the totally real case requires most of the work.

In closing we describe how these three cases influence the structure of $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W)$. The normalizer $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W)$ is the direct product of $\mathcal{N}_{\mathrm{Sp}(n)}(W) := \mathcal{N}_{\mathrm{Sp}(n)\times\{e\}}(W)$ and a normal subgroup L of $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W)$ that is isomorphic to the image of $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W)$ under the projection $\pi_2\colon \mathrm{Sp}(n) \times \mathrm{Sp}(1) \to \mathrm{Sp}(1)$ given by $(g, h) \mapsto h$, see [Kna16, Lemma 7.13]. Note that L contains $\mathcal{N}_{\mathrm{Sp}(1)}(W) := \mathcal{N}_{\{e\}\times\mathrm{Sp}(1)}(W)$. Moreover, if $\mathfrak{l} \cong \mathfrak{sp}(1)$, then $\mathcal{N}_{\mathrm{Sp}(1)}(W)$ is either trivial or coincides with L, see [Kna16, Lemma 7.13].

The group $\mathcal{N}_{\mathrm{Sp}(1)}(W)$ is of real dimension 3 if and only if W is quaternionic. It has real dimension 1 if and only if W is not quaternionic but invariant under the subfield $F_\lambda = \mathbb{R}\oplus\mathbb{R}\lambda$ of $\mathbb{H}$ for some $\lambda \in \mathrm{Sp}(1)\cap\mathrm{Im}(\mathbb{H})$. The group $\mathcal{N}_{\mathrm{Sp}(1)}(W)$ is zero-dimensional if and only if W is totally real, see [Kna16, Lemma 7.16].

7.3. **The quaternionic case.** The following lemma gives a normal form for a quaternionic subspace $W \subset \mathbb{H}^n$ and its normalizer. For a proof see [Kna16, Lemma 7.17].

Lemma 7.3. *Let W be any quaternionic subspace of $\mathbb{H}^n$. Then $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W)$ coincides with $(\mathrm{Sp}(W)\times\mathrm{Sp}(W^\perp))\times\mathrm{Sp}(1)$ and acts transitively on the spheres in W. Moreover, there is an orthonormal basis $(v_1,\ldots,v_n)$ of $\mathbb{H}^n$ such that $W = v_1\cdot\mathbb{H}\oplus\cdots\oplus v_l\cdot\mathbb{H}$ where $l = \dim_{\mathbb{H}} W$.*

7.4. **The F_λ-complex case.** Let us now suppose that W is not quaternionic but that there exists a $\lambda \in \mathrm{Sp}(1)\cap\mathrm{Im}(\mathbb{H})$ such that W is invariant under the subfield F_λ. In that case the assumption that

$$\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)} = \mathcal{N}_{\mathrm{Sp}(n)}(W)\times\underbrace{(\mathrm{Sp}(1)\cap F_\lambda)}_{=L\cong S^1}$$

acts transitively on the spheres in W implies that W is an F_λ-irreducible representation of $\mathcal{N}_{\mathrm{Sp}(n)}(W)$, see [Kna16, Lemma 7.18]. We complete $(1,\lambda)$ to a $\mathrm{Re}\, q_{\mathbb{H}}$-orthonormal basis $(1,\lambda,\mu,\lambda\mu)$ over $\mathbb{R}$ of $\mathbb{H}$. Then $V := W\oplus W\mu \subset \mathbb{H}^n$ is the quaternionification of W on which $\mathcal{N}_{\mathrm{Sp}(n)}(W)\subset\mathrm{Sp}(n)$ acts $\mathbb{H}$-linearly. The idea is to forget the F_λ-structure on W and $\mathbb{H}^n$, i.e., to consider W as maximal totally real subspace of the complex space V (with respect to F_μ), and then to use the same ideas as in Section 6.

In particular we consider the Hermitian form h_μ given by projecting q to F_μ, i.e., $h_\mu = s+\mu\omega_\mu$, where $s = \mathrm{Re}\, q$ is a real inner product on $\mathbb{H}^n$ and $\omega_\mu = \mathrm{Re}(-\mu\cdot q)$ is a symplectic form on $\mathbb{H}^n$ (see Lemma 7.1). We then define the $\mathbb{R}$-linear map $\mathcal{I}_\mu : W\to W$ by the identity

$$s(\mathcal{I}_\mu w_1, w_2) := \omega_\mu(w_1,w_2) \quad \forall\, w_1,w_2\in W, \tag{7.2}$$

and extend it F_μ-linearly to $V = W\oplus W\mu$.

Since $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ is a subgroup of $\mathcal{N}_{\mathrm{U}(F_\mu^n,h_\mu)}(W)$ we obtain as in Section 6 that V decomposes as

$$V = \ker\mathcal{I}_\mu \oplus \bigoplus_{\xi\in\mu\mathbb{R}^*} V_\xi$$

into $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-invariant and h_μ-orthogonal eigenspaces of $\mathcal{I}_\mu$ to eigenvalues in $\mu\mathbb{R}$ such that $\sigma_\mu(V_\xi) = V_{-\xi}$ holds where σ_μ denotes complex conjugation of V with respect to W. Furthermore, the eigenspaces are quaternionic subspaces that are q-orthogonal (see [Kna16, Lemma 7.25]). Thus we obtain the $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-invariant decomposition

$$W = \ker\mathcal{I}_\mu \oplus \bigoplus_{\xi\in\mu\mathbb{R}^{>0}} (V_\xi\oplus\sigma_\mu(V_\xi))^{\sigma_\mu}.$$

Since the group $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts irreducibly on W by assumption, we have either $W = \ker(\mathcal{I}_\mu)$ or $W = (V_\xi\oplus\sigma_\mu(V_\xi))^{\sigma_\mu}$ for some $\xi\in\mu\mathbb{R}^{>0}$. This discussion proves part of the following proposition.

Proposition 7.4. *Let W be an F_λ-invariant subspace of $\mathbb{H}^n$ with quaternionification $V = W\oplus W\mu$. If $K\subset\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts irreducibly on W, then*

(i) *either $W\subset V$ is Lagrangian with respect to η_λ, V is a quaternionic irreducible $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representation and $\mathcal{N}_{\mathrm{Sp}(n)}(W) = \mathrm{U}_{h_\lambda}(W)\times\mathrm{Sp}(W^{\perp_q})$.*

(ii) *or $W \subset V$ is symplectic with respect to η_λ, V decomposes as $V = V_\xi \oplus V_{-\xi}$ into two, q-orthogonal, quaternionic irreducible $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations which are as F_λ-representations isomorphic to W and*

$$\mathcal{N}_{\mathrm{Sp}(n)}(W) = \big\{(g, \psi(g)) \in \mathrm{Sp}(V_\xi) \times \mathrm{Sp}(V_{-\xi})\big\} \times \mathrm{Sp}(W^{\perp_q}),$$

where $\psi : \mathrm{Sp}(V_\xi) \to \mathrm{Sp}(V_{-\xi})$ is a Lie group isomorphism.

In both cases $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts transitively on the spheres in W.

Proof. If $W \subset V$ is Lagrangian with respect to η_λ, then W and $W\mu$ are orthogonal with respect to h_λ. We now want to show that $\mathcal{N}_{\mathrm{Sp}(n)}(W) = \mathrm{U}_{h_\lambda}(W) \times \mathrm{Sp}(W^{\perp_q})$ holds, where $\mathrm{U}_{h_\lambda}(W) := \{g \in \mathrm{GL}_{F_\lambda}(W) : h_\lambda(gv, gw) = h_\lambda(v, w)\ \forall\ v, w \in W\}$. Any element in $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ leaves W, $W^{\perp_q}$ and $h_\lambda|_{W\times W}$ invariant and lies therefore in $\mathrm{U}_{h_\lambda}(W) \times \mathrm{Sp}(W^{\perp_q})$. Conversely any element in $\mathrm{U}_{h_\lambda}(W)$ acts $\mathbb{H}$-linearly on V. Again, due to the fact that W is Lagrangian in V, the quaternionic inner product q is $\mathrm{U}_{h_\lambda}(W)$-invariant. Hence $\mathrm{U}_{h_\lambda}(W) \times \mathrm{Sp}(W^{\perp_q}) \subset \mathcal{N}_{\mathrm{Sp}(n)}(W)$ and therefore $\mathcal{N}_{\mathrm{Sp}(n)}(W) = \mathrm{U}_{h_\lambda}(W) \times \mathrm{Sp}(W^{\perp_q})$ holds. The group $\mathcal{N}_{\mathrm{Sp}(n)}(W) = \mathrm{U}_{h_\lambda}(W) \times \mathrm{Sp}(W^{\perp_q})$ acts transitively on the spheres in W and quaternionic irreducibly on V (see Remark 7.2).

Let us now consider the case that $V = V_\xi \oplus V_{-\xi}$ is the decomposition of V into $\mathcal{I}_\mu$-eigenspaces corresponding to the eigenvalues $\pm\xi \in \mu\mathbb{R}\backslash\{0\}$. Recall that this decomposition is q-orthogonal and that the F_μ-anti-linear involution σ_μ on V yields an isomorphism between V_ξ and $V_{-\xi}$. Hence, we obtain F_λ-linear, $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-equivariant isomorphisms $g_\pm$ between $V_{\pm\xi}$ and W given by $g_\pm(v) = \frac{1}{2}\big(v + \sigma_\mu(v)\big)$. In particular, the quaternionic spaces $V_{\pm\xi}$ are F_λ-irreducible $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations isomorphic to W. We define another quaternionic inner product q_{σ_μ} on $V_{-\xi}$ by

$$q_{\sigma_\mu}(v, u) = \sigma_{\mathbb{H},\mu}\big(q(\sigma_\mu(v), \sigma_\mu(u))\big) \quad \forall\ v, u \in V_{-\xi} = \sigma_\mu(V_\xi).$$

One verifies directly that q_{σ_μ} and $q|_{V_{-\xi}\times V_{-\xi}}$ are both $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-invariant. Thus there exists a $t \in \mathbb{R}^{>0}$ such that $q_{\sigma_\mu} = t^2 q|_{V_{-\xi}\times V_{-\xi}}$. Suppose for a moment that $t = 1$, i.e., that σ_μ is an isometry of h_λ. Consequently, W is orthogonal to $W\mu$ with respect to h_λ, hence a Lagrangian subspace of V with respect to η_λ, contradicting our assumption. Thus we conclude $t \neq 1$.

We define the map $\psi : \mathrm{Sp}(V_\xi) \to \mathrm{GL}(V_{-\xi})$ by $\psi(g)v = \sigma_\mu\big(g \cdot \sigma_\mu^{-1}(v)\big)$ for all $v \in V_{-\xi}$. A direct calculation shows that ψ is a Lie group isomorphism from $\mathrm{Sp}(V_\xi)$ to $\mathrm{Sp}(V_{-\xi})$. This observation allows us to show that

$$\mathcal{N}_{\mathrm{Sp}(n)}(W) = \big\{(g, \psi(g)) \in \mathrm{Sp}(V_\xi) \times \mathrm{Sp}(V_{-\xi})\big\} \times \mathrm{Sp}(W^{\perp_q}).$$

Indeed any element in $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts as element of $\mathrm{Sp}(V_\xi) \times \mathrm{Sp}(V_{-\xi})$ on $V_\xi \times V_{-\xi}$. Using the fact that any element in $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ stabilizes $W = V^{\sigma_\mu}$ we obtain

$$(k_1, k_2) \cdot \big(v + \sigma_\mu(v)\big) = k_1 v + k_2 \sigma_\mu(v) = k_1 v + \sigma_\mu\big(\psi(k_2)v\big)$$

has to lie in W for all $(k_1, k_2) \in \mathrm{Sp}(V_\xi)\times\mathrm{Sp}(V_{-\xi})$ and for all $v \in V_\xi$. This implies $k_1 v = \psi(k_2)v$ for all $v \in V_\xi$ forcing $k_2 = \psi(k_1)$ and thus

$$\mathcal{N}_{\mathrm{Sp}(n)}(W) \subset \big\{(g, \psi(g)) \in \mathrm{Sp}(V_\xi) \times \mathrm{Sp}(V_{-\xi})\big\} \times \mathrm{Sp}(W^{\perp_q}).$$

The converse conclusion is elementary to check. □

Introducing coordinates we obtain the following normal form of these real subspaces W of $(\mathbb{H}^n, q)$, up to the action of $\mathrm{Sp}(n)$.

Corollary 7.5. *Let W be a real subspace of $\mathbb{H}^n$, that has no quaternionic subspace but is invariant under F_λ for some $\lambda \in \mathrm{Sp}(1) \cap \mathrm{Im}(\mathbb{H})$. If some compact subgroup of $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts irreducibly on W, then W lies in the same $\mathrm{Sp}(n)$-orbit as one of the following*

$$W_{F_\lambda,l} := F_\lambda^l \times \{0\}^{n-l}, \qquad 0 \leq l \leq n$$
$$W_{F_\lambda,l,t} := \left\{ \begin{pmatrix} z+\mu w \\ t(z-\mu w) \\ 0 \end{pmatrix} : z, w \in F_\lambda^l \right\}, \qquad 0 \leq l \leq \lfloor \tfrac{n}{2} \rfloor$$

for some $t \in (0,1)$. The corresponding normalizers are given by

$$\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W_{F_\lambda,l}) := \big(\mathrm{U}_{h_\lambda}(l) \times \mathrm{Sp}(n-l)\big) \times (\mathrm{Sp}(1) \cap F_\lambda),$$
$$\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W_{F_\lambda,l,t}) := \left\{ \left(\begin{pmatrix} A & & \\ & \psi(A) & \\ & & B \end{pmatrix}, a \right) \in \mathrm{Sp}(n) \times \mathrm{Sp}(1) : \ A \in \mathrm{Sp}(l), B \in \mathrm{Sp}(n-2l), a \in F_\lambda \right\}$$
$$\cong (\mathrm{Sp}(l) \times \mathrm{Sp}(n-2l)) \times S^1.$$

In both cases $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts transitively on the connected components of the spheres in W.

Proof. If W is F_λ-invariant and Lagrangian in $V = W \oplus W\mu$ with respect to η_λ, then W is orthogonal to $W\mu$ with respect to h_λ. Therefore any h_λ-orthonormal basis over F_λ of W is a q-orthonormal basis over $\mathbb{H}$ of V. In particular, if we choose an h_λ-orthonormal basis $(w_1, \ldots, w_l)$ of W and complete it to a q-orthonormal basis $(w_1, \ldots, w_n)$ of $\mathbb{H}^n$, we obtain an element $(w_1 \cdots w_n) \in \mathrm{Sp}(n)$ that maps $W_{F_\lambda,l}$ to W.

If W is symplectic with respect to η_λ, then we saw in the proof of Proposition 7.4 that there exists a $t \in \mathbb{R}^{>0}\backslash\{1\}$ such that $\sigma_{\mathbb{H},\mu}\big(q(\sigma_\mu(v), \sigma_\mu(u))\big) = t^2 q(v,u)$ for all $v, u \in V_{-\xi}$. If we choose a q-orthonormal basis $(v_1, \ldots, v_{l/2})$ over $\mathbb{H}$ of V_ξ and set $v_{l/2+j} := \frac{1}{t}\sigma_\mu(v_j)$ for all $1 \leq j \leq l/2$ we obtain the q-orthonormal basis $(v_1, \ldots, v_l)$ of $V = V_\xi \oplus V_{-\xi}$. Completing it to a q-orthonormal basis $(v_1, \ldots, v_n)$ of $\mathbb{H}^n$ the element $(v_1 \cdots v_n) \in \mathrm{Sp}(n)$ maps $W_{F_\lambda,l,t}$ to W. Note that t may be chosen between 0 and 1, because the element $\left(\begin{smallmatrix} 0 & \mathrm{id} & 0 \\ \mathrm{id} & 0 & 0 \\ 0 & 0 & \mathrm{id} \end{smallmatrix}\right) \in \mathrm{Sp}(n)$ maps $W_{F_\lambda,l,t}$ to $W_{F_\lambda,l,1/t}$. □

7.5. **The totally real case.** Let us now consider the case that W is a totally real subspace of $\mathbb{H}^n$, i.e., that there exists no $\lambda \in \mathrm{Sp}(1)\backslash\{\pm 1\}$ such that $W = W\lambda$.

Again we want to determine those real subspaces $W \subset (\mathbb{H}^n, q)$ such that $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W)$ acts transitively on the spheres in W. Recall that $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W) = \mathcal{N}_{\mathrm{Sp}(n)}(W) \times L$.

Theorem 4.2 implies that if $\dim_{\mathbb{R}}(W) > 4$ then already $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ has to act transitively on the spheres in W. We therefore consider first totally real subspaces W of arbitrary dimension which satisfy the condition that $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts irreducibly on W. The only spherical subalgebras that we are possibly missing are those for wich $\dim_{\mathbb{R}}(W) \leq 4$. We will consider these cases at the end of this subsection.

We first give a normal form of totally real W up to the action of $\mathrm{GL}(\mathbb{H}^n)$.

Lemma 7.6. *Let W be a real subspace of $\mathbb{H}^n$ such that $\langle W \rangle_{\mathbb{H}} = W \oplus Wi \oplus Wj \oplus Wk$. Then W lies in the same $\mathrm{GL}(\mathbb{H}^n)$-orbit as $W_{\mathbb{R},l} := \mathbb{R}^l \times \{0\}^{n-l}$, where $l = \dim_{\mathbb{R}}(W)$ and $\mathfrak{l}$ acts trivially on W.*

Proof. Let $(v_1, \ldots, v_l)$ be an s-orthonormal real basis of W. Then $\langle W \rangle_{\mathbb{H}} = \langle v_1, \ldots, v_l \rangle_{\mathbb{H}}$. In particular $(v_1, \ldots, v_l)$ is a quaternionic basis of $\langle W \rangle_{\mathbb{H}}$ that we complete to a quaternionic basis $(v_1, \ldots, v_n)$ of $\mathbb{H}^n$. Now $g = (v_1 \cdots v_n)$ is an element in $\mathrm{GL}(\mathbb{H}^n)$ that maps $W_{\mathbb{R},l}$ to W.

This implies that

$$\begin{aligned}
\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W) &\subset \mathrm{Ad}(g)(\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W_{\mathbb{R},l}))\\
\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W_{\mathbb{R},l}) &= \{(A,\xi)\in\mathfrak{sp}(n)\oplus\mathfrak{sp}(1) : Aw - w\xi \in W_0\ \forall\, w\in W_0\}\\
&= \left\{(A,\xi)\in\mathfrak{sp}(n)\oplus\mathfrak{sp}(1) : A = \left(\begin{smallmatrix} A_1 & 0\\ 0 & *\end{smallmatrix}\right), A_1 - \xi\,\mathrm{id}\in\mathbb{R}^{l\times l}\right\}\\
&= \underbrace{\left\{(A,0)\in\mathfrak{sp}(n)\oplus\mathfrak{sp}(1) : A = \left(\begin{smallmatrix} A_1 & 0\\ 0 & *\end{smallmatrix}\right), A_1 \in\mathbb{R}^{l\times l}\right\}}_{\mathcal{N}_{\mathfrak{sp}(n)}(W_{\mathbb{R},l})}\\
&\quad\oplus \underbrace{\left\{(A,\xi)\in\mathfrak{sp}(n)\oplus\mathfrak{sp}(1) : A = \left(\begin{smallmatrix} \xi\,\mathrm{id} & 0\\ 0 & 0\end{smallmatrix}\right)\right\}}_{\tilde{\mathfrak{l}}}.
\end{aligned}$$

Therefore, the action of $\tilde{\mathfrak{l}}$ on $W_{\mathbb{R},l}$ and thus on W is trivial. □

In order to give a normal form of W up to the action of $\mathrm{Sp}(n)$, we will analyze the position of W in $\mathbb{H}^n$ with respect to the various symplectic forms listed in Lemma 7.1 which are invariant under $\mathrm{Sp}(n)$. We will see that under the condition that $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts irreducibly on W, these symplectic forms contain information about the division algebra $\mathrm{End}_{\mathcal{N}_{\mathrm{Sp}(n)}(W)}(W)$. Hence, we will now divide such totally real W into three subcases, corresponding to the three possibilities $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ for $\mathrm{End}_{\mathcal{N}_{\mathrm{Sp}(n)}(W)}(W)$.

We define the $\mathbb{R}$-linear maps $\mathcal{I}_i, \mathcal{I}_j, \mathcal{I}_k : W\to W$ by the identity

$$s(\mathcal{I}_m w_1, w_2) = \omega_m(w_1,w_2)\quad \forall\, w_1,w_2\in W,\ \forall\, m\in\{i,j,k\}. \tag{7.3}$$

Let us consider the map

$$\begin{aligned}
\chi : \mathbb{H} &\to \mathrm{End}_{\mathcal{N}_{\mathrm{Sp}(n)}(W)}(W)\\
a+bi+cj+dk &\mapsto a\,\mathrm{id} + b\mathcal{I}_i + c\mathcal{I}_j + d\mathcal{I}_k.
\end{aligned}$$

Note that the kernel of χ lies in $\mathrm{Im}(\mathbb{H})$, i.e., that $\chi|_{\mathbb{R}}$ is injective and that

$$s(\chi(z)w_1, w_2) = s(w_1 z, w_2)$$

holds for all $z\in\mathbb{H}$ and $w_1,w_2\in W$. In particular, if $\lambda\in\mathrm{Im}(\mathbb{H})\cap\mathrm{Sp}(1)$, then $s(\chi(\lambda)w_1,w_2) = s(w_1\lambda, w_2) = \omega_\lambda(w_1,w_2)$ (see Lemma 7.1), i.e., $\chi(\lambda) = \mathcal{I}_\lambda$, where $\mathcal{I}_\lambda$ is defined by the equation

$$s(\mathcal{I}_\lambda w_1, w_2) = \omega_\lambda(w_1,w_2) \tag{7.4}$$

for all $w_1,w_2\in W$. The $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-equivariant $\mathbb{R}$-linear endomorphisms of W form a division algebra over $\mathbb{R}$ that is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We are going to successively consider the cases that $\dim_{\mathbb{R}}(\chi(\mathbb{H}))$ equals 1 (see Lemma 7.7), 2 (see Lemma 7.8) or is bigger than or equal to 3 (see Lemma 7.9). We will see in Remark 7.10 that these cases do indeed correspond to the cases that $\mathrm{End}_{\mathcal{N}_{\mathrm{Sp}(n)}(W)}(W)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ respectively.

Let us start with the case that $\dim_{\mathbb{R}}(\chi(\mathbb{H}))=1$. Since we already noted that the kernel of χ lies in $\mathrm{Im}(\mathbb{H})$, we obtain that $\omega_i|_{W\times W} = \omega_j|_{W\times W} = \omega_k|_{W\times W} = 0$ in this case.

Lemma 7.7. *Let W be a totally real subspace of $\mathbb{H}^n$ of real dimension l with quaternionification $U := W\oplus Wi\oplus Wj\oplus Wk$. If $W\subset U$ is isotropic with respect to ω_i, ω_j and ω_k and $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts irreducibly on W, then U is a quaternionic irreducible $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representation, $\mathcal{N}_{\mathrm{Sp}(n)}(W) = \mathrm{O}(W)\times\mathrm{Sp}(W^{\perp_q})$ acts transitively on the spheres in W, $\mathfrak{l}\cong\mathfrak{sp}(1)$ acts trivially on W, and W lies in the same $\mathrm{Sp}(n)$-orbit as $W_{\mathbb{R},l} := \mathbb{R}^l\times\{0\}^{n-l}$.*

Proof. Since $W\subset U$ is isotropic with respect to ω_i the spaces W and $W\cdot i$ are s-orthogonal. Any element in $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ respects W, $W^{\perp_q}$ and leaves $s|_{W\times W}$ invariant and lies therefore in $\mathrm{O}(W)\times\mathrm{Sp}(W^{\perp_q})$. Conversely, any element in $\mathrm{O}(W)$ acts $\mathbb{H}$-linearly on U. Again, due to the fact that W is isotropic with respect to ω_i, ω_j and ω_k, the quaternionic inner product $q|_{U\times U}$ is $\mathrm{O}(W)$-invariant, which implies $\mathrm{O}(W)\times\mathrm{Sp}(W^{\perp_q})\subset\mathcal{N}_{\mathrm{Sp}(n)}(W)$. This shows $\mathcal{N}_{\mathrm{Sp}(n)}(W)=\mathrm{O}(W)\times\mathrm{Sp}(W^{\perp_q})$. Consequently, Remark 7.2 implies that $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts transitively on the spheres in W and quaternionic irreducibly on U.

If we choose an s-orthonormal basis $(w_1,\dots,w_l)$ over $\mathbb{R}$ of W then $(w_1,\dots,w_l)$ is a q-orthonormal basis over $\mathbb{H}$ of U. Completing it to a q-orthonormal basis $(w_1,\dots,w_n)$ of $\mathbb{H}^n$ we obtain the element $(w_1\cdots w_n)\in\mathrm{Sp}(n)$ that maps $W_{\mathbb{R},l}$ to W. A direct calculation yields

$$\begin{aligned}\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W_{\mathbb{R},l})&=\{(A,\xi)\in\mathfrak{sp}(n)\oplus\mathfrak{sp}(1):Aw-w\xi\in W_{\mathbb{R},l}\ \forall\ w\in W_{\mathbb{R},l}\}\\&=\left\{(A,\xi)\in\mathfrak{sp}(n)\oplus\mathfrak{sp}(1):A=\left(\begin{smallmatrix}A_1&0\\0&*\end{smallmatrix}\right),A_1-\xi\,\mathrm{id}\in\mathbb{R}^{l\times l}\right\}\\&=\underbrace{\left\{(A,0)\in\mathfrak{sp}(n)\oplus\mathfrak{sp}(1):A=\left(\begin{smallmatrix}A_1&0\\0&*\end{smallmatrix}\right),A_1\in\mathbb{R}^{l\times l}\right\}}_{=\mathfrak{so}(l)\oplus\mathfrak{sp}(n-l)}\\&\qquad\oplus\underbrace{\left\{(A,\xi)\in\mathfrak{sp}(n)\oplus\mathfrak{sp}(1):A=\left(\begin{smallmatrix}\xi\,\mathrm{id}&0\\0&*\end{smallmatrix}\right)\right\}}_{\cong\mathfrak{sp}(1)},\end{aligned}$$

i.e., $\mathfrak{l}\cong\mathfrak{sp}(1)$ in this case and $\mathfrak{l}$ acts trivially on $W_{\mathbb{R},l}$ (see Lemma 7.6). □

Let us now assume that $\dim_{\mathbb{R}}(\chi(\mathbb{H}))=2$, i.e., the kernel of χ is a 2-dimensional subspace of $\mathrm{Im}(\mathbb{H})$. Its orthogonal complement in $\mathbb{H}$ is also 2-dimensional and contains $\mathbb{R}$. Let us assume that it is equal to $F_\lambda=\mathbb{R}\oplus\mathbb{R}\lambda$ for some $\lambda\in\mathrm{Im}(\mathbb{H})\cap\mathrm{Sp}(1)$, i.e., $\chi|_{F_\lambda}:F_\lambda\to\mathbb{R}\,\mathrm{id}\oplus\mathbb{R}\mathcal{I}_\lambda$ is an isomorphism. We choose the $\mathrm{Re}\,q_{\mathbb{H}}$-orthonormal basis $(1,\lambda,\mu,\lambda\mu)$ of $\mathbb{H}$. Then the kernel of χ is equal to $\mathbb{R}\mu\oplus\mathbb{R}\lambda\mu=F_\lambda\mu$. We set $U=W\oplus W\lambda\oplus W\mu\oplus W\lambda\mu$ and denote by $\sigma_\lambda:U\to U$ the F_λ-anti-linear and F_μ-linear involution that fixes W pointwise. Analogously we let $\sigma_\mu:U\to U$ denote the F_μ-anti-linear and F_λ-linear involution that fixes W pointwise. The two involutions σ_λ and σ_μ are $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-equivariant and commute. Furthermore W is the joint set of fixed points of σ_λ and σ_μ in U, i.e., $W=(U^{\sigma_\lambda})^{\sigma_\mu}=(U^{\sigma_\mu})^{\sigma_\lambda}$. The map $f_{\sigma_\mu}:U\to U^{\sigma_\mu}$ defined by $u\mapsto\frac{1}{2}(u+\sigma_\mu(u))$ is F_λ-linear, $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-equivariant, surjective and has kernel $W\mu\oplus W\lambda\mu$.

Lemma 7.8. *Let W be a totally real subspace of $\mathbb{H}^n$ of real dimension l with quaternionification $U:=W\oplus W\lambda\oplus W\mu\oplus W\lambda\mu$ such that $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts irreducibly on W. We assume furthermore that $\mathcal{I}_\lambda\neq0$ on W and $\omega_\mu|_{W\times W}=\omega_{\mu\lambda}|_{W\times W}=0$, where $\mathcal{I}_\lambda$ is defined by (7.4). Then U decomposes q-orthogonally as $U=U_{\mathcal{I}_\lambda,\xi}\oplus U_{\mathcal{I}_\lambda,-\xi}$, where $U_{\mathcal{I}_\lambda,\xi}$ and $U_{\mathcal{I}_\lambda,-\xi}$ are quaternionic irreducible $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations while $f_{\sigma_\mu}(U_{\mathcal{I}_\lambda,\xi})$ and $f_{\sigma_\mu}(U_{\mathcal{I}_\lambda,-\xi})$ are F_λ-irreducible $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations that are as real representations isomorphic to W. Furthermore*

$$\begin{aligned}&\mathcal{N}_{\mathrm{Sp}(n)}(W)=\{(g,\varphi(g))\in\mathrm{U}(f_{\sigma_\mu}(U_{\mathcal{I}_\lambda,\xi}),h_\lambda)\times\mathrm{U}(f_{\sigma_\mu}(U_{\mathcal{I}_\lambda,-\xi}),h_\lambda)\}\times\mathrm{Sp}(W^{\perp_q}),\\&\mathfrak{l}\cong\mathfrak{u}(1),\end{aligned}$$

where $\varphi:\mathrm{U}(f_{\sigma_\mu}(U_{\mathcal{I}_\lambda,\xi}),h_\lambda)\to\mathrm{U}(f_{\sigma_\mu}(U_{\mathcal{I}_\lambda,-\xi}),h_\lambda)$ defined by $\varphi(g)(v)=\sigma_\lambda(g\cdot\sigma_\lambda(v))$ for all $v\in f_{\sigma_\mu}(U_{\mathcal{I}_\lambda,-\xi})$ is a Lie group isomorphism and W lies in the same $\mathrm{Sp}(n)$-orbit as $W_{\mathbb{R},l,r,F_\lambda}:=\left\{\begin{pmatrix}x+\lambda y\\r(x-\lambda y)\\0\end{pmatrix}:x,y\in\mathbb{R}^{l/2}\right\}$ for some $r\in(0,1)$. Furthermore $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts transitively on the spheres in W while $\mathfrak{l}$ acts trivially on W.

Proof. We denote by V_λ the complexification $W\oplus W\lambda$ of W. Note that $V_\lambda=U^{\sigma_\mu}$. The assumption $\omega_\mu|_{W\times W}=\omega_{\mu\lambda}|_{W\times W}=0$ implies $\omega_\mu|_{V_\lambda\times V_\lambda}=\omega_{\mu\lambda}|_{V_\lambda\times V_\lambda}=0$. This gives in turn that V_λ is h_λ-orthogonal to $V_\lambda\cdot\mu$. Therefore any h_λ-orthonormal basis over F_λ of V_λ is a q-orthonormal basis over $\mathbb{H}$ of U. In particular, if we choose an h_λ-orthonormal basis of V_λ and complete it to a q-orthonormal basis of $\mathbb{H}^n$, we obtain an element in $\mathrm{Sp}(n)$ that maps $V_{F_\lambda,l} := F_\lambda^l \times \{0\}^{n-l}$ to V_λ. This implies that $\mathcal{N}_{\mathrm{Sp}(n)}(V_\lambda) = \mathrm{U}(V_\lambda, h_\lambda) \times \mathrm{Sp}(U^{\perp_q})$ acts F_λ-irreducibly on V_λ and quaternionic irreducibly on U (see Remark 7.2). Since $\mathrm{Sp}(n)$ acts $\mathbb{H}$-linear on $\mathbb{H}^n$ any element in $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ leaves V_λ invariant. As a result we obtain

$$\mathcal{N}_{\mathrm{Sp}(n)}(W) \subset \mathcal{N}_{\mathrm{Sp}(n)}(V_\lambda) = \mathrm{U}(V_\lambda, h_\lambda) \times \mathrm{Sp}(U^{\perp_q})$$

and the action of $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ on W is orbit equivalent to the action of the first factor, i.e., the action of $\mathcal{N}_{\mathrm{U}(V_\lambda,h_\lambda)}(W)$. Therefore $\mathcal{N}_{\mathrm{U}(V_\lambda,h_\lambda)}(W)$ acts irreducibly on W.

Since $\mathcal{I}_\lambda$ is defined as in Section 6 we obtain that the kernel of $\mathcal{I}_\lambda : W \to W$ is equal to the maximal subspace of W on which ω_λ is degenerate. Since W is a maximal totally real subspace of the Hermitian vector space (V_λ, h_λ) and $\mathrm{Sp}(n)$ is a subgroup of $\mathrm{U}(F_\lambda^{2n}, h_\lambda)$, the requirements of Proposition 6.4 are met and we conclude (since $\mathcal{I}_\lambda \neq 0$) that $W \subset V_\lambda$ is symplectic with respect to ω_λ, V_λ decomposes h_λ-orthogonally as $V_\lambda = V_{\lambda,\xi} \oplus V_{\lambda,-\xi}$ into two F_λ-irreducible $\mathcal{N}_{\mathrm{U}(F_\lambda^{2n},h_\lambda)}(W)$-representations that are as real representations isomorphic to W and which are eigenspaces of the F_λ-linear continuation of $\mathcal{I}_\lambda : W \to W$ to V_λ for the eigenvalues $\pm\xi \in \lambda\mathbb{R}\backslash\{0\}$. The involution σ_λ induces an $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-equivariant and F_λ-anti-linear isomorphism between $V_{\lambda,\xi}$ and $V_{\lambda,-\xi}$ and

$$\mathcal{N}_{\mathrm{U}(V_\lambda,h_\lambda)}(W) = \{(g, \varphi(g)) \in \mathrm{U}(V_{\lambda,\xi}) \times \mathrm{U}(V_{\lambda,-\xi})\},$$

where the Lie group isomorphism $\varphi : \mathrm{U}(V_{\lambda,\xi}) \to \mathrm{U}(V_{\lambda,-\xi})$ is given by $\varphi(g)(v) = \sigma_\lambda(g\sigma_\lambda(v))$ for all $g \in \mathrm{U}(V_{\lambda,\xi})$ and $v \in V_{\lambda,-\xi}$.

We already stated that the action of $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ on W and V_λ is orbit equivalent to the action of $\mathcal{N}_{\mathrm{U}(V_\lambda,h_\lambda)}(W)$ on W and V_λ. As a result the subspaces $V_{\lambda,\pm\xi}$ are two F_λ-irreducible $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations that are as real representations isomorphic to W and

$$\mathcal{N}_{\mathrm{Sp}(n)}(W) = \{(g, \varphi(g)) \in \mathrm{U}(V_{\lambda,\xi}, h_\lambda) \times \mathrm{U}(V_{\lambda,-\xi}, h_\lambda)\} \times \mathrm{Sp}(U^{\perp_q}).$$

Note that due to Remark 7.2 the induced representations of $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ on $U_{\mathcal{I}_\lambda,\pm\xi} := \langle V_{\lambda,\pm\xi}\rangle_{\mathbb{H}}$ are as quaternionic representations irreducible. We conclude that $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts transitively on the spheres in W, F_λ-irreducible on $V_{\lambda,\pm\xi}$ and quaternionic irreducibly on $U_{\mathcal{I}_\lambda,\pm\xi}$.

If we map V_λ to $V_{F_\lambda,l}$ by an element in $\mathrm{Sp}(n)$ as described above, Corollary 6.5 yields that W lies in the same $\mathrm{U}(V_\lambda, h_\lambda)$-orbit as $W_{\mathbb{R},l,r,F_\lambda}$ for some $r \in (0, 1)$. A direct calculation shows $\mathfrak{l} \cong \mathfrak{u}(1)$. The action of $\mathfrak{l}$ on W is trivial (see Lemma 7.6). □

Let us now assume $\dim_{\mathbb{R}}(\chi(\mathbb{H})) \geq 3$. In particular $\mathrm{End}_{\mathcal{N}_{\mathrm{Sp}(n)}(W)}(W) \cong \mathbb{H}$. In order to simplify this case we are not only going to assume that $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts irreducibly on W but we are going to use the stronger condition that $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts transitively on the spheres in W. As we noted at the begin of this section this is always the case if $\mathcal{N}_M(W)$ acts transitively on the spheres in W and $\dim_{\mathbb{R}}(W) > 4$. The case $\dim_{\mathbb{R}}(W) \leq 4$ will be considered later.

Lemma 7.9. *Let W be a totally real subspace of $\mathbb{H}^n$ of real dimension l with quaternionification $U := W \oplus W\lambda \oplus W\mu \oplus W\lambda\mu$ such that $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts transitively on the spheres in W. We assume $\mathrm{End}_{\mathcal{N}_{\mathrm{Sp}(n)}(W)}(W) \cong \mathbb{H}$. Then $U = \langle W\rangle_{\mathbb{H}}$ decomposes into four isomorphic quaternionic irreducible $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations that are as real $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations isomorphic to W, $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ is isomorphic to $\mathrm{Sp}(W) \times \mathrm{Sp}(W^{\perp_q})$ and its action on W is given by the standard action of $\mathrm{Sp}(W)$ on W. In this case W lies in the same $\mathrm{Sp}(n)$-orbit as*

$$W_{\mathbb{R},l,z} := \left\{ \begin{pmatrix} u \\ uz_2 \\ uz_3 \\ uz_4 \\ 0 \end{pmatrix} : u \in \mathbb{H}^{l/4} \right\}$$

for some $z \in \mathcal{P}$, where

$$\mathcal{P} = \left\{ (z_2, z_3, z_4)^t \in (\mathbb{H}^*)^3 : \langle 1, z_2, z_3, z_4 \rangle_{\mathbb{R}} = \mathbb{H}, \right.$$
$$\left. \dim_{\mathbb{R}}(\mathrm{Im}(\langle \xi + \overline{z_2}\xi z_2 + \overline{z_3}\xi z_3 + \overline{z_4}\xi z_4 : \xi \in \mathbb{H} \rangle_{\mathbb{R}})) \geq 2 \right\}.$$

Furthermore $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W_{\mathbb{R},l,z}) = \mathcal{N}_{\mathrm{Sp}(n)}(W_{\mathbb{R},l,z})$ for all $z \in \mathcal{P}$.

Proof. Since $\mathrm{End}_{\mathcal{N}_{\mathrm{Sp}(n)}(W)}(W)$ is assumed to be isomorphic to $\mathbb{H}$, Theorem [BtD85, Theorem II.6.7] implies that $W \in \mathrm{Irr}(\mathcal{N}_{\mathrm{Sp}(n)}(W), \mathbb{R})_{\mathbb{H}}$. This in turn implies that the complexification $V := \mathbb{C} \otimes_{\mathbb{R}} W$ decomposes as $\mathbb{C} \otimes_{\mathbb{R}} W = V_1 \oplus V_2$ into two complex irreducible $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations V_1, V_2, which are as complex $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations isomorphic and as real $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations isomorphic to W (see Proposition [BtD85, Proposition II.6.6 (iii)]). Moreover, for $m \in \{1, 2\}$, the quaternionification $\mathbb{H} \otimes_{\mathbb{C}} V_m$ of V_m decomposes as $\mathbb{H} \otimes_{\mathbb{C}} V_m = U_{m,1} \oplus U_{m,2}$ into two quaternionic irreducible $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations, which are as quaternionic $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations isomorphic and as real $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations isomorphic to W (see Proposition [BtD85, Proposition II.6.6 (ix)]). Overall this shows that the quaternionification $U := \mathbb{H} \otimes_{\mathbb{R}} W = \mathbb{H} \otimes_{\mathbb{C}} V$ decomposes as $U_{1,1} \oplus U_{1,2} \oplus U_{2,1} \oplus U_{2,2}$ into four quaternionic irreducible $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations, which are as quaternionic $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations isomorphic and as real $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations isomorphic to W.

Let $g : W \to U_{1,1}$ be an $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-equivariant $\mathbb{R}$-linear isomorphism. Since we assumed that $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts transitively on the spheres in W, it also acts transitively on the spheres in $U_{1,1}$. Furthermore $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ respect the restriction $q|_{U_{1,1}\times U_{1,1}}$ and therefore acts as a subgroup of $\mathrm{Sp}(U_{1,1})$ on $U_{1,1}$. Due to Onishchik's classification (see Theorem 4.3) there exists no proper subgroup of $\mathrm{Sp}(U_{1,1})$ that acts transitively on the spheres in $U_{1,1}$. Therefore the action of $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ on $U_{1,1}$ is the action of $\mathrm{Sp}(U_{1,1})$ on $U_{1,1}$. If we pull back the quaternionic structure on $U_{1,1}$ via $g : W \to U_{1,1}$ to W we obtain a quaternionic structure on W that is respected by $\mathcal{N}_{\mathrm{Sp}(n)}(W)$. Moreover the action of $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ on W is the action of $\mathrm{Sp}(W)$ on W. Since $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ leaves $W^{\perp_q}$ invariant we conclude $\mathcal{N}_{\mathrm{Sp}(n)}(W) = \mathrm{Sp}(W) \times \mathrm{Sp}(W^{\perp_q})$.

Since the four quaternionic irreducible $\mathcal{N}_{\mathrm{Sp}(n)}(W)$-representations $U_{1,1}, U_{1,2}, U_{2,1}, U_{2,2}$ are isomorphic, we may assume them to be q-orthogonal and thus W lies in the same $\mathrm{Sp}(n)$-orbit as $W_\psi = \left\{ \begin{pmatrix} u \\ \psi_2(u) \\ \psi_3(u) \\ \psi_4(u) \\ 0 \end{pmatrix} : u \in \mathbb{H}^{l/4} \right\}$, for some $\psi_2, \psi_3, \psi_4 \in \mathrm{End}_{\mathrm{Sp}(l/4)}(\mathbb{H}^{l/4}) \backslash \{0\} \cong \mathbb{H}^*$. A direct calculation then shows that the isomorphisms have to fulfill the constraints given by the set $\mathcal{P}$ in order for W_ψ to have the assumed properties. For the technical details of the proof of the normal form we refer the reader to [Kna16, Proposition 7.42]. $\square$

Remark 7.10. Proposition 6.6 and Theorem 6.7 in [BtD85] imply that Lemmas 7.7, 7.8 and 7.9 cover the cases that the division algebra $\mathrm{End}_{\mathcal{N}_{\mathrm{Sp}(n)}(W)}(W)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ respectively.

The following well-known Lemma 7.11 implies that the element $\lambda \in \mathrm{Im}(\mathbb{H}) \cap \mathrm{Sp}(1)$ in Proposition 7.4, Lemmas 7.5, 7.8 and 7.9 may be chosen to be i (while μ may be chosen as j).

For the proof of the following result we refer the reader to [Kna16, Lemma 7.45].

Lemma 7.11. *Any ring automorphism ψ of $\mathbb{H}$ is inner, i.e., for any ψ there exists a $q_\psi \in \mathbb{H} \backslash \{0\}$ such that $\psi(z) = q_\psi z q_\psi^{-1}$. Furthermore q_ψ can be chosen in $\mathrm{Sp}(1)$.*

Furthermore the action of any $p \in \mathrm{Sp}(1)$ on $\mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$ given by pvp^{-1} is a special orthogonal transformation, i.e., the map $\mathrm{Sp}(1) \to \mathrm{SO}(3)$, $p \mapsto (v \mapsto pvp^{-1})$ for all $v \in \mathrm{Im}(\mathbb{H})$ is a group homomorphism with kernel $\{\pm 1\}$.

At this point we have found all quaternionic and complex subspaces W such that the normalizer $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W)$ acts irreducibly on W and all real subspaces W of $\mathbb{H}^n$ such that $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ acts transitively on the spheres in W. Furthermore we have shown that for all such W the normalizer $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ (and therefore also $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W)$) acts transitively on the connected components of the spheres in W.

The goal of this section is to find all real subspaces W of $\mathbb{H}^n$ such that $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W)$ acts transitively on the connected components of the spheres in W. If $\mathcal{N}_{\mathrm{Sp}(n)}(W)$ does not act irreducibly on W (in particular if $\dim_{\mathbb{R}}(W) > 1$) then L has to act transitively on the spheres in W forcing $\dim_{\mathbb{R}}(W) \leq 4$. This is the reason why we will consider the cases $2 \leq \dim_{\mathbb{R}}(W) \leq 4$ from now on. Even though we could assume that L acts transitively on the spheres in W we rather just assume $\dim_{\mathbb{R}}(W) \leq 4$ and see what this implies for the L-action on W. Note that we still assume that $W \subset \mathbb{H}^n$ is a real subspace such that $W \neq W\lambda$ for all $\lambda \in \mathbb{H}\backslash\mathbb{R}$ and that

$$\mathfrak{l} = \{(\phi(\alpha), \alpha) \in \mathfrak{sp}(n) \times \mathfrak{sp}(1) : \alpha \in \pi_2(\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W))\}$$

for a Lie algebra homomorphism $\phi : \pi_2(\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W)) \to \mathfrak{sp}(n)$, which is injective by the assumption $\mathcal{N}_{\{0\}\oplus\mathfrak{sp}(1)}(W) = \{0\}$. If $\pi_2(\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W)) = \mathfrak{sp}(1)$ then the simply connectedness of $\mathrm{Sp}(1)$ implies that ϕ lifts to a Lie group homomorphism.

Note that $\dim_{\mathbb{R}}(\langle W\rangle_{\mathbb{H}}) \leq 4\dim_{\mathbb{R}}(W)$ and that $\dim_{\mathbb{R}}(\langle W\rangle_{\mathbb{H}})$ is divisible by four. Therefore we have the following cases

$\dim_{\mathbb{R}}(W)$	$\dim_{\mathbb{H}}(\langle W\rangle_{\mathbb{H}})$
4	$1,\ 2^{\bigstar},\ 3^{\bigstar},\ 4^*$
3	$1,\ 2^{\bigstar},\ 3^*$
2	$1,\ 2^*,$

where the case $\dim_{\mathbb{R}}(W) = 4$, $\dim_{\mathbb{H}}(\langle W\rangle_{\mathbb{H}}) = 1$ and the case $\dim_{\mathbb{R}}(W) = 2$, $\dim_{\mathbb{H}}(\langle W\rangle_{\mathbb{H}}) = 1$ are excluded by the assumption $W \neq W\lambda$ for all $\lambda \in \mathbb{H}\backslash\mathbb{R}$.

The cases * are excluded by Lemma 7.6, while the cases with $^{\bigstar}$ are excluded by [Kna16, Lemma 7.48, Lemma 7.49]).

Hence we only need to consider the case where $\dim_{\mathbb{R}}(W) = 3$ and $\dim_{\mathbb{H}}(\langle W\rangle_{\mathbb{H}}) = 1$.

Lemma 7.12. *Let* $\dim_{\mathbb{R}}(W) = 3$ *and* $\dim_{\mathbb{H}}(\langle W\rangle_{\mathbb{H}}) = 1$. *Then* W *lies in the same* $\mathrm{Sp}(n)$*-orbit as* $W_{\mathbb{R},3,1} := \mathrm{Im}(\mathbb{H}) \times \{0\}^{n-1}$ *and the action of* $L \cong \mathrm{Sp}(1)$ *on* W *is isomorphic to the defining representation of* $\mathrm{SO}(3)$ *on* $\mathbb{R}^3$ *and* $\mathcal{N}_{\mathfrak{sp}(n)}(W) = \{0\} \times \mathfrak{sp}(W^{\perp_q})$ *acts trivially on* W.

Proof. Since L respects the real inner product s, the real 1-dimensional subspace $W^{\perp_s} \cap \langle W\rangle_{\mathbb{H}}$ is invariant under the action of L. Let us choose an element $v_1 \in W^{\perp_s} \cap \langle W\rangle_{\mathbb{H}}$ with $q(v_1, v_1) = 1$. Then we can complete v_1 to a quaternionic basis of $\mathbb{H}^n$ and obtain an element in $\mathrm{Sp}(n)$ that maps $W_{\mathbb{R},3,1}$ to W. Then

$$\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W_{\mathbb{R},3,1}) = \left\{ (A, \xi) \in \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) : A = \begin{pmatrix} \xi & 0 \dots 0 \\ 0 & \\ \vdots & * \\ 0 & \end{pmatrix} \right\}$$

$$\mathcal{N}_{\mathfrak{sp}(n)}(W_{\mathbb{R},3,1}) = \left\{ A \in \mathfrak{sp}(n) : A = \begin{pmatrix} 0 & \dots 0 \\ \vdots & * \\ 0 & \end{pmatrix} \right\}$$

$$\mathfrak{l} = \left\{ (A, \xi) \in \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) : A = \begin{pmatrix} \xi & 0 \dots 0 \\ 0 & \\ \vdots & 0 \\ 0 & \end{pmatrix} \right\} \cong \mathfrak{sp}(1).$$

Now the proof follows from Lemma 7.11. $\square$

We summarize the work of this section in the following theorem.

Theorem 7.13. *Let W be a real subspace of $\mathbb{H}^n$. If $\mathcal{N}_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)}(W)$ acts irreducibly on W then W lies in the same $\mathrm{Sp}(n)\times\mathrm{Sp}(1)$-orbit as one of the following*

$$\begin{aligned}
W_{\mathbb{H},l} &= \mathbb{H}^l \times \{0\}^{n-l}, && 0 \le l \le n\\
W_{\mathbb{C},l} &= \mathbb{C}^l \times \{0\}^{n-l}, && 0 < l \le n\\
W_{\mathbb{C},2l,t} &= \left\{ \begin{pmatrix} z+jw \\ t(z-jw) \\ 0 \end{pmatrix} : z,w \in \mathbb{C}^l \right\}, && 0 < l \le \lfloor \tfrac{n}{2} \rfloor\\
W_{\mathbb{R},l} &= \mathbb{R}^l \times \{0\}^{n-l}, && 0 < l \le n\\
W_{\mathbb{R},2l,t,\mathbb{C}} &= \left\{ \begin{pmatrix} z \\ t\overline{z} \\ 0 \end{pmatrix} : z \in \mathbb{C}^l \right\}, && 0 < l \le \lfloor \tfrac{n}{2} \rfloor\\
W_{\mathbb{R},4l,z} &= \left\{ \begin{pmatrix} v \\ vz_2 \\ vz_3 \\ vz_4 \\ 0 \end{pmatrix} : v \in \mathbb{H}^l \right\}, && 0 < l \le \lfloor \tfrac{n}{4} \rfloor\\
W_{\mathbb{R},3,1} &= \mathrm{Im}(\mathbb{H}) \times \{0\}^{n-1},
\end{aligned}$$

for some $t \in \mathbb{R}^{>0}\backslash\{1\}$, $z \in \mathcal{P}$. Furthermore

$$\begin{aligned}
\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W_{\mathbb{H},l}) &= (\mathfrak{sp}(l) \oplus \mathfrak{sp}(n-l)) \oplus \mathfrak{sp}(1),\\
\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W_{\mathbb{C},l}) &= (\mathfrak{u}(l) \oplus \mathfrak{sp}(n-l)) \oplus \mathfrak{u}(1),\\
\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W_{\mathbb{C},2l,t}) &\cong (\mathfrak{sp}(l) \oplus \mathfrak{sp}(n-2l)) \oplus \mathfrak{u}(1),\\
\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W_{\mathbb{R},l}) &\cong (\mathfrak{so}(l) \oplus \mathfrak{sp}(n-l)) \oplus \mathfrak{sp}(1),\\
\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W_{\mathbb{R},2l,t,\mathbb{C}}) &\cong (\mathfrak{u}(l) \oplus \mathfrak{sp}(n-2l)) \oplus \mathfrak{u}(1),\\
\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W_{\mathbb{R},4l,z}) &\cong (\mathfrak{sp}(l) \oplus \mathfrak{sp}(n-4l)) \oplus \{0\},\\
\mathcal{N}_{\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)}(W_{\mathbb{R},3,1}) &\cong \mathfrak{sp}(n-1) \oplus \mathfrak{sp}(1).
\end{aligned}$$

Proof. The proof for $W_{\mathbb{H},l}$ follows directly from Lemma 7.3. Using Lemma 7.11 the proof for $W_{\mathbb{C},l}$ and $W_{\mathbb{C},2l,t}$ follows from Corollary 7.5, while the proof for $W_{\mathbb{R},l}$, $W_{\mathbb{R},2l,t,\mathbb{C}}$ and $W_{\mathbb{R},4l,z}$ follow from Lemmas 7.7, 7.8 and 7.9 respectively. The proof for $W_{\mathbb{R},3,1}$ follows from Lemma 7.12. □

7.6. **Non-reductive spherical subalgebras.** In this subsection we describe all non-reductive spherical algebraic subalgebras of $\mathfrak{sp}(n,1)$, up to the action of M. Their unipotent radicals are of the form $W^\perp \oplus \mathfrak{g}_{2\alpha}$, where $W \subset \mathfrak{g}_\alpha$ is one of the subspaces described in Theorem 7.13. In order to find all possibilities for their maximal compact subalgebras we apply Onishchik's Theorem 4.3 to their normalizers and thus obtain (with Remark 4.4) the following.

Theorem 7.14. *Every spherical non-reductive algebraic subalgebra of $\mathfrak{g} = \mathfrak{sp}(n,1)$, $n \ge 2$, is G-conjugate to one in the following list, where $\mathfrak{b}_j \subset \mathfrak{sp}(n-1-j)$, $\mathfrak{c} \subset \mathfrak{sp}(1)$ and $\mathfrak{d} \subset \mathfrak{u}(1)$ are* *arbitrary (under the condition displayed in italic in Remark 5.3 adapted to this situation).*

$\mathfrak{l}_H \oplus \mathfrak{n}$	$\mathfrak{l}_H \subset \mathfrak{m} \oplus \mathfrak{a}$ *arbitrary*
$\mathfrak{sp}(k) \oplus \mathfrak{b}_l \oplus \mathfrak{c} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{H},l}$	$1 \leq l \leq n-1,$
$\mathfrak{c} \oplus \mathfrak{b}_1 \oplus \mathfrak{sp}(1) \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{H},1}$	
$\mathfrak{u}(l) \oplus \mathfrak{b}_l \oplus \mathfrak{d} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{C},l}$	$1 \leq l \leq n-1$
$\mathfrak{su}(l) \oplus \mathfrak{b}_l \oplus \mathfrak{d} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{C},l}$	$2 \leq l \leq n-1$
$\mathfrak{sp}(m) \oplus \mathfrak{u}(1) \oplus \mathfrak{b}_l \oplus \mathfrak{d} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{C},l}$	$1 \leq l \leq n-1, l = 2m, m \geq 2$
$\mathfrak{sp}(m) \oplus \mathfrak{b}_l \oplus \mathfrak{d} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{C},l}$	$1 \leq l \leq n-1, l = 2m, m \geq 2$
$\mathfrak{sp}(l) \oplus \mathfrak{b}_{2l} \oplus \mathfrak{d} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{C},2l,t}$	$1 \leq 2l \leq n-1, l = 2m, t \in \mathbb{R}^{>0}\backslash\{1\}$
$\mathfrak{so}(l) \oplus \mathfrak{b}_l \oplus \mathfrak{c} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},l}$	$1 \leq l \leq n-1$
$\mathfrak{u}(m) \oplus \mathfrak{b}_l \oplus \mathfrak{c} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},l}$	$1 \leq l \leq n-1, l = 2m, m \geq 3$
$\mathfrak{su}(m) \oplus \mathfrak{b}_l \oplus \mathfrak{c} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},l}$	$1 \leq l \leq n-1, l = 2m, m \geq 3$
$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1) \oplus \mathfrak{b}_l \oplus \mathfrak{c} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},l}$	$1 \leq l \leq n-1, l = 4m, m \geq 2$
$\mathfrak{sp}(m) \oplus \mathfrak{u}(1) \oplus \mathfrak{b}_l \oplus \mathfrak{c} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},l}$	$1 \leq l \leq n-1, l = 4m, m \geq 1$
$\mathfrak{sp}(m) \oplus \mathfrak{b}_l \oplus \mathfrak{c} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},l}$	$1 \leq l \leq n-1, l = 4m, m \geq 1$
$\mathfrak{so}(9) \oplus \mathfrak{b}_{16} \oplus \mathfrak{c} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},16}$	
$\mathfrak{so}(7) \oplus \mathfrak{b}_8 \oplus \mathfrak{c} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},8}$	
$\mathfrak{g}_2 \oplus \mathfrak{b}_7 \oplus \mathfrak{c} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},7}$	
$\mathfrak{u}(l) \oplus \mathfrak{b}_{2l} \oplus \mathfrak{d} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},2l,t,\mathbb{C}}$	$2 \leq 2l \leq n-1, t \in \mathbb{R}^{>0}\backslash\{1\}$
$\mathfrak{su}(l) \oplus \mathfrak{b}_{2l} \oplus \mathfrak{d} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},2l,t,\mathbb{C}}$	$4 \leq 2l \leq n-1, t \in \mathbb{R}^{>0}\backslash\{1\}$
$\mathfrak{sp}(m) \oplus \mathfrak{u}(1) \oplus \mathfrak{b}_{2l} \oplus \mathfrak{d} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},2l,t,\mathbb{C}}$	$2 \leq 2l \leq n-1, l = 2m, m \geq 2, t \in \mathbb{R}^{>0}\backslash\{1\}$
$\mathfrak{sp}(m) \oplus \mathfrak{b}_{2l} \oplus \mathfrak{d} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},2l,t,\mathbb{C}}$	$2 \leq 2l \leq n-1, l = 2m, m \geq 2, t \in \mathbb{R}^{>0}\backslash\{1\}$
$\mathfrak{sp}(l) \oplus \mathfrak{b}_{4l} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},4l,z}$	$2 \leq 4l \leq n-1, z \in \mathcal{P}$
$\mathfrak{sp}(1) \oplus \mathfrak{b}_1 \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathbb{R},3,1}$	

Proof. Here we have written $\mathfrak{n}_{\mathbb{H},l}$ to denote $W_{\mathbb{H},l}^{\perp} \oplus \mathfrak{g}_{2\alpha}$ and so on. Note that $\mathfrak{n}_{\mathbb{R},1}$ is 1-codimensional and that $\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_{\mathbb{R},1}) \cong \mathfrak{sp}(n-2) \oplus \mathfrak{sp}(1)$. □

7.7. **Reductive spherical subalgebras.** In this subsection we classify the reductive spherical subalgebras $\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H$ of $\mathfrak{sp}(n,1)$ (for $n > 1$, since $\mathfrak{sp}(1,1) \cong \mathfrak{so}(4,1)$). Since the K-action on $\mathfrak{p}$ is essentially the same as the M-action on $\mathfrak{g}_\alpha$, we may apply again Theorem 7.13 in order to obtain all the candidates for the subspace $\mathfrak{p}_H^{\perp} \subset \mathfrak{p}$. We use the fact, that the Lie bracket on $\mathfrak{p}$ is given by

$$\left[\left[\left(\begin{smallmatrix} 0 & z \\ \overline{z}^t & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & w \\ \overline{w}^t & 0 \end{smallmatrix}\right)\right], \left(\begin{smallmatrix} 0 & v \\ \overline{v}^t & 0 \end{smallmatrix}\right)\right] = \left(\begin{smallmatrix} 0 & (z\overline{w}^t - w\overline{z}^t)v - v(\overline{z}^t w - \overline{w}^t z) \\ (\overline{z}^t w - \overline{w}^t z)\overline{v}^t - \overline{v}^t(z\overline{w}^t - w\overline{z}^t) & 0 \end{smallmatrix}\right)$$

in order to single out those Lie algebras that fulfill the additional restriction that $[\mathfrak{p}_H, \mathfrak{p}_H] \subset \mathfrak{k}_H \subset \mathcal{N}_{\mathfrak{k}}(\mathfrak{p}_H)$ (see (4.2)) has to hold.

If $\mathfrak{p}_H^{\perp}$ is equal to $W_{\mathbb{C},2l,t}$ or $W_{\mathbb{R},2l,t,\mathbb{C}}$ (for some $1 \leq l \leq \lfloor \frac{n}{2} \rfloor$ and some $t \in \mathbb{R}^{>0}\backslash\{1\}$) or $W_{\mathbb{R},4l,z}$ (for some $1 \leq l \leq \lfloor \frac{n}{4} \rfloor$, $z \in \mathcal{P}$) or $W_{\mathbb{R},l}$ for some $0 < l \leq n$ or $W_{\mathbb{R},3,1}$ then $\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H$ cannot be spherical in $\mathfrak{sp}(n,1)$ if $n > 1$, see [Kna16, Lemma 7.55-7.57].

If $\mathfrak{p}_H^{\perp} = W_{\mathbb{C},l}$ for some $0 < l \leq n$, then $\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H$ can only be spherical in $\mathfrak{sp}(n,1)$ for $n > 1$ if $l = n$. In that case $\mathfrak{h}$ is K-conjugate to $\mathfrak{su}(n,1)$ or $\mathfrak{u}(n,1)$, see [Kna16, Lemma 7.54].

If $\mathfrak{p}_H^\perp = W_{\mathbb{H},l}$ for some $0 \le l \le n$, then it is shown in [Kna16, Lemma 7.53] that the only reductive spherical Lie algebras $\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H$ of $\mathfrak{sp}(n,1)$ for $n > 1$ are the following

$$\begin{array}{lr} \mathfrak{sp}(n)\oplus\mathfrak{sp}(1), \quad \mathfrak{sp}(n)\oplus\mathfrak{u}(1), \quad \mathfrak{sp}(n)\oplus\{0\}, & l=n \\ \mathfrak{sp}(l)\oplus\mathfrak{sp}(n-l,1), & 0\le l<n \\ \mathfrak{u}(1)\oplus\mathfrak{sp}(n-1,1), \quad \mathfrak{sp}(n-1,1). & \end{array}$$

Hence, combining these results with Proposition 3.1 and Theorem 4.3 we obtain the following list of all spherical reductive algebraic subalgebras of $\mathfrak{sp}(n,1)$ up to conjugacy in G.

Theorem 7.15. *All spherical, reductive subalgebras $\mathfrak{h}$ of $\mathfrak{sp}(n,1)$ where $n > 1$ are (up to conjugation in G) one of the following*

$\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)$ $\mathfrak{sp}(n)\oplus\mathfrak{u}(1)$ $\mathfrak{sp}(n)\oplus\{0\}$ $\mathfrak{sp}(l)\oplus\mathfrak{sp}(n-l,1)$ $\mathfrak{u}(1)\oplus\mathfrak{sp}(n-1,1)$ $\mathfrak{sp}(n-1,1)$	$0\le l<n$
$\mathfrak{su}(n,1)$ $\mathfrak{u}(n,1)$	

According to [Ber57, Table 2]*, the symmetric subalgebras are $\mathfrak{sp}(l)\oplus\mathfrak{sp}(n-l,1)$ for $0 \le l < n$, $\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)$ and $\mathfrak{u}(n,1)$.*

8. The exceptional group $G = F_4$

8.1. **Reductive spherical subalgebras.** We want to classify the spherical reductive algebraic subalgebras $\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H$ of $\mathfrak{g} = \mathfrak{f}_4$. Using the classification of maximal reductive subalgebras of $\mathfrak{g}$, Krötz and Schlichtkrull have shown that a spherical reductive subalgebra of $\mathfrak{g}$ must be contained in a symmetric one, see [KS16a, Lemma 6.2]. According to [Ber57, Table 2], the symmetric proper subalgebras of $\mathfrak{f}_4$ are $\mathfrak{so}(9)$, $\mathfrak{so}(8,1)$ and $\mathfrak{sp}(2,1)\oplus\mathfrak{sp}(1)$.

We first determine $\mathfrak{p}_H$.

Lemma 8.1. *Let $G^\sigma = K^\sigma \exp(\mathfrak{p}^\sigma)$ be a symmetric subgroup of G (defined as the fixed point set of an involution σ that commutes with the Cartan involution) and $H = K_H \exp(\mathfrak{p}_H)$ a subgroup of G^σ. The adjoint K_H-action on $\mathfrak{p}$ can only be transitive on the spheres in $\mathfrak{p}_H^\perp \subset \mathfrak{p}$ if $\mathfrak{p}_H = \mathfrak{p}^\sigma$.*

Proof. Since $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}^\sigma$, $\mathfrak{p}_H$ lies in $\mathfrak{p}^\sigma$ which implies that $\mathfrak{p}_H^\perp \subset \mathfrak{p} = \mathfrak{p}^\sigma \oplus \mathfrak{p}^{-\sigma}$ is the direct sum of $\mathfrak{p}^{-\sigma}$ and $\mathfrak{p}_H^\perp \cap \mathfrak{p}^\sigma$. Since the adjoint $\mathfrak{k}^\sigma$ action on $\mathfrak{p}$ stabilizes $\mathfrak{p}^\sigma$ and $\mathfrak{p}^{-\sigma}$, K_H can only act irreducibly on $\mathfrak{p}_H^\perp \subset \mathfrak{p}$ if $\mathfrak{p}_H = \mathfrak{p}^\sigma$. □

If $\mathfrak{g}^\sigma$ is simple and non-compact, then we have $\mathfrak{k}^\sigma = [\mathfrak{p}^\sigma, \mathfrak{p}^\sigma]$. We are now in the position to prove the main statement of this section.

Theorem 8.2. *Up to conjugation by an element of G, the following table exhausts the spherical, reductive subalgebras $\mathfrak{h} = \mathfrak{k}_H \oplus \mathfrak{p}_H$ of $\mathfrak{f}_4$.*

$\mathfrak{f}_4$
$\mathfrak{so}(9)$
$\mathfrak{so}(8,1)$
$\mathfrak{sp}(2,1)$
$\mathfrak{sp}(2,1) \oplus \mathfrak{u}(1)$
$\mathfrak{sp}(2,1) \oplus \mathfrak{sp}(1)$

According to [Ber57, Table 2]*, the symmetric ones are $\mathfrak{f}_4$, $\mathfrak{so}(9)$, $\mathfrak{so}(8,1)$ and $\mathfrak{sp}(2,1)\oplus\mathfrak{sp}(1)$.*

Proof. Let us first consider the case that $\mathfrak{g}^\sigma$ equals $\mathfrak{f}_4$ or $\mathfrak{so}(8,1)$. Since $\mathfrak{p}_H = \mathfrak{p}^\sigma$ by Lemma 8.1 and $\mathfrak{g}^\sigma$ is simple non-compact, we conclude that $\mathfrak{h}$ has to contain $\mathfrak{k}^\sigma = [\mathfrak{p}^\sigma, \mathfrak{p}^\sigma]$. Hence $\mathfrak{h} = \mathfrak{g}^\sigma$ is the only spherical subalgebra of $\mathfrak{g}$ that lies in $\mathfrak{g}^\sigma$ in these two cases.

Let us now consider the case that $\mathfrak{h} \subset \mathfrak{so}(9)$. In this case $\mathfrak{h} = \mathfrak{k}_H$ and $\mathfrak{p}_H^\perp = \mathfrak{p} \cong \mathbb{R}^{16}$. We are therefore looking for connected subgroups of Spin(9) that act transitively on the spheres in $\mathbb{R}^{16} \cong \mathfrak{p}$ with respect to the adjoint action. Due to Theorem 4.3 there are none except for Spin(9) itself. This gives $\mathfrak{h} = \mathfrak{so}(9)$.

If $\mathfrak{g}^\sigma = \mathfrak{sp}(2,1) \oplus \mathfrak{sp}(1)$ then $\mathfrak{k}^\sigma = \mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ and $\dim_{\mathbb{R}}(\mathfrak{p}^\sigma) = 8$. Since $\mathfrak{p}_H = \mathfrak{p}^\sigma$ by Lemma 8.1 and $\mathfrak{sp}(2,1)$ is simple non-compact, $\mathfrak{h} \subset \mathfrak{g}^\sigma$ has to contain $\mathfrak{sp}(2,1)$. Hence $\mathfrak{h} = \mathfrak{sp}(2,1) \oplus \mathfrak{b}$, where $\mathfrak{b}$ is a subalgebra of $\mathfrak{sp}(1)$. Note that we have

$$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \subset \mathfrak{k}_H \subset \mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) = \mathfrak{k}^\sigma.$$

The condition for H to be spherical in G is that K_H acts transitively on the spheres in $\mathfrak{p}_H^\perp \cong \mathfrak{p}^{-\sigma} \cong \mathbb{R}^8$.

Let θ denote a Cartan involution on $\mathfrak{g}$ that commutes with σ. Then $\theta\sigma$ is another involution on $\mathfrak{g}$ defining the Lie algebra $\mathfrak{g}^{\theta\sigma} = \mathfrak{k}^\sigma \oplus \mathfrak{p}^{-\sigma}$ as fixed point set. Note that $\mathfrak{g}^{\theta\sigma}$ has real rank 1 since $\mathfrak{g}$ has real rank 1. Therefore $(K^\sigma)^\circ = \mathrm{Sp}(2) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ acts transitively on the spheres in $\mathfrak{p}^{-\sigma}$. In particular the adjoint K^σ-representation on $\mathfrak{p}^{-\sigma}$ is irreducible. It is well known that then $\mathfrak{p}^{-\sigma}$ decomposes into a tensor product of three D-vector spaces that are irreducible representations for Sp(2), Sp(1) and Sp(1) respectively, where $D \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Since the adjoint K^σ-representation on $\mathfrak{p}^{-\sigma}$ is not only irreducible but transitive on the spheres this tensor product has two factors that are equal to D. Using again the fact that K^σ acts transitively on the sphere $S^7 \in \mathbb{R}^8 \cong \mathfrak{p}^{-\sigma}$, and the fact that $\dim_{\mathbb{R}} \mathrm{Sp}(1) = 3 < 7 = \dim_{\mathbb{R}} S^7$ we obtain the following three possibilities to decompose $\mathfrak{p}^{-\sigma}$

$$\mathfrak{p}^{-\sigma} \cong \begin{cases} \mathbb{R}^8 \otimes \mathbb{R} \otimes \mathbb{R}, \\ \mathbb{C}^4 \otimes \mathbb{C} \otimes \mathbb{C}, \\ \mathbb{H}^2 \otimes \mathbb{H} \otimes \mathbb{H}. \end{cases}$$

Note that in each case Sp(2) acts by the standard action on $\mathfrak{p}^{-\sigma} \cong \mathbb{R}^8 \cong \mathbb{C}^4 \cong \mathbb{H}^2$. Hence, already $\mathrm{Sp}(2) \times \{e\} \times \{e\}$ acts transitively on the spheres in $\mathfrak{p}_H^\perp$. This implies that $\mathfrak{b} \subset \mathfrak{sp}(1)$ is arbitrary. $\square$

8.2. **Non-reductive spherical subalgebras.** We start by analyzing the structure of a minimal parabolic subgroup $Q_0 = MAN$ of $G = F_4$. According to [Ara62, p. 32-33] we have that $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ is the restricted root space decomposition, where $\dim \mathfrak{g}_\alpha = 8$ and $\dim \mathfrak{g}_{2\alpha} = 7$. Due to Corollary 3.3 we have $\mathfrak{n}_H = W^\perp \oplus \mathfrak{g}_{2\alpha}$ where $W \subset \mathfrak{g}_\alpha$ is a real subspace such that M_H acts transitively on the spheres in W.

First we are going to identify the M-representation on $\mathfrak{g}_\alpha$. Note that it has to be irreducible due to [Wol84, Chapter 8.13].

Lemma 8.3. *The M-representation on $\mathfrak{g}_\alpha$ is induced by the embedding $\varphi : \mathfrak{so}(7) \hookrightarrow \mathfrak{so}(8)$ given by*

$$\begin{pmatrix} 0 & -a & -b & -c & -d & -e & -f \\ a & 0 & -g & -h & -i & -j & -k \\ b & g & 0 & -l & -m & -n & -p \\ c & h & l & 0 & -q & -r & -s \\ d & i & m & q & 0 & -t & -u \\ e & j & n & r & t & 0 & -v \\ f & k & p & s & u & v & 0 \end{pmatrix} \mapsto$$

$$\begin{pmatrix} 0 & -a+s-t & -b-r-u & -c-k+n & -d+j+p & -e-i-l & -f+h-m & g+q-v \\ a-s+t & 0 & -g+q-v & f-h-m & -e-i+l & d-j+p & -c-k-n & -b+r+u \\ b+r+u & g-q+v & 0 & -e+i-l & -f-h-m & c-k-n & d+j-p & a+s-t \\ c+k-n & -f+h+m & e-i+l & 0 & g-q-v & -b-r+u & a-s-t & -d-j-p \\ d-j-p & e+i-l & f+h+m & -g+q+v & 0 & -a-s-t & -b+r-u & c-k+n \\ e+i+l & -d+j-p & -c+k+n & b+r-u & a+s+t & 0 & -g-q-v & f+h-m \\ f-h+m & c+k+n & -d-j+p & -a+s+t & b-r+u & g+q+v & 0 & -e+i+l \\ -g-q+v & b-r-u & -a-s+t & d+j+p & -c+k-n & -f-h+m & e-i-l & 0 \end{pmatrix}.$$

Proof. The embedding from Spin(7) into SO(8) is described in [Oni94, Chapter 1.5.3] as follows. If $1 \leq i < j \leq 7$, then $(-E_{ij} + E_{ji})$ forms a basis of $\mathfrak{so}(7)$, where E_{ij} is the matrix with a 1 in line i and column j and zeros everywhere else.

We take the standard orthogonal basis $\{e_0, e_1, \dots, e_7\}$ of octonions over $\mathbb{R}$ with the following multiplication table.

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Note that the space of pure imaginary octonions is spanned by $\{e_1, \dots, e_7\}$. The Lie algebra $\mathfrak{spin}(7)$ is spanned by elements e_ie_j with $1 \leq i < j \leq 7$. The map $2(-E_{ij} + E_{ji}) \mapsto e_ie_j$ for all $1 \leq i < j \leq 7$ is an isomorphism from $\mathfrak{so}(7)$ to $\mathfrak{spin}(7)$. The map $\lambda : e_ie_j \mapsto (x \mapsto e_i(e_jx))$ is an embedding of $\mathfrak{spin}(7)$ into $\mathfrak{so}(8)$, where we choose the basis $\{e_1, \dots, e_7, e_0\}$ of $\mathbb{R}^8$. Direct calculation with the multiplication table shows that the map $\lambda : \mathfrak{spin}(7) = \mathfrak{so}(7) \to \mathfrak{so}(8)$ is equal to the map φ from the statement of this proposition.

Note that the induced action of Spin(7) on $\mathbb{R}^8$ is transitive on the spheres. This follows since $\lambda(\mathfrak{spin}(7))(e_0) = T_{e_0}(\mathrm{Spin}(7) \cdot e_0)$ coincides (by the multiplication table) with the purely imaginary octonions, which equals $T_{e_0}(S^7)$. Therefore the orbit of Spin(7) is open in S^7. Since Spin(7) and hence also its orbits are compact, it coincides with S^7. In particular, this representation is irreducible.

In order to finish the proof we note that a direct application of the Weyl Dimension formula (see e.g. [Kna02, Chapter V.6]) shows that the irreducible $\mathfrak{so}(7)$-representation in dimension 8 is unique up to isomorphism. For the complete argument we refer the reader to [Kna16, p. 78]. □

Now that we identified the M-representation on $\mathfrak{g}_\alpha$ (see Lemma 8.3) we want to give a normal form for the subspace $\mathfrak{n}_H \subset \mathfrak{n}$. Before doing so we state the following remark.

Remark 8.4. Recall that $\mathfrak{so}(4)=\mathfrak{so}(3)\oplus\mathfrak{so}(3)$. Moreover, both $\mathfrak{so}(3)$ factors are conjugate under an element in $\mathrm{O}(4)$. The maps

$$\begin{pmatrix} 0 & -a & -b & c \\ a & 0 & -c & -b \\ b & c & 0 & a \\ -c & b & -a & 0 \end{pmatrix} \mapsto \begin{pmatrix} ia & -b-ic \\ b-ic & -ia \end{pmatrix} \mapsto ia + j(b-ic).$$

define the Lie algebra isomorphisms from the ideal $\mathfrak{so}(3)$ in $\mathfrak{so}(4)$ to $\mathfrak{su}(2)$ and $\mathfrak{sp}(1)$.

Lemma 8.5. *Let* $\mathrm{Spin}(7)$ *act irreducibly on* $\mathbb{R}^8$ *and* W *be an* l*-dimensional subspace. Then* W *lies in the same* $\mathrm{Spin}(7)$*-orbit as* $\mathbb{R}^l\times\{0\}^{8-l}$ *or as the real span of* e_1,e_2,e_3,xe_4+e_8 *for some* $x\in\mathbb{R}$*, where* $(e_1,\dots,e_8)$ *is the standard basis of* $\mathbb{R}^8$*. The latter case can only occur if* $l=4$*.*

Proof. The proof of Lemma 8.3 shows that $\mathrm{Spin}(7)$ acts transitively on the spheres in $\mathbb{R}^8$. In particular all its stabilizers are conjugate to each other. The stabilizer of $\mathrm{Spin}(7)$ in $e_8:=e_0$ is equal to G_2 ([Oni94, Chapter 1 §5]). Due to Onishchik's classification ([Oni94, Table 8]) the group G_2 acts transitively on the spheres in $\mathbb{R}^7$ and has stabilizer equal to $\mathrm{SU}(3)$. The group $\mathrm{SU}(3)$ acts transitively on the sphere $S^5\subset\mathbb{R}^6\cong\mathbb{C}^3$ but its isotropy in one point (being isomorphic to $\mathrm{SU}(2)$) acts reducibly on $\mathbb{R}^5\cong(i\mathbb{R})\times\mathbb{C}^2$.

Now let $(v_1,\dots,v_l)$ be an orthonormal basis of W. If $l\le 3$ we can map v_1 to e_1 with $\mathrm{Spin}(7)$, v_2 to e_2 with G_2 and v_3 to e_3 with $\mathrm{SU}(3)$.

If $l>4$ the orthogonal complement of W has dimension less or equal to 3. By the previous argument we can map the orthogonal complement of W to $\{0\}^l\times\mathbb{R}^{8-l}$, thus mapping W to the span of $e_1,\dots,e_l$.

Now let $l=4$. As before we use $\mathrm{Spin}(7)$, G_2 and $\mathrm{SU}(3)$ to map v_1 to e_1, v_2 to e_2 and v_3 to e_3. Recall that the action of $\mathrm{Spin}(7)$ on $\mathbb{R}^8$ is induced by the map $\varphi:\mathfrak{so}(7)\to\mathfrak{so}(8)$. A direct calculation shows that

$$\big((\varphi(\mathfrak{so}(7))_{e_1})_{e_2}\big)_{e_3} = \left\{ \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&-v&u&-t&0\\ 0&0&0&v&0&-t&-u&0\\ 0&0&0&-u&t&0&-v&0\\ 0&0&0&t&u&v&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} : t,u,v\in\mathbb{R} \right\} \overset{R.8.4}{\cong} \mathfrak{so}(3).$$

Therefore $\big((\varphi(\mathfrak{so}(7))_{e_1})_{e_2}\big)_{e_3}\cong\mathfrak{so}(3)\cong\mathfrak{su}(2)$ acts trivially on $\mathbb{R}e_8$ and transitively on the spheres in $\{0\}^3\times\mathbb{R}^4\times\{0\}$.

If v_4 is an element of $\{0\}^3\times\mathbb{R}^4\times\{0\}$ then W lies in the same $\mathrm{Spin}(7)$-orbit as the span of $e_1,\dots,e_4$.

If v_4 is in $\{0\}^3\times\mathbb{R}^5\backslash(\{0\}^3\times\mathbb{R}^4\times\{0\})$ we can use the described $\mathrm{SU}(2)$-action to map it to ae_4+be_8, where $a\in\mathbb{R}$ and $b\in\mathbb{R}\backslash\{0\}$. The assumption follows since $\frac{1}{b}(ae_4+be_8)=\frac{a}{b}e_4+e_8=:xe_4+e_8$ holds for some $x\in\mathbb{R}$. □

Now we arrive at the main result of this section. We assume $\mathfrak{n}_H=W\oplus\mathfrak{g}_{2\alpha}$, where W is a real subspace of $\mathfrak{g}_\alpha$. Due to Lemma 8.5 we may assume (after conjugation in M) that $\mathfrak{n}_H=\mathfrak{n}_l:=(\mathbb{R}^l\times\{0\}^{8-l})\oplus\mathfrak{g}_{2\alpha}$ or $\mathfrak{n}_H=\mathfrak{n}_{4,x}:=\langle e_1,e_2,e_3,xe_4+e_8\rangle_{\mathbb{R}}\oplus\mathfrak{g}_{2\alpha}$ for some $x\in\mathbb{R}$, where the latter case can only occur if $\dim_{\mathbb{R}}(W)=4$. We consider the map $\varphi:\mathfrak{so}(7)\hookrightarrow\mathfrak{so}(8)$ defined in Lemma 8.3. Its restriction to $\mathcal{N}_{\mathfrak{m}}(W)\subset\mathfrak{m}=\mathfrak{so}(7)$ yields a map $\varphi|_{\mathcal{N}_{\mathfrak{m}}(W)}:\mathcal{N}_{\mathfrak{m}}(W)\to\mathcal{N}_{\mathfrak{so}(8)}(W)$ and the action of $\mathcal{N}_M(W)$ on $W^\perp$ is induced by the standard action of $\varphi(\mathcal{N}_{\mathfrak{m}}(W))\subset\mathcal{N}_{\mathfrak{so}(8)}(W)$ on $W^\perp$.

Theorem 8.6. *Every non-reductive spherical algebraic subalgebra of* $\mathfrak{f}_4$ *is* G*-conjugate to one in the following table where* $\mathfrak{l}_1\subset\mathfrak{sp}(1)$ *and* $\mathfrak{l}_2\subsetneq\mathfrak{sp}(1)$ *are arbitrary (under the condition that* *the maximal compact subalgebra is a Lie algebra, see Remark 5.3).*

$\mathfrak{l}_H \oplus \mathfrak{n}$	$\mathfrak{l}_H \subset \mathfrak{m} \oplus \mathfrak{a}$ *arbitrary*
$\mathfrak{so}(7) \oplus \mathfrak{a} \oplus \mathfrak{n}_0$	
$\mathfrak{g}_2 \oplus \mathfrak{a} \oplus \mathfrak{n}_1$	
$\mathfrak{u}(3) \oplus \mathfrak{a} \oplus \mathfrak{n}_2$ $\mathfrak{su}(3) \oplus \mathfrak{a} \oplus \mathfrak{n}_2$	
$\mathfrak{so}(4) \oplus \mathfrak{a} \oplus \mathfrak{n}_l$ $\mathfrak{sp}(1) \oplus \mathfrak{l}_2 \oplus \mathfrak{a} \oplus \mathfrak{n}_l$ $\mathfrak{l}_2 \oplus \mathfrak{sp}(1) \oplus \mathfrak{a} \oplus \mathfrak{n}_l$	$l = 4, 5$ $l = 4, 5$ $l = 4$
$\mathfrak{l}_1 \oplus \mathfrak{so}(4) \oplus \mathfrak{a} \oplus \mathfrak{n}_{4,0}$ $\mathfrak{l}_1 \oplus \mathfrak{sp}(1) \oplus \mathfrak{l}_2 \oplus \mathfrak{a} \oplus \mathfrak{n}_{4,0}$ $\mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \mathfrak{sp}(1) \oplus \mathfrak{a} \oplus \mathfrak{n}_{4,0}$	
$\mathfrak{so}(4) \oplus \mathfrak{a} \oplus \mathfrak{n}_{4,x}$ $\mathfrak{sp}(1) \oplus \mathfrak{l}_2 \oplus \mathfrak{a} \oplus \mathfrak{n}_{4,x}$ $\mathfrak{l}_2 \oplus \mathfrak{sp}(1) \oplus \mathfrak{a} \oplus \mathfrak{n}_{4,x}$	$x \neq 0$ $x \neq 0$ $x \neq 0$
$\mathfrak{u}(1) \oplus \mathfrak{k}_2 \oplus \mathfrak{a} \oplus \mathfrak{n}_6$	
$\mathfrak{m}' \oplus \mathfrak{a} \oplus \mathfrak{n}_7$	$\mathfrak{m}' \subset \mathfrak{g}_2 = \mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_7)$ *arbitrary*

where $\mathfrak{k}_2$ *is isomorphic to* $\{0\}$, $\mathfrak{u}(1)^k$ *for some* $1 \leq k \leq 2$, $\mathfrak{su}(2) \oplus \mathfrak{l}_1$ *or* $\mathfrak{su}(3)$.

Proof. We assume $\mathfrak{n}_H = W \oplus \mathfrak{g}_{2\alpha}$. Lemma 8.5 shows that W may (after conjugation in M) be chosen as $\mathbb{R}^l \times \{0\}^{8-l}$ or as the real span of $e_1, e_2, e_3, xe_4 + e_8$ if $\dim_{\mathbb{R}}(W) = 4$, where $x \in \mathbb{R}$. The case that W is the real span of $e_1, e_2, e_3, xe_4 + e_8$ will be considered at the end of this proof. Due to Proposition 3.2 it suffices to consider the case $\dim W = l < 8$.

If $W = \mathbb{R}^l \times \{0\}^{8-l}$ we have

$$\mathfrak{n}_H = \mathfrak{n}_l := (\mathbb{R}^l \times \{0\}^{8-l}) \oplus \mathfrak{g}_{2\alpha},$$

where $0 \leq l \leq 7$ is the dimension of $\mathfrak{n}_H \cap \mathfrak{g}_\alpha$. A direct calculation yields

$$\mathcal{N}_M(\mathfrak{n}_l) = \left\{ A \in \mathrm{Spin}(7) : \varphi(A) = \left(\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix} \right), A_1 \in \mathbb{R}^{l \times l}, A_2 \in \mathbb{R}^{(8-l)\times(8-l)} \right\}$$

and the action of $\mathcal{N}_M(\mathfrak{n}_l)$ on $\mathfrak{n}_l^\perp = \{0\}^l \times \mathbb{R}^{8-l}$ is given by

$$\varphi(A) \cdot \left(\begin{smallmatrix} 0 \\ v \end{smallmatrix} \right) = \left(\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix} \right) \cdot \left(\begin{smallmatrix} 0 \\ v \end{smallmatrix} \right) = \left(\begin{smallmatrix} 0 \\ A_2 v \end{smallmatrix} \right).$$

Onishchik's classification Theorem 4.3 shows that the subgroup M_H of $\mathcal{N}_M(\mathfrak{n}_l)$ can only act transitively on the spheres in $\mathfrak{n}_l^\perp$ if the projection of $\varphi(\mathfrak{m}_H) \subset \mathfrak{so}(l) \oplus \mathfrak{so}(8-l)$ onto $\mathfrak{so}(8-l)$ is one of the following Lie algebras, where $\mathfrak{l}_2 \subsetneq \mathfrak{sp}(1)$ is arbitrary.

l	
0	$\mathfrak{so}(8), \mathfrak{so}(7), \mathfrak{u}(4), \mathfrak{su}(4), \mathfrak{sp}(2) \oplus \mathfrak{u}(1), \mathfrak{sp}(2)$
1	$\mathfrak{so}(7), \mathfrak{g}_2$
2	$\mathfrak{so}(6), \mathfrak{u}(3), \mathfrak{su}(3)$
3	$\mathfrak{so}(5)$
4	$\mathfrak{so}(4), \mathfrak{sp}(1) \oplus \mathfrak{l}_2, \mathfrak{l}_2 \oplus \mathfrak{sp}(1)$
5	$\mathfrak{so}(3)$
6	$\mathfrak{so}(2)$

Since $\varphi(\mathfrak{m}_H) \subset \varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_l))$ holds, our next step is to take each of the Lie algebras in the table above and check if it is contained in the projection onto $\mathfrak{so}(8-l)$ of

$$\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_l)) = \left\{ \left(\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix} \right) \in \varphi(\mathfrak{so}(7)) : A_1 \in \mathbb{R}^{l\times l}, A_2 \in \mathbb{R}^{(8-l)\times(8-l)} \right\}.$$

We start with $l = 0$. Note that this implies $\mathfrak{n}_H = \mathfrak{g}_{2\alpha}$ and $\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_0) = \mathfrak{m} = \mathfrak{so}(7)$. It is clear that $\mathfrak{so}(8)$ is not contained in $\varphi(\mathfrak{so}(7)) \cong \mathfrak{so}(7)$, since the dimension of $\mathfrak{so}(7)$ is smaller than the dimension of $\mathfrak{so}(8)$. Due to Onishchik's classification Theorem 4.3 there exists no subgroup of $\mathrm{Spin}(7)$ that acts transitively on the spheres in $\mathbb{R}^8$. In particular $\mathrm{Spin}(7)$ is the only subgroup of $\varphi(\mathcal{N}_M(\mathfrak{n}_0)) \cong \mathrm{Spin}(7)$ that acts transitively on the spheres in $\mathfrak{n}$ (with respect to the action defined by φ).

Now let $l = 1$. Since $G_2 = (\mathrm{Spin}(7))_{e_0}$ (with respect to the action defined by φ, i.e. $e_0 = e_8$) it is clear that $\mathfrak{g}_2$ is contained in $\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_1)) \cong \mathfrak{g}_2$. The dimension $\dim(\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_1))) = \dim(\mathfrak{g}_2) = 14$ is smaller than the dimension of $\mathfrak{so}(7)$. Therefore the projection of $\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_1))$ onto $\mathfrak{so}(7)$ can not contain $\mathfrak{so}(7)$.

If $l = 2$ then

$$\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_2)) = \left\{ \begin{pmatrix} 0 & a-s+t & 0 & 0 & 0 & 0 & 0 & 0 \\ -a+s-t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & -h & -k & j & a \\ 0 & 0 & -i & 0 & -v & u & s & -j \\ 0 & 0 & h & v & 0 & t & -u & -k \\ 0 & 0 & k & -u & -t & 0 & -v & h \\ 0 & 0 & -j & -s & u & v & 0 & i \\ 0 & 0 & -a & j & k & -h & -i & 0 \end{pmatrix} : a,h,i,j,k,s,t,u,v \in \mathbb{R} \right\}.$$

In particular its projection onto $\mathfrak{so}(6)$ cannot contain $\mathfrak{so}(6)$ for dimensional reasons. If we conjugate its projection onto $\mathfrak{so}(6)$ with the matrix $\left(\begin{smallmatrix} 1&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&-1&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&1&0&0&0&0 \end{smallmatrix}\right)$ in $\mathrm{O}(6)$ we obtain

$$\left\{ \begin{pmatrix} 0 & a & k & -h & j & i \\ -a & 0 & h & k & -i & j \\ -k & -h & 0 & t & v & u \\ h & -k & -t & 0 & -u & v \\ -j & i & -v & u & 0 & -s \\ -i & -j & -u & -v & s & 0 \end{pmatrix} : a,h,i,j,k,s,t,u,v \in \mathbb{R} \right\},$$

which is the natural embedding of $\mathfrak{u}(3)$ into $\mathfrak{so}(6)$. Note that this implies $\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_2) \cong \mathfrak{u}(3)$. Since $\mathfrak{su}(3)$ is a subalgebra of $\mathfrak{u}(3)$ this shows that the projection of $\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_2))$ onto $\mathfrak{so}(6)$ contains $\mathfrak{u}(3)$ and $\mathfrak{su}(3)$.

If $l = 3$ the projection of

$$\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_3)) = \left\{ \begin{pmatrix} 0 & s-t & -r-u & 0 & 0 & 0 & 0 & 0 \\ -s+t & 0 & q-v & 0 & 0 & 0 & 0 & 0 \\ r+u & -q+v & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q & -r & -s & 0 \\ 0 & 0 & 0 & q & 0 & -t & -u & 0 \\ 0 & 0 & 0 & r & t & 0 & -v & 0 \\ 0 & 0 & 0 & s & u & v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} : q,r,s,t,u,v \in \mathbb{R} \right\}.$$

onto $\mathfrak{so}(5)$ does not contain $\mathfrak{so}(5)$.

If $l = 4$ the projection of

$$\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_4)) = \left\{ \begin{pmatrix} 0 & -t & -u & n & 0 & 0 & 0 & 0 \\ t & 0 & -v & -m & 0 & 0 & 0 & 0 \\ u & v & 0 & i & 0 & 0 & 0 & 0 \\ -n & m & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -t & -u & n \\ 0 & 0 & 0 & 0 & t & 0 & -v & -m \\ 0 & 0 & 0 & 0 & u & v & 0 & i \\ 0 & 0 & 0 & 0 & -n & m & -i & 0 \end{pmatrix} : i,m,n,t,u,v \in \mathbb{R} \right\} \cong \mathfrak{so}(4)$$

onto $\mathfrak{so}(4)$ is a Lie algebra isomorphism. This shows furthermore that the projection of $\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_4))$ onto $\mathfrak{so}(4)$ also contains $\mathfrak{sp}(1) \oplus \mathfrak{l}_2$ and $\mathfrak{l}_2 \oplus \mathfrak{sp}(1)$, where $\mathfrak{l}_2 \subsetneq \mathfrak{sp}(1)$ is arbitrary.

If $l = 5$ the projection of

$$\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_5)) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -g & -m & l & 0 & 0 & 0 \\ 0 & g & 0 & i & -h & 0 & 0 & 0 \\ 0 & m & -i & 0 & -q & 0 & 0 & 0 \\ 0 & -l & h & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -g-q & h-m \\ 0 & 0 & 0 & 0 & 0 & g+q & 0 & i+l \\ 0 & 0 & 0 & 0 & 0 & -h+m & -i-l & 0 \end{pmatrix} : g, h, i, l, m, q \in \mathbb{R} \right\} \cong \mathfrak{so}(4)$$

onto $\mathfrak{so}(3)$ is surjective. The projection of $f_1(\mathfrak{so}(3)) \subset \mathfrak{so}(4) \cong \varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_5))$ (i.e., $m = -h$, $i = l$, $g = q$) onto $\mathfrak{so}(3)$ is still surjective, while the projection of $f_2(\mathfrak{so}(3)) \subset \mathfrak{so}(4) \cong \varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_5))$ (i.e., $m = h$, $i = -l$, $g = -q$) onto $\mathfrak{so}(3)$ equals $\{0\}$. Therefore $\mathfrak{m}_H$ is isomorphic to $\mathfrak{so}(4)$ or $\mathfrak{sp}(1) \oplus \mathfrak{l}_2$ for arbitrary $\mathfrak{l}_2 \subsetneq \mathfrak{sp}(1)$ in this case.

If $l = 6$ the projection of

$$\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_6)) = \left\{ \begin{pmatrix} 0 & -a & -b & -c & -d & -e & 0 & 0 \\ a & 0 & -g & -h & -w & d & 0 & 0 \\ b & g & 0 & -l & -h & c & 0 & 0 \\ c & h & l & 0 & g & -b & 0 & 0 \\ d & w & h & -g & 0 & -a & 0 & 0 \\ e & -d & -c & b & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e-w-l \\ 0 & 0 & 0 & 0 & 0 & 0 & -e+w+l & 0 \end{pmatrix} : a, b, c, d, e, g, h, l, w \in \mathbb{R} \right\}$$

onto $\mathfrak{so}(2)$ is surjective. A direct calculation shows furthermore that

$$\begin{aligned} \mathrm{Ad} \begin{pmatrix} 1 & & & & & & & \\ & & & & & 1 & & \\ & & & -1 & & & & \\ & & -1 & & & & & \\ & & & & 1 & & & \\ & 1 & & & & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix} (\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_6))) &= \left\{ \begin{pmatrix} 0 & -e & c & b & -d & -a & 0 & 0 \\ e & 0 & -b & c & a & -d & 0 & 0 \\ -c & b & 0 & l & -g & -h & 0 & 0 \\ -b & -c & -l & 0 & h & -g & 0 & 0 \\ d & -a & g & -h & 0 & w & 0 & 0 \\ a & d & h & g & -w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e-w-l \\ 0 & 0 & 0 & 0 & 0 & 0 & -e+w+l & 0 \end{pmatrix} \in \mathfrak{so}(8) \right\} \\ &\cong \left\{ \begin{pmatrix} ie & c-ib & -d+ia & 0 \\ -c-ib & -il & -g+ih & 0 \\ d+ia & g+ih & -iw & 0 \\ 0 & 0 & 0 & -ie+il+iw \end{pmatrix} \in \mathfrak{u}(4) \right\} \\ &= \left\{ \begin{pmatrix} A & 0 \\ 0 & -\operatorname{Tr}(A) \end{pmatrix} \in \mathfrak{u}(4) : A \in \mathfrak{u}(3) \right\} \cong \mathfrak{u}(3) \end{aligned}$$

holds. Therefore $\mathfrak{m}_H$ is any subalgebra of $\mathfrak{u}(3)$ that does not lie in $\mathfrak{su}(3)$. If we denote the projection $\mathfrak{u}(3) \to \mathcal{Z}(\mathfrak{u}(3)) = \mathfrak{u}(1)$ by π then its restriction to $\mathfrak{m}_H$ is surjective. The kernel $\ker \pi|_{\mathfrak{m}_H} =: \mathfrak{k}_2$ is an ideal in $\mathfrak{m}_H$ and equals $\mathfrak{m}_H \cap \mathfrak{su}(3)$. We obtain that the reductive subalgebra $\mathfrak{m}_H$ of $\mathfrak{u}(3)$ decomposes as $\mathfrak{m}_H = \mathfrak{k}_1 \oplus \mathfrak{k}_2$, where $\mathfrak{k}_1 \cong \mathfrak{u}(1)$. Any subalgebra of $\mathfrak{su}(3)$ has at most dimension 8 and is at most of rank 2. The simple algebras of rank 1 and 2 are $\mathfrak{su}(2) \cong \mathfrak{so}(3) = \mathfrak{sp}(1)$, $\mathfrak{su}(3)$, $\mathfrak{so}(5)$ and $\mathfrak{g}_2$. Since $\dim_{\mathbb{R}}(\mathfrak{so}(5)) = 10$ and $\dim_{\mathbb{R}}(\mathfrak{g}_2) = 14$ they can not be subalgebras of $\mathfrak{k}_2$. We therefore obtain the following list of possibilities for $\mathfrak{m}_H = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ up to isomorphism:

$$\begin{gathered} \mathfrak{u}(1), \quad \mathfrak{u}(1) \oplus \mathfrak{u}(1), \quad \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \\ \mathfrak{u}(1) \oplus \mathfrak{su}(2), \quad \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(2), \quad \mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2), \\ \mathfrak{u}(1) \oplus \mathfrak{su}(3), \end{gathered}$$

i.e., $\mathfrak{m}_H$ is isomorphic to $\mathfrak{u}(1)^k$, $\mathfrak{u}(2) \oplus \mathfrak{l}_1$ or $\mathfrak{u}(3)$, where $1 \leq k \leq 3$ and $\mathfrak{l}_1 \subset \mathfrak{su}(2) \cong \mathfrak{sp}(1)$ is arbitrary.

If $l = 7$ then $\mathfrak{n}_7$ has codimension 1 in $\mathfrak{n}$. Since $\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_7) = \mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_7^{\perp})$ and $\mathfrak{n}_7^{\perp} = \mathbb{R}e_8$ we obtain $\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_7) = \mathcal{N}_{\mathfrak{m}}(\mathbb{R}e_8) = \mathfrak{g}_2$. Due to Proposition 3.2 the subalgebra $\mathfrak{m}_H \subset \mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_7) = \mathfrak{g}_2$ is arbitrary in this case.

To summarize, we have seen that if $\mathfrak{h}$ is a spherical algebraic subalgebra of $\mathfrak{g}$ then $\mathfrak{m}_H$ has to be conjugate to one of the following, where $\mathfrak{l}_1 \subset \mathfrak{sp}(1)$ and $\mathfrak{l}_2 \subsetneq \mathfrak{sp}(1)$ are arbitrary.

l	$\mathfrak{m}_H$
0	$\mathfrak{so}(7)$
1	$\mathfrak{g}_2$,
2	$\mathfrak{u}(3),\ \mathfrak{su}(3)$
4	$\mathfrak{so}(4),\ \mathfrak{sp}(1)\oplus\mathfrak{l}_2,\ \mathfrak{l}_2\subset\mathfrak{sp}(1)$
5	$\mathfrak{so}(4),\ \mathfrak{sp}(1)\oplus\mathfrak{l}_2$
6	$\mathfrak{u}(1)\oplus\mathfrak{k}_2$
7	any subalgebra of $\mathfrak{g}_2$

and $\mathfrak{k}_2$ is isomorphic to $\{0\}$, $\mathfrak{u}(1)^k$ where $1 \le k \le 2$, $\mathfrak{su}(2)\oplus\mathfrak{l}_1$ or $\mathfrak{su}(3)$. Now let us consider the case that $\mathfrak{n}_H = \mathfrak{n}_{4,x}$, i.e., $\mathfrak{n}_H = W \oplus \mathfrak{g}_{2\alpha}$ and $W \subset \mathfrak{g}_\alpha = \mathbb{R}^8$ is the real span of $e_1, e_2, e_3, xe_4 + e_8$ for some $x \in \mathbb{R}$. Then $\mathfrak{n}_{4,x}^\perp \subset \mathfrak{g}_\alpha$ is the real span of $e_5, e_6, e_7, e_4 - xe_8$ and

$$\mathcal{N}_{\mathfrak{so}(8)}(W) = \left\{ \begin{pmatrix} 0 & * & * & -v_1 & 0 & 0 & 0 & -w_1 \\ * & 0 & * & -v_2 & 0 & 0 & 0 & -w_2 \\ * & * & 0 & -v_3 & 0 & 0 & 0 & -w_3 \\ v_1 & v_2 & v_3 & 0 & z_1 & z_2 & z_3 & 0 \\ 0 & 0 & 0 & -z_1 & 0 & * & * & -y_1 \\ 0 & 0 & 0 & -z_2 & * & 0 & * & -y_2 \\ 0 & 0 & 0 & -z_3 & * & * & 0 & -y_3 \\ w_1 & w_2 & w_3 & 0 & y_1 & y_2 & y_3 & 0 \end{pmatrix} \in \mathfrak{so}(8), v_i = xw_i, -xz_i = y_i\ \forall\ 1 \le i \le 3 \right\}$$

holds. A direct calculation shows

$$\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_{4,0}) = \left\{ \begin{pmatrix} 0 & -a & -b & 0 & 0 & 0 & 0 \\ a & 0 & -g & 0 & 0 & 0 & 0 \\ b & g & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q & -r & -s \\ 0 & 0 & 0 & q & 0 & -t & -u \\ 0 & 0 & 0 & r & t & 0 & -v \\ 0 & 0 & 0 & s & u & v & 0 \end{pmatrix} : a, b, g, q, r, s, t, u, v \in \mathbb{R} \right\} \cong \mathfrak{so}(3) \oplus \mathfrak{so}(4)$$

and

$$\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_{4,x\neq 0}) = \left\{ \begin{pmatrix} 0 & -a & -b & 0 & 0 & \frac{x}{2}(a-t) & \frac{x}{2}(b-u) \\ a & 0 & -g & 0 & -\frac{x}{2}(a-t) & 0 & \frac{x}{2}(g-v) \\ b & g & 0 & 0 & -\frac{x}{2}(b-u) & -\frac{x}{2}(g-v) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{x}{2}(a-t) & \frac{x}{2}(b-u) & 0 & 0 & -t & -u \\ -\frac{x}{2}(a-t) & 0 & \frac{x}{2}(g-v) & 0 & t & 0 & -v \\ -\frac{x}{2}(b-u) & \frac{x}{2}(g-v) & 0 & 0 & u & v & 0 \end{pmatrix} : a, b, g, t, u, v \in \mathbb{R} \right\}$$

in the case that $x \neq 0$. In order to understand the Lie algebra structure of $\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_{4,x\neq 0})$ and the action of $\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_{4,x})$ on $\mathfrak{n}_{4,x}^\perp = W^\perp \subset \mathbb{R}^8$ for arbitrary x we calculate the image of $\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_{4,x})$ under φ.

$$\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_{4,x})) = \begin{cases} \left\{ \begin{pmatrix} 0 & a & b & 0 & 0 & 0 & 0 & c+d-i \\ -a & 0 & c & 0 & 0 & 0 & 0 & -b+e+h \\ -b & -c & 0 & 0 & 0 & 0 & 0 & a+f-g \\ 0 & 0 & 0 & 0 & d & e & f & 0 \\ 0 & 0 & 0 & -d & 0 & g & h & 0 \\ 0 & 0 & 0 & -e & -g & 0 & i & 0 \\ 0 & 0 & 0 & -f & -h & -i & 0 & 0 \\ -c-d+i & b-e-h & -a-f+g & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(8) \right\}, & x = 0, \\ \left\{ \begin{pmatrix} 0 & d & e & xa & 0 & 0 & 0 & a \\ -d & 0 & f & xb & 0 & 0 & 0 & b \\ -e & -f & 0 & xc & 0 & 0 & 0 & c \\ -xa & -xb & -xc & 0 & a & b & c & 0 \\ 0 & 0 & 0 & -a & 0 & d & e & xa \\ 0 & 0 & 0 & -b & -d & 0 & f & xb \\ 0 & 0 & 0 & -c & -e & -f & 0 & xc \\ -a & -b & -c & 0 & -xa & -xb & -xc & 0 \end{pmatrix} : a, b, c, d, e, f \in \mathbb{R} \right\}, & x \neq 0. \end{cases}$$

Let π denote the projection of $\mathcal{N}_{\mathfrak{so}(8)}(W)$ to $\mathfrak{so}(W^\perp) \cong \mathfrak{so}(4)$ given by

$$\begin{pmatrix} 0 & * & * & -xg & 0 & 0 & 0 & -g \\ * & 0 & * & -xh & 0 & 0 & 0 & -h \\ * & * & 0 & -xj & 0 & 0 & 0 & -j \\ xg & xh & xj & 0 & a & b & c & 0 \\ 0 & 0 & 0 & -a & 0 & d & e & ax \\ 0 & 0 & 0 & -b & -d & 0 & f & bx \\ 0 & 0 & 0 & -c & -e & -f & 0 & cx \\ g & h & j & 0 & -ax & -bx & -cx & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & c & 0 \\ 0 & 0 & 0 & -a & 0 & d & e & ax \\ 0 & 0 & 0 & -b & -d & 0 & f & bx \\ 0 & 0 & 0 & -c & -e & -f & 0 & cx \\ 0 & 0 & 0 & 0 & -ax & -bx & -cx & 0 \end{pmatrix}.$$

Then its restriction $\pi|_{\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_{4,x}))} : \varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_{4,x})) \to \mathfrak{so}(W^\perp) \cong \mathfrak{so}(4)$ is a surjective Lie algebra homomorphism which is an isomorphism if $x \neq 0$. This implies that the action of $\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_{4,x})$ on $W^\perp \cong \mathbb{R}^4$ is the standard action of $\mathfrak{so}(4)$ on $\mathbb{R}^4$. Hence $\mathcal{N}_M(\mathfrak{n}_{4,x})$ acts transitively on the spheres in $W^\perp$ and the only proper subgroups of $N_M(\mathfrak{n}_{4,x})$ that act transitively on the spheres in $W^\perp$ have Lie algebras that lie in the preimage

$$\left(\pi|_{\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_{4,x}))}\right)^{-1}\big(\underbrace{\mathfrak{sp}(1) \oplus \mathfrak{l}_2}_{\subset \mathfrak{so}(4)}\big) \quad \text{or} \quad \left(\pi|_{\varphi(\mathcal{N}_{\mathfrak{m}}(\mathfrak{n}_{4,x}))}\right)^{-1}\big(\underbrace{\mathfrak{l}_2 \oplus \mathfrak{sp}(1)}_{\subset \mathfrak{so}(4)}\big),$$

where $\mathfrak{l}_2 \subset \mathfrak{sp}(1)$ is arbitrary. □

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