HAL open science

Geometry of invariant domains in complex semi-simple Lie groups

Christian Miebach

▶ To cite this version:

Christian Miebach. Geometry of invariant domains in complex semi-simple Lie groups. Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, 2009, 8 (3), pp.509-541. 10.2422/2036-2145.2009.3.05 . hal-03600270

HAL Id: hal-03600270

https://hal.science/hal-03600270v1

Submitted on 7 Mar 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

GEOMETRY OF INVARIANT DOMAINS IN COMPLEX SEMI-SIMPLE LIE GROUPS

CHRISTIAN MIEBACH

Abstract. We investigate the joint action of two real forms of a semi-simple complex Lie group $U^{\mathbb{C}}$ by left and right multiplication. After analyzing the orbit structure, we study the CR structure of closed orbits. The main results are an explicit formula of the Levi form of closed orbits and the determination of the Levi cone of generic orbits. Finally, we apply these results to prove q–completeness of certain invariant domains in $U^{\mathbb{C}}$.

1. Introduction

Let $U^{\mathbb{C}}$ be a connected semi-simple complex Lie group with compact real form U which is given by the Cartan involution θ. Let us assume that there are two anti-holomorphic involutive automorphisms σ_1 and σ_2 of $U^{\mathbb{C}}$ which both commute with θ and let $G_j = \mathrm{Fix}(\sigma_j)$, $j = 1, 2$, be the corresponding real forms of $U^{\mathbb{C}}$. The group $G_1 \times G_2$ acts on $U^{\mathbb{C}}$ by $(g_1, g_2) \cdot z := g_1 z g_2^{-1}$. In this paper we investigate complex-analytic properties of certain $(G_1 \times G_2)$–invariant domains in $U^{\mathbb{C}}$ through the intrinsic Levi form of closed $(G_1 \times G_2)$–orbits.

If $\sigma_1 = \sigma_2 = \theta$, then we discuss the $(U \times U)$–action on $U^{\mathbb{C}}$ by left and right multiplication. Every $(U \times U)$–orbit intersects the set $\exp(i\mathfrak{t})$ in the orbit of the Weyl group $W := \mathcal{N}_U(\mathfrak{t})/\mathcal{Z}_U(\mathfrak{t})$ where $\mathfrak{t}$ is a maximal torus in $\mathfrak{u}$. In [Las78] Lassalle showed that every bi-invariant domain $\Omega \subset U^{\mathbb{C}}$ is of the form $U \exp(i\omega)U$ for a W–invariant domain $\omega \subset \mathfrak{t}$ and that Ω is a domain of holomorphy if and only if ω is convex. In [AL92] Azad and Loeb proved the stronger statement that a $(U \times U)$–invariant function Φ on Ω is plurisubharmonic if and only if the W–invariant function

$$\varphi \colon \mathfrak{t} \to \mathbb{R}, \quad \varphi(\eta) := \Phi\big(\exp(i\eta)\big),$$

is convex.

In the case that $\sigma_1 = \sigma_2$ and G_1 is a real semi-simple Lie group of Hermitian type there is a distinguished $(G_1 \times G_1)$–invariant in $U^{\mathbb{C}}$, namely the open complex Ol'shanskiĭ semi-group. According to a result of Neeb ([Nee98]) the open Ol'shanskiĭ semi-group is a domain of holomorphy.

Although the above results are statements about complex-analytic properties of domains in complex Stein manifolds, the method of their proofs is representation-theoretic. A different approach to the study of $(G_1 \times G_1)$–invariant domains in $U^{\mathbb{C}}$ by analytic methods was made by Fels an Geatti in [FG98]. There, Fels and Geatti gave explicit formulas for the intrinsic Levi form of a closed orbit $M_z := (G_1 \times G_1) \cdot z$ of maximal dimension in $U^{\mathbb{C}}$ (in the following called a generic orbit) and determined the Levi cone of M_z, which enabled them to decide whether or not there may exist a bi-invariant domain of holomorphy containing z in its boundary.

The main results in this paper are an explicit formula for the intrinsic Levi form of an arbitrary closed $(G_1 \times G_2)$–orbit in $U^{\mathbb{C}}$ and the determination of the Levi cone of a generic orbit. We use a theorem of Matsuki ([Mat97]) in order to obtain a parameterization of closed $(G_1 \times G_2)$–orbits by certain Cartan algebras in the Lie algebra $\mathfrak{u}^{\mathbb{C}} = \mathrm{Lie}(U^{\mathbb{C}})$. More precisely, there are finitely many Cartan algebras $\mathfrak{c}_j$ such that the closed orbits are precisely those intersecting a set of the form $C_j = n \exp(i\mathfrak{c}_j)$, where the element n can be chosen from a fixed torus in U. It turns out that the weight space decomposition of $\mathfrak{u}^{\mathbb{C}}$ with respect to $\mathfrak{c}_j$ is well-suited to describe the CR structure of closed orbits intersecting C_j. In particular, the complex tangent space of such an orbit can be identified with a direct sum of weight spaces and the intrinsic Levi form of a closed orbit is determined by the Lie bracket of certain weight vectors together with a coefficient which depends on the intersection of the orbit with C_j (Theorem 3.11). From this

fact it can be derived that the CR structures of closed orbits which belong to the same set C_j have very similar properties.

The method used here for the derivation of explicit formulas for the Levi form is different from the one used in [FG98]. While Fels and Geatti found explicit local extensions of complex tangent vectors to CR vector fields on a generic orbit and computed their Lie brackets, the approach used here avoids these technical difficulties by pulling back the CR structure of the orbit into the Lie algebra of $G_1 \times G_2$ where the Levi form can be determined via Lie-theoretic methods. In particular, we obtain a new proof for their results in the case $G_1 = G_2$.

A finer analysis of the weight space decomposition of $\mathfrak{u}^{\mathbb{C}}$ with respect to $\mathfrak{c}_j$ reveals that it has properties very close to a root space decomposition. The most important one is the existence of $\mathfrak{sl}(2)$–triples which enables us to determine the Levi cone of generic orbits by essentially the same method as in [FG98] (Theorem 3.17).

In Section 4 we give several applications of the results obtained so far. First we use the criterion from [FG98] together with the knowledge of the Levi cone in order to decide which $(G_1 \times G_2)$–invariant domains containing a generic orbit in their boundary can be Stein. Secondly, we classify and study the rank one case in some detail since this case provides a class of examples where the methods and results become most transparent. This is due to the facts that complex-analytic properties of smooth domains in Stein manifolds are determined by the classical Levi form of their boundaries and that in the rank one case the boundaries of almost all invariant domains coincide with orbits of hypersurface type. Finally, we define a $(G_1 \times G_2)$–invariant domain $\Omega \subset U^{\mathbb{C}}$ which is the right analogon of the open complex Ol'shanskiĭ semi-group in the case $G_1 \neq G_2$. We prove that the classical Levi form of a $(G_1 \times G_2)$–invariant smooth function at a point $z \in \Omega$ splits into a contribution coming from the complex tangent space of $(G_1 \times G_2) \cdot z$ and a contribution due to a transversal slice. Via this splitting we construct a strictly q–convex function on Ω which goes to infinity at $\partial\Omega$, and hence conclude that Ω is q–complete.

Notation. If φ is an automorphism of a Lie group G, then by abuse of notation we write φ also for the derived automorphism of $\mathfrak{g} = \mathrm{Lie}(G)$.

Acknowledgment. This paper is a modified version of the author's Ph.D. thesis [Mie07]. The support by a Promotionsstipendium of the Studienstiftung des deutschen Volkes and by SFB/TR 12 of the DFG is gratefully acknowledged.

2. The $(G_1 \times G_2)$–Action on $U^{\mathbb{C}}$

2.1. **Compatible real forms.** Let U be a connected semi-simple compact Lie group. Then its universal complexification $U^{\mathbb{C}}$ is a connected semi-simple complex Lie group, and hence carries a unique structure of a linear algebraic group (compare [Che70]). The map $\Phi\colon U \times i\mathfrak{u} \to U^{\mathbb{C}}$, $(u, \xi) \mapsto u \exp(\xi)$, is a real-analytic diffeomorphism, called the Cartan decomposition of $U^{\mathbb{C}}$. Furthermore, the map $\theta\colon U^{\mathbb{C}} \to U^{\mathbb{C}}$, $\theta\big(u \exp(\xi)\big) := u \exp(-\xi)$, is an anti-holomorphic involutive automorphism with $U = \mathrm{Fix}(\theta)$, called the Cartan involution of $U^{\mathbb{C}}$ corresponding to the compact real form U. Proofs of these facts can be found e. g. in [Kna02].

Let σ_1 and σ_2 be two anti-holomorphic involutive automorphisms of $U^{\mathbb{C}}$, which both commute with the Cartan involution θ. The fixed point set $G_j := \mathrm{Fix}(\sigma_j)$ is a real form of $U^{\mathbb{C}}$ for $j = 1, 2$. The assumption that σ_j commutes with θ implies that the Cartan decomposition of $U^{\mathbb{C}}$ restricts to a real-analytic diffeomorphism $K_j \times \mathfrak{p}_j \to G_j$, where $K_j := G_j \cap U$ and $\mathfrak{p}_j := \mathfrak{g}_j \cap i\mathfrak{u}$ hold. Thus the real form G_j is a compatible subgroup of $U^{\mathbb{C}}$ in the sense of [HS07]. In particular, K_j is a deformation retract of G_j.

Remark. Since G_j is closed, the group K_j is compact and hence a maximal compact subgroup of G_j. Thus G_j has only finitely many connected components. If the group $U^{\mathbb{C}}$ is simply-connected, it follows from [Ste68] that G_j is connected.

The product group $G_1 \times G_2$ acts on $U^{\mathbb{C}}$ by left and right multiplication, i. e. we define

$$(g_1, g_2) \cdot z := g_1 z g_2^{-1}$$

where $g_j \in G_j$ and $z \in U^{\mathbb{C}}$.

Definition 2.1. We say that an element $z \in U^{\mathbb{C}}$ is regular (with respect to $G_1 \times G_2$) if the orbit $(G_1 \times G_2) \cdot z$ has maximal dimension. The element z is called strongly regular (with respect to $G_1 \times G_2$), if it is regular and if $(G_1 \times G_2) \cdot z$ is closed. We write $U_r^{\mathbb{C}}$ and $U_{sr}^{\mathbb{C}}$ for the sets of regular and strongly regular elements, respectively. Finally, we call the orbit $(G_1 \times G_2) \cdot z$ generic if z is strongly regular.

Remark. (a) If we consider the action of $U^{\mathbb{C}}$ on itself given by conjugation, then Definition 2.1 yields the usual notion of (strongly) regular elements in linear algebraic groups (compare [Hum95]).
(b) The subsets $U_r^{\mathbb{C}}$ and $U_{sr}^{\mathbb{C}}$ are invariant under $G_1 \times G_2$. The set $U_{sr}^{\mathbb{C}}$ can be proven to be open and dense in $U^{\mathbb{C}}$ which justifies the terminology "generic orbit".
(c) In [Mat97] an element $z \in U^{\mathbb{C}}$ is called regular semi-simple if the automorphism $\mathrm{Ad}(z^{-1})\sigma_1 \mathrm{Ad}(z)\sigma_2$ is semi-simple and if the Lie algebra $\mathfrak{g}_2 \cap \mathrm{Ad}(z^{-1})\mathfrak{g}_1$ is Abelian. It can be shown that an element is regular semi-simple in Matsuki's sense if and only if it is strongly regular.

2.2. **The isotropy representation.** The following proposition is crucial. For convenience of the reader we give a short proof which makes use of the complex-analytic structure of $U^{\mathbb{C}}$.

Proposition 2.2. *Let $z \in U^{\mathbb{C}}$ be a point such that the orbit $M_z := (G_1 \times G_2) \cdot z$ is closed. Then the isotropy group $(G_1 \times G_2)_z$ is real-reductive and the isotropy representation of $(G_1 \times G_2)_z$ on $T_z U^{\mathbb{C}}$ is completely reducible.*

Proof. Since $U^{\mathbb{C}}$ is a Stein manifold, there exists a smooth strictly plurisubharmonic exhaustion function $\rho\colon U^{\mathbb{C}} \to \mathbb{R}$. By compactness of U we can average ρ using the Haar measure and hence assume that ρ is $(U \times U)$–invariant. It follows that $\omega := i\partial\overline{\partial}\rho$ is a $(U \times U)$–invariant Kähler form on $U^{\mathbb{C}}$ with respect to which $U \times U$ acts in a Hamiltonian fashion. The last statement means that there exists a $(U \times U)$–equivariant momentum map $\mu\colon U^{\mathbb{C}} \to \mathfrak{u}^* \oplus \mathfrak{u}^*$. Since the group $G_1 \times G_2$ is compatible with the Cartan decomposition of $U^{\mathbb{C}} \times U^{\mathbb{C}}$, we can restrict μ to the subspace $(i\mathfrak{p}_1)^* \oplus (i\mathfrak{p}_2)^*$ and obtain the restricted momentum map $\mu_{i\mathfrak{p}}\colon U^{\mathbb{C}} \to (i\mathfrak{p}_1)^* \oplus (i\mathfrak{p}_2)^*$. According to [HS07] this restricted momentum map encodes a lot of information about the $(G_1 \times G_2)$–action on $U^{\mathbb{C}}$ from which we need the following.

(a) A $(G_1 \times G_2)$–orbit is closed in $U^{\mathbb{C}}$ if and only if it intersects $\mathcal{M}_{i\mathfrak{p}} := \mu_{i\mathfrak{p}}^{-1}(0)$ non-trivially (Proposition 11.2 in [HS07]).
(b) If $z \in \mathcal{M}_{i\mathfrak{p}}$, then the isotropy group $(G_1 \times G_2)_z$ is a compatible subgroup of $U^{\mathbb{C}} \times U^{\mathbb{C}}$ and hence real-reductive (Lemma 5.5 in [HS07]). Together with the previous statement this implies that isotropy groups of closed orbits are real-reductive.
(c) If $z \in \mathcal{M}_{i\mathfrak{p}}$, then the isotropy representation is completely reducible (Corollary 14.9 in [HS07]).

Hence, the proposition is proven. □

In the rest of this subsection we will have a closer look at the isotropy representation. Every element $(\xi_1, \xi_2) \in \mathfrak{g}_1 \oplus \mathfrak{g}_2$ induces the tangent vector

$$\begin{aligned} \frac{d}{dt}\Big|_0 \big(\exp(t\xi_1) z \exp(-t\xi_2)\big) &= \frac{d}{dt}\Big|_0 \big(z \exp(t\,\mathrm{Ad}(z^{-1})\xi_1) \exp(-t\xi_2)\big) \\ &= (\ell_z)_* \big(\mathrm{Ad}(z^{-1})\xi_1 - \xi_2\big) \in T_z U^{\mathbb{C}}, \end{aligned} \tag{2.1}$$

where ℓ_z denotes left multiplication with $z \in U^{\mathbb{C}}$. These tangent vectors span the tangent space of the $(G_1 \times G_2)$–orbit through z, i. e. we obtain $T_z M_z = (\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cdot z = \big\{(\ell_z)_*\xi;\ \xi \in \mathfrak{g}_2 + \mathrm{Ad}(z^{-1})\mathfrak{g}_1\big\}$.

Let ρ denote the isotropy representation of $(G_1 \times G_2)_z$ on $T_z U^{\mathbb{C}}$. One checks directly that the isotropy group at $z \in U^{\mathbb{C}}$ is given by

$$(G_1 \times G_2)_z = \big\{(z g_2 z^{-1}, g_2);\ g_2 \in G_2 \cap z^{-1} G_1 z\big\}.$$

Consequently, we may identify $(G_1 \times G_2)_z$ with $G_2 \cap z^{-1}G_1 z$ via the isomorphism $\Phi\colon G_2 \cap z^{-1}G_1 z \to (G_1 \times G_2)_z$, $g \mapsto (zgz^{-1}, g)$. Similarly, we will identify the tangent space $T_z M_z$ with $\mathfrak{g}_2 + \mathrm{Ad}(z^{-1})\mathfrak{g}_1$ via

$(\ell_z)_*$. We conclude from

$$\begin{aligned}\rho\big(\Phi(g)\big)(\ell_z)_*\xi &= \frac{d}{dt}\Big|_0 (zgz^{-1}, g)\cdot\big(z\exp(t\xi)\big)\\ &= \frac{d}{dt}\Big|_0 \big(zg\exp(t\xi)g^{-1}\big)\\ &= \frac{d}{dt}\Big|_0 z\exp\big(t\operatorname{Ad}(g)\xi\big) = (\ell_z)_*\operatorname{Ad}(g)\xi\end{aligned}$$

that the map $(\ell_z)_*$ intertwines the adjoint representation of $G_2\cap z^{-1}G_1z$ on $\mathfrak{u}^{\mathbb{C}}$ with the isotropy representation of $(G_1\times G_2)_z$ on $T_zU^{\mathbb{C}}$ modulo Φ. We summarize our considerations in the following

Proposition 2.3. *Modulo the isomorphism Φ the isotropy representation of $(G_1\times G_2)_z$ on $T_zU^{\mathbb{C}}$ is equivalent to the adjoint representation of $G_2\cap z^{-1}G_1z$ on $\mathfrak{u}^{\mathbb{C}}$.*

2.3. **The orbit structure theorem.** We review the main results of [Mat97] in order to describe the orbit structure of the $(G_1\times G_2)$–action on $U^{\mathbb{C}}$. A proof of Matsuki's theorem which relies on the momentum map techniques developed in [HS07] can be found in [Mie07].

Let $\mathfrak{a}_0$ be a maximal Abelian subspace of $\mathfrak{p}_1\cap\mathfrak{p}_2$ and let $\mathfrak{t}_0$ be a maximal torus in the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}_1\cap\mathfrak{k}_2$. It follows that $\mathfrak{c}_0 := \mathfrak{t}_0\oplus\mathfrak{a}_0$ is a maximally non-compact θ–invariant Cartan subalgebra of $\mathfrak{g}_1\cap\mathfrak{g}_2$.

Remark. By maximality of $\mathfrak{a}_0$ the group $A_0^c := \exp(i\mathfrak{a}_0)$ is a compact torus in U.

Definition 2.4. A subset of the form $C = n\exp(i\mathfrak{c})\subset U^{\mathbb{C}}$ is called a standard Cartan subset, if $n\in A_0^c$ and $\mathfrak{c}=\mathfrak{t}\oplus\mathfrak{a}$ is a θ–stable Cartan subalgebra of $\mathfrak{g}_2\cap\operatorname{Ad}(n^{-1})\mathfrak{g}_1$ such that $\mathfrak{t}\supset\mathfrak{t}_0$, $\mathfrak{a}\subset\mathfrak{a}_0$ and $\dim\mathfrak{c}=\dim\mathfrak{c}_0$ hold. The standard Cartan subset $C_0 := \exp(i\mathfrak{c}_0)$ is called the fundamental Cartan subset.

We call two standard Cartan subsets equivalent if there is a generic $(G_1\times G_2)$–orbit which intersects both non-trivially. Let $\{C_j\}_{j\in J}$ be a complete set of representatives for the equivalence classes. For each $j\in J$ we define the groups

$$\begin{aligned}\mathcal{N}_{K_1\times K_2}(C_j) &:= \big\{(k_1,k_2)\in K_1\times K_2;\ k_1C_jk_2^{-1}=C_j\big\},\\ \mathcal{Z}_{K_1\times K_2}(C_j) &:= \big\{(k_1,k_2)\in K_1\times K_2;\ k_1zk_2^{-1}=z \text{ for all } z\in C_j\big\},\end{aligned}$$

and $W_{K_1\times K_2}(C_j) := \mathcal{N}_{K_1\times K_2}(C_j)/\mathcal{Z}_{K_1\times K_2}(C_j)$.

Remark. The group $W_{K_1\times K_2}(C_j)$ is finite for each $j\in J$.

Theorem 2.5 (Matsuki)**.** *The set J is finite and we have*

$$U^{\mathbb{C}}_{cl} = \bigcup_{j\in J} G_1C_jG_2 \quad \textit{and} \quad U^{\mathbb{C}}_{sr} = \dot{\bigcup}_{j\in J} G_1(C_j\cap U^{\mathbb{C}}_{sr})G_2,$$

where $U^{\mathbb{C}}_{cl} := \{z\in U^{\mathbb{C}};\ (G_1\times G_2)\cdot z$ is closed$\}$. Moreover, each generic $(G_1\times G_2)$–orbit intersects C_j in a $W_{K_1\times K_2}(C_j)$–orbit.

Remark. If $G_1=G_2$, then let $\mathfrak{c}_0,\dots,\mathfrak{c}_k$ be a complete set of representatives for the equivalence classes of Cartan subalgebras of $\mathfrak{g}_1$. We can assume without loss of generality that each $\mathfrak{c}_j$ is θ–stable. Let $\{n_{j,l}\}$ be a set of representatives for the Weyl group corresponding to $\mathfrak{c}_j$. It can be shown that the sets $n_{j,l}\exp(i\mathfrak{c}_j)$ exhaust the equivalence classes of standard Cartan subsets for the $(G_1\times G_1)$–action on $U^{\mathbb{C}}$. Hence, we obtain Bremigan's theorem ([Bre96]).

2.4. **The weight space decomposition.** Let $C = n\exp(i\mathfrak{c})$ be a standard Cartan subset. In this subsection we discuss the weight space decomposition

$$\mathfrak{u}^{\mathbb{C}} = \bigoplus_{\lambda\in\Lambda}\mathfrak{u}^{\mathbb{C}}_\lambda$$

of $\mathfrak{u}^{\mathbb{C}}$ with respect to the Cartan subalgebra $\mathfrak{c}\subset\mathfrak{g}_2\cap\operatorname{Ad}(n^{-1})\mathfrak{g}_1$. Here, we have written $\Lambda=\Lambda(\mathfrak{u}^{\mathbb{C}},\mathfrak{c})$ for the set of weights and $\mathfrak{u}^{\mathbb{C}}_\lambda$ for the weight space corresponding to the weight λ. We say that the weight

λ is real (respectively imaginary) if $\lambda \neq 0$ and $\lambda(\mathfrak{c}) \subset \mathbb{R}$ (respectively $\lambda(\mathfrak{c}) \subset i\mathbb{R}$) holds. A non-zero weight which is neither real nor imaginary is called complex. We write Λ_r, Λ_i and Λ_c for the sets of real, imaginary and complex weights, and obtain

$$\Lambda \setminus \{0\} = \Lambda_r \,\dot\cup\, \Lambda_i \,\dot\cup\, \Lambda_c.$$

Remark. We extend the weight λ by $\mathbb{C}$–linearity to the complexified Cartan algebra $\mathfrak{c}^{\mathbb{C}}$. Since $\lambda(\mathfrak{t}) \subset i\mathbb{R}$ and $\lambda(\mathfrak{a}) \subset \mathbb{R}$ hold for all $\lambda \in \Lambda$, we conclude that the weights are real-valued on $i\mathfrak{t} \oplus \mathfrak{a}$.

Since $n \in A_0^c$, the automorphism $\tau_n := \mathrm{Ad}(n^{-1})\sigma_1 \,\mathrm{Ad}(n)\sigma_2 \in \mathrm{Aut}(\mathfrak{u}^{\mathbb{C}})$ is unitary with respect to the Hermitian inner product $\langle \xi_1, \xi_2 \rangle := -B_{\mathfrak{u}^{\mathbb{C}}}\big(\xi_1, \theta(\xi_2)\big)$, where $B_{\mathfrak{u}^{\mathbb{C}}}$ is the Killing form of $\mathfrak{u}^{\mathbb{C}}$. Consequently, τ_n is semi-simple with eigenvalues in the unit circle S^1. Since τ_n leaves $\mathfrak{c}$ pointwise fixed, each weight space $\mathfrak{u}^{\mathbb{C}}_\lambda$ is invariant under τ_n. Hence, following [Mat97] we obtain the finer decomposition

$$\mathfrak{u}^{\mathbb{C}} = \bigoplus_{(\lambda,a)\in\widetilde{\Lambda}} \mathfrak{u}^{\mathbb{C}}_{\lambda,a}, \tag{2.2}$$

where $\mathfrak{u}^{\mathbb{C}}_{\lambda,a} := \big\{\xi \in \mathfrak{u}^{\mathbb{C}}_\lambda;\ \tau_n(\xi) = a\xi\big\}$ and $\widetilde{\Lambda} := \big\{(\lambda, a) \in \Lambda \times S^1;\ \mathfrak{u}^{\mathbb{C}}_{\lambda,a} \neq \{0\}\big\}$. The elements of $\widetilde{\Lambda}$ are called the extended weights, and (2.2) is called the extended weight space decomposition.

Remark. Since $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{g}_2 \cap \mathrm{Ad}(n^{-1})\mathfrak{g}_1$, we conclude $\mathfrak{u}^{\mathbb{C}}_{0,1} = \mathfrak{c}^{\mathbb{C}}$.

We collect some properties of the extended weight space decomposition in the following

Lemma 2.6. *(1) The Cartan involution θ maps $\mathfrak{u}^{\mathbb{C}}_{\lambda,a}$ onto $\mathfrak{u}^{\mathbb{C}}_{-\lambda,a^{-1}}$. In particular, if (λ, a) is an extended weight, then $(-\lambda, a^{-1})$ is an extended weight, too.*
(2) We have $B_{\mathfrak{u}^{\mathbb{C}}}\big(\mathfrak{u}^{\mathbb{C}}_{\lambda,a}, \mathfrak{u}^{\mathbb{C}}_{\mu,b}\big) = 0$ unless $(\lambda, a) = (-\mu, b^{-1}) \in \widetilde{\Lambda}$.
(3) Let $\xi_{\lambda,a} \in \mathfrak{u}^{\mathbb{C}}_{\lambda,a}$ with $\|\xi_{\lambda,a}\| = 1$ be given and let $\eta_{\lambda,a} := -\big[\xi_{\lambda,a}, \theta(\xi_{\lambda,a})\big]$. Then we have $B_{\mathfrak{u}^{\mathbb{C}}}(\eta_{\lambda,a}, \eta) = \lambda(\eta)$ for all $\eta \in \mathfrak{c}$. In particular, $\eta_{\lambda,a}$ does not depend on the element $a \in S^1$, i. e. $\eta_{\lambda,a} = \eta_{\lambda,a'} =: \eta_\lambda$ for all $(\lambda, a), (\lambda, a') \in \widetilde{\Lambda}$.
(4) We have $[\xi_{\lambda,a}, \xi] = B_{\mathfrak{u}^{\mathbb{C}}}(\xi_{\lambda,a}, \xi)\eta_\lambda$ for all $\xi \in \mathfrak{u}^{\mathbb{C}}_{-\lambda,a^{-1}}$.

Proof. In order to prove the first claim let $\eta = \eta_{\mathfrak{t}} + \eta_{\mathfrak{a}} \in \mathfrak{t} \oplus \mathfrak{a} = \mathfrak{c}$ and $\xi \in \mathfrak{u}^{\mathbb{C}}_{\alpha,\lambda}$ be given and consider

$$\big[\eta, \theta(\xi)\big] = \theta\big[\theta(\eta), \xi\big] = \theta[\eta_{\mathfrak{t}} - \eta_{\mathfrak{a}}, \xi] = \theta\big(\lambda(\eta_{\mathfrak{t}})\xi\big) - \theta\big(\lambda(\eta_{\mathfrak{a}})\xi\big) = -\lambda(\eta)\theta(\xi).$$

Here we used the facts that $\lambda(\mathfrak{t}) \subset i\mathbb{R}$ while $\lambda(\mathfrak{a}) \subset \mathbb{R}$ and that θ is $\mathbb{C}$–anti-linear. Since θ commutes with τ_n, we conclude

$$\tau_n\theta(\xi) = \theta\tau_n(\xi) = \theta(a\xi) = \overline{a}\theta(\xi) = a^{-1}\theta(\xi),$$

which proves the first claim.

The second claim follows from the fact that the Killing form $B_{\mathfrak{u}^{\mathbb{C}}}$ is invariant under $\mathrm{Aut}(\mathfrak{u}^{\mathbb{C}})$.

In order to prove the third one we compute

$$\begin{aligned} B_{\mathfrak{u}^{\mathbb{C}}}(\eta_{\lambda,a}, \eta) = -B_{\mathfrak{u}^{\mathbb{C}}}\big([\xi_{\lambda,a}, \theta(\xi_{\lambda,a})], \eta\big) &= B_{\mathfrak{u}^{\mathbb{C}}}\big(\theta(\xi_{\lambda,a}), [\xi_{\lambda,a}, \eta]\big) \\ &= -\lambda(\eta)B_{\mathfrak{u}^{\mathbb{C}}}\big(\xi_{\lambda,a}, \theta(\xi_{\lambda,a})\big) = \lambda(\eta)\|\xi_{\lambda,a}\|^2 = \lambda(\eta). \end{aligned}$$

The last claim is proven in the same way as Lemma 2.18(a) in [Kna02]. □

Standard arguments from Lie theory (see for example Chapter II.4 in [Kna02]) lead to the following result.

Proposition 2.7. *(1) Let $\lambda \neq 0$. After a suitable normalization the elements η_λ, $\xi_{\lambda,a}$ and $\theta(\xi_{\lambda,a})$ form an $\mathfrak{sl}(2)$–triple.*
(2) If $\lambda \neq 0$, then we have $\dim_{\mathbb{C}} \mathfrak{u}^{\mathbb{C}}_{\lambda,a} = 1$ and $\dim_{\mathbb{C}} \mathfrak{u}^{\mathbb{C}}_{m\lambda,a^m} = 0$ for all $m \geq 2$.
(3) The set $\Lambda \setminus \{0\}$ of non-zero weights fulfills the axioms of an abstract root system in $(i\mathfrak{t} \oplus \mathfrak{a})^*$.
(4) Let $\lambda, \mu \in \Lambda \setminus \{0\}$ such that $\lambda + \mu \in \Lambda \setminus \{0\}$ holds. Then we have $[\mathfrak{u}^{\mathbb{C}}_\lambda, \mathfrak{u}^{\mathbb{C}}_\mu] = \mathfrak{u}^{\mathbb{C}}_{\lambda+\mu}$.

3. CR Geometry of Closed Orbits

3.1. **Preliminaries from CR geometry.** In this subsection we will review the basic definitions and facts from the theory of CR submanifolds as far as they are needed later on. For more details and complete proofs we refer the reader to the textbooks [BER99] and [Bog91].

Let Z be a complex manifold with complex structure J. A real submanifold M of Z is called a Cauchy-Riemann or CR submanifold if the dimension of the complex tangent space $H_pM := T_pM \cap J_pT_pM$ does not depend on the point $p \in M$. In this case, the set $HM := \bigcup_{p\in M} H_pM$ is a smooth subbundle of the tangent bundle TM invariant under the complex structure J, called the complex tangent bundle of M. A CR submanifold $M \subset Z$ is called generic if $T_pM + J_pT_pM = T_pZ$ holds for all $p \in M$. For example, every smooth real hypersurface in Z is a generic CR submanifold of Z.

Remark. Since the group $G_1 \times G_2$ acts by holomorphic transformations on $U^{\mathbb{C}}$, each closed $(G_1 \times G_2)$–orbit is a CR submanifold of $U^{\mathbb{C}}$. Since the $(G_1 \times G_2)$–action extends to a transitive $(U^{\mathbb{C}} \times U^{\mathbb{C}})$–action on $U^{\mathbb{C}}$, each closed orbit is moreover generic as a CR submanifold.

A smooth section in HM is called a CR vector field on M. A smooth map f from M into a CR submanifold $M' \subset (Z', J')$ is called a CR map if f_* maps HM into HM' and if $f_*J = J'f_*$ holds. A CR function on M is a CR map $M \to \mathbb{C}$, where $\mathbb{C}$ is equipped with its usual structure as complex manifold.

For each CR submanifold $M \subset Z$ one can define the intrinsic Levi form, which generalizes the classical Levi form of a smooth hypersurface.

Definition 3.1. The Levi form of M at the point p is the map $\mathcal{L}_p\colon H_pM \times H_pM \to T_p^{\mathbb{C}}M/H_p^{\mathbb{C}}M$ defined by

$$\mathcal{L}_p(v,w) := \left(\frac{i}{2}[V,W]_p - \frac{1}{2}[V,JW]_p\right) \bmod H_p^{\mathbb{C}}M,$$

where V and W are CR vector fields on M with $V_p = v$ and $W_p = w$.

Remark. One can show that the intrinsic Levi form is well-defined, i. e. that it does not depend on the choice of CR extensions of $v, w \in H_pM$ (compare [Bog91]).

The Levi cone $\mathcal{C}_p$ of M at p is by definition the closed convex cone generated by the vectors $\mathcal{L}_z(v,v)$ where v runs through H_pM. Because of $\mathcal{L}_p(v,w) = \overline{\mathcal{L}_p(w,v)}$ the Levi cone is contained in T_pM/H_pM. The Levi cone generalizes the signature of the classical Levi form of a hypersurface. Its significance stems from the fact that it governs the local extension of CR functions on M to holomorphic functions on Z.

Theorem 3.2 (Boggess, Polking)**.** *Let M be a generic CR submanifold of a complex manifold Z and let us assume that the Levi cone at some point $p \in M$ satisfies $\mathcal{C}_p(M) = T_pM/H_pM$. Then, for each neighborhood ω of p in M there exists a neighborhood Ω of p in Z satisfying $\Omega \cap M \subset \omega$ which has the property that every CR function on $\Omega \cap M$ extends to a unique holomorphic function on Ω.*

A proof of this theorem can be found in [Bog91].

3.2. **The complex tangent space of a closed orbit.** Let $z \in U^{\mathbb{C}}$ be given such that the orbit $M_z = (G_1 \times G_2)\cdot z$ is closed in $U^{\mathbb{C}}$. By Matsuki's theorem we can assume that there is a standard Cartan subset $C = n\exp(i\mathfrak{c})$ which contains $z = n\exp(i\eta)$. We define

$$\widetilde{\Lambda}(z) := \left\{(\lambda,a) \in \widetilde{\Lambda};\ ae^{-2i\lambda(\eta)} = 1\right\}$$

and set $\tau_z := \mathrm{Ad}(z^{-1})\sigma_1\,\mathrm{Ad}(z)\sigma_2 \in \mathrm{Aut}(\mathfrak{u}^{\mathbb{C}})$.

Lemma 3.3. *The automorphism τ_z is semi-simple and we have*

$$\mathrm{Fix}(\tau_z) = \left(\mathfrak{g}_2 \cap \mathrm{Ad}(z^{-1})\mathfrak{g}_1\right)^{\mathbb{C}} = \bigoplus_{(\lambda,a)\in\widetilde{\Lambda}(z)} \mathfrak{u}^{\mathbb{C}}_{\lambda,a}.$$

Proof. The first equality is a consequence of [Mat97], p. 57. In order to prove the second one let $\xi = \sum_{(\lambda,a)} \xi_{\lambda,a}$ be an arbitrary element of $\mathfrak{u}^{\mathbb{C}}$. Then we have

$$\begin{aligned}\tau_z(\xi) = \operatorname{Ad}(z^{-1})\sigma_1 \operatorname{Ad}(z)\sigma_2(\xi) &= \operatorname{Ad}\big(\exp(-i\eta)\big)\tau_n \operatorname{Ad}\big(\exp(-i\eta)\big)\xi \\ &= \operatorname{Ad}\big(\exp(-i\eta)\big)\tau_n \left(\sum_{(\lambda,a)} e^{-i\lambda(\eta)}\xi_{\lambda,a}\right) \\ &= \operatorname{Ad}\big(\exp(-i\eta)\big) \sum_{(\lambda,a)} a e^{-i\lambda(\eta)}\xi_{\lambda,a} = \sum_{(\lambda,a)} a e^{-2i\lambda(\eta)}\xi_{\lambda,a}.\end{aligned}$$

This proves that τ_z is semi-simple. Moreover, $\tau_z(\xi) = \xi$ holds if and only if $\xi_{\lambda,a} = 0$ for all $(\lambda, a) \notin \widetilde{\Lambda}(z)$. □

Since $\mathfrak{g}_2 \cap \operatorname{Ad}(z^{-1})\mathfrak{g}_1$ is isomorphic to the Lie algebra of $(G_1 \times G_2)_z$, we obtain the following characterization of strongly regular elements in terms of the extended weights as a corollary.

Theorem 3.4. *We have* $\operatorname{codim}_{\mathbb{R}}(G_1 \times G_2) \cdot z = \dim_{\mathbb{R}} \mathfrak{c} + (\#\widetilde{\Lambda}(z) - 1)$. *The element z is strongly regular if and only if* $\widetilde{\Lambda}(z) = \big\{(0,1)\big\}$ *holds. This implies that the codimension of a generic orbit coincides with the rank of the real-reductive Lie algebra* $\mathfrak{g}_1 \cap \mathfrak{g}_2$.

Finally we describe the tangent space T_zM_z in terms of the extended weight space decomposition.

Theorem 3.5. *Under the map* $(\ell_z)_*$ *the tangent space* T_zM_z *is isomorphic to*

$$\mathfrak{g}_2 + \operatorname{Ad}(z^{-1})\mathfrak{g}_1 = \big(\mathfrak{g}_2 \cap \operatorname{Ad}(z^{-1})\mathfrak{g}_1\big) \oplus \bigoplus_{(\lambda,a)\notin\widetilde{\Lambda}(z)} \mathfrak{u}^{\mathbb{C}}_{\lambda,a}.$$

In particular, the complex tangent space of $(G_1 \times G_2) \cdot z$ *is isomorphic to* $\bigoplus_{(\lambda,a)\notin\widetilde{\Lambda}(z)} \mathfrak{u}^{\mathbb{C}}_{\lambda,a}$.

Remark. From now on we will identify the quotient $T^{\mathbb{C}}_zM/H^{\mathbb{C}}_zM$ with $R^{\mathbb{C}}_zM := (\ell_z)_*\big(\mathfrak{g}_2 \cap \operatorname{Ad}(z^{-1})\mathfrak{g}_1\big)^{\mathbb{C}}$. It follows that these spaces are isomorphic as $(G_1 \times G_2)_z$–modules.

Proof of Theorem 3.5. Since τ_z is semi-simple, we conclude from Lemma 1(i) in [Mat97] that

$$\mathfrak{u}^{\mathbb{C}} = i\big(\mathfrak{g}_2 \cap \operatorname{Ad}(z^{-1})\mathfrak{g}_1\big) \oplus \big(\mathfrak{g}_2 + \operatorname{Ad}(z^{-1})\mathfrak{g}_1\big)$$

holds. Moreover, one checks directly that this decomposition is orthogonal with respect to the real part of the Killing form $B_{\mathfrak{u}^{\mathbb{C}}}$. Similarly, we have the decomposition

$$\mathfrak{u}^{\mathbb{C}} = \operatorname{Fix}(\tau_z) \oplus \operatorname{Fix}(\tau_z)^{\perp}, \quad \operatorname{Fix}(\tau_z) = \big(\mathfrak{g}_2 \cap \operatorname{Ad}(z^{-1})\mathfrak{g}_1\big)^{\mathbb{C}},$$

where the orthogonal complement $\operatorname{Fix}(\tau_z)^{\perp}$ with respect to $B_{\mathfrak{u}^{\mathbb{C}}}$ is the sum of the τ_z–eigenspaces corresponding to eigenvalues $\neq 1$. These observations imply

$$\mathfrak{g}_2 + \operatorname{Ad}(z^{-1})\mathfrak{g}_1 = \big(\mathfrak{g}_2 \cap \operatorname{Ad}(z^{-1})\mathfrak{g}_1\big) \oplus \operatorname{Fix}(\tau_z)^{\perp}.$$

Since the same argument as the one in the proof of Lemma 3.3 implies the equality

$$\operatorname{Fix}(\tau_z)^{\perp} = \bigoplus_{(\lambda,a)\notin\widetilde{\Lambda}(z)} \mathfrak{u}^{\mathbb{C}}_{\lambda,a},$$

the theorem is proven. □

3.3. **Pulling back the Levi form into the Lie algebra.** As abbreviation we put $G := G_1 \times G_2$ in this subsection. Consequently, we have $\mathfrak{g} := \operatorname{Lie}(G) = \mathfrak{g}_1 \oplus \mathfrak{g}_2$.

As we have remarked above, every closed G–orbit $M_z = G \cdot z$ is a generic CR submanifold of $U^{\mathbb{C}}$. Let $\pi_z \colon \mathfrak{g} \to \mathfrak{g} \cdot z = T_zM_z$ be the differential of the orbit map. By Equation (2.1) the map π_z is given by

$$\pi_z(\xi_1, \xi_2) = (\ell_z)_*\big(\operatorname{Ad}(z^{-1})\xi_1 - \xi_2\big).$$

In this subsection we will pull back the CR structure of M_z into the Lie algebra $\mathfrak{g}$ and compute the Levi form of M_z via this pull back. The following proposition is essential.

Proposition 3.6. *We have the G_z–invariant decomposition $\mathfrak{g} = \mathfrak{g}_z \oplus \mathfrak{q}_z$, and $\mathfrak{q}_z$ and T_zM_z are isomorphic as G_z–spaces where the isomorphism is given by $\widetilde{\pi}_z := \pi_z|_{\mathfrak{q}_z}$. Since the complex tangent space H_zM_z is invariant under G_z, we obtain the G_z–invariant decomposition $\mathfrak{q}_z = R(\mathfrak{q}_z) \oplus H(\mathfrak{q}_z)$ where $H(\mathfrak{q}_z) := \widetilde{\pi}_z^{-1}(H_zM_z)$ and $R(\mathfrak{q}_z) := \widetilde{\pi}_z^{-1}(R_zM_z)$.*

Proof. We only have to show that the adjoint representation of G_z on $\mathfrak{g}$ is completely reducible. This follows from Proposition 2.2 since G_z is conjugate to a compatible subgroup of $U^{\mathbb{C}} \times U^{\mathbb{C}}$ if the orbit $G \cdot z = M_z$ is closed. □

Proposition 3.7. *The Levi form $\mathcal{L}_z \colon H_zM_z \times H_zM_z \to R_z^{\mathbb{C}}M_z$ is given by*

$$\mathcal{L}_z(v,w) = \pi_z\left(\frac{i}{2}\left[\widetilde{\pi}_z^{-1}(v), \widetilde{\pi}_z^{-1}(w)\right] - \frac{1}{2}\left[\widetilde{\pi}_z^{-1}(v), \widetilde{\pi}_z^{-1}(iw)\right]\right) \bmod H_z^{\mathbb{C}}M_z. \tag{3.1}$$

Proof. Let $v, w \in H_zM_z$ be given and let V, W be CR vector fields on M_z with $V_z = v$ and $W_z = w$. Since the orbit map $G \to M_z$ is a G_z–principal bundle, there exist projectable vector fields $\widetilde{V}$ and $\widetilde{W}$ on G with $\widetilde{V}_e = \widetilde{\pi}_z^{-1}(v)$ and $\widetilde{W}_e = \widetilde{\pi}_z^{-1}(w)$ such that $\pi_z\widetilde{V} = V$ and $\pi_z\widetilde{W} = W$ hold. For a proof of this fact and more details about projectable vector fields we refer the reader to [KN63]. Although it is in general not possible to choose the vector fields $\widetilde{V}$ and $\widetilde{W}$ to be left-invariant, the same argument which proves well-definedness of the intrinsic Levi form applies to show that

$$\left(\frac{i}{2}[\widetilde{V}, \widetilde{W}]_e - \frac{1}{2}[\widetilde{V}, \widetilde{JW}]_e\right) \bmod H^{\mathbb{C}}(\mathfrak{q}_z)$$

does only depend on the values $\widetilde{V}_e$ and $\widetilde{W}_e$ (compare the proof of Lemma 1 in Chapter 10.1 of [Bog91]). Therefore we conclude

$$\left(\frac{i}{2}[\widetilde{V}, \widetilde{W}]_e - \frac{1}{2}[\widetilde{V}, \widetilde{JW}]_e\right) \bmod H^{\mathbb{C}}(\mathfrak{q}_z) = \left(\frac{i}{2}\left[\widetilde{\pi}_z^{-1}(v), \widetilde{\pi}_z^{-1}(w)\right] - \frac{1}{2}\left[\widetilde{\pi}_z^{-1}(v), \widetilde{\pi}_z^{-1}(iw)\right]\right) \bmod H^{\mathbb{C}}(\mathfrak{q}_z),$$

and obtain

$$\begin{aligned}
\mathcal{L}_z(v,w) &= \left(\frac{i}{2}[V,W]_z - \frac{1}{2}[V,JW]_z\right) \bmod H_z^{\mathbb{C}}M_z \\
&= \left(\frac{i}{2}[\pi_z\widetilde{V}, \pi_z\widetilde{W}]_z - \frac{1}{2}[\pi_z\widetilde{V}, \pi_z\widetilde{JW}]_z\right) \bmod H_z^{\mathbb{C}}M_z \\
&= \left(\frac{i}{2}\pi_z[\widetilde{V}, \widetilde{W}]_e - \frac{1}{2}\pi_z[\widetilde{V}, \widetilde{JW}]_e\right) \bmod H_z^{\mathbb{C}}M_z \\
&= \pi_z\left(\frac{i}{2}[\widetilde{V}, \widetilde{W}]_e - \frac{1}{2}[\widetilde{V}, \widetilde{JW}]_e \bmod H^{\mathbb{C}}(\mathfrak{q}_z)\right) \\
&= \pi_z\left(\frac{i}{2}\left[\widetilde{\pi}_z^{-1}(v), \widetilde{\pi}_z^{-1}(w)\right] - \frac{1}{2}\left[\widetilde{\pi}_z^{-1}(v), \widetilde{\pi}_z^{-1}(iw)\right] \bmod H^{\mathbb{C}}(\mathfrak{q}_z)\right) \\
&= \pi_z\left(\frac{i}{2}\left[\widetilde{\pi}_z^{-1}(v), \widetilde{\pi}_z^{-1}(w)\right] - \frac{1}{2}\left[\widetilde{\pi}_z^{-1}(v), \widetilde{\pi}_z^{-1}(iw)\right]\right) \bmod H_z^{\mathbb{C}}M_z.
\end{aligned}$$

This finishes the proof. □

In the next subsection we will use the weight space decomposition in order to determine the map $\widetilde{\pi}_z^{-1}$ explicitely.

3.4. **The Levi form of a closed orbit.** This rather technical subsection contains the computations which are necessary to achieve the final formula of the Levi form.

Lemma 3.8. *We have*

$$\mathfrak{g}_z = \ker(\pi_z) = \left\{(\mathrm{Ad}(z)\xi, \xi);\ \xi \in \mathfrak{g}_2 \cap \mathrm{Ad}(z^{-1})\mathfrak{g}_1\right\}.$$

Proof. One checks directly that $\left\{(\mathrm{Ad}(z)\xi, \xi);\ \xi \in \mathfrak{g}_2 \cap \mathrm{Ad}(z^{-1})\mathfrak{g}_1\right\} \subset \ker(\pi_z)$ holds. The other inclusion follows for dimensional reasons. □

Lemma 3.9. *The subspace $\mathfrak{q}_z = R(\mathfrak{q}_z) \oplus H(\mathfrak{q}_z)$ is determined by the following.*

(i) We have

$$R(\mathfrak{q}_z) = \widetilde{\pi}_z^{-1}\big(\mathfrak{g}_2 \cap \mathrm{Ad}(z^{-1})\mathfrak{g}_1\big) = \big\{(\mathrm{Ad}(z)\xi, -\xi);\ \xi \in \mathfrak{g}_2 \cap \mathrm{Ad}(z^{-1})\mathfrak{g}_1\big\}.$$

(ii) We have

$$(\widetilde{\pi}_z)^{-1}(\mathfrak{u}^{\mathbb{C}}_{\lambda,a}) = \big\{\big(\mathrm{Ad}(z)\sigma_2(\xi) + \sigma_1\big(\mathrm{Ad}(z)\sigma_2(\xi)\big), \xi + \sigma_2(\xi)\big);\ \xi \in \mathfrak{u}^{\mathbb{C}}_{\lambda,a}\big\}$$

for all $(\lambda, a) \in \widetilde{\Lambda} \setminus \widetilde{\Lambda}(z)$.

Proof. Firstly, we have to show that $\big\{(\mathrm{Ad}(z)\xi, -\xi);\ \xi \in \mathfrak{g}_2 \cap \mathrm{Ad}(z^{-1})\mathfrak{g}_1\big\}$ is contained in $\mathfrak{q}_z = \mathfrak{g}_z^{\perp}$ where the orthogonal complement is taken with respect to the Killing form of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Hence, let $\xi, \xi' \in \mathfrak{g}_2 \cap \mathrm{Ad}(z^{-1})\mathfrak{g}_1$ and consider

$$\begin{aligned} B_{\mathfrak{g}_1\oplus\mathfrak{g}_2}\big((\mathrm{Ad}(z)\xi, \xi), (\mathrm{Ad}(z)\xi', -\xi')\big) &= B_{\mathfrak{g}_1}(\mathrm{Ad}(z)\xi, \mathrm{Ad}(z)\xi') - B_{\mathfrak{g}_2}(\xi, \xi') \\ &= B_{\mathfrak{u}^{\mathbb{C}}}(\xi, \xi') - B_{\mathfrak{u}^{\mathbb{C}}}(\xi, \xi') = 0. \end{aligned}$$

A simple computation shows $\big\{(\mathrm{Ad}(z)\xi, -\xi);\ \xi \in \mathfrak{g}_2 \cap \mathrm{Ad}(z^{-1})\mathfrak{g}_1\big\} \subset R(\mathfrak{q}_z)$. Since the converse inclusion follows for dimensional reasons, the first claim is proven.

A similar argument as above implies that $\big\{\big(\mathrm{Ad}(z)\sigma_2(\xi) + \sigma_1\big(\mathrm{Ad}(z)\sigma_2(\xi)\big), \xi + \sigma_2(\xi)\big);\ \xi \in \mathfrak{u}^{\mathbb{C}}_{\lambda,a}\big\}$ lies in $\mathfrak{q}_z$. In order to prove the second assertion let $\xi \in \mathfrak{u}^{\mathbb{C}}_{\lambda,a}$ be given and consider

$$\begin{aligned} \widetilde{\pi}_z\big(\mathrm{Ad}(z)\sigma_2(\xi) + \sigma_1\big(\mathrm{Ad}(z)\sigma_2(\xi)\big), \xi + \sigma_2(\xi)\big) &= \sigma_2(\xi) + \mathrm{Ad}(z^{-1})\sigma_1\,\mathrm{Ad}(z)\sigma_2(\xi) - \xi - \sigma_2(\xi) \\ &= \tau_z(\xi) - \xi = \big(ae^{-2i\lambda(\eta)} - 1\big)\xi =: \varphi_{\lambda,a}(\xi). \end{aligned}$$

Since $\varphi_{\lambda,a}(\xi) \in \mathfrak{u}^{\mathbb{C}}_{\lambda,a}$ holds, the lemma is proven. □

Remark. Note that the map $\varphi_{\lambda,a}\colon \mathfrak{u}^{\mathbb{C}}_{\lambda,a} \to \mathfrak{u}^{\mathbb{C}}_{\lambda,a}$ is an isomorphism if and only if $(\lambda, a) \notin \widetilde{\Delta}(z)$ holds. In this case the inverse map is given by

$$\varphi^{-1}_{\lambda,a}(\xi) = \frac{1}{ae^{-2i\lambda(\eta)} - 1}\xi.$$

Definition 3.10. A Levi basis of H_zM_z is a basis $(\xi_{\lambda,a})_{(\lambda,a)}$ of H_zM_z such that $\xi_{\lambda,a} \in \mathfrak{u}^{\mathbb{C}}_{\lambda,a}$ and $\sigma_2(\xi_{\lambda,a}) = \xi_{\sigma_2(\lambda),a}$ hold for all $(\lambda, a) \in \widetilde{\Lambda} \setminus \widetilde{\Lambda}(z)$.

From now on we fix a Levi basis $(\xi_{\lambda,a})$ of H_zM_z.

Theorem 3.11. *We obtain the following formula for the Levi form of* M_z*:*

$$\mathcal{L}_z(\xi_{\lambda,a}, \xi_{\mu,b}) = \begin{cases} \frac{i}{ae^{-2i\lambda(\eta)}-1}\big[\xi_{\lambda,a}, \xi_{\sigma_2(\mu),b}\big] & \text{if } (\lambda + \sigma_2(\mu), ab) \in \widetilde{\Lambda}(z) \\ 0 & \text{else} \end{cases}.$$

Proof. We will start by computing $\widetilde{\pi}_z^{-1}(\xi_{\lambda,a})$ for $\xi_{\lambda,a} \in \mathfrak{u}^{\mathbb{C}}_{\lambda,a}$. Lemma 3.9 gives us

$$\begin{aligned} \widetilde{\pi}_z^{-1}(\xi) &= \Big(\mathrm{Ad}(z)\sigma_2(\varphi^{-1}_{\lambda,a}\xi) + \sigma_1\big(\mathrm{Ad}(z)\sigma_2(\varphi^{-1}_{\lambda,a}\xi)\big), \varphi^{-1}_{\lambda,a}\xi + \sigma_2(\varphi^{-1}_{\lambda,a}\xi)\Big) \\ &= \Big(\mathrm{Ad}(z)\big(\sigma_2(\varphi^{-1}_{\lambda,a}\xi) + \tau_z(\varphi^{-1}_{\lambda,a}\xi)\big), \varphi^{-1}_{\lambda,a}\xi + \sigma_2(\varphi^{-1}_{\lambda,a}\xi)\Big) \\ &= \Big(\mathrm{Ad}(z)\big(\varphi^{-1}_{\lambda,a}\xi + \sigma_2(\varphi^{-1}_{\lambda,a}\xi)\big) + \mathrm{Ad}(z)\xi, \varphi^{-1}_{\lambda,a}\xi + \sigma_2(\varphi^{-1}_{\lambda,a}\xi)\Big) \end{aligned}$$

for any $\xi \in \mathfrak{u}^{\mathbb{C}}_{\lambda,a}$. In the next step we determine the Lie bracket $\big[\widetilde{\pi}_z^{-1}(\xi_{\lambda,a}), \widetilde{\pi}_z^{-1}(\xi_{\mu,b})\big]$. Since the Lie bracket of $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is defined component-wise, we consider

$$\begin{aligned} &\Big[\mathrm{Ad}(z)\big(\varphi^{-1}_{\lambda,a}\xi_{\lambda,a} + \sigma_2(\varphi^{-1}_{\lambda,a}\xi_{\lambda,a})\big) + \mathrm{Ad}(z)\xi_{\alpha,\lambda},\ \mathrm{Ad}(z)\big(\varphi^{-1}_{\mu,b}\xi_{\mu,b} + \sigma_2(\varphi^{-1}_{\mu,b}\xi_{\mu,b})\big) + \mathrm{Ad}(z)\xi_{\mu,b}\Big] \\ &\qquad = \mathrm{Ad}(z)\left[\varphi^{-1}_{\lambda,a}\xi_{\lambda,a} + \sigma_2(\varphi^{-1}_{\lambda,a}\xi_{\lambda,a}) + \xi_{\lambda,a},\ \varphi^{-1}_{\mu,b}\xi_{\mu,b} + \sigma_2(\varphi^{-1}_{\mu,b}\xi_{\mu,b}) + \xi_{\mu,b}\right] \end{aligned}$$

and

$$\left[\varphi^{-1}_{\lambda,a}\xi_{\lambda,a} + \sigma_2(\varphi^{-1}_{\alpha,\lambda}\xi_{\lambda,a}),\ \varphi^{-1}_{\mu,b}\xi_{\mu,b} + \sigma_2(\varphi^{-1}_{\mu,b}\xi_{\mu,b})\right].$$

The application of π_z to the element in $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ whose components are given by the above gives

$$\Big[\varphi_{\lambda,a}^{-1}\xi_{\lambda,a} + \sigma_2(\varphi_{\alpha,\lambda}^{-1}\xi_{\lambda,a}), \xi_{\mu,b}\Big] + \Big[\xi_{\lambda,a}, \varphi_{\mu,b}^{-1}\xi_{\mu,b} + \sigma_2(\varphi_{\mu,b}^{-1}\xi_{\mu,b})\Big] + [\xi_{\lambda,a}, \xi_{\mu,b}]. \tag{3.2}$$

By the same computation we obtain for $\pi_z\big([\widetilde{\pi}_z^{-1}(\xi_1), \widetilde{\pi}_z^{-1}(i\xi_2)]\big)$ the following expression:

$$\Big[\varphi_{\lambda,a}^{-1}\xi_{\lambda,a} + \sigma_2(\varphi_{\alpha,\lambda}^{-1}\xi_{\lambda,a}), i\xi_{\mu,b}\Big] + \Big[\xi_{\lambda,a}, \varphi_{\mu,b}^{-1}i\xi_{\mu,b} + \sigma_2(\varphi_{\mu,b}^{-1}i\xi_{\mu,b})\Big] + [\xi_{\lambda,a}, i\xi_{\mu,b}]. \tag{3.3}$$

To arrive at the Levi form, we have to multiply (3.2) by $\frac{i}{2}$ and subtract (3.3) multiplied by $\frac{1}{2}$. Due to the facts that $\varphi_{\lambda,a}$ and $\varphi_{\mu,b}$ are complex-linear, while σ_2 is anti-linear over $\mathbb{C}$, this leads to

$$i\big[\xi_{\lambda,a}, \sigma_2(\varphi_{\mu,b}^{-1}\xi_{\mu,b})\big].$$

Inserting the concrete expression for $\varphi_{\mu,b}^{-1}$ yields

$$\pi_z\left(\frac{i}{2}\big[\widetilde{\pi}_z^{-1}(\xi_{\lambda,a}), \widetilde{\pi}_z^{-1}(\xi_{\mu,b})\big] - \frac{1}{2}\big[\widetilde{\pi}_z^{-1}(\xi_{\lambda,a}), \widetilde{\pi}_z^{-1}(i\xi_{\mu,b})\big]\right) = \frac{i}{\mu^{-1}e^{2i\overline{\beta(\eta)}} - 1}\big[\xi_{\lambda,a}, \sigma_2(\xi_{\mu,b})\big].$$

To arrive at the Levi form, we have to project this element onto $\big(\mathfrak{g}_2 \cap \mathrm{Ad}(z^{-1})\mathfrak{g}_1\big)^{\mathbb{C}}$. Consequently, we only obtain a nonzero contribution if $\big[\xi_{\lambda,a}, \sigma_2(\xi_{\mu,b})\big] \in \mathrm{Fix}(\tau_z)$ holds. By the definition of a Levi basis this condition translates into the one formulated in the theorem. This finishes the proof. □

3.5. **The quadratic Levi form.** In this subsection we will derive explicit formulas for the quadratic Levi form of a generic orbit $M_z = (G_1 \times G_2) \cdot z$ from Theorem 3.11. For $(\lambda, a) \in \widetilde{\Lambda}$ we define

$$\mathfrak{u}^{\mathbb{C}}[\lambda, a] := \mathfrak{u}^{\mathbb{C}}_{\lambda,a} + \mathfrak{u}^{\mathbb{C}}_{\sigma_2(\lambda),a} + \mathfrak{u}^{\mathbb{C}}_{-\lambda,a^{-1}} + \mathfrak{u}^{\mathbb{C}}_{-\sigma_2(\lambda),a^{-1}}.$$

Since the $\mathfrak{u}^{\mathbb{C}}[\lambda, a] \perp \mathfrak{u}^{\mathbb{C}}[\mu, b]$ with respect to the Levi form $\mathcal{L}_z$ for $\big(\lambda + \sigma_2(\mu), ab\big) \notin \widetilde{\Lambda}(z) = \big\{(0,1)\big\}$, the Levi form is determined by its restriction to these spaces for which explicit formulas are given in the next proposition. We make use of the partition $\Lambda \setminus \{0\} = \Lambda_r \,\dot\cup\, \Lambda_i \,\dot\cup\, \Lambda_c$.

Proposition 3.12. *Let $(\lambda, a) \in \widetilde{\Lambda}$ be given.*

(1) For $\lambda \in \Lambda_r$ we obtain

$$\widehat{\mathcal{L}}_z(r_\lambda\xi_{\lambda,a} + r_{-\lambda}\xi_{-\lambda,a^{-1}}) = -2\,\mathrm{Im}\left(\frac{r_\lambda\overline{r}_{-\lambda}}{ae^{-2i\lambda(\eta)} - 1}\right)[\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}].$$

(2) For $\lambda \in \Lambda_i$ and $a = 1$ we obtain

$$\widehat{\mathcal{L}}_z(r_\lambda\xi_{\lambda,1} + r_{-\lambda}\xi_{-\lambda,1}) = \left(\frac{|r_\lambda|^2}{e^{-2i\lambda(\eta)} - 1} - \frac{|r_{-\lambda}|^2}{e^{2i\lambda(\eta)} - 1}\right) i[\xi_{\lambda,1}, \xi_{-\lambda,1}].$$

(3) For $\lambda \in \Lambda_i$ and $a = -1$ we obtain

$$\widehat{\mathcal{L}}_z(r_\lambda\xi_{\lambda,-1} + r_{-\lambda}\xi_{-\lambda,-1}) = -\left(\frac{|r_\lambda|^2}{e^{-2i\lambda(\eta)} + 1} - \frac{|r_{-\lambda}|^2}{e^{2i\lambda(\eta)} + 1}\right) i[\xi_{\lambda,-1}, \xi_{-\lambda,-1}].$$

(4) For $\lambda \in \Lambda_i$ and $a \neq \pm 1$ we obtain

$$\begin{aligned}\widehat{\mathcal{L}}_z(&r_{\lambda,a}\xi_{\lambda,a} + r_{\lambda,a^{-1}}\xi_{\lambda,a^{-1}} + r_{-\lambda,a}\xi_{-\lambda,a} + r_{-\lambda,a^{-1}}\xi_{-\lambda,a^{-1}}) \\ &= 2\,\mathrm{Re}\left(\frac{ir_{\lambda,a}\overline{r}_{\lambda,a^{-1}}}{ae^{-2i\lambda(\eta)} - 1}[\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}]\right) + 2\,\mathrm{Re}\left(\frac{ir_{-\lambda,a}\overline{r}_{-\lambda,a^{-1}}}{ae^{2i\lambda(\eta)} - 1}[\xi_{-\lambda,a}, \xi_{\lambda,a^{-1}}]\right).\end{aligned}$$

(5) For $\lambda \in \Lambda_c$ and $a = 1$ we obtain

$$\begin{aligned}\widehat{\mathcal{L}}_z\big(&r_\lambda\xi_{\lambda,1} + s_\lambda\sigma_2(\xi_{\lambda,1}) + r_{-\lambda}\xi_{-\lambda,1} + s_{-\lambda}\sigma_2(\xi_{-\lambda,1})\big) \\ &= 2\,\mathrm{Re}\left(\left(\frac{ir_\lambda\overline{s}_{-\lambda}}{e^{-2i\lambda(\eta)} - 1} - \frac{ir_{-\lambda}\overline{s}_\lambda}{e^{2i\lambda(\eta)} - 1}\right)[\xi_{\lambda,1}, \xi_{-\lambda,1}]\right).\end{aligned}$$

(6) For $\lambda \in \Lambda_c$ and $a = -1$ we obtain

$$\widehat{\mathcal{L}}_z\big(r_\lambda \xi_{\lambda,-1} + s_\lambda \sigma_2(\xi_{\lambda,-1}) + r_{-\lambda}\xi_{-\lambda,-1} + s_{-\lambda}\sigma_2(\xi_{-\lambda,-1})\big) = 2\,\mathrm{Re}\left(\left(\frac{ir_{-\lambda}\overline{s}_\lambda}{e^{2i\lambda(\eta)}+1} - \frac{ir_\lambda \overline{s}_{-\lambda}}{e^{-2i\lambda(\eta)}+1}\right)[\xi_{\lambda,-1}, \xi_{-\lambda,-1}]\right).$$

(7) For $\lambda \in \Lambda_c$ and $a \neq \pm 1$ we obtain

$$\widehat{\mathcal{L}}_z\big(r_{\lambda,a}\xi_{\lambda,a} + s_{\lambda,a}\sigma_2(\xi_{\lambda,a}) + r_{-\lambda,a^{-1}}\xi_{-\lambda,a^{-1}} + s_{-\lambda,a^{-1}}\sigma_2(\xi_{-\lambda,a^{-1}})\big) = 2\,\mathrm{Re}\left(\frac{ir_{\lambda,a}\overline{s}_{-\lambda,a^{-1}}}{ae^{-2i\lambda(\eta)}-1}[\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}]\right) + 2\,\mathrm{Re}\left(\frac{ir_{-\lambda,a^{-1}}\overline{s}_{\lambda,a}}{a^{-1}e^{2i\lambda(\eta)}-1}[\xi_{-\lambda,a^{-1}}, \xi_{\lambda,a}]\right).$$

Proof. The proof is a straightforward application of Theorem 3.11. As illustration we will prove the first assertion. If λ is a real weight, we have $\sigma_2(\lambda) = \lambda$ and therefore $\mathfrak{u}^{\mathbb{C}}[\lambda, a] = \mathfrak{u}^{\mathbb{C}}_{\lambda,a} \oplus \mathfrak{u}^{\mathbb{C}}_{-\lambda,a^{-1}}$. For arbitrary numbers $r_\lambda, r_{-\lambda} \in \mathbb{C}$ we obtain

$$\begin{aligned}
\widehat{\mathcal{L}}_z(r_\lambda\xi_{\lambda,a} + r_{-\lambda}\xi_{-\lambda,a^{-1}}) &= |r_\lambda|^2\widehat{\mathcal{L}}_z(\xi_{\lambda,a}) + r_\lambda\overline{r}_{-\lambda}\mathcal{L}_z(\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}) \\
&\quad + \overline{r}_\lambda r_{-\lambda}\mathcal{L}_z(\xi_{-\lambda,a^{-1}}, \xi_{\lambda,a}) + |r_{-\lambda}|^2\widehat{\mathcal{L}}_z(\xi_{-\lambda,a^{-1}}) \\
&= 2\,\mathrm{Re}\big(r_\lambda\overline{r}_{-\lambda}\mathcal{L}_z(\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}})\big) \\
&= 2\,\mathrm{Re}\left(\frac{ir_\lambda\overline{r}_{-\lambda}}{ae^{-2i\lambda(\eta)}-1}[\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}]\right) \\
&= 2\,\mathrm{Re}\left(\frac{ir_\lambda\overline{r}_{-\lambda}}{ae^{-2i\lambda(\eta)}-1}\right)[\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}],
\end{aligned}$$

since $\sigma_2[\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}] = [\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}]$ for real weights λ. □

3.6. **Reduction to the (σ_1, σ_2)–irreducible case.** In this subsection we will introduce the appropriate reduction method in order to facilitate the determination of the Levi cone.

Definition 3.13. We say that $\mathfrak{u}^{\mathbb{C}}$ is (σ_1, σ_2)–irreducible if there is no non-trivial ideal in $\mathfrak{u}^{\mathbb{C}}$ which is invariant under σ_1 and σ_2.

Remark. Let $n \in A_0^c \subset U$ and $\sigma_1' := \mathrm{Ad}(n^{-1})\sigma_1\,\mathrm{Ad}(n)$. Then σ_1' is again a $\mathbb{C}$–anti-linear involutive automorphism of $\mathfrak{u}^{\mathbb{C}}$ commuting with θ and $\mathfrak{u}^{\mathbb{C}}$ is (σ_1', σ_2)–irreducible if and only if it is (σ_1, σ_2)–irreducible.

The next lemma characterizes (σ_1, σ_2)–irreducibility in terms of the set of weights $\Lambda = \Lambda(\mathfrak{u}^{\mathbb{C}}, \mathfrak{c}_0)$ where $\mathfrak{c}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is the fundamental Cartan subalgebra of $\mathfrak{g}_1 \cap \mathfrak{g}_2$.

Lemma 3.14. *The Lie algebra $\mathfrak{u}^{\mathbb{C}}$ is (σ_1, σ_2)–irreducible if and only if the root system $\Delta := \Lambda \setminus \{0\} \subset (i\mathfrak{t}_0 \oplus \mathfrak{a}_0)^*$ is irreducible.*

Proof. Let us assume that $\mathfrak{u}^{\mathbb{C}}$ is (σ_1, σ_2)–irreducible. If the root system Δ is not irreducible, there is a decomposition $\Delta = \Delta_1 \dot\cup \Delta_2$ into non-empty subsystems Δ_1, Δ_2 such that for all $\lambda_j \in \Delta_j$ neither of $\lambda_1 \pm \lambda_2$ is a root. It follows that

$$\mathfrak{u}^{\mathbb{C}}_j := \mathfrak{u}^{\mathbb{C}}_{0,j} \oplus \bigoplus_{\lambda \in \Delta_j} \mathfrak{u}^{\mathbb{C}}_\lambda,$$

where $\mathfrak{u}^{\mathbb{C}}_{0,j} := \mathrm{Span}\big\{[\mathfrak{u}^{\mathbb{C}}_\lambda, \mathfrak{u}^{\mathbb{C}}_{-\lambda}];\ \lambda \in \Delta_j\big\}$, is a non-trivial ideal invariant under σ_1 and σ_2, which contradicts the fact that $\mathfrak{u}^{\mathbb{C}}$ is (σ_1, σ_2)–irreducible.

In order to prove the converse, let us assume that $\mathfrak{u}^{\mathbb{C}}_1$ is a non-trivial ideal in $\mathfrak{u}^{\mathbb{C}}$ invariant under σ_1 and σ_2. Consequently, its orthogonal complement $\mathfrak{u}^{\mathbb{C}}_2$ with respect to the Killing form $B_{\mathfrak{u}^{\mathbb{C}}}$ is also a non-trivial σ_1– and σ_2–stable ideal and $\mathfrak{u}^{\mathbb{C}} = \mathfrak{u}^{\mathbb{C}}_1 \oplus \mathfrak{u}^{\mathbb{C}}_2$. It is not hard to see that this decomposition induces similar decompositions of $\mathfrak{g}_1 \cap \mathfrak{g}_2$, $\mathfrak{c}_0$, and hence also of the root system Δ which contradicts the fact that Δ is irreducible. □

Since the computation of the Levi form is local and the Levi form is invariant under local biholomorphisms, it does no harm to go over to coverings. Hence, we assume that $U^{\mathbb{C}}$ is simply connected.

Theorem 3.15. *There exists an up to re-ordering unique decomposition $\mathfrak{u}^{\mathbb{C}} = \mathfrak{u}_1^{\mathbb{C}} \oplus \cdots \oplus \mathfrak{u}_N^{\mathbb{C}}$ into (σ_1, σ_2)–irreducible ideals. If $U^{\mathbb{C}}$ is simply-connected, we have the corresponding decomposition of $U^{\mathbb{C}}$, of the real forms G_1 and G_2, and of the orbits and their (complex) tangent spaces. This decomposition of the complex tangent space of a closed orbit is orthogonal with respect to its Levi form. Consequently, the Levi cone is the direct product of the Levi cones of each factor.*

Proof. Since $\mathfrak{u}^{\mathbb{C}}$ is semi-simple, it is the direct sum of its simple ideals, and each of these is θ–invariant. Since σ_1 and σ_2 are automorphisms of $\mathfrak{u}^{\mathbb{C}}$, they map simple ideals onto simple ideals. This observation proves that $\mathfrak{u}^{\mathbb{C}}$ has a unique decomposition into (σ_1, σ_2)–irreducible ideals. Moreover, the simple ideals which appear in one (σ_1, σ_2)–irreducible ideal must be all isomorphic.

Let $\mathfrak{u}^{\mathbb{C}} = \mathfrak{u}_1^{\mathbb{C}} \oplus \cdots \oplus \mathfrak{u}_N^{\mathbb{C}}$ denote this decomposition and let $U_k^{\mathbb{C}}$ be the subgroup of $U^{\mathbb{C}}$ with Lie algebra $\mathfrak{u}_k^{\mathbb{C}}$. Since $U^{\mathbb{C}}$ is simply-connected, we obtain

$$U^{\mathbb{C}} \cong U_1^{\mathbb{C}} \times \cdots \times U_N^{\mathbb{C}},$$

and since each semi-simple normal subgroup $U_k^{\mathbb{C}}$ is invariant under σ_1 and σ_2, we have similar decompositions

$$G_j \cong (G_j)_1 \times \cdots \times (G_j)_N$$

for $j = 1, 2$. Here, $(G_j)_k$ is the fixed point set of $\sigma_j|_{U_k^{\mathbb{C}}}$. It follows that the $(G_1 \times G_2)$–orbits are also direct products of their intersections with the normal subgroups $U_k^{\mathbb{C}}$. Since the $\mathfrak{u}_k^{\mathbb{C}}$ are ideals, we have a corresponding decomposition of the set of weights into strongly orthogonal subsystems. Finally, the computation of the Levi form in Theorem 3.11 tells us that the respective parts of the complex tangent spaces are Levi-orthogonal. □

3.7. **The Levi cone.** In this subsection we will determine the full Levi cone of a generic $(G_1 \times G_2)$–orbit. We assume that $U^{\mathbb{C}}$ is simply-connected and that $\mathfrak{u}^{\mathbb{C}}$ is (σ_1, σ_2)–irreducible.

Let $z = n\exp(i\eta)$ be a regular element contained in the standard Cartan slice $C := n\exp(i\mathfrak{c})$. Since z is regular, we have $e^{2i\lambda(\eta)} \neq 1$ for all $(\lambda, 1) \in \widetilde{\Lambda}$. Hence, we conclude $\lambda(\eta) \neq 0$ for all $(\lambda, 1) \in \widetilde{\Lambda}_i$. The following lemma is then a direct consequence of Proposition 3.12.

Lemma 3.16. *The Levi cone $\mathcal{C}_z$ of the generic orbit $(G_1 \times G_2) \cdot z$ is generated by*

(1) $\pm[\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}]$ *for* $(\lambda, a) \in \widetilde{\Lambda}_r$,
(2) $-i[\xi_{\lambda,1}, \xi_{-\lambda,1}]$ *for* $(\lambda, 1) \in \widetilde{\Lambda}_i$ *with* $\lambda(\eta) > 0$,
(3) $i[\xi_{\lambda,1}, \xi_{-\lambda,1}]$ *for* $(\lambda, 1) \in \widetilde{\Lambda}_i$ *with* $\lambda(\eta) < 0$,
(4) $\pm i[\xi_{\lambda,-1}, \xi_{-\lambda,-1}]$ *for* $(\lambda, -1) \in \widetilde{\Lambda}_i$,
(5) $\pm\operatorname{Re}\big([\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}]\big)$ *and* $\pm\operatorname{Im}\big([\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}]\big)$ *for* $(\lambda, a) \in \widetilde{\Lambda}_i$ *with* $\lambda \neq \pm 1$, *and*
(6) $\pm\operatorname{Re}\big([\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}]\big)$ *and* $\pm\operatorname{Im}\big([\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}]\big)$ *for* $(\lambda, a) \in \widetilde{\Lambda}_c$.

Remark. Since we have defined the real structure on $\mathfrak{c}^{\mathbb{C}}$ via σ_2, we obtain

$$\begin{aligned}\operatorname{Re}\big([\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}]\big) &= [\xi_{\lambda,\lambda}, \xi_{-\lambda,a^{-1}}] + \sigma_2\big([\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}]\big) \\ &= [\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}] + [\xi_{\sigma_2(\lambda),\lambda}, \xi_{-\sigma_2(\lambda),\lambda^{-1}}].\end{aligned}$$

The imaginary part $\operatorname{Im}\big([\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}]\big)$ can be expressed by an analogous formula.

In order to state the main theorem we have to review some properties of real simple Lie algebras of Hermitian type. For a more detailed exposition of these topics we refer the reader to [HN93] and [Nee00].

Recall that a simple real Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is said to be of Hermitian type if the center of $\mathfrak{k}$ is non-trivial. This condition implies that a maximal torus $\mathfrak{t} \subset \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$. Then every root α in $\Delta = \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t})$ is imaginary, and either $\mathfrak{g}_\alpha^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}}$ or $\mathfrak{g}_\alpha^{\mathbb{C}} \subset \mathfrak{p}^{\mathbb{C}}$ holds. In the first case we call α a compact root, while in the second case α is said to be non-compact. We write $\Delta_{\mathfrak{k}}$ and $\Delta_{\mathfrak{p}}$ for the sets of compact and non-compact roots, respectively. Since $\mathfrak{g}$ is Hermitian, the root system Δ possesses a good ordering, i. e. there is a choice of the set Δ^+ of positive roots such that each positive non-compact root is larger than every compact root. This is equivalent to the fact that the set $\Delta_{\mathfrak{p}}^+$ is invariant under the

Weyl group $W(\Delta_{\mathfrak{k}})$. Therefore there are two natural $W(\Delta_{\mathfrak{k}})$–invariant cones $C_{\min} \subset C_{\max}$, where $C_{\min}$ is the closed convex cone generated by

$$\{-i[\xi_\alpha, \sigma(\xi_\alpha)];\ \xi_\alpha \in \mathfrak{g}_\alpha^{\mathbb{C}}, \alpha \in \Delta_{\mathfrak{p}}^+\} \subset \mathfrak{t}$$

and

$$C_{\max} := \{\eta \in \mathfrak{t};\ i\alpha(\eta) \geq 0 \text{ for all } \alpha \in \Delta_{\mathfrak{p}}^+\}.$$

Let $C_{\max}^0$ be the interior of $C_{\max}$. Then the open subset $G\exp(iC_{\max}^0)G \subset U^{\mathbb{C}}$ is closed under multiplication and hence a semi-group, called the open complex Ol'shanskiĭ semi-group.

Theorem 3.17. *Let $\mathfrak{u}^{\mathbb{C}}$ be (σ_1, σ_2)–irreducible and let $(G_1 \times G_2) \cdot z$ be a generic orbit where $z = n\exp(i\eta)$ lies in the standard Cartan slice $C := n\exp(i\mathfrak{c})$.*

(1) If the standard Cartan subset $\mathfrak{c}$ is non-compact, then $\mathcal{C}_z = \mathfrak{c}$ holds.
(2) If $\mathfrak{c}$ is compact and if $a \neq 1$ for some $(\lambda, a) \in \widetilde{\Lambda}$, then we have $\mathcal{C}_z = \mathfrak{c}$.
(3) If $\mathfrak{c}$ is compact and if $a = 1$ for all generalized weights, then $\sigma_1 = \sigma_2$ holds and there are the following cases.
  (i) If $\mathfrak{g}_1 = \mathfrak{g}_2 =: \mathfrak{g}$ is of Hermitian type and if η lies in $C_{\max}$, then the Levi cone $\mathcal{C}_z$ is isomorphic to the dual of the positive Weyl chamber defined by Λ^+. In particular, the Levi cone is pointed.
  (ii) If $\mathfrak{g}$ is of Hermitian type and $\eta \notin C_{\max}$, then $\mathcal{C}_z = \mathfrak{c}$.
  (iii) If $\mathfrak{g}$ is not of Hermitian type, then $\mathcal{C}_z = \mathfrak{c}$.

Remark. The reader will note that the statement of Theorem 3.17 differs also for the case $\sigma_1 = \sigma_2$ from the corresponding Theorem 5.3 in [FG98]. Indeed, as L. Geatti has kindly pointed out, the formulation of the third part of Theorem 5.3 in [FG98] is not correct. The correct statement in Theorem 3.17 and its proof in the case $\sigma_1 = \sigma_2$ are due to an unpublished erratum written by L. Geatti.

It will turn out to be convenient to express the generators of the Levi cone in terms of the coroots $\eta_\lambda \in i\mathfrak{t} \oplus \mathfrak{a}$. Therefore we will identify $\mathfrak{c} = \mathfrak{t} \oplus \mathfrak{a}$ with $i\mathfrak{t} \oplus \mathfrak{a}$ via the map $(\eta_1, \eta_2) \mapsto (i\eta_1, \eta_2)$. By abuse of notation, we denote the image of the Levi cone under this map again by $\mathcal{C}_z \subset i\mathfrak{t} \oplus \mathfrak{a}$. According to Lemma 2.6 we have

$$[\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}] = B_{\mathfrak{u}^{\mathbb{C}}}(\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}})\eta_\lambda \in \mathbb{C}\eta_\lambda.$$

Hence, we can normalize the $\xi_{\lambda,a}$ such that $[\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}] = \eta_\lambda$ holds for all $\lambda \in \Lambda^+ \backslash \Lambda_i$ and $[\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}] = \pm\eta_\lambda$ holds for $\lambda \in \Lambda_i^+$ depending on the sign of $B_{\mathfrak{u}^{\mathbb{C}}}(\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}})$.

Remark. In the case where $\mathfrak{g}_1 = \mathfrak{g}_2 =: \mathfrak{g}$ and $\mathfrak{t}$ is a compact Cartan subalgebra of $\mathfrak{g}$, we obtain after the above normalization

$$[\xi_\alpha, \xi_{-\alpha}] = \begin{cases} \eta_\alpha & \text{for } \alpha \in \Delta_{\mathfrak{p}}^+ \\ -\eta_\alpha & \text{for } \alpha \in \Delta_{\mathfrak{k}}^+ \end{cases},$$

since the real part of $B_{\mathfrak{u}^{\mathbb{C}}}$ is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$.

Proof of Theorem 3.17. **(1)** Let $\mathfrak{c}$ be non-compact. Since $\Lambda \setminus \{0\}$ satisfies the axioms for an abstract root system, we may choose a set $\Pi \subset \Lambda^+$ of simple weights. By Lemma 3.16 we know that $\pm\eta_\lambda$ lies in $\mathcal{C}_z$ for $\lambda \in \Lambda \setminus \Lambda_i$, and we have to show that $\pm\eta_\lambda \in \mathcal{C}_z$ holds for all $\lambda \in \Lambda$. It is enough to prove this fact for all η_λ with $\lambda \in \Pi_i := \Pi \cap \Lambda_i$.

If $\lambda, \mu \in \Pi_i$ with $\lambda + \mu \in \Lambda$ are given, then $\lambda + \mu \in \Lambda_i^+$ holds. Since $\mathfrak{c}$ is non-compact, this observation implies that there exists an element $\mu \in \Pi \setminus \Pi_i$. Let $\lambda \in \Pi_i$ be arbitrary (if $\Pi_i = \emptyset$, the proof is finished). Since $\Lambda \setminus \{0\}$ is irreducible by Lemma 3.14, its Dynkin diagram is connected and hence we find a sequence $\lambda = \lambda_1, \ldots, \lambda_N = \mu$ of simple roots which are adjacent in the Dynkin diagram. Consequently, we obtain $\lambda_j + \cdots + \lambda_N \in \Lambda \setminus \Lambda_i$ for all $0 \leq j \leq N-1$. This implies

$$\pm\eta_{\lambda_j + \cdots + \lambda_N} = \pm(\eta_{\lambda_j + \cdots + \lambda_{N-1}} + \eta_{\lambda_N}) \in \mathcal{C}_z$$

for all $0 \leq j \leq N-1$. Since $\pm\eta_{\lambda_N}$ lies in $\mathcal{C}_z$, we conclude $\pm\eta_{\lambda_j + \cdots + \lambda_{N-1}} \in \mathcal{C}_z$ for all j. Iterating this argument we finally arrive at $\pm\eta_\lambda \in \mathcal{C}_z$ which was to be shown.

(2) Let us assume that $\mathfrak{c}$ is compact and that there exists $(\lambda, a) \in \widetilde{\Lambda}$ with $a \neq 1$. In this case we have $\Lambda = \Lambda_i$ and $\pm\eta_\lambda \in \mathcal{C}_z$ for all λ such that there exists $a \neq 1$ with $(\lambda, a) \in \widetilde{\Lambda}$. If there are two weights

$\lambda_1, \lambda_2 \in \Lambda$ such that $(\lambda_j, a) \in \widetilde{\Lambda}$ implies $a = 1$ for $j = 1, 2$ and such that $\lambda_1 + \lambda_2$ is again a weight, then we conclude from Proposition 2.7 that

$$(\lambda_1 + \lambda_2, a) \in \widetilde{\Lambda} \Longrightarrow a = 1$$

holds. Consequently, each set $\Pi \subset \Lambda^+$ of simple roots must contain a root μ with $\pm\eta_\mu \in \mathcal{C}_z$. Now the claim follows from the same argument as above.

(3) Let $\mathfrak{c} = \mathfrak{t}$ be a compact Cartan subalgebra of $\mathfrak{g}_2 \cap \mathrm{Ad}(n^{-1})\mathfrak{g}_1$ such that $a = 1$ holds for each extended weight $(\lambda, a) \in \widetilde{\Lambda}$. It is enough to prove that this assumption implies $\mathfrak{g}_1 = \mathfrak{g}_2$ since then the claim follows from [FG98] and Geatti's erratum.

The proof of $\mathfrak{g}_1 = \mathfrak{g}_2$ relies on the comparison of the weight space decompositions

$$\mathfrak{u}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\lambda \in \Lambda \setminus \{0\}} \mathfrak{u}^{\mathbb{C}}_{\lambda,1} \quad \text{and} \quad \big(\mathfrak{g}_2 \cap \mathrm{Ad}(n^{-1})\mathfrak{g}_1\big)^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\lambda \in \Lambda'} \mathfrak{u}^{\mathbb{C}}_{\lambda,1},$$

where Λ' denotes the set of non-zero weights λ for which $\mathfrak{u}^{\mathbb{C}}_{\lambda,1}$ is contained in $\big(\mathfrak{g}_2 \cap \mathrm{Ad}(n^{-1})\mathfrak{g}_1\big)^{\mathbb{C}}$. Note that this is well-defined since $\dim \mathfrak{u}^{\mathbb{C}}_{\lambda,1} = 1$ by Proposition 2.7. Since the weight space decomposition is in both cases defined with respect to $\mathfrak{t}$, a basis of Λ' has to be a basis of $\Lambda \setminus \{0\}$, too. Since the root system $\Lambda \setminus \{0\}$ is reduced by Proposition 2.7, we conclude

$$\mathfrak{u}^{\mathbb{C}} = \big(\mathfrak{g}_2 \cap \mathrm{Ad}(n^{-1})\mathfrak{g}_1\big)^{\mathbb{C}},$$

and hence, that $\mathfrak{g}_2 \cap \mathrm{Ad}(n^{-1})\mathfrak{g}_1$ is a real form of $\mathfrak{u}^{\mathbb{C}}$. For dimensional reasons this implies

$$\mathfrak{g}_2 = \mathfrak{g}_2 \cap \mathrm{Ad}(n^{-1})\mathfrak{g}_1 = \mathrm{Ad}(n^{-1})\mathfrak{g}_1,$$

i. e. $\sigma_2 = \mathrm{Ad}(n^{-1})\sigma_1 \,\mathrm{Ad}(n)$.

By the definition of a standard Cartan subset, the fundamental Cartan subalgebra $\mathfrak{c}_0 \subset \mathfrak{g}_1 \cap \mathfrak{g}_2$ has the same dimension as $\mathfrak{t}$. Therefore, we obtain

$$\mathrm{rk}(\mathfrak{g}_1 \cap \mathfrak{g}_2) = \dim \mathfrak{c}_0 = \dim \mathfrak{t} = \mathrm{rk}\big(\mathfrak{g}_2 \cap \mathrm{Ad}(n^{-1})\mathfrak{g}_1\big)^{\mathbb{C}} = \mathrm{rk}\,\mathfrak{u}^{\mathbb{C}},$$

which implies in the same way as above that $\mathfrak{g}_1 \cap \mathfrak{g}_2$ is a real form of $\mathfrak{u}^{\mathbb{C}}$ and hence that $\mathfrak{g}_1 = \mathfrak{g}_2$ holds. □

4. Applications

4.1. **The criterion of Fels and Geatti.** We restate Corollary 5.6 from [FG98] whose proof relies on Theorem 3.2.

Theorem 4.1 (Fels, Geatti)**.** *Let Z be a complex manifold on which the Lie group G acts by holomorphic transformations. Let the orbit $M_z = G \cdot z$ be a generic CR submanifold such that $\mathcal{C}_z = T_z M_z / H_z M_z$ holds. Then there exists no G–invariant Stein domain in Z which contains M_z in its boundary. Furthermore, there is no non-constant G–invariant plurisubharmonic function which is defined in a neighborhood of M_z.*

Theorem 4.1 gives a necessary condition for an invariant domain with a generic orbit in its boundary to be Stein. In our situation we obtain the following result.

Theorem 4.2. *Let $C = n\exp(i\mathfrak{c})$ be a standard Cartan subset and let Ω be a connected component of the open set $G_1(C \cap U^{\mathbb{C}}_{sr})G_2$. Then Ω does not contain any proper $(G_1 \times G_2)$–invariant Stein subdomain unless $\mathfrak{c}$ is compact and τ_n has $a = 1$ as only eigenvalue. In this case $G_1 = G_2$ must be of Hermitian type and Ω is a translate of the open Ol'shanskiĭ semi-group in $U^{\mathbb{C}}$.*

Consequently, we see that in the case $G_1 \neq G_2$ there are no invariant Stein subdomains in $U^{\mathbb{C}}$ in whose boundary a generic orbit lies. The reader should note that there are only finitely many $(G_1 \times G_2)$–invariant domains whose boundaries consist entirely of non-generic orbits.

4.2. **q–pseudo-convex functions and q–completeness.** In this subsection we review quickly the notions of q–pseudo-convex functions and q–complete complex manifolds. Let Ω be a domain in a complex manifold Z. We call a smooth function on Ω strictly q–pseudo-convex if its Levi form has at least $n-q$ positive eigenvalues, $n := \dim_{\mathbb{C}} \Omega$, at each point of Ω. Hence, a strictly 0–pseudo-convex function is the same as a strictly plurisubharmonic function. If Ω admits a strictly q–pseudo-convex exhaustion function, we say that Ω is q–complete. The solution of the Levi problem implies that a domain is Stein if and only if it is 0–complete. For more properties of q–complete complex spaces we refer the reader to [AG62].

Remark. A standard argument of complex analysis (compare Corollary XIII.5.4 in [Nee00] for the case $q = 0$ and [Dem] for the generalization to $q > 0$) shows that a domain Ω in a Stein manifold Z is q–complete if and only if there exists a strictly q–pseudo-convex function Φ on Ω with the property $\Phi(z_n) \to \infty$ whenever $z_n \to z \in \partial\Omega$.

A domain $\Omega \subset Z$ with smooth boundary is called Levi–q–convex, if the boundary $\partial\Omega$ can locally be defined by a function whose Levi form has at most q negative eigenvalues when restricted to the complex tangent space at any point of $\partial\Omega$.

By a theorem of Docquier and Grauert ([DG60]) a domain Ω with smooth boundary in a Stein manifold Z is Stein if and only if it is Levi–0–convex. In [ES80] this result is generalized to arbitrary q.

Theorem 4.3 (Oka, Docquier-Grauert, Eastwood-Suria)**.** *Let Z be a Stein manifold and let $\Omega \subset Z$ be a domain with smooth boundary. Then Ω is strictly q–complete if and only if Ω is Levi–q–convex.*

4.3. **The rank one case.** If the closed orbit $M_z = (G_1 \times G_2) \cdot z$ is a hypersurface in $U^{\mathbb{C}}$, its intrinsic Levi form coincides with the classical Levi form of that hypersurface, and hence the signature of $\widehat{\mathcal{L}}_z$ is defined. According to Theorem 4.3 this signature encodes information about complex-analytic properties of the domains bounded by M_z. In this subsection we will use Matsuki's classification of pairs of involutive automorphisms of simply-connected compact Lie groups in order to classify the triples $(U^{\mathbb{C}}, G_1, G_2)$ where $U^{\mathbb{C}}$ is simply-connected and the generic $(G_1 \times G_2)$–orbit is a hypersurface. Moreover, we will determine the signature of the Levi form of each generic hypersurface orbit.

In [Mat02] pairs of involutive automorphisms of simply-connected semi-simple compact Lie groups are classified under the following notion of equivalence.

Definition 4.4. Let U be a simply-connected semi-simple compact Lie group. Two pairs of involutive automorphisms (σ_1, σ_2) and (σ_1', σ_2') are called equivalent if there exist an automorphism $\varphi \in \mathrm{Aut}(U)$ and an element $u \in U$ such that

$$\sigma_1' = \varphi\sigma_1\varphi^{-1} \quad \text{and} \quad \sigma_2' = \mathrm{Int}(u)\varphi\sigma_2\varphi^{-1}\,\mathrm{Int}(u)^{-1}$$

hold.

Since in our case the involutive automorphisms $\sigma_1, \sigma_2 \colon U^{\mathbb{C}} \to U^{\mathbb{C}}$ commute with θ and are anti-holomorphic, they are completely determined by their restrictions to U. Therefore, we may apply the classification result from [Mat02].

Theorem 4.5 (Matsuki)**.** *Let $U^{\mathbb{C}}$ be simply-connected. If the generic $(G_1 \times G_2)$–orbit is a hypersurface in $U^{\mathbb{C}}$, then $U^{\mathbb{C}}$ is of the form*

$$U^{\mathbb{C}} = \underbrace{S \times \cdots \times S}_{k \text{ times}},$$

where S is a θ–stable normal subgroup of $U^{\mathbb{C}}$ either isomorphic to $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{SL}(3,\mathbb{C})$. Let σ and τ be anti-holomorphic involutive automorphisms of S commuting with $\theta|_S$. If k is odd, then we have

$$\begin{aligned}\sigma_1(g_1,\dots,g_k) &= \big(\sigma(g_1), \theta(g_3), \theta(g_2), \dots, \theta(g_k), \theta(g_{k-1})\big)\\ \sigma_2(g_1,\dots,g_k) &= \big(\theta(g_2), \theta(g_1), \dots, \theta(g_{k-1}), \theta(g_{k-2}), \tau(g_k)\big),\end{aligned}$$

and if k is even, then

$$\begin{aligned}\sigma_1(g_1,\dots,g_k) &= \big(\sigma(g_1), \theta(g_3), \theta(g_2), \dots, \theta(g_{k-1}), \theta(g_{k-2}), \tau(g_k)\big)\\ \sigma_2(g_1,\dots,g_k) &= \big(\theta(g_2), \theta(g_1), \dots, \theta(g_k), \theta(g_{k-1})\big)\end{aligned}$$

holds. If $S = \mathrm{SL}(2,\mathbb{C})$, then the pair (σ,τ) is equivalent to one of $\big\{(\sigma_{1,1},\sigma_{1,1}),(\sigma_{1,1},\theta),(\theta,\theta)\big\}$, where $\sigma_{1,1}$ is the involution defining the non-compact real form $\mathrm{SU}(1,1)$ of $\mathrm{SL}(2,\mathbb{C})$. If $S = \mathrm{SL}(3,\mathbb{C})$, then the only possibility for (σ,τ) up to equivalence is $\big(\sigma(g),\tau(g)\big) = \big(\overline{g}, I_{2,1}\theta(g)I_{2,1}\big)$ with $I_{2,1} := \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$.

Proof. Since the semi-simple complex Lie group $U^{\mathbb{C}}$ is assumed to be simply-connected, we can identify the automorphism group $\mathrm{Aut}(U^{\mathbb{C}})$ with $\mathrm{Aut}(\mathfrak{u}^{\mathbb{C}})$. By Proposition 2.2 in [Mat02] there exists a θ–invariant decomposition

$$\mathfrak{u}^{\mathbb{C}} = \mathfrak{u}_1^{\mathbb{C}} \oplus \cdots \oplus \mathfrak{u}_N^{\mathbb{C}}$$

into σ_1– and σ_2–invariant semi-simple ideals $\mathfrak{u}_j^{\mathbb{C}}$. Moreover, each $\mathfrak{u}_j^{\mathbb{C}}$ is of the form

$$\mathfrak{u}_j^{\mathbb{C}} = \underbrace{\mathfrak{s}_j \oplus \cdots \oplus \mathfrak{s}_j}_{k_j \text{ times}},$$

where $\mathfrak{s}_j$ is a θ–stable simple ideal in $\mathfrak{u}_j^{\mathbb{C}}$, such that the restriction of the pair (σ_1,σ_2) (or (σ_2,σ_1)) to $\mathfrak{u}_j^{\mathbb{C}}$ is equivalent to one of the following three types:

(1) The number k_j is even and

$$\begin{aligned}\sigma_1(\xi_1,\dots,\xi_k) &= \big(\varphi(\xi_k),\theta(\xi_3),\theta(\xi_2),\dots,\theta(\xi_{k-1}),\theta(\xi_{k-2}),\varphi^{-1}(\xi_1)\big)\\ \sigma_2(\xi_1,\dots,\xi_k) &= \big(\theta(\xi_2),\theta(\xi_1),\dots,\theta(\xi_k),\theta(\xi_{k-1})\big),\end{aligned}$$

for some $\mathbb{C}$–anti-linear automorphism φ of $\mathfrak{s}_j$ commuting with $\theta|_{\mathfrak{s}_j}$.

(2) The number k is even and

$$\begin{aligned}\sigma_1(\xi_1,\dots,\xi_k) &= \big(\sigma(\xi_1),\theta(\xi_3),\theta(\xi_2),\dots,\theta(\xi_{k-1}),\theta(\xi_{k-2}),\tau(\xi_k)\big)\\ \sigma_2(\xi_1,\dots,\xi_k) &= \big(\theta(\xi_2),\theta(\xi_1),\dots,\theta(\xi_k),\theta(\xi_{k-1})\big),\end{aligned}$$

where σ and τ are $\mathbb{C}$–anti-linear involutive automorphisms of $\mathfrak{s}_j$ commuting with $\theta|_{\mathfrak{s}_j}$.

(3) The number k is odd and

$$\begin{aligned}\sigma_1(\xi_1,\dots,\xi_k) &= \big(\sigma(\xi_1),\theta(\xi_3),\theta(\xi_2),\dots,\theta(\xi_k),\theta(\xi_{k-1})\big)\\ \sigma_2(\xi_1,\dots,\xi_k) &= \big(\theta(\xi_2),\theta(\xi_1),\dots,\theta(\xi_{k-1}),\theta(\xi_{k-2}),\tau(\xi_k)\big),\end{aligned}$$

where σ and τ are $\mathbb{C}$–anti-linear involutive automorphisms of $\mathfrak{s}_j$ commuting with $\theta|_{\mathfrak{s}_j}$.

The condition that the generic $(G_1 \times G_2)$–orbit is a hypersurface is equivalent to $\mathrm{rk}(\mathfrak{g}_1 \cap \mathfrak{g}_2) = 1$. In particular, this condition implies that $\mathfrak{u}^{\mathbb{C}}$ is (σ_1,σ_2)–irreducible.

If $\mathfrak{u}^{\mathbb{C}}$ is of the first type, one checks directly that

$$\mathfrak{g}_1 \cap \mathfrak{g}_2 \cong (\mathfrak{s}_j^{\theta\varphi})^{\mathbb{R}}$$

holds. Consequently $\mathrm{rk}(\mathfrak{g}_1 \cap \mathfrak{g}_2)$ is even and in particular larger than 1. This excludes the first type.

Let $\mathfrak{u}^{\mathbb{C}}$ be of the second or the third type. Again it is not hard to see that

$$\mathfrak{g}_1 \cap \mathfrak{g}_2 \cong \mathfrak{s}_j^{\sigma} \cap \mathfrak{s}_j^{\tau}$$

holds. It follows that the simple complex Lie algebra $\mathfrak{s}_j$ contains the complex subalgebra $(\mathfrak{g}_1 \cap \mathfrak{g}_2)^{\mathbb{C}}$ which is given as the set of fixed points of the $\mathbb{C}$–linear semi-simple automorphism $\sigma\tau$. Using the classification of semi-simple automorphisms of simple complex Lie algebras we see that the only possibilities for $\mathfrak{s}$ are $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sl}(3,\mathbb{C})$. Then the claim follows from Proposition 2.1 in [Mat02] where the pairs of involutions on the classical Lie algebras are classified up to equivalence. □

Remark. Let $U^{\mathbb{C}} = S \times \cdots \times S$ (k times) with $S = \mathrm{SL}(2,\mathbb{C})$, and let us consider the involutions σ_1 and σ_2 on $U^{\mathbb{C}}$ corresponding to $(\sigma_{1,1},\theta)$ in the way described in Theorem 4.5. In this case we see that $\mathfrak{g}_1 \cap \mathfrak{g}_2 =: \mathfrak{t}_0 \cong \mathfrak{so}(2,\mathbb{R})$ is one-dimensional and compact. Hence, the fundamental Cartan subset $C_0 = \exp(i\mathfrak{t}_0)$ is an exact slice for the $(G_1 \times G_2)$–action on $U^{\mathbb{C}}$, i. e. every $(G_1 \times G_2)$–orbit intersects C_0 in exactly one point. In particular, we conclude that each element $z \in U^{\mathbb{C}}$ is strongly regular and that $G_1 \times G_2$ acts properly on $U^{\mathbb{C}}$.

In the following let us consider a point $z \in U^{\mathbb{C}}$ such that the orbit $M_z = (G_1 \times G_2) \cdot z$ is a closed hypersurface in $U^{\mathbb{C}}$. Without loss of generality we take z to be of the form $z = \exp(i\eta) \in C$ for some standard Cartan subset $C = \exp(i\mathfrak{c})$. Because of $\mathrm{rk}(\mathfrak{g}_1 \cap \mathfrak{g}_2) = 1$ the Cartan subalgebra $\mathfrak{c} \subset \mathfrak{g}_1 \cap \mathfrak{g}_2$ is one-dimensional and hence either $\mathfrak{c} = \mathfrak{t}$ is a maximal torus in $\mathfrak{k}_1 \cap \mathfrak{k}_2$ or $\mathfrak{c} = \mathfrak{a}$ is a maximal Abelian subspace of $\mathfrak{p}_1 \cap \mathfrak{p}_2$. Let

$$\mathfrak{u}^{\mathbb{C}} = \bigoplus_{(\lambda,a)\in\widetilde{\Lambda}} \mathfrak{u}^{\mathbb{C}}_{\lambda,a}$$

be the extended weight space decomposition of $\mathfrak{u}^{\mathbb{C}}$ with respect to $\mathfrak{c}$.

Let us first assume that $\mathfrak{c} = \mathfrak{a}$ is non-compact. In this case every weight is real and we conclude from Theorem 3.11 that the only non-zero contributions to the Levi form of M_z stem from the terms

$$\mathcal{L}_z(\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}) = \frac{i}{ae^{-2i\lambda(\eta)} - 1}[\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}],$$

where $\lambda \neq 0$ and $\xi_{\lambda,a}$ is a non-zero element in $\mathfrak{u}^{\mathbb{C}}_{\lambda,a}$ with $\sigma_2(\xi_{\lambda,a}) = \xi_{\lambda,a}$. Consequently, the restriction of the Levi form to

$$\mathfrak{u}^{\mathbb{C}}[\lambda, a] = \mathfrak{u}^{\mathbb{C}}_{\lambda,a} \oplus \mathfrak{u}^{\mathbb{C}}_{-\lambda,a^{-1}}, \quad \lambda \in \Lambda^+,$$

has with respect to the bases $(\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}})$ of $\mathfrak{u}^{\mathbb{C}}[\lambda, a]$ and $[\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}]$ of $\mathfrak{a}$ the matrix

$$\begin{pmatrix} 0 & \frac{i}{ae^{-2i\lambda(\eta)}-1} \\ -\frac{i}{a^{-1}e^{2i\lambda(\eta)}-1} & 0 \end{pmatrix},$$

which has the eigenvalues $\pm\frac{1}{|ae^{-2i\lambda(\eta)}-1|}$. Hence, we obtain a pair of one positive and one negative eigenvalue of the Levi form on $\mathfrak{u}^{\mathbb{C}}[\lambda, a]$ for each $\lambda \in \Lambda^+$.

If $\mathfrak{c} = \mathfrak{t}$ is compact, each weight is imaginary and we have to handle the cases $a = \pm 1$ and $a \neq \pm 1$ separately. If $a = \pm 1$, then $a = a^{-1}$ and consequently

$$\mathfrak{u}^{\mathbb{C}}[\lambda, a] = \mathfrak{u}^{\mathbb{C}}_{\lambda,a} \oplus \mathfrak{u}^{\mathbb{C}}_{-\lambda,a}, \quad \lambda \in \Lambda^+,$$

holds. As basis of $\mathfrak{u}^{\mathbb{C}}[\lambda, a]$ we choose $(\xi_{\lambda,a}, \xi_{-\lambda,a})$ with $\sigma_2(\xi_{\lambda,a}) = \xi_{-\lambda,a}$, and as basis of $\mathfrak{t}$ we take $i[\xi_{\lambda,a}, \xi_{-\lambda,a}]$. Then the Levi form has with respect to these bases the matrix

$$\begin{pmatrix} \frac{1}{ae^{-2i\lambda(\eta)}-1} & 0 \\ 0 & -\frac{1}{ae^{2i\lambda(\eta)}-1} \end{pmatrix}.$$

If $a = 1$, then both eigenvalues have the same sign, and if $a = -1$, then the eigenvalues have different sign.

For $a \neq \pm 1$ we have

$$\mathfrak{u}^{\mathbb{C}}[\lambda, a] = \mathfrak{u}^{\mathbb{C}}_{\lambda,a} \oplus \mathfrak{u}^{\mathbb{C}}_{-\lambda,a^{-1}} \oplus \mathfrak{u}^{\mathbb{C}}_{\lambda,a^{-1}} \oplus \mathfrak{u}^{\mathbb{C}}_{-\lambda,a}$$

and take $(\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}, \xi_{\lambda,a^{-1}}, \xi_{-\lambda,a})$ as a basis of $\mathfrak{u}^{\mathbb{C}}[\lambda, a]$. Under the assumption $B_{\mathfrak{u}^{\mathbb{C}}}(\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}) = B_{\mathfrak{u}^{\mathbb{C}}}(\xi_{\lambda,a^{-1}}, \xi_{-\lambda,a})$ we obtain $i[\xi_{\lambda,a}, \xi_{-\lambda,a^{-1}}] = i[\xi_{\lambda,a^{-1}}, \xi_{-\lambda,a}]$ which we take as a basis of $\mathfrak{t}$. With respect to these bases the restriction of the Levi form has the matrix

$$\begin{pmatrix} 0 & \frac{1}{ae^{-2i\lambda(\eta)}-1} & 0 & 0 \\ \frac{1}{a^{-1}e^{-2i\lambda(\eta)}-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a^{-1}e^{-2i\lambda(\eta)}-1} \\ 0 & 0 & \frac{1}{ae^{-2i\lambda(\eta)}-1} & 0 \end{pmatrix},$$

whose eigenvalues are given by $\pm\frac{1}{|ae^{-2i\lambda(\eta)}-1|}$ and $\pm\frac{1}{|a^{-1}e^{-2i\lambda(\eta)}-1|}$.

We summarize these results in the following

Theorem 4.6. *If $\mathfrak{c} = \mathfrak{a}$ is non-compact, then each generic orbit M_z with $z \in \exp(i\mathfrak{a})$ is Levi–q–convex with*

$$q = \#\Lambda^+.$$

If $\mathfrak{c} = \mathfrak{t}$ is compact, let us choose an ordering on the set of weights such that $\lambda(\eta) < 0$ for all $\lambda \in \Lambda^+$ and $z = \exp(i\eta)$. Then each generic orbit M_z with $z \in \exp(i\mathfrak{t})$ is Levi–q–convex with

$$q = \#\{(\lambda, -1) \in \widetilde{\Lambda};\ \lambda \in \Lambda^+\} + \#\{(\lambda, a) \in \widetilde{\Lambda};\ a \neq \pm 1\}$$

Moreover, this numbers for q are sharp, i. e. M_z is not Levi–q'–convex for any $q' < q$.

Proof. The only claim which is left to show is the fact that the multiplicity of the eigenvalue 0 of $\mathcal{L}_z$ is given by $\mathrm{rk}(\mathfrak{u}^{\mathbb{C}}) - 1$.

We conclude from Theorem 3.11 that the nullspace of the Levi form of a generic orbit coincides with

$$\bigoplus_{(0,a)\in\widetilde{\Lambda}:a\neq 1} \mathfrak{u}^{\mathbb{C}}_{0,a}$$

According to Lemma 5.1 in [Mat02] the subalgebra $\mathfrak{u}^{\mathbb{C}}_0$ is a Cartan subalgebra of $\mathfrak{u}^{\mathbb{C}}$. Since we assume that the generic $(G_1 \times G_2)$–orbit is a hypersurface, we conclude $\dim_{\mathbb{C}} \mathfrak{u}^{\mathbb{C}}_{0,1} = 1$ which finishes the proof. □

Theorem 4.3 and Theorem 4.6 yield the following result.

Theorem 4.7. *Let M_z be a closed hypersurface orbit where $z \in C = \exp(i\mathfrak{c})$ and let Ω be an invariant domain with $\partial\Omega = M_z$.*

(1) If $\mathfrak{c} = \mathfrak{a}$ is non-compact, then Ω or $U^{\mathbb{C}} \setminus \overline{\Omega}$ is strictly q–complete with

$$q = \#\Lambda^+,$$

and this q is optimal.

(2) If $\mathfrak{c} = \mathfrak{t}$ is compact, then Ω or $U^{\mathbb{C}} \setminus \overline{\Omega}$ is strictly q–complete with

$$q = \#\big\{(\lambda, -1) \in \widetilde{\Lambda};\ \lambda \in \Lambda^+\big\} + \#\big\{(\lambda, a) \in \widetilde{\Lambda};\ a \neq \pm 1\big\}$$

and this q is optimal.

4.4. **The Levi form of invariant functions and q–complete domains.** Let us assume in this subsection that the intersection $\mathfrak{g}_1 \cap \mathfrak{g}_2$ contains a compact Cartan subalgebra $\mathfrak{t}$. By Matsuki's result a generic orbit $M_z = (G_1 \times G_2) \cdot z$ intersects the corresponding standard Cartan subset $C = \exp(i\mathfrak{t})$ in an orbit of the group $W_{K_1\times K_2}(C)$.

Remark. It can be shown that there exists a group isomorphism from

$$W := W_{K_1\cap K_2}(\mathfrak{t}) := \mathcal{N}_{K_1\cap K_2}(\mathfrak{t})/\mathcal{Z}_{K_1\cap K_2}(\mathfrak{t})$$

onto $W_{K_1\times K_2}(C)$ such that the diffeomorphism $\mathfrak{t} \to C$, $\eta \mapsto \exp(i\eta)$, intertwines the W–action on $\mathfrak{t}$ with the $W_{K_1\times K_2}(C)$–action on C modulo this isomorphism ([Mie07]).

We say that a non-zero weight $\lambda \in \Lambda = \Lambda(\mathfrak{u}^{\mathbb{C}}, \mathfrak{t})$ is compact, if the reflection with respect to the hypersurface $(i\eta_\lambda)^\perp \subset \mathfrak{t}$ belongs to W. Otherwise, λ is called non-compact. Let us assume that there exists a good ordering Λ^+ on the set of non-zero weights, i. e. that that each positive non-compact weight is larger than every compact weight. This implies that the convex cone

$$C_{\max} := \big\{\eta \in \mathfrak{t};\ i\lambda(\eta) \geq 0 \text{ for all non-compact } \lambda \in \Lambda^+\big\} \subset \mathfrak{t}$$

is W–invariant.

Let $\Omega := G_1 \exp(iC^0_{\max})G_2$, where $C^0_{\max}$ is the interior of $C_{\max}$. Then $G_1 \times G_2$ acts properly on Ω. Moreover, the mapping $\mathcal{R}\colon \mathcal{C}^\infty(\Omega)^{G_1\times G_2} \to \mathcal{C}^\infty(C^0_{\max})^W$,

$$\Phi \mapsto \varphi : \eta \mapsto \Phi\big(\exp(i\eta)\big),$$

is an isomorphism (compare [FJ78]). The inverse $\mathcal{E} := \mathcal{R}^{-1}$ is called the extension operator.

One would expect that the Levi form of an invariant smooth function on Ω is determined by the direction tangent to the $(G_1 \times G_2)$–orbits and by a direction transversal to the orbit. The following propositions explains how the Levi form $L(\Phi)(z)$ is influenced by the complex tangent space of $(G_1 \times G_2) \cdot z = M_z$.

Lemma 4.8. *Let $\Phi \in \mathcal{C}^\infty(\Omega)^{G_1\times G_2}$ be given. If $v, w \in H_zM_z \subset T_z\Omega$, then we have*

$$L(\Phi)(z)(v, w) = -d^c\Phi(z)\mathcal{L}_z(v, w),$$

where $\mathcal{L}_z$ is the Levi form of M_z.

Proof. By definition, the Levi form of $\Phi \in \mathcal{C}^\infty(\Omega)$ at the point $z \in \Omega$ is the Hermitian form $L(\Phi)(z)$ on $T_z\Omega$ associated to the $(1,1)$–form $\omega := -\frac{1}{2}dd^c\Phi$. We use the formula

$$d\omega(V,W) = V\big(\omega(W)\big) - W\big(\omega(V)\big) - \omega\big([V,W]\big)$$

and extend v to a CR vector field V on M_z to compute as follows:

$$\begin{aligned} -dd^c\Phi(z)(v, J_z v) &= -v\big(d^c\Phi(JV)\big) + J_z v\big(d^c\Phi(V)\big) + d^c\Phi(z)[V, JV] \\ &= v\big(d\Phi(V)\big) + J_z v\big(d\Phi(JV)\big) + d^c\Phi(z)[V, JV] \\ &= v\big(V(\Phi)\big) + J_z v\big(JV(\Phi)\big) + d^c\Phi(z)[V, JV]. \end{aligned}$$

Since the vector fields V and JV are tangent to the orbit and since Φ is constant along the orbit, we obtain

$$L(\Phi)(z)(v,v) = -\tfrac{1}{2}dd^c\Phi(z)(v, Jv) = \tfrac{1}{2}d\Phi(z)J[V, JV]_z = -d\Phi(z)J\widehat{\mathcal{L}}_z(v).$$

Thus the claim follows from the polarization identities. □

Proposition 4.9. *Let $z \in \Omega \cap U^{\mathbb{C}}_{sr}$ and let $T_z\Omega = T_zU^{\mathbb{C}}$ be identified with*

$$\mathfrak{u}^{\mathbb{C}} = \bigoplus_{(\lambda,a)\in\widetilde{\Lambda}} \mathfrak{u}^{\mathbb{C}}_{\lambda,a} \tag{4.1}$$

via $(\ell_z)_*$. Let $\varphi \in \mathcal{C}^\infty(C^0_{\max})^W$ be given and let $\Phi := \mathcal{E}(\varphi)$ be its extension to a smooth $(G_1 \times G_2)$–invariant function on Ω. Then the decomposition (4.1) *is orthogonal with respect to the Levi form $L(\Phi)(z)$.*

Proof. In view of Lemma 4.8 it is enough to show that $\mathfrak{t}^{\mathbb{C}}$ and $H_zM_z \cong \bigoplus_{(\lambda,a)\neq(0,1)} \mathfrak{u}^{\mathbb{C}}_{\lambda,a}$ are orthogonal with respect to $L(\Phi)(z)$. Thus let $v \in \mathfrak{t}$ and $w \in \mathfrak{u}^{\mathbb{C}}_{\lambda,a}$ be given. Since Jv and w are tangent to $M_z = (G_1 \times G_2)\cdot z$, there are elements $\eta, \xi \in \mathfrak{q}_z \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $Jv = \eta_\Omega(z)$ and $w = \xi_\Omega(z)$ hold, where η_Ω and ξ_Ω are the corresponding vector fields on Ω. Using the same arguments as in the proof of Lemma 4.8 together with the invariance of Φ we obtain

$$L(\Phi)(z)(v,w) = d^c\Phi(z)[\eta_\Omega, \xi_\Omega](z) - id^c\Phi(z)[\eta_\Omega, \xi_\Omega](z).$$

Since $[\eta_\Omega, \xi_\Omega](z) = [\eta, \xi]_\Omega(z) \in H_zM_z$, the invariance of Φ implies $d^c\Phi(z)[\eta_\Omega, \xi_\Omega](z) = 0$, which finishes the proof. □

We will apply Proposition 4.9 in order to establish existence of a strictly q–pseudo-convex exhaustion function on Ω. The following theorem extends Neeb's result on open complex Ol'shanskiĭ semi-groups to the case $G_1 \neq G_2$.

Theorem 4.10. *The domain Ω is q–complete for*

$$q = \mathrm{rk}(\mathfrak{g}_1 \cap \mathfrak{g}_2) + \#\big\{(\lambda,1) \in \widetilde{\Lambda};\ \lambda \in \Lambda^+\big\} + \#\big\{(\lambda,-1) \in \widetilde{\Lambda};\ \lambda \in \Lambda^+\big\} + \#\big\{(\lambda,a) \in \widetilde{\Lambda};\ a \neq \pm 1\big\}.$$

Proof. Let $\varphi\colon C^0_{\max} \to \mathbb{R}^{\geq 0}$ be smooth, W–invariant, and strictly convex with the property that $\varphi(x_n) \to \infty$ whenever $x_n \to x \in \partial C^0_{\max}$, and let $\Phi := \mathcal{E}(\varphi)$ be the corresponding smooth $(G_1 \times G_2)$–invariant function in Ω. Let $z \in \Omega \cap U^{\mathbb{C}}_{sr}$. Due to Proposition 4.9 we may compute the Levi form $L(\Phi)(z)$ on each $\mathfrak{u}^{\mathbb{C}}[\lambda, a]$ separately.

We start with the case $a \neq 1$. Then our considerations from the rank one case imply together with Lemma 4.8 that we obtain for each $\mathfrak{u}^{\mathbb{C}}[\lambda, -1]$ a pair of one positive and one negative eigenvalue and for each $\mathfrak{u}^{\mathbb{C}}[\lambda, a]$, $a \neq -1$, two pairs of positive and negative eigenvalues in the Levi form $L(\Phi)(z)$.

Thus let $a = 1$. In this case all computations take place in $(\mathfrak{g}_1 \cap \mathfrak{g}_2)^{\mathbb{C}}$ and hence the whole question is reduced to the case that Ω is an open Ol'shanskiĭ semi-group in $(G_1 \cap G_2)^{\mathbb{C}}$. In [Nee00] it is proven that in this case the extension of a strictly convex function is strictly plurisubharmonic. Hence, we see that in our case the extension Φ is strictly q–pseudo-convex for

$$q = \dim \mathfrak{t}^{\mathbb{C}} + \#\big\{(\lambda,a) \in \widetilde{\Lambda};\ a = 1\big\} + \#\big\{(\lambda,-1) \in \widetilde{\Lambda};\ \lambda \in \Lambda^+\big\} + \#\big\{(\lambda,a) \in \widetilde{\Lambda};\ a \neq \pm 1\big\}$$

If $z_n \to z \in \partial\Omega$, then $\Phi(z_n) \to \infty$ holds by construction and hence we conclude that Ω is q–complete for the above q. □

References

[AG62] Aldo Andreotti and Hans Grauert, *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193–259.

[AL92] H. Azad and J.-J. Loeb, *Plurisubharmonic functions and Kählerian metrics on complexification of symmetric spaces*, Indag. Math. (N.S.) **3** (1992), no. 4, 365–375.

[BER99] M. Salah Baouendi, Peter Ebenfelt, and Linda Preiss Rothschild, *Real submanifolds in complex space and their mappings*, Princeton Mathematical Series, vol. 47, Princeton University Press, Princeton, NJ, 1999.

[Bog91] Albert Boggess, *CR manifolds and the tangential Cauchy-Riemann complex*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1991.

[Bre96] R. J. Bremigan, *Invariant analytic domains in complex semisimple groups*, Transform. Groups **1** (1996), no. 4, 279–305.

[Che70] Claude Chevalley, *Theory of Lie groups. I*, Princeton University Press, Princeton, N. J., 1946 [Eighth Printing, 1970].

[Dem] Jean-Pierre Demailly, *Complex analytic and algebraic geometry*, available at http://www-fourier.ujf-grenoble.fr/ demailly/books.html.

[DG60] Ferdinand Docquier and Hans Grauert, *Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten*, Math. Ann. **140** (1960), 94–123.

[ES80] Michael G. Eastwood and Giuseppe Vigna Suria, *Cohomologically complete and pseudoconvex domains*, Comment. Math. Helv. **55** (1980), no. 3, 413–426.

[FG98] Gregor Fels and Laura Geatti, *Geometry of biinvariant subsets of complex semisimple Lie groups*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), no. 2, 329–356.

[FJ78] Mogens Flensted-Jensen, *Spherical functions of a real semisimple Lie group. A method of reduction to the complex case*, J. Funct. Anal. **30** (1978), no. 1, 106–146.

[HN93] Joachim Hilgert and Karl-Hermann Neeb, *Lie semigroups and their applications*, Lecture Notes in Mathematics, vol. 1552, Springer-Verlag, Berlin, 1993.

[HS07] Peter Heinzner and Gerald W. Schwarz, *Cartan decomposition of the moment map*, Math. Ann. **337** (2007), no. 1, 197–232.

[Hum95] James E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Mathematical Surveys and Monographs, vol. 43, American Mathematical Society, Providence, RI, 1995.

[KN63] Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of differential geometry. Vol I*, Interscience Publishers, a division of John Wiley & Sons, New York-Lond on, 1963.

[Kna02] Anthony W. Knapp, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston Inc., Boston, MA, 2002.

[Las78] Michel Lassalle, *Sur la transformation de Fourier-Laurent dans un groupe analytique complexe réductif*, Ann. Inst. Fourier (Grenoble) **28** (1978), no. 1, x, 115–138.

[Mat97] Toshihiko Matsuki, *Double coset decompositions of reductive Lie groups arising from two involutions*, J. Algebra **197** (1997), no. 1, 49–91.

[Mat02] ———, *Classification of two involutions on compact semisimple Lie groups and root systems*, J. Lie Theory **12** (2002), no. 1, 41–68.

[Mie07] Christian Miebach, *Geometry of invariant domains in complex semi-simple Lie groups*, Dissertation, Bochum, 2007.

[Nee98] Karl-Hermann Neeb, *On the complex and convex geometry of Ol′shanskiĭ semigroups*, Ann. Inst. Fourier (Grenoble) **48** (1998), no. 1, 149–203.

[Nee00] ———, *Holomorphy and convexity in Lie theory*, de Gruyter Expositions in Mathematics, vol. 28, Walter de Gruyter & Co., Berlin, 2000.

[Ste68] Robert Steinberg, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society, No. 80, American Mathematical Society, Providence, R.I., 1968.

Author's adress:
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstrasse 150
44780 Bochum
Germany

e-mail:
christian.miebach@ruhr-uni-bochum.de