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# A central limit theorem for continuous-time Markov processes conditioned not to be absorbed

William Oçafrain\*

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## Abstract

This paper aims to establish a central limit theorem for Markov processes conditioned not to be absorbed under a very general assumption on quasi-stationarity for the underlying process. To do so, a central limit theorem has been established for ergodic Markov processes. The conditional central limit theorem is then obtained by applying the central limit theorem to the  $Q$ -process.

*Key words: Quasi-stationary distribution; Quasi-stationarity; Quasi-ergodic distribution; Central limit theorem;  $Q$ -process.*

## Notation

- $\mathcal{M}_1(E)$ : Set of the probability measures defined on  $E$ .
- For any  $\mu \in \mathcal{M}_1(E)$  and measurable function  $f$  such that  $\int_E f(x)\mu(dx)$  is well-defined,

$$\mu(f) := \int_E f(x)\mu(dx).$$

- For a given positive function  $\psi$ ,  $\mathbb{L}^\infty(\psi)$  is the set of functions  $f$  such that  $f/\psi$  is bounded, endowed with the norm

$$\|f\|_{\mathbb{L}^\infty(\psi)} := \|f/\psi\|_\infty.$$

- For any positive measurable function  $\psi$ , for any  $\mu, \nu \in \mathcal{M}_1(E)$ ,

$$\|\mu - \nu\|_\psi := \sup_{\|f\|_{\mathbb{L}^\infty(\psi)} \leq 1} |\mu(f) - \nu(f)|.$$

- For any nonnegative measurable function  $f$  and  $\mu \in \mathcal{M}_1(E)$  such that  $\mu(f) \in (0, +\infty)$ ,

$$f \circ \mu(dx) := \frac{f(x)\mu(dx)}{\mu(f)}.$$

- Kolmogorov distance: For any  $\mu, \nu \in \mathcal{M}_1(\mathbb{R})$ ,

$$d_{\text{Kolm}}(\mu, \nu) := \sup_{x \in \mathbb{R}} |\mu((-\infty, x]) - \nu((-\infty, x])|.$$

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# 1 Introduction

## 1.1 Introduction to quasi-stationarity

Let  $(X_t)_{t \geq 0}$  be a time-homogeneous continuous-time Markov process living on a state space  $(E \cup \{\partial\}, \mathcal{E})$ , where  $\partial \notin E$  is an absorbing state for the process  $X$ , which means that  $X_t = \partial$  conditioned to  $\{X_s = \partial\}$  for all  $s \leq t$ , and  $\mathcal{E}$  is a  $\sigma$ -field associated to the state space  $E^1$ . Denote by  $\tau_\partial$  the hitting time of  $\partial$  by the process  $X$ . We associate to the process  $X$  a family of probability measure  $(\mathbb{P}_x)_{x \in E \cup \{\partial\}}$  such that  $\mathbb{P}_x[X_0 = x] = 1$  for any  $x \in E \cup \{\partial\}$ . For any probability measure  $\mu \in \mathcal{M}_1(E \cup \{\partial\})$ , define  $\mathbb{P}_\mu := \int_{E \cup \{\partial\}} \mu(dx) \mathbb{P}_x$ , and denote  $\mathbb{E}_x$  and  $\mathbb{E}_\mu$  the associated expectations. Moreover, denote by  $(\mathcal{F}_t)_{t \geq 0}$  the natural filtration of the process  $X$ .

In this paper, we assume that the process  $X$  admits a *quasi-stationary distribution*, defined as a probability measure  $\alpha \in \mathcal{M}_1(E)$  such that, for all  $t \geq 0$ ,

$$\mathbb{P}_\alpha[X_t \in \cdot | \tau_\partial > t] = \alpha. \quad (1)$$

Such a probability measure is also a *quasi-limiting distribution*, defined as a probability measure such that there exists a subset  $\mathcal{D}(\alpha) \subset \mathcal{M}_1(E)$ , called *domain of attraction of  $\alpha$* , such that, for all  $\mu \in \mathcal{D}(\alpha)$  and  $A \in \mathcal{E}$ ,

$$\mathbb{P}_\mu[X_t \in A | \tau_\partial > t] \xrightarrow[t \rightarrow \infty]{} \alpha(A).$$

In particular, if  $\alpha$  is a quasi-stationary distribution,  $\alpha \in \mathcal{D}(\alpha)$  by (1). Conversely, we can show that any quasi-limiting distributions for  $X$  satisfy (1) for all  $t \geq 0$  (see [26, Proposition 1]). In other terms, quasi-stationary and quasi-limiting distributions are equivalent notions.

Denote by  $\lambda_0 := -\log(\mathbb{P}_\alpha[\tau_\partial > 1])$ . Then, it is well-known (see [26, Proposition 2] for example) that, for all  $t \geq 0$ ,

$$\mathbb{P}_\alpha[\tau_\partial > t] = e^{-\lambda_0 t}, \quad \forall t \geq 0.$$

A consequence of this property coupled with (1) is that, for all  $t \geq 0$ ,

$$\mathbb{P}_\alpha[X_t \in \cdot, \tau_\partial > t] = e^{-\lambda_0 t} \alpha(\cdot). \quad (2)$$

Conversely, if a probability measure  $\alpha$  satisfies (2) for a given  $\lambda_0 > 0$ , then  $\alpha$  is a quasi-stationary distribution for the process  $X$ . In that respect, the quasi-stationary distributions for  $X$  are exactly the probability left eigenmeasures for the semigroup  $(P_t)_{t \geq 0}$  defined by

$$P_t f(x) := \mathbb{E}_x(f(X_t) \mathbb{1}_{\tau_\partial > t}),$$

for all  $t \geq 0$ ,  $f$  belonging to a Banach space and  $x \in E$ . In what follows, we will use the notation

$$\mu P_t := \mathbb{P}_\mu(X_t \in \cdot, \tau_\partial > t).$$

Also, we assume that the process  $X$  admits a nonnegative function  $\eta$  defined on  $E$ , vanishing at  $\partial$  and satisfying  $\alpha(\eta) = 1$ , such that, for all  $x \in E$  and  $t \geq 0$ ,

$$\mathbb{E}_x[\eta(X_t) \mathbb{1}_{\tau_\partial > t}] = e^{-\lambda_0 t} \eta(x).$$

$\eta$  is therefore a right eigenfunction for the semigroup  $(P_t)_{t \geq 0}$ , associated to the eigenvalues  $(e^{-\lambda_0 t})_{t \geq 0}$ .

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<sup>1</sup> $X$  is assumed to satisfy the time-homogeneous Markov property with respect to its natural filtration.  $E$  and sample paths of  $X$  are assumed to be equipped with appropriate  $\sigma$ -fields; for instance, and most usually, there may exist a Polish topology on  $E$  under which  $X$  is cadlag, although topology plays no role in the present work.

## 1.2 The main assumption and the $Q$ -process

The main assumption on this process is the following.

**Assumption 1.** *There exists a function  $\psi_1 : E \rightarrow [1, +\infty)$ , such that  $\alpha(\psi_1) < +\infty$  and  $\eta \in \mathbb{L}^\infty(\psi_1)$ , as well as two constants  $C, \gamma > 0$  such that, for any  $\mu \in \mathcal{M}_1(E)$  and  $t \geq 0$ ,*

$$\|e^{\lambda_0 t} \mu P_t - \mu(\eta)\alpha\|_{\psi_1} \leq C\mu(\psi_1)e^{-\gamma t}. \quad (3)$$

This assumption is satisfied under the general criteria Assumption (F) of [9]. In particular, it is shown in [9] that Assumption 1 is satisfied for a lot of processes such as multidimensional elliptic diffusion processes or processes defined in discrete state space. In particular, we refer the reader to [9, Sections 4 and 5] for examples for which Assumption 1 holds true. Assumption 1 is also satisfied for general strongly Feller processes, as shown in [16], and for some degenerate diffusion processes, as studied in [3, 23]. We refer the reader to [6, 15, 30, 2, 28] for alternative criteria ensuring Assumption 1.

We can show (a short proof is provided later in the appendix of this paper) that Assumption 1 implies the following one.

**Assumption 2.** *i) Denoting  $E' := \{x \in E : \eta(x) > 0\}$ , the family of probability measures  $(\mathbb{Q}_x)_{x \in E'}$  defined by*

$$\mathbb{Q}_x(\Gamma) := \lim_{T \rightarrow \infty} \mathbb{P}_x(\Gamma | \tau_\partial > T), \quad \forall t \geq 0, \forall \Gamma \in \mathcal{F}_t,$$

*is well-defined.*

*ii) Under  $(\mathbb{Q}_x)_{x \in E'}$ ,  $X$  is a Markov process on  $E'$  admitting  $\beta(dx) := \eta(x)\alpha(dx)$  as an invariant probability measure. Moreover, denoting*

$$\psi(x) := \frac{\psi_1(x)}{\eta(x)},$$

*$\beta(\psi) < +\infty$  and, for all  $t \geq 0$  and  $x \in E'$ ,*

$$\|\mathbb{Q}_x(X_t \in \cdot) - \beta\|_\psi \leq C\psi(x)e^{-\gamma t}, \quad (4)$$

*where  $C, \gamma > 0$  are the same constants as in (3).*

Since the process  $X$  under  $(\mathbb{Q}_x)_{x \in E'}$  is a Markov process, the family of operators  $(Q_t)_{t \geq 0}$  defined by

$$Q_t f(x) := \mathbb{E}_x^{\mathbb{Q}}(f(X_t)), \quad \forall t \geq 0, \forall x \in E', \forall f \in \mathbb{L}^\infty(\psi_1/\eta),$$

where  $\mathbb{E}_x^{\mathbb{Q}}$  is the expectation associated to  $\mathbb{Q}_x$ , is a semigroup. In the literature (see for example [9, Theorem 2.7]), the Markov process associated to this semigroup is called the  $Q$ -process.

Independently on the satisfaction of Assumption 1, (4) is satisfied when the  $Q$ -process satisfies the assumptions 1 and 2 in [18]. Moreover, the inequality (4) implies, since  $\eta \in \mathbb{L}^\infty(\psi_1)$ , that, for all  $x \in E'$  and  $t \geq 0$ ,

$$\|\delta_x Q_t - \beta\|_{TV} \leq C\|\eta\|_{\mathbb{L}^\infty(\psi_1)} \frac{\psi_1(x)}{\eta(x)} e^{-\gamma t},$$

where  $\|\cdot\|_{TV}$  denotes the total variation norm.

### 1.3 The main result

A consequence of Assumption 1 is that the probability measure  $\beta$  is a *quasi-ergodic distribution* for the process  $X$ . That is, for all bounded measurable function  $f$  and  $\mu \in \mathcal{M}_1(E)$  satisfying  $\mu(\psi_1) < +\infty$  and  $\mu(\eta) > 0$ , the convergence

$$\mathbb{E}_\mu \left[ \frac{1}{t} \int_0^t f(X_s) ds \middle| \tau_\partial > t \right] \xrightarrow[t \rightarrow \infty]{} \beta(f) \quad (5)$$

holds true. This property is a consequence of the following lemma, whose the proof is postponed to the appendix of this paper.

**Lemma 1.** *For all  $x \in E'$ ,  $t \geq 0$  and  $\Gamma \in \mathcal{F}_t$ ,*

$$\mathbb{Q}_x(\Gamma) = e^{\lambda_0 t} \frac{P_t[\eta \mathbb{1}_\Gamma](x)}{\eta(x)}. \quad (6)$$

Moreover, there exists a constant  $C' > 0$  such that, for all  $\mu \in \mathcal{M}_1(E)$ ,  $0 \leq t \leq T$  and  $\Gamma \in \mathcal{F}_t$ ,

$$|\mathbb{Q}_{\eta \circ \mu}(\Gamma) - \mathbb{P}_\mu(\Gamma | \tau_\partial > T)| \leq C' \frac{\mu(\psi_1)}{\mu(\eta)} e^{-\gamma(T-t)}. \quad (7)$$

In particular, the property (7) implies (5) as shown in [7]. More precisely, we can show that the previous lemma implies the corollary below:

**Corollary 1.** *For all  $\mu \in \mathcal{M}_1(E)$  such that  $\mu(\psi_1) < +\infty$  and  $\mu(\eta) > 0$ , for all  $f$  bounded,*

$$\mathbb{E}_\mu \left( \left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right|^2 \middle| \tau_\partial > t \right) \xrightarrow[t \rightarrow \infty]{} 0,$$

implying (5) and that, for all  $\mu \in \mathcal{M}_1(E)$  and  $f$  bounded, for all  $\epsilon > 0$ ,

$$\mathbb{P}_\mu \left( \left| \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right| \geq \epsilon \middle| \tau_\partial > t \right) \xrightarrow[t \rightarrow \infty]{} 0.$$

Provided Lemma 1, a short proof can be obtained adapting the proofs in [27] or [19], providing even  $1/t$  as speed of convergence.

The aim of this paper is to prove a central limit theorem for processes satisfying Assumption 1, conditioned not to be absorbed up to the time  $t$ . Existing results stating a conditional central limit theorem for absorbing discrete-time Markov chains can be found in [10, 25, 1, 29, 4, 20]. In particular, in [10, Section 3.6], it is stated that, for any Markov chain  $(X_n)_{n \in \mathbb{Z}_+}$  defined on a finite state space  $E \cup \{\partial\}$  (absorbed at  $\partial$ ) whose matrix  $(\mathbb{P}_i(X_1 = j))_{i,j \in E}$  is irreducible and aperiodic, one has that, for all function  $f$  such that  $\beta(f) = 0$ , the limit

$$\theta^2 := \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_\alpha \left( \left( \sum_{k=0}^n f(X_k) \right)^2 \middle| \tau_\partial > n \right)$$

is well-defined. If moreover  $\theta^2 \neq 0$ , one obtains

$$\lim_{n \rightarrow \infty} \mathbb{P}_\alpha \left[ \frac{1}{\sqrt{n}} \sum_{k=0}^n f(X_k) \leq y \middle| \tau_\partial > n \right] = \int_{-\infty}^y \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{x^2}{2\theta^2}} dx,$$

for all  $y \in \mathbb{R}$ . This result is extended to all initial distributions in [25], where it is also claimed that the limiting Gaussian distribution is the same as the one obtained in the central limit theorem applied to the  $Q$ -process (i.e. same limiting variance).

The main result of this paper is then the following.

**Theorem 1.** Assume that the process  $(X_t)_{t \geq 0}$  satisfies Assumption 1.

Then, for all  $f \in \mathbb{L}^\infty(\mathbb{1}_E)$  such that  $\sigma_f^2 > 0$  and  $\mu \in \mathcal{M}_1(E)$  such that  $\mu(\psi_1) < +\infty$  and  $\mu(\eta) > 0$ ,

$$\mathbb{P}_\mu \left( \sqrt{t} \left[ \frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right] \in \cdot \mid \tau_\partial > t \right) \xrightarrow[t \rightarrow +\infty]{w} \mathcal{N}(0, \sigma_f^2),$$

where  $w$  refers to the weak convergence of measures, where  $\mathcal{N}(0, \sigma_f^2)$  refers to the centered Gaussian variable of variance

$$\sigma_f^2 := 2 \int_0^\infty \mathbf{Cov}_\beta^\mathbb{Q}(f(X_0), f(X_s)) ds, \quad (8)$$

where  $\mathbf{Cov}_\beta^\mathbb{Q}$  refers to the covariance with respect to the probability measure  $\mathbb{Q}_\beta := \int_{E'} \beta(dx) \mathbb{Q}_x$ .

In particular, (4) implies that  $\sigma_f^2 < +\infty$  for any  $f$  bounded by 1, since, assuming without loss of generality that  $\beta(f) = 0$ , for all  $k \geq 0$ ,

$$|\mathbb{E}_\beta^\mathbb{Q}(f(X_0)f(X_k))| = |\mathbb{E}_\beta^\mathbb{Q}(f(X_0)\mathbb{E}_{X_0}^\mathbb{Q}(f(X_k)))| \leq C\beta(\psi_1/\eta)e^{-\gamma k}. \quad (9)$$

This paper is only interested in processes conditioned not to be absorbed by absorbing states. Nevertheless, the following proofs can be adapted to general non-conservative semigroups satisfying Assumption 1. Some examples of such semigroups have been studied in [15, 2, 8, 31].

Theorem 1 will be proved at the third section. To prove it, we first need to show a central limit theorem for the  $Q$ -process satisfying (4). In particular, up to my knowledge, the papers dealing with central limit theorems for Markov processes require stronger hypotheses than (4) (see the references provided in Section 2). That is why the second section aims to prove a central limit theorem for general ergodic Markov processes, which could be interesting and useful beyond the framework of quasi-stationarity.

To conclude, the paper ends with an appendix showing the implication Assumption 1  $\Rightarrow$  Assumption 2 and Lemma 1 stated above. In particular, even if the existence of a quasi-ergodic distribution is quite classical assuming that Lemma 1 holds true (see for example [7] for a simple proof of this statement), the lemma itself is not clearly stated in the literature for processes satisfying Assumption 1 ([7], for example, states it under stronger conditions). This is why a short proof is provided in this appendix.

## 2 Central limit theorem for Markov processes

This section aims to establish a central limit theorem for Markov processes satisfying the condition (4). In the literature, central limit theorems for continuous-time Markov processes have, among others, been established in [21, 5, 24]. In particular, the papers [21, 5] made use of central limit theorems for martingales; the paper [24] used Kato's theory applied to analytically perturbed operators.

In this paper, a central limit theorem will be proved for Markov processes studying the convergence of the moments of  $\frac{1}{\sqrt{t}} \int_0^t f(X_s) ds$ , for bounded functions  $f$  such that  $\beta(f) = 0$  and  $\sigma_f^2 > 0$ . Up to my knowledge, this method to establish a central limit theorem for (non-stationary) Markov processes is new. However, this method is difficult to apply for discrete-time processes; we refer to [13, 12, 22, 17, 11] for central limit theorems for discrete-time Markov chains.

In all this section, we deal with a general Markov process  $(X_t)_{t \geq 0}$  defined on a state space  $E'$ . We denote by  $(\mathbb{Q}_x)_{x \in E'}$  a family of probability measure such that, for all  $x \in E'$ ,  $\mathbb{Q}_x(X_0 = x) = 1$ , for all probability measure  $\mu \in \mathcal{M}_1(E')$ ,  $\mathbb{Q}_\mu := \int_{E'} \mathbb{Q}_x \mu(dx)$ . We denote by  $\mathbb{E}^\mathbb{Q}$  and  $\mathbf{Cov}^\mathbb{Q}$  the expectation and the covariance associated to the probability measure  $\mathbb{Q}$ , respectively.

We emphasize that this section can be read independently on the rest of the paper. In particular,  $(X_t)$  has no link with the  $Q$ -process with this section.

In all what follows, we denote by  $\mathcal{B}_1(E')$  the set of the bounded by 1 measurable functions defined over  $E'$ .

We introduce now the only assumption used all along this section:

**Assumption 3.** *The process  $(X_t)_{t \geq 0}$  admits an invariant measure  $\beta$ , and there exists a function  $\psi : E' \rightarrow [c, +\infty)$  ( $c > 0$ ) and two constants  $C, \gamma > 0$  such that, for all  $x \in E'$  and  $t \geq 0$ ,*

$$\|\mathbb{Q}_x(X_t \in \cdot) - \beta\|_\psi \leq C e^{-\gamma t} \psi(x).$$

In accordance with the introduction, we introduce, for all bounded function  $f$ , the variance

$$\sigma_f^2 := 2 \int_0^\infty \text{Cov}_\beta^\mathbb{Q}(f(X_0), f(X_s)) ds.$$

## 2.1 Convergence of the moments of $\frac{1}{\sqrt{t}} \int_0^t f(X_s) ds$

In this subsection, the following theorem will be proved.

**Theorem 2.** *Assume that  $(X_t)_{t \geq 0}$  satisfies Assumption 3. Then there exist a positive constants  $C_1$  and a sequence of positive constant  $(D_k)_{k \in \mathbb{N}}$  such that, for all  $k \in \mathbb{Z}_+$ ,  $\mu \in \mathcal{M}_1(E')$  such that  $\mu(\psi) < +\infty$ ,  $f \in \mathcal{B}_1(E')$  such that  $\beta(f) = 0$  and  $t > 0$ ,*

$$\left| \mathbb{E}_\mu^\mathbb{Q} \left( \frac{1}{t^k} \left( \int_0^t f(X_s) ds \right)^{2k} \right) - \frac{(2k)!}{k!} \frac{\sigma_f^{2k}}{2^k} \right| \leq (2k)! D_k C_1 \frac{k}{(k-1)!} \frac{\mu(\psi)}{t} \quad (10)$$

and

$$\lim_{t \rightarrow \infty} \mathbb{E}_\mu^\mathbb{Q} \left( \frac{1}{t^k \sqrt{t}} \left( \int_0^t f(X_s) ds \right)^{2k+1} \right) = 0.$$

In particular, for all  $\mu \in \mathcal{M}_1(E')$  such that  $\mu(\psi) < +\infty$  and  $f \in \mathcal{B}_1(E')$  such that  $\beta(f) = 0$  and  $\sigma_f^2 > 0$ ,

$$\mathbb{Q}_\mu \left( \frac{1}{\sqrt{t}} \int_0^t f(X_s) ds \in \cdot \right) \xrightarrow[t \rightarrow \infty]{w} \mathcal{N}(0, \sigma_f^2).$$

Moreover, a suitable sequence  $(D_k)_{k \in \mathbb{N}}$  satisfying the inequalities (10) is the one defined as:

$$D_k := \left( \left( \frac{C}{\gamma} \left( 1 + \frac{\beta(\psi)}{c} \right) \right)^{k-1} \vee 1 \right) \times \left( \frac{C^2}{c} \vee \frac{C\beta(\psi)}{c^2\gamma} \right), \quad \forall k \in \mathbb{N}.$$

Before proving Theorem 2, we need to prove two lemmata.

**Lemma 2.** *There exists a sequence of positive constants  $(D_k)_{k \in \mathbb{N}}$  such that, for all  $f \in \mathcal{B}_1(E')$  such that  $\beta(f) = 0$  and  $\sigma_f^2 > 0$ , for all  $k \in \mathbb{N}$ ,  $\mu \in \mathcal{M}_1(E')$  and  $s_2 \leq \dots \leq s_{2k}$ ,*

$$\left| \mathbb{E}_\mu^\mathbb{Q} \left( \left[ \int_0^{s_2} f(X_{s_1}) ds_1 \right] f(X_{s_2}) \dots \left[ \int_{s_{2k-2}}^{s_{2k}} f(X_{s_{2k-1}}) ds_{2k-1} \right] f(X_{s_{2k}}) \right) - \frac{\sigma_f^{2k}}{2^k} \right| \leq D_k \mu(\psi) \sum_{i=0}^{k-1} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})}, \quad (11)$$

where  $s_0 = 0$  by convention.

*Proof.* We prove it by induction on  $k$ . We begin by showing the case  $k = 1$ . For all  $\mu \in \mathcal{M}_1(E')$  and  $f \in \mathcal{B}_1(E')$  and  $t \geq 0$ ,

$$\begin{aligned} \mathbb{E}_\mu^\mathbb{Q} \left( \left[ \int_0^t f(X_s) ds \right] f(X_t) \right) &= \int_0^t \mathbb{E}_\mu^\mathbb{Q}(f(X_s)f(X_t)) ds \\ &= \int_0^t \mathbb{E}_{\mu_{Q_{t-s}}}^\mathbb{Q}(f(X_0)f(X_s)) ds, \end{aligned} \quad (12)$$

where we denote by  $(Q_t)_{t \geq 0}$  the semigroup for the process  $(X_t)_{t \geq 0}$ . By Assumption 3, for all  $x \in E'$  and  $f \in \mathcal{B}_1(E')$  such that  $\beta(f) = 0$ ,

$$|\mathbb{E}_x^\mathbb{Q}(f(X_0)f(X_s))| \leq \mathbb{E}_x^\mathbb{Q}(|\mathbb{E}_{X_0}^\mathbb{Q}(f(X_s))|) \leq \frac{C}{c} \psi(x) e^{-\gamma s}. \quad (13)$$

Hence, by (12), Assumption 3 and this last inequality, for all  $t \geq 0$ ,  $\mu \in \mathcal{M}_1(E')$  such that  $\mu(\psi) < +\infty$  and  $f \in \mathcal{B}_1(E')$  such that  $\beta(f) = 0$ ,

$$\begin{aligned} \left| \mathbb{E}_\mu^\mathbb{Q} \left( \left[ \int_0^t f(X_s) ds \right] f(X_t) \right) - \int_0^t \mathbb{E}_\beta^\mathbb{Q}(f(X_0)f(X_s)) ds \right| &\leq C\mu(\psi) \int_0^t e^{-\gamma(t-s)} \|\mathbb{E}^\mathbb{Q}(f(X_0)f(X_s))\|_{L^\infty(\psi)} ds \\ &\leq C\mu(\psi) \int_0^t e^{-\gamma(t-s)} \frac{C}{c} e^{-\gamma s} ds \\ &\leq \frac{C^2}{c} \mu(\psi) t e^{-\gamma t}. \end{aligned} \quad (14)$$

Moreover, since  $\beta(f) = 0$ , by (13), for all  $t \geq 0$ ,

$$\left| \int_t^\infty \mathbb{E}_\beta^\mathbb{Q}(f(X_0)f(X_s)) ds \right| \leq \int_t^\infty \frac{C}{c} \beta(\psi) e^{-\gamma s} ds \leq \frac{C\beta(\psi)}{c\gamma} e^{-\gamma t} \leq \frac{C\beta(\psi)}{c^2\gamma} \mu(\psi) e^{-\gamma t}. \quad (15)$$

Hence, by definition of  $\sigma_f^2$ , there exists  $D_1 > 0$  such that

$$\left| \mathbb{E}_\mu^\mathbb{Q} \left( \left[ \int_0^t f(X_s) ds \right] f(X_t) \right) - \frac{\sigma_f^2}{2} \right| \leq D_1 \mu(\psi) (t+1) e^{-\gamma t}.$$

In particular, one can choose here  $D_1 := \frac{C^2}{c} \vee \frac{C\beta(\psi)}{c^2\gamma}$ . This concludes the base case.

Let  $k-1 \in \mathbb{N}$  be such that the hypothesis of induction is satisfied. Then, by the Markov property,

$$\begin{aligned} &\mathbb{E}_\mu^\mathbb{Q} \left( \int_0^{s_2} f(X_{s_1}) ds_1 f(X_{s_2}) \dots \int_{s_{2k-2}}^{s_{2k}} f(X_{s_{2k-1}}) ds_{2k-1} f(X_{s_{2k}}) \right) \\ &= \mathbb{E}_\mu^\mathbb{Q} \left( \int_0^{s_2} f(X_{s_1}) ds_1 f(X_{s_2}) \mathbb{E}_{X_{s_2}}^\mathbb{Q} \left( \int_{s_2}^{s_4} f(X_{s_3-s_2}) ds_3 f(X_{s_4-s_2}) \dots \int_{s_{2k-2}}^{s_{2k}} f(X_{s_{2k-1}-s_2}) ds_{2k-1} f(X_{s_{2k}-s_2}) \right) \right) \\ &= \mathbb{E}_\mu^\mathbb{Q} \left( \int_0^{s_2} f(X_{s_1}) ds_1 f(X_{s_2}) \mathbb{E}_{X_{s_2}}^\mathbb{Q} \left( \int_0^{s_4-s_2} f(X_{s_3}) ds_3 f(X_{s_4-s_2}) \dots \int_{s_{2k-2}-s_2}^{s_{2k}-s_2} f(X_{s_{2k-1}}) ds_{2k-1} f(X_{s_{2k}-s_2}) \right) \right). \end{aligned} \quad (16)$$

By hypothesis, for all  $s_2 \leq s_4 \dots \leq s_{2k}$ ,

$$\begin{aligned} &\left| \mathbb{E}_{X_{s_2}}^\mathbb{Q} \left( \int_0^{s_4-s_2} f(X_{s_3-s_2}) ds_3 f(X_{s_4-s_2}) \dots \int_{s_{2k-2}-s_2}^{s_{2k}-s_2} f(X_{s_{2k-1}}) ds_{2k-1} f(X_{s_{2k}-s_2}) \right) - \frac{\sigma_f^{2k-2}}{2^{k-1}} \right| \\ &\leq D_{k-1} \psi(X_{s_2}) \sum_{i=1}^{k-1} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})}. \end{aligned} \quad (17)$$



Moreover, since  $\beta(f) = 0$ , for all  $\mu \in \mathcal{M}_1(E')$ , for all  $s_2 \geq 0$ , for all  $h \in \mathbb{L}^\infty(\psi)$ ,

$$\left| \mathbb{E}_\mu^\mathbb{Q} \left( \int_0^{s_2} f(X_s) ds f(X_{s_2}) h(X_{s_2}) \right) \right| \leq \frac{C}{\gamma} \left( 1 + \frac{\beta(\psi)}{c} \right) \mu(\psi) \|h\|_{\mathbb{L}^\infty(\psi)}, \quad (18)$$

where  $C$  is the constant implied in Assumption 3. Indeed, for all  $t \geq 0$ ,  $\mu \in \mathcal{M}_1(E')$  and  $f, g \in \mathbb{L}^\infty(\psi)$ ,

$$\int_0^t \mathbb{E}_\mu^\mathbb{Q} [f(X_s) g(X_t)] ds = \int_0^t \mathbb{E}_\mu^\mathbb{Q} [f(X_s) \mathbb{E}_{X_s} (g(X_{t-s}))] ds.$$

Thus, by Assumption 3, for all  $t \geq 0$ ,  $\mu \in \mathcal{M}_1(E')$ ,  $f \in \mathcal{B}_1(E')$  and  $g \in \mathbb{L}^\infty(\psi)$ ,

$$\begin{aligned} \left| \int_0^t \mathbb{E}_\mu^\mathbb{Q} [f(X_s) g(X_t)] ds - \int_0^t \mathbb{E}_\mu^\mathbb{Q} [f(X_s)] \beta(g) ds \right| &\leq C \|g\|_{\mathbb{L}^\infty(\psi)} \int_0^t \mu(\psi) e^{-\gamma(t-s)} ds \\ &\leq \frac{C}{\gamma} \|g\|_{\mathbb{L}^\infty(\psi)} \mu(\psi). \end{aligned}$$

Moreover, again by Assumption 3, for all  $t \geq 0$ ,  $\mu \in \mathcal{M}_1(E')$  and  $f \in \mathcal{B}_1(E')$  such that  $\beta(f) = 0$ ,

$$\left| \int_0^t \mathbb{E}_\mu^\mathbb{Q} [f(X_s)] ds \right| \leq \frac{C}{c} \mu(\psi) \int_0^t e^{-\gamma s} ds \leq \frac{C}{c\gamma} \mu(\psi).$$

These two last inequalities applied to  $g = f \times h$  imply (18).

Now, denote

$$h : x \mapsto \mathbb{E}_x^\mathbb{Q} \left( \int_0^{s_4-s_2} f(X_{s_3-s_2}) ds_3 f(X_{s_4-s_2}) \dots \int_{s_{2k-2}-s_2}^{s_{2k}-s_2} f(X_{s_{2k-1}}) ds_{2k-1} f(X_{s_{2k}-s_2}) \right) - \frac{\sigma_f^{2k-2}}{2^{k-1}}.$$

Then, by (17),  $h \in \mathbb{L}^\infty(\psi)$  and

$$\|h\|_{\mathbb{L}^\infty(\psi)} \leq D_{k-1} \sum_{i=1}^{k-1} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})}.$$

Hence, by (16), (18) and this last inequality,

$$\begin{aligned} &\left| \mathbb{E}_\mu^\mathbb{Q} \left( \int_0^{s_2} f(X_{s_1}) ds_1 f(X_{s_2}) \dots \int_{s_{2k-2}}^{s_{2k}} f(X_{s_{2k-1}}) ds_{2k-1} f(X_{s_{2k}}) \right) - \frac{\sigma_f^{2(k-1)}}{2^{k-1}} \mathbb{E}_\mu^\mathbb{Q} \left( \int_0^{s_2} f(X_{s_1}) ds_1 f(X_{s_2}) \right) \right| \\ &\leq \frac{C}{\gamma} \left( 1 + \frac{\beta(\psi)}{c} \right) \mu(\psi) \|h\|_{\mathbb{L}^\infty(\psi)} \leq \frac{C}{\gamma} \left( 1 + \frac{\beta(\psi)}{c} \right) \mu(\psi) D_{k-1} \sum_{i=1}^{k-1} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})}. \end{aligned}$$

This and the case  $k = 1$  conclude the induction setting

$$D_k := \left[ D_{k-1} \times \left( \frac{C}{\gamma} \left( 1 + \frac{\beta(\psi)}{c} \right) \right) \right] \vee D_1.$$

□

We need also the following lemma.

**Lemma 3.** *For all  $k \in \mathbb{Z}_+$ , there exists  $C_k \in (0, +\infty)$  such that, for  $t \geq 1$ ,*

$$\int_{0 \leq s_2 \dots \leq s_{2k} \leq t} \sum_{i=0}^{k-1} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})} ds_2 \dots ds_{2k} \leq C_k t^{k-1}. \quad (19)$$

*Proof.* We prove (19) by induction on  $k$ . The case  $k = 1$  can easily be obtained by the reader for a given constant  $C_1 > 0$ . Now, assume that (19) holds true for  $k - 1 \in \mathbb{N}$ . For all  $t \geq 0$ ,

$$\begin{aligned}
& \sum_{i=0}^{k-1} \int_{0 \leq s_2 \leq \dots \leq s_{2k} \leq t} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})} ds_2 \dots ds_{2k} \\
&= \int_{0 \leq s_2 \leq \dots \leq s_{2k} \leq t} \sum_{i=0}^{k-2} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})} ds_2 \dots ds_{2k} \\
&\quad + \int_{0 \leq s_2 \leq \dots \leq s_{2k} \leq t} (s_{2k} - s_{2(k-1)} + 1) e^{-\gamma(s_{2k} - s_{2(k-1)})} ds_2 \dots ds_{2k} \\
&= \int_0^t \left[ \int_{0 \leq s_2 \leq \dots \leq s_{2(k-1)} \leq s_{2k}} \sum_{i=0}^{k-2} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})} ds_2 \dots ds_{2(i-1)} \right] ds_{2k} \\
&\quad + \int_{0 \leq s_2 \leq \dots \leq s_{2k} \leq t} (s_{2k} - s_{2(k-1)} + 1) e^{-\gamma(s_{2k} - s_{2(k-1)})} ds_2 \dots ds_{2k}. \quad (20)
\end{aligned}$$

By hypothesis, for all  $t \geq 0$ ,

$$\begin{aligned}
& \int_0^t \left[ \int_{0 \leq s_2 \leq \dots \leq s_{2(k-1)} \leq s_{2k}} \sum_{i=0}^{k-2} (s_{2(i+1)} - s_{2i}) e^{-\gamma(s_{2(i+1)} - s_{2i})} ds_2 \dots ds_{2(i-1)} \right] ds_{2k} \\
&\leq \int_0^t C_{k-1} s_{2k}^{k-2} ds_{2k} = C_{k-1} \frac{t^{k-1}}{k-1}.
\end{aligned}$$

For all  $t \geq 0$ , the second term of (20) is equal to

$$\begin{aligned}
& \int_{0 \leq s_{2(k-1)} \leq s_{2k} \leq t} (s_{2k} - s_{2(k-1)} + 1) e^{-\gamma(s_{2k} - s_{2(k-1)})} \left[ \int_{0 \leq s_2 \leq \dots \leq s_{2(k-2)}} ds_2 \dots ds_{2(k-2)} \right] ds_{2(k-1)} ds_{2k} \\
&= \int_{0 \leq r \leq s \leq t} (s - r + 1) e^{-\gamma(s-r)} \frac{r^{k-2}}{(k-2)!} dr ds \\
&= \int_0^t \left[ \int_r^t (s - r + 1) e^{-\gamma(s-r)} ds \right] \frac{r^{k-2}}{(k-2)!} dr \\
&= \int_0^t \left( \int_0^{t-r} (u + 1) e^{-\gamma u} du \right) \frac{r^{k-2}}{(k-2)!} dr \leq \frac{C_1}{(k-1)!} t^{k-1},
\end{aligned}$$

where  $C_1 < +\infty$  is exactly the same constant as for the case  $k = 1$ . Hence, (19) is proved with  $C_k$  satisfying the relation  $C_k = \frac{C_{k-1}}{k-1} + \frac{C_1}{(k-1)!}$ . By induction, for all  $k \geq 2$ ,

$$C_k = \frac{C_1}{(k-1)!} + \frac{C_1}{(k-2)!}.$$

□

We can now prove Theorem 2.

*Proof of Theorem 2.* We begin with the convergence of the even moment. For all  $\mu \in$

$\mathcal{M}_1(E')$ ,  $t \geq 0$ ,  $f \in \mathcal{B}_1(E')$  and  $k \in \mathbb{Z}_+$ ,

$$\begin{aligned} & \mathbb{E}_\mu^\mathbb{Q} \left( \left( \int_0^t f(X_s) ds \right)^{2k} \right) \\ &= (2k)! \int_{0 \leq s_2 \leq \dots \leq s_{2k} \leq t} \int_0^{s_2} \int_{s_2}^{s_4} \dots \int_{s_{2k-2}}^{s_{2k}} \mathbb{E}_\mu^\mathbb{Q}(f(X_{s_1})f(X_{s_2}) \dots f(X_{s_{2k-1}})f(X_{s_{2k}})) ds_1 \dots ds_{2k}. \end{aligned} \quad (21)$$

Then, assuming moreover that  $\beta(f) = 0$ , by (21),(11) and (19),

$$\left| \mathbb{E}_\mu^\mathbb{Q} \left( \left( \int_0^t f(X_s) ds \right)^{2k} \right) - \frac{(2k)!}{k!} t^k \frac{\sigma_f^{2k}}{2^k} \right| \leq (2k)! D_k \mu(\psi) \times C_k t^{k-1}, \quad (22)$$

which implies (10). Now, for all  $\mu \in \mathcal{M}_1(E')$ ,  $t \geq 0$ ,  $k \in \mathbb{Z}_+$  and  $f \in \mathcal{B}_1(E')$  such that  $\beta(f) = 0$ ,

$$\begin{aligned} & \mathbb{E}_\mu^\mathbb{Q} \left( \left( \int_0^t f(X_s) ds \right)^{2k+1} \right) \\ &= (2k+1)! \int_0^t \mathbb{E}_\mu^\mathbb{Q} \left( f(X_s) \mathbb{E}_{X_s}^\mathbb{Q} \left[ \int_{0 \leq s_2 \leq \dots \leq s_{2k+1} \leq t-s} f(X_{s_2}) \dots f(X_{s_{2k+1}}) ds_2 \dots ds_{2k+1} \right] \right) ds \\ &= (2k+1) \int_0^t \mathbb{E}_\mu^\mathbb{Q} \left( f(X_s) \mathbb{E}_{X_s}^\mathbb{Q} \left( \left( \int_0^{t-s} f(X_u) du \right)^{2k} \right) \right) ds \\ &= (2k+1) \int_0^t \mathbb{E}_\mu^\mathbb{Q} \left( f(X_{t-s}) \mathbb{E}_{X_{t-s}}^\mathbb{Q} \left( \left( \int_0^s f(X_u) du \right)^{2k} \right) \right) ds. \end{aligned}$$

By (22) and using that  $\mathbb{E}_\mu^\mathbb{Q}[\psi(X_{t-s})] \leq \frac{\beta(\psi)}{c} + C$  for all  $\mu \in \mathcal{M}_1(E')$  and  $s \leq t$  (this is a consequence of Assumption (3)), there exists  $\hat{C} > 0$  such that

$$\left| \mathbb{E}_\mu^\mathbb{Q} \left( \left( \int_0^t f(X_s) ds \right)^{2k+1} \right) - \frac{(2k+1)!}{k!} \frac{\sigma_f^{2k}}{2^k} \int_0^t s^k \mathbb{E}_\mu^\mathbb{Q}(f(X_{t-s})) ds \right| \leq D_k \hat{C} \frac{k(2k+1)}{(k-1)!} \mu(\psi) \frac{t^k}{k}. \quad (23)$$

Since  $\beta(f) = 0$ , by Assumption 3, for all  $\mu \in \mathcal{M}_1(E')$  and  $s \leq t$ ,

$$|\mathbb{E}_\mu^\mathbb{Q}[f(X_{t-s})]| \leq C \mu(\psi) e^{-\gamma(t-s)}. \quad (24)$$

For all  $t > 0$  and  $k \in \mathbb{Z}_+$ ,

$$\frac{1}{t^{k+\frac{1}{2}}} \int_0^t s^k e^{-\gamma(t-s)} ds = \frac{e^{-\gamma t}}{t^{k+\frac{1}{2}}} \int_0^t s^k e^{\gamma s} ds \leq \frac{e^{-\gamma t}}{\sqrt{t}} \int_0^t e^{\gamma s} ds \leq \frac{1}{\gamma \sqrt{t}}. \quad (25)$$

We deduce from (23), (24) and (25) that there exists  $\hat{C} > 0$  (different from the previous one) such that, for all  $\mu \in \mathcal{M}_1(E')$  such that  $\mu(\psi) < +\infty$  and  $f \in \mathcal{B}_1(E')$  such that  $\beta(f) = 0$ ,

$$\left| \frac{1}{t^{k+\frac{1}{2}}} \mathbb{E}_\mu^\mathbb{Q} \left( \left( \int_0^t f(X_s) ds \right)^{2k+1} \right) \right| \leq D_k \times \hat{C} \left[ \frac{(2k+1)!}{2^k k!} + \frac{2k+1}{(k-1)!} \right] \frac{\mu(\psi)}{\sqrt{t}}. \quad (26)$$

The central limit theorem is deduced from the method of moments. Now, concerning a suitable candidate for the sequence  $(D_k)_{k \in \mathbb{N}}$ , as proven in Lemma 1, a suitable candidate is the sequence defined recursively by

$$D_1 := \frac{C^2}{c} \vee \frac{C\beta(\psi)}{c^2\gamma},$$

$$D_k := \left[ D_{k-1} \times \left( \frac{C}{\gamma} \left( 1 + \frac{\beta(\psi)}{c} \right) \right) \right] \vee D_1, \quad \forall k \geq 2,$$

in other words the sequence defined in Theorem 2.  $\square$

## 2.2 A quantitative uniform CLT

The aim of this subsection is to prove the following result, which can be seen as an improved central limit theorem for  $(X_t)_{t \geq 0}$ .

**Theorem 3.** *Assume that  $(X_t)_{t \geq 0}$  satisfies Assumption 3. Then, for all  $\mu \in \mathcal{M}_1(E')$  such that  $\mu(\psi) < \infty$ ,  $f \in \mathcal{B}_1(E')$  such that  $\beta(f) = 0$  and  $\sigma_f^2 > 0$ , and  $\omega \in \mathbb{R}$ ,*

$$\lim_{t \rightarrow \infty} \sup_{g \in \mathbb{L}^\infty(\psi) : \|g\|_{\mathbb{L}^\infty(\psi)} \leq 1} \left| \mathbb{E}_\mu^\mathbb{Q} \left[ e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} g(X_t) \right] - \beta(g) e^{-\frac{\sigma_f^2 \omega^2}{2}} \right| = 0. \quad (27)$$

Moreover, for all  $t > 0$  and  $\mu \in \mathcal{M}_1(E')$ , one has

$$\sup_{g \in \mathbb{L}^\infty(\psi) : \|g\|_{\mathbb{L}^\infty(\psi)} \leq 1} \left| \mathbb{E}_\mu^\mathbb{Q} \left[ e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} g(X_t) \right] - \beta(g) \mathbb{E}_\mu^\mathbb{Q} \left( e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} \right) \right| \leq C\mu(\psi) e^{-\gamma t} + \frac{C|\omega|}{\sqrt{t}} \frac{\beta(\psi) + C\mu(\psi)}{\gamma}.$$

*Proof of Theorem 3.* For all  $\mu \in \mathcal{M}_1(E')$ ,  $f \in \mathcal{B}(E')$  such that  $\beta(f) = 0$ ,  $t \geq 0$ ,  $k \in \mathbb{Z}_+$  and  $g \in \mathbb{L}^\infty(\psi)$ ,

$$\begin{aligned} \mathbb{E}_\mu^\mathbb{Q} \left( \left( \int_0^t f(X_s) ds \right)^k g(X_t) \right) &= k \int_0^t \mathbb{E}_\mu^\mathbb{Q} \left( \left[ \int_0^s f(X_u) du \right]^{k-1} f(X_s) g(X_t) \right) ds \\ &= k \int_0^t \mathbb{E}_\mu^\mathbb{Q} \left( \left[ \int_0^s f(X_u) du \right]^{k-1} f(X_s) \mathbb{E}_{X_s}^\mathbb{Q} (g(X_{t-s})) \right) ds. \end{aligned}$$

Hence, for all  $\mu \in \mathcal{M}_1(E')$ ,  $t \geq 0$ ,  $f \in \mathcal{B}_1(E')$  such that  $\beta(f) = 0$ ,  $k \in \mathbb{Z}_+$  and  $g \in \mathbb{L}^\infty(\psi)$ ,

$$\begin{aligned} \mathbb{E}_\mu^\mathbb{Q} \left( \left( \int_0^t f(X_s) ds \right)^k g(X_t) \right) - \beta(g) \mathbb{E}_\mu^\mathbb{Q} \left( \left( \int_0^t f(X_s) ds \right)^k \right) \\ = k \int_0^t \mathbb{E}_\mu^\mathbb{Q} \left( \left( \int_0^s f(X_u) du \right)^{k-1} f(X_s) [\mathbb{E}_{X_s}^\mathbb{Q} (g(X_{t-s})) - \beta(g)] \right) ds. \end{aligned}$$

Thus, using that  $e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} = \sum_{k=0}^{\infty} \frac{i^k \omega^k}{t^{k/2} k!} \left( \int_0^t f(X_s) ds \right)^k$  for all  $t \geq 0$ ,  $\omega \in \mathbb{R}$ , and  $f \in \mathcal{B}_1(E')$  such that  $\beta(f) = 0$ , then, using the above equality, for all  $\mu \in \mathcal{M}_1(E')$  and

$g \in \mathbb{L}^\infty(\psi)$ ,

$$\begin{aligned}
& \mathbb{E}_\mu^\mathbb{Q} \left( e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} g(X_t) \right) - \beta(g) \mathbb{E}_\mu^\mathbb{Q} \left( e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} \right) \\
&= \sum_{k=0}^{\infty} \left( \frac{i\omega}{\sqrt{t}} \right)^k \frac{1}{k!} \left\{ \mathbb{E}_\mu^\mathbb{Q} \left( \left( \int_0^t f(X_s) ds \right)^k g(X_t) \right) - \beta(g) \mathbb{E}_\mu^\mathbb{Q} \left( \left( \int_0^t f(X_s) ds \right)^k \right) \right\} \\
&= \mathbb{E}_\mu^\mathbb{Q}(g(X_t)) - \beta(g) + \sum_{k=1}^{\infty} \left( \frac{i\omega}{\sqrt{t}} \right)^k \frac{1}{k!} \left\{ \mathbb{E}_\mu^\mathbb{Q} \left( \left( \int_0^t f(X_s) ds \right)^k g(X_t) \right) - \beta(g) \mathbb{E}_\mu^\mathbb{Q} \left( \left( \int_0^t f(X_s) ds \right)^k \right) \right\} \\
&= \mathbb{E}_\mu^\mathbb{Q}(g(X_t)) - \beta(g) + \frac{i\omega}{\sqrt{t}} \int_0^t \mathbb{E}_\mu^\mathbb{Q} \left( e^{\frac{i\omega}{\sqrt{s}} \int_0^s f(X_u) du} f(X_s) [\mathbb{E}_{X_s}^\mathbb{Q}(g(X_{t-s})) - \beta(g)] \right) ds.
\end{aligned}$$

By Assumption 3 one has, for all  $\mu \in \mathcal{M}_1(E')$ ,  $\mathbb{Q}_\mu$ -almost surely and for all  $s \leq t$  and  $g \in \mathbb{L}^\infty(\psi)$ ,

$$|\mathbb{E}_{X_s}^\mathbb{Q}(g(X_{t-s})) - \beta(g)| \leq C \|g\|_{\mathbb{L}^\infty(\psi)} \psi(X_s) e^{-\gamma(t-s)}.$$

Thus, for all  $\mu \in \mathcal{M}_1(E')$ ,  $t > 0$ ,  $\omega \in \mathbb{R}$ ,  $g \in \mathbb{L}^\infty(\psi)$  and  $f \in \mathcal{B}_1(E')$  such that  $\beta(f) = 0$ ,

$$\begin{aligned}
& \left| \mathbb{E}_\mu^\mathbb{Q} \left( e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} g(X_t) \right) - \beta(g) \mathbb{E}_\mu^\mathbb{Q} \left( e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} \right) \right| \\
& \leq C \mu(\psi) \|g\|_{\mathbb{L}^\infty(\psi)} e^{-\gamma t} + \frac{C \|g\|_{\mathbb{L}^\infty(\psi)} |\omega|}{\sqrt{t}} \int_0^t e^{-\gamma(t-s)} \mathbb{E}_\mu^\mathbb{Q}(\psi(X_s)) ds.
\end{aligned}$$

Since  $|\mathbb{E}_\mu^\mathbb{Q}(\psi(X_s)) - \beta(\psi)| \leq C \mu(\psi) e^{-\gamma s}$  for all  $s \geq 0$ , one has that

$$\sup_{t \geq 0} \int_0^t e^{-\gamma(t-s)} \mathbb{E}_\mu^\mathbb{Q}(\psi(X_s)) ds \leq \sup_{t \geq 0} \int_0^t e^{-\gamma(t-s)} [\beta(\psi) + C \mu(\psi) e^{-\gamma s}] ds \leq \frac{\beta(\psi) + C \mu(\psi)}{\gamma}.$$

These two last inequalities and Theorem 2 imply (27) and conclude the proof.  $\square$

### 3 Proof of Theorem 1

In this section,  $(X_t)_{t \geq 0}$  refers again to the process living in  $E \cup \{\partial\}$  and absorbed at  $\partial$ .

We now prove Theorem 1, divided in three steps.

Step 1. For all  $\mu \in \mathcal{M}_1(E)$ ,  $t \geq 0$  and  $f \in \mathcal{B}_1(E)$ ,  $g \in \mathbb{L}^\infty(\psi_1)$  and  $k \in \mathbb{Z}_+$ ,

$$\begin{aligned}
& \mathbb{E}_\mu \left( \left[ \int_0^t f(X_s) ds \right]^k g(X_t) \Big| \tau_\partial > t \right) = k! \int_{0 \leq s_1 \leq \dots \leq s_k \leq t} \mathbb{E}_\mu(f(X_{s_1}) f(X_{s_2}) \dots f(X_{s_k}) g(X_t) | \tau_\partial > t) ds_1 \dots ds_k \\
&= k! \int_0^t \mathbb{E}_\mu \left( \left[ \int_{0 \leq s_1 \leq \dots \leq s_{k-1} \leq s} f(X_{s_1}) \dots f(X_{s_{k-1}}) ds_1 \dots ds_{k-1} \right] f(X_s) g(X_t) \Big| \tau_\partial > t \right) ds \\
&= k \int_0^t \mathbb{E}_\mu \left( \left[ \int_0^s f(X_u) du \right]^{k-1} f(X_s) g(X_t) \Big| \tau_\partial > t \right) ds \\
&= k \int_0^t \frac{1}{\mathbb{P}_\mu(\tau_\partial > t)} \mathbb{E}_\mu \left( \left[ \int_0^s f(X_u) du \right]^{k-1} f(X_s) \mathbb{E}_{X_s}(g(X_{t-s}) \mathbb{1}_{\tau_\partial > t-s}) \mathbb{1}_{\tau_\partial > s} \right) ds.
\end{aligned}$$

For all  $s \leq t$ ,  $\mu \in \mathcal{M}_1(E)$ ,  $g \in \mathbb{L}^\infty(\psi_1)$  and  $x \in E$ , denote

$$C_{\mu,g}(s, t, x) := \frac{\mu(\eta)}{e^{\lambda_0 s}} e^{\gamma(t-s)} \left\{ \frac{\mathbb{E}_x(g(X_{t-s}) \mathbb{1}_{\tau_\partial > t-s})}{\mathbb{P}_\mu(\tau_\partial > t)} - \frac{e^{\lambda_0 s} \eta(x) \alpha(g)}{\mu(\eta)} \right\}.$$

Thus, for all  $\mu \in \mathcal{M}_1(E)$ ,  $f \in \mathcal{B}_1(E)$  such that  $\beta(f) = 0$ ,  $g \in \mathbb{L}^\infty(\psi_1)$ ,  $k \in \mathbb{Z}_+$  and  $t \geq 0$ ,

$$\begin{aligned} & \mathbb{E}_\mu \left( \left[ \int_0^t f(X_s) ds \right]^k \middle| g(X_t) \middle| \tau_\partial > t \right) - \alpha(g) \mathbb{E}_{\eta \circ \mu}^{\mathbb{Q}} \left( \left[ \int_0^t f(X_s) ds \right]^k \right) \\ &= k \times e^{-\gamma t} \int_0^t e^{\gamma s} \mathbb{E}_{\eta \circ \mu}^{\mathbb{Q}} \left( \left( \int_0^s f(X_u) du \right)^{k-1} \frac{f(X_s) C_{\mu, g}(s, t, X_s)}{\eta(X_s)} \right) ds. \end{aligned}$$

Similarly to the proof of Theorem 3, for all  $t > 0$ ,  $\omega \in \mathbb{R}$ ,  $\mu \in \mathcal{M}_1(E)$  and  $f$  such that  $\beta(f) = 0$ ,

$$\begin{aligned} & \mathbb{E}_\mu \left( e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} \middle| \tau_\partial > t \right) - \mathbb{E}_{\eta \circ \mu}^{\mathbb{Q}} \left[ e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} \right] \\ &= \sum_{k=1}^{\infty} \left( \frac{i\omega}{\sqrt{t}} \right)^k \frac{1}{k!} \left\{ \mathbb{E}_\mu \left( \left[ \int_0^t f(X_s) ds \right]^k \middle| \tau_\partial > t \right) - \mathbb{E}_{\eta \circ \mu}^{\mathbb{Q}} \left( \left[ \int_0^t f(X_s) ds \right]^k \right) \right\} \\ &= \sum_{k=1}^{\infty} \frac{i^k \omega^k}{t^{k/2}} \frac{1}{(k-1)!} e^{-\gamma t} \int_0^t e^{\gamma s} \mathbb{E}_{\eta \circ \mu}^{\mathbb{Q}} \left( \left( \int_0^s f(X_u) du \right)^{k-1} \frac{f(X_s) C_{\mu, \mathbb{1}_E}(s, t, X_s)}{\eta(X_s)} \right) ds \\ &= e^{-\gamma t} \int_0^t e^{\gamma s} \frac{i\omega}{\sqrt{t}} \mathbb{E}_{\eta \circ \mu}^{\mathbb{Q}} \left( e^{\frac{i\omega}{\sqrt{s}} \int_0^s f(X_u) du} \frac{f(X_s) C_{\mu, \mathbb{1}_E}(s, t, X_s)}{\eta(X_s)} \right) ds. \quad (28) \end{aligned}$$

*Step 2.* By triangular inequality, for all  $s \leq t$  and  $x \in E$ ,

$$\begin{aligned} |C_{\mu, g}(s, t, x)| &\leq \frac{\mu(\eta)}{e^{\lambda_0 s}} e^{\gamma(t-s)} \left\{ \left| \frac{\mathbb{E}_x[g(X_{t-s}) \mathbb{1}_{\tau_\partial > t-s}]}{\mathbb{P}_\mu(\tau_\partial > t)} - \frac{e^{-\lambda_0(t-s)} \eta(x) \alpha(g)}{\mathbb{P}_\mu(\tau_\partial > t)} \right| \right. \\ &\quad \left. + \left| \frac{e^{-\lambda_0(t-s)} \eta(x) \alpha(g)}{\mathbb{P}_\mu(\tau_\partial > t)} - \frac{e^{\lambda_0 s} \eta(x) \alpha(g)}{\mu(\eta)} \right| \right\}. \quad (29) \end{aligned}$$

By (3),

$$\frac{\mu(\eta)}{e^{\lambda_0 s}} e^{\gamma(t-s)} \left| \frac{\mathbb{E}_x[g(X_{t-s}) \mathbb{1}_{\tau_\partial > t-s}]}{\mathbb{P}_\mu(\tau_\partial > t)} - \frac{e^{-\lambda_0(t-s)} \eta(x) \alpha(g)}{\mathbb{P}_\mu(\tau_\partial > t)} \right| \leq C \|g\|_{\mathbb{L}^\infty(\psi_1)} \psi_1(x) \mu(\eta) \frac{e^{-\lambda_0 t}}{\mathbb{P}_\mu(\tau_\partial > t)}.$$

Again by (3),

$$e^{\lambda_0 t} \mathbb{P}_\mu(\tau_\partial > t) \geq \mu(\eta) - C\mu(\psi_1) e^{-\gamma t}.$$

Hence, for all  $t \geq \frac{1}{\gamma} \log \left( \frac{2C\mu(\psi_1)}{\mu(\eta)} \right)$ ,

$$\begin{aligned} \frac{\mu(\eta)}{e^{\lambda_0 t} \mathbb{P}_\mu(\tau_\partial > t)} &\leq \frac{1}{1 - C \frac{\mu(\psi_1)}{\mu(\eta)} e^{-\gamma t}} \\ &\leq 1 + 2C \frac{\mu(\psi_1)}{\mu(\eta)} e^{-\gamma t} \leq 2. \end{aligned}$$

For the second part of the right-hand side of the inequality (29),

$$\begin{aligned} \frac{\mu(\eta)}{e^{\lambda_0 s}} e^{\gamma(t-s)} \left| \frac{e^{-\lambda_0(t-s)} \eta(x) \alpha(g)}{\mathbb{P}_\mu(\tau_\partial > t)} - \frac{e^{\lambda_0 s} \eta(x) \alpha(g)}{\mu(\eta)} \right| &= \eta(x) |\alpha(g)| e^{\gamma(t-s)} \left| \frac{\mu(\eta)}{e^{\lambda_0 t} \mathbb{P}_\mu(\tau_\partial > t)} - 1 \right| \\ &\leq C \eta(x) |\alpha(g)| e^{-\gamma s} \frac{C\mu(\psi_1)}{e^{\lambda_0 t} \mathbb{P}_\mu(\tau_\partial > t)} \\ &\leq C \eta(x) |\alpha(g)| 2C \frac{\mu(\psi_1)}{\mu(\eta)}. \end{aligned}$$

Hence, these inequalities, the fact that  $|\alpha(g)| \leq \|g\|_{\mathbb{L}^\infty(\psi_1)} \alpha(\psi_1)$  and (29) imply the existence of a constant  $C' > 0$  such that, for all  $s \leq t$  such that  $t \geq \frac{1}{\gamma} \log \left( \frac{2C\mu(\psi_1)}{\mu(\eta)} \right)$  and  $x \in E$ ,

$$|C_{\mu,g}(s, t, x)| \leq C' \|g\|_{\mathbb{L}^\infty(\psi_1)} \left[ \psi_1(x) + \frac{\mu(\psi_1)}{\mu(\eta)} \eta(x) \right]. \quad (30)$$

*Last step.* By using this last inequality (30) in (28), one obtains that, for all  $f \in \mathcal{B}(E)$  such that  $\beta(f) = 0$ ,  $\omega \in \mathbb{R}$  and  $t \geq \frac{1}{\gamma} \log \left( \frac{2C\mu(\psi_1)}{\mu(\eta)} \right)$ ,

$$\begin{aligned} \left| \mathbb{E}_\mu \left( e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} \middle| \tau_\partial > t \right) - \mathbb{E}_{\eta \circ \mu}^\mathbb{Q} \left[ e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} \right] \right| &\leq e^{-\gamma t} \int_0^t e^{\gamma s} \frac{|\omega|}{\sqrt{t}} \mathbb{E}_{\eta \circ \mu}^\mathbb{Q} \left( \frac{|C_{\mu, \mathbb{1}_E}(s, t, X_s)|}{\eta(X_s)} \right) ds \\ &\leq C' \|\mathbb{1}_E\|_{\mathbb{L}^\infty(\psi_1)} e^{-\gamma t} \int_0^t e^{\gamma s} \frac{|\omega|}{\sqrt{t}} \left( \mathbb{E}_{\eta \circ \mu}^\mathbb{Q}(\psi(X_s)) + \frac{\mu(\psi_1)}{\mu(\eta)} \right) ds \\ &\leq C'' \frac{|\omega|}{\sqrt{t}} \frac{\mu(\psi_1)}{\mu(\eta)}, \end{aligned} \quad (31)$$

where  $C'' > 0$ . This, combined with Theorem 2, proves Theorem 1.

**Remark 1.** *The presence of  $1/\sqrt{t}$  in the last inequality suggests the idea that a Berry-Esseen inequality holds true for Markov processes conditioned not to be absorbed satisfying Assumption 1. In reality, this last upper-bound does not allow directly to deduce such a result.*

*As a matter of fact, an approach would be to consider the inequality (from [14]),*

$$\begin{aligned} d_{Kolm} \left( \mathbb{P}_\mu \left( \frac{1}{\sqrt{t}} \int_0^t f(X_s) ds \in \cdot \middle| \tau_\partial > t \right), \mathbb{Q}_{\eta \in \mu} \left( \frac{1}{\sqrt{t}} \int_0^t f(X_s) ds \in \cdot \right) \right) \\ \leq \int_{-W}^W \frac{\left| \mathbb{E}_\mu \left( e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} \middle| \tau_\partial > t \right) - \mathbb{E}_{\eta \circ \mu}^\mathbb{Q} \left[ e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} \right] \right|}{|\omega|} d\omega + \frac{\tilde{C}}{W}, \end{aligned} \quad (32)$$

where  $\tilde{C} > 0$ , holding true for all  $W > 0$ . In particular, it is visible that the inequality (31) and Theorem 3 are not enough to deduce a Berry-Esseen theorem.

*The inequality (32) is in particular used in [24] to state a Berry-Esseen theorem for reversible Markov processes, using spectral arguments applied to perturbed operators. This approach could certainly be adapted to prove a similar result in the quasi-stationary framework.*

## Appendix: Proof of Lemma 1 and Assumption 1 $\Rightarrow$ Assumption 2.

This little section is devoted to the proof of Lemma 1, needed to justify the existence of a quasi-ergodic distribution and the convergence (5). The implication Assumption 1  $\Rightarrow$  Assumption 2 will also be proved in this short proof.

*Proof of Lemma 1.* Assume Assumption 1. Let  $t \geq 0$  and  $\Gamma \in \mathcal{F}_t$ . Then, for all  $T \geq t$  and  $x \in E$ ,

$$e^{\lambda_0 T} \mathbb{P}_x(\Gamma, \tau_\partial > T) = e^{\lambda_0 T} \mathbb{E}_x(\mathbb{1}_{\Gamma, \tau_\partial > t} \mathbb{P}_{X_t}(\tau_\partial > T-t)) = e^{\lambda_0 t} \mathbb{E}_x(\mathbb{1}_{\Gamma, \tau_\partial > t} e^{\lambda_0(T-t)} \mathbb{P}_{X_t}(\tau_\partial > T-t)).$$

By Assumption 1, since  $\psi_1 \geq 1$ , for all  $T \geq t$  and  $x \in E$ ,

$$\left| e^{\lambda_0 T} \mathbb{P}_x(\Gamma, \tau_\partial > T) - e^{\lambda_0 t} \mathbb{E}_x(\mathbb{1}_{\Gamma, \tau_\partial > t} \eta(X_t)) \right| \leq C e^{\lambda_0 t} \mathbb{E}_x(\psi_1(X_t) \mathbb{1}_{\tau_\partial > t}) e^{-\gamma(T-t)}. \quad (33)$$

Assumption 1 implies that

$$e^{\lambda_0 t} P_t \psi_1(x) \leq \alpha(\psi_1) + C \psi_1(x) e^{-\gamma t} < +\infty, \quad (34)$$

so that the previous inequality entails that, for all  $x \in E$ ,

$$\lim_{T \rightarrow \infty} e^{\lambda_0 T} \mathbb{P}_x(\Gamma, \tau_\partial > T) = e^{\lambda_0 t} \mathbb{E}_x(\mathbb{1}_{\Gamma, \tau_\partial > t} \eta(X_t)).$$

Since the previous inequality holds true for  $\Gamma = \mathbb{1}_E$  and  $t = 0$ , we deduce that, for all  $x \in E'$ ,

$$\lim_{T \rightarrow \infty} \mathbb{P}_x(\Gamma | \tau_\partial > T) = e^{\lambda_0 t} \frac{\mathbb{E}_x(\eta(X_t) \mathbb{1}_{\Gamma, \tau_\partial > t})}{\eta(x)},$$

which proves the first point of Assumption 2 setting, for all  $x \in E'$  and  $\Gamma \in \mathcal{F}_t$ ,

$$\mathbb{Q}_x(\Gamma) := \mathbb{E}_x(e^{\lambda_0 t} \eta(X_t) \mathbb{1}_{\Gamma, \tau_\partial > t}) / \eta(x),$$

proving therefore the equality (6). By definition of  $\eta$  and  $\alpha$ , it is easy to check that  $\beta$  is an invariant measure for the  $Q$ -process. Moreover, by Assumption 1, for all  $f \in \mathbb{L}^\infty(\psi)$  and  $x \in E'$ ,

$$|\mathbb{E}_x^{\mathbb{Q}}(f(X_t)) - \beta(f)| = \frac{|e^{\lambda_0 t} P_t(f\eta)(x) - \eta(x)\alpha(f\eta)|}{\eta(x)} \leq C \psi(x) e^{-\gamma t} \|f\|_{\mathbb{L}^\infty(\psi)},$$

which confirms therefore the implication Assumption 1  $\Rightarrow$  Assumption 2. It remains therefore to show the exponential convergence of the function  $T \mapsto \mathbb{P}_\mu(\Gamma | \tau_\partial > T)$  to  $\mathbb{Q}_{\eta \circ \mu}(\Gamma)$ , for all  $\mu \in \mathcal{M}_1(E)$  such that  $\mu(\eta) > 0$  and  $\mu(\psi_1) < +\infty$ . To do so, fix such a probability measure  $\mu$ . Integrating the inequality (33) over  $\mu(dx)$ , for all  $T \geq \frac{1}{\gamma} \log(2C\mu(\psi_1)/\mu(\eta))$ ,

$$\frac{e^{\lambda_0 t} \mathbb{E}_\mu(\mathbb{1}_{\Gamma, \tau_\partial > t} \eta(X_t)) - C e^{\lambda_0 t} \mu P_t \psi_1 e^{-\gamma(T-t)}}{\mu(\eta) + C\mu(\psi_1)e^{-\gamma T}} \leq \mathbb{P}_\mu(\Gamma | \tau_\partial > T) \leq \frac{e^{\lambda_0 t} \mathbb{E}_\mu(\mathbb{1}_{\Gamma, \tau_\partial > t} \eta(X_t)) + C e^{\lambda_0 t} \mu P_t \psi_1 e^{-\gamma(T-t)}}{\mu(\eta) - C\mu(\psi_1)e^{-\gamma T}}.$$

Since  $1/(1-x) \leq 1+2x$  for all  $x \in (0, 1/2]$ , for all  $T \geq \log\left(\frac{2C\mu(\psi_1)}{\mu(\eta)}\right)$ ,

$$\begin{aligned} & \frac{e^{\lambda_0 t} \mathbb{E}_\mu(\mathbb{1}_{\Gamma, \tau_\partial > t} \eta(X_t)) + C e^{\lambda_0 t} \mu P_t \psi_1 e^{-\gamma(T-t)}}{\mu(\eta) - C\mu(\psi_1)e^{-\gamma T}} \\ & \leq \left( \frac{e^{\lambda_0 t} \mathbb{E}_\mu(\mathbb{1}_{\Gamma, \tau_\partial > t} \eta(X_t))}{\mu(\eta)} + C \frac{e^{\lambda_0 t} \mu P_t \psi_1 e^{-\gamma(T-t)}}{\mu(\eta)} \right) \left( 1 + 2C \frac{\mu(\psi_1) e^{-\gamma T}}{\mu(\eta)} \right) \\ & \leq \frac{e^{\lambda_0 t} \mathbb{E}_\mu(\mathbb{1}_{\Gamma, \tau_\partial > t} \eta(X_t))}{\mu(\eta)} + 2C \frac{\mu(\psi_1)}{\mu(\eta)} e^{-\gamma T} + 2C \frac{e^{\lambda_0 t} \mu P_t \psi_1 e^{-\gamma(T-t)}}{\mu(\eta)} \\ & \leq \frac{e^{\lambda_0 t} \mathbb{E}_\mu(\mathbb{1}_{\Gamma, \tau_\partial > t} \eta(X_t))}{\mu(\eta)} + 2C(\alpha(\psi_1) + C + 1) \frac{\mu(\psi_1) e^{-\gamma(T-t)}}{\mu(\eta)}, \end{aligned}$$

where we used (34). In the same vein, there exists a constant  $C' > 0$  such that, for all  $T \geq \frac{1}{\gamma} \log(2C\mu(\psi_1)/\mu(\eta))$ ,

$$\frac{e^{\lambda_0 t} \mathbb{E}_\mu(\mathbb{1}_{\Gamma, \tau_\partial > t} \eta(X_t)) - C e^{\lambda_0 t} \mu P_t \psi_1 e^{-\gamma(T-t)}}{\mu(\eta) + C e^{\lambda_0 t} \mu P_t \psi_1 e^{-\gamma(T-t)}} \geq \frac{e^{\lambda_0 t} \mathbb{E}_\mu(\mathbb{1}_{\Gamma, \tau_\partial > t} \eta(X_t))}{\mu(\eta)} - C' \frac{\mu(\psi_1) e^{-\gamma(T-t)}}{\mu(\eta)}.$$

In conclusion, there exists a constant  $C' > 0$  such that, for all  $\mu \in \mathcal{M}_1(E)$  such that  $\mu(\psi_1) < +\infty$  and  $\mu(\eta) > 0$ ,  $t \geq 0$ ,  $\Gamma \in \mathcal{F}_t$  and  $T \geq t \vee \frac{1}{\gamma} \log(2C\mu(\psi_1)/\mu(\eta))$ ,

$$|\mathbb{P}_\mu(\Gamma | \tau_\partial > T) - \mathbb{Q}_{\eta \circ \mu}(\Gamma)| \leq C' \frac{\mu(\psi_1) e^{-\gamma(T-t)}}{\mu(\eta)}.$$

To generalize this inequality for all  $T \geq t$ , it is enough to remark that, if  $t \leq T$  and  $T \leq \frac{1}{\gamma} \log(2C\mu(\psi_1)/\mu(\eta))$ , then for all  $\Gamma \in \mathcal{F}_t$ ,

$$|\mathbb{P}_\mu(\Gamma | \tau_\partial > t) - \mathbb{Q}_{\eta \circ \mu}(\Gamma)| \leq 2 \leq 2 \times 2C \frac{\mu(\psi_1)}{\mu(\eta)} \times e^{-\gamma(T-t)},$$

which concludes the proof.  $\square$



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