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A central limit and Berry-Esseen theorem for continuous-time Markov processes conditioned not to be absorbed

William Oçafrain*

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Abstract

This paper aims to establish a central limit and Berry-Esseen-like theorem for Markov processes conditioned not to be absorbed. First, we prove that a central limit theorem holds true for the Q -process, under general criteria on its exponential ergodicity. Then, we prove that the Kolmogorov distance between the conditional distribution of the renormalized centered empirical mean for the absorbed process and the one for the Q -process decays as $1/\sqrt{t}$.

Key words: Quasi-stationary distribution; Quasi-stationarity; Quasi-ergodic distribution; Central limit theorem; Berry-Esseen theorem; Q-process.

Notation

- $\mathcal{M}_1(E)$: Set of the probability measures defined on E .
- For any $\mu \in \mathcal{M}_1(E)$ and measurable function f such that $\int_E f(x)\mu(dx)$ is well-defined,

$$\mu(f) := \int_E f(x)\mu(dx).$$

- For a given positive function ψ , $\mathbb{L}^\infty(\psi)$ is the set of functions f such that f/ψ is bounded, endowed with the norm

$$\|f\|_{\mathbb{L}^\infty(\psi)} := \|f/\psi\|_\infty.$$

- For any positive measurable function ψ , for any $\mu, \nu \in \mathcal{M}_1(E)$,

$$\|\mu - \nu\|_\psi := \sup_{\|f\|_{\mathbb{L}^\infty(\psi)} \leq 1} |\mu(f) - \nu(f)|.$$

- For any nonnegative measurable function f and $\mu \in \mathcal{M}_1(E)$ such that $\mu(f) \in (0, +\infty)$,

$$f \circ \mu(dx) := \frac{f(x)\mu(dx)}{\mu(f)}.$$

- Kolmogorov distance: For any $\mu, \nu \in \mathcal{M}_1(\mathbb{R})$,

$$d_{\text{Kolm}}(\mu, \nu) := \sup_{x \in \mathbb{R}} |\mu((-\infty, x]) - \nu((-\infty, x])|.$$

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1 Introduction

1.1 Introduction to quasi-stationarity

Let $(X_t)_{t \geq 0}$ be a continuous-time Markov process living on a state space $(E \cup \{\partial\}, \mathcal{E})$, where $\partial \notin E$ is an absorbing state for the process X , which means that $X_t = \partial$ conditioned to $\{X_s = \partial\}$ for all $s \leq t$, and \mathcal{E} is a σ -field associated to the state space E . Denote by τ_∂ the hitting time of ∂ by the process X . We associate to the process X a family of probability measure $(\mathbb{P}_x)_{x \in E \cup \{\partial\}}$ such that $\mathbb{P}_x[X_0 = x] = 1$ for any $x \in E \cup \{\partial\}$. For any probability measure $\mu \in \mathcal{M}_1(E \cup \{\partial\})$, define $\mathbb{P}_\mu := \int_{E \cup \{\partial\}} \mu(dx) \mathbb{P}_x$, and denote \mathbb{E}_x and \mathbb{E}_μ the associated expectations. Moreover, denote by $(\mathcal{F}_t)_{t \geq 0}$ the natural filtration of the process X .

In this paper, we assume that the process X admits a *quasi-stationary distribution*, defined as a probability measure $\alpha \in \mathcal{M}_1(E)$ such that, for all $t \geq 0$,

$$\mathbb{P}_\alpha[X_t \in \cdot | \tau_\partial > t] = \alpha. \quad (1)$$

Such a probability measure is also a *quasi-limiting distribution*, defined as a probability measure such that there exists a subset $\mathcal{D}(\alpha) \subset \mathcal{M}_1(E)$, called *domain of attraction of α* , such that, for all $\mu \in \mathcal{D}(\alpha)$ and $A \in \mathcal{E}$,

$$\mathbb{P}_\mu[X_t \in A | \tau_\partial > t] \xrightarrow[t \rightarrow \infty]{} \alpha(A).$$

In particular, if α is a quasi-stationary distribution, $\alpha \in \mathcal{D}(\alpha)$ by (1). Conversely, we can show that any quasi-limiting distributions for X satisfy (1) for all $t \geq 0$ (see [25, Proposition 1]). In other terms, quasi-stationary and quasi-limiting distributions are equivalent notions.

Denote by $\lambda_0 := -\log(\mathbb{P}_\alpha[\tau_\partial > 1])$. Then, it is well-known (see [25, Proposition 2] for example) that, for all $t \geq 0$,

$$\mathbb{P}_\alpha[\tau_\partial > t] = e^{-\lambda_0 t}, \quad \forall t \geq 0.$$

A consequence of this property coupled with (1) is that, for all $t \geq 0$,

$$\mathbb{P}_\alpha[X_t \in \cdot, \tau_\partial > t] = e^{-\lambda_0 t} \alpha(\cdot). \quad (2)$$

Conversely, if a probability measure α satisfies (2) for a given $\lambda_0 > 0$, then α is a quasi-stationary distribution for the process X . In that respect, the quasi-stationary distributions for X are exactly the probability left eigenmeasures for the semigroup $(P_t)_{t \geq 0}$ defined by

$$P_t f(x) := \mathbb{E}_x(f(X_t) \mathbb{1}_{\tau_\partial > t}),$$

for all $t \geq 0$, f belonging to a Banach space and $x \in E$. In what follows, we will use the notation

$$\mu P_t := \mathbb{P}_\mu(X_t \in \cdot, \tau_\partial > t).$$

Also, we assume that the process X admits a nonnegative function η defined on E , vanishing at ∂ and satisfying $\alpha(\eta) = 1$, such that, for all $x \in E$ and $t \geq 0$,

$$\mathbb{E}_x[\eta(X_t) \mathbb{1}_{\tau_\partial > t}] = e^{-\lambda_0 t} \eta(x).$$

η is therefore a right eigenfunction for the semigroup $(P_t)_{t \geq 0}$, associated to the eigenvalues $(e^{-\lambda_0 t})_{t \geq 0}$.

1.2 The main assumption and the Q -process

The main assumption on this process is the following.

Assumption 1. *There exists a function $\psi_1 : E \rightarrow [1, +\infty)$, such that $\alpha(\psi_1) < +\infty$ and $\eta \in \mathbb{L}^\infty(\psi_1)$, as well as two constants $C, \gamma > 0$ such that, for any $\mu \in \mathcal{M}_1(E)$ and $t \geq 0$,*

$$\|e^{\lambda_0 t} \mu P_t - \mu(\eta)\alpha\|_{\psi_1} \leq C\mu(\psi_1)e^{-\gamma t}. \quad (3)$$

This assumption is satisfied under the general criteria Assumption (F) of [8] or Assumption (G) of [10]. In particular, it is shown in [8] that Assumption 1 is satisfied for a lot of processes such as multidimensional elliptic diffusion processes or processes defined in discrete state space. In particular, we refer the reader to [8, Sections 4 and 5] for examples for which Assumption 1 holds true. Assumption 1 is also satisfied for general strongly Feller processes, as shown in [15], and for some degenerate diffusion processes, as studied in [3, 22]. We refer the reader to [7, 14, 29, 2, 27] for alternative criteria ensuring Assumption 1.

We can show that Assumption 1 is equivalent to the following one.

Assumption 2. *Denoting $E' := \{x \in E : \eta(x) > 0\}$, the family of probability measures $(\mathbb{Q}_x)_{x \in E'}$ defined by*

$$\mathbb{Q}_x(\Gamma) := \lim_{T \rightarrow \infty} \mathbb{P}_x(\Gamma | \tau_\partial > T), \quad \forall t \geq 0, \forall \Gamma \in \mathcal{F}_t,$$

is well-defined. Moreover, for all $x \in E'$, $t \geq 0$ and $\Gamma \in \mathcal{F}_t$,

$$\mathbb{Q}_x(\Gamma) = \mathbb{E}_x \left(\mathbb{1}_{\Gamma, \tau_\partial > t} e^{\lambda_0 t} \frac{\eta(X_t)}{\eta(x)} \right).$$

Under $(\mathbb{Q}_x)_{x \in E'}$, X is a Markov process on E' admitting $\beta(dx) := \eta(x)\alpha(dx)$ as an invariant probability measure and, for all $t \geq 0$ and $x \in E'$,

$$\|\mathbb{Q}_x(X_t \in \cdot) - \beta\|_{\frac{\psi_1}{\eta}} \leq C \frac{\psi_1(x)}{\eta(x)} e^{-\gamma t}, \quad (4)$$

where $C, \gamma > 0$ are the same constants as in (3). Moreover, for all $x \in E'$, $t \geq 0$, $T \geq t$ and $\Gamma \in \mathcal{F}_t$,

$$|\mathbb{Q}_x(\Gamma) - \mathbb{P}_x(\Gamma | \tau_\partial > T)| \leq C \frac{\psi_1(x)}{\eta(x)} e^{-\gamma(T-t)}. \quad (5)$$

Moreover, $\beta(\psi_1/\eta) < +\infty$.

Since the process X under $(\mathbb{Q}_x)_{x \in E'}$ is a Markov process, the family of operators $(Q_t)_{t \geq 0}$ defined by

$$Q_t f(x) := \mathbb{E}_x^{\mathbb{Q}}(f(X_t)), \quad \forall t \geq 0, \forall x \in E', \forall f \in \mathbb{L}^\infty(\psi_1/\eta),$$

where $\mathbb{E}_x^{\mathbb{Q}}$ is the expectation associated to \mathbb{Q}_x , is a semigroup. In the literature (see for example [8, Theorem 2.7]), the Markov process associated to this semigroup is called the Q -process.

One has more precisely that Assumption 1 implies Assumption 2 and that the property (4) implies Assumption 1. In particular, Assumption 1 (equivalently Assumption 2) is satisfied when the Q -process satisfies the assumptions 1 and 2 in [17]. Moreover, the inequality (4) implies, since $\eta \in \mathbb{L}^\infty(\psi_1)$, the existence of a constant $C > 0$ such that, for all $x \in E'$ and $t \geq 0$,

$$\|\delta_x Q_t - \beta\|_{TV} \leq C \frac{\psi_1(x)}{\eta(x)} e^{-\gamma t}, \quad (6)$$

where $\|\cdot\|_{TV}$ denotes the total variation norm.

1.3 The main result

A consequence of (5) (see for example [9] for a proof of this statement) is that

$$\mathbb{E}_\mu \left[\frac{1}{t} \int_0^t f(X_s) ds \middle| \tau_\partial > t \right] \xrightarrow[t \rightarrow \infty]{} \beta(f), \quad (7)$$

for all bounded measurable function f and $\mu \in \mathcal{M}_1(E)$ satisfying $\mu(\psi_1) < +\infty$ and $\mu(\eta) > 0$. A probability measure β satisfying (7) is called a *quasi-ergodic distribution*.

The aim of this paper is to prove a central limit and Berry-Esseen-like theorem for processes satisfying Assumption 1, conditioned not to be absorbed up to the time t . Existing results stating a conditional central limit theorem for absorbing discrete-time Markov chains can be found in [11, 24, 1, 28, 4, 18]. In particular, in [11, Section 3.6], it is stated that, for any Markov chain $(X_n)_{n \in \mathbb{Z}_+}$ defined on a finite state space $E \cup \{\partial\}$ (absorbed at ∂) whose the matrix $(\mathbb{P}_i(X_1 = j))_{i,j \in E}$ is irreducible and aperiodic, one has that, for all function f such that $\beta(f) = 0$, the limit

$$\theta^2 := \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_\alpha \left(\left(\sum_{k=0}^n f(X_k) \right)^2 \middle| \tau_\partial > n \right)$$

is well-defined. If moreover $\theta^2 \neq 0$, one obtains

$$\lim_{n \rightarrow \infty} \mathbb{P}_\alpha \left[\frac{1}{\sqrt{n}} \sum_{k=0}^n f(X_k) \leq y \middle| \tau_\partial > n \right] = \int_{-\infty}^y \frac{1}{\sqrt{2\pi\theta^2}} e^{-\frac{x^2}{2\theta^2}} dx,$$

for all $y \in \mathbb{R}$. Up to my knowledge, there does not exist any results on a Berry-Esseen-like theorem associated to these conditional central limit theorems.

The main result of this paper is then the following.

Theorem 1. *Assume that the process $(X_t)_{t \geq 0}$ satisfies Assumption 1 (or equivalently Assumption 2).*

Then,

- *for all $f \in \mathbb{L}^\infty(\mathbb{1}_E)$ such that $\sigma_f^2 > 0$ and $\mu \in \mathcal{M}_1(E)$ such that $\mu(\psi_1) < +\infty$ and $\mu(\eta) > 0$,*

$$\mathbb{P}_\mu \left(\sqrt{t} \left[\frac{1}{t} \int_0^t f(X_s) ds - \beta(f) \right] \in \cdot \middle| \tau_\partial > t \right) \xrightarrow[t \rightarrow +\infty]{w} \mathcal{N}(0, \sigma_f^2),$$

where w refers to the weak convergence of measures, where $\mathcal{N}(0, \sigma_f^2)$ refers to the centered Gaussian variable of variance

$$\sigma_f^2 := 2 \int_0^\infty \mathbf{Cov}_\beta^\mathbb{Q}(f(X_0) f(X_s)) ds, \quad (8)$$

where $\mathbf{Cov}_\beta^\mathbb{Q}$ refers to the covariance with respect to the probability measure $\mathbb{Q}_\beta := \int_{E'} \beta(dx) \mathbb{Q}_x$.

- *Moreover, there exists $C > 0$ (different from the constant stated in Assumptions 1 and 2) such that, for all $f \in \mathbb{L}^\infty(\mathbb{1}_E)$ such that $\beta(f) = 0$, $t > 0$ and $\mu \in \mathcal{M}_1(E)$ such that $\mu(\psi_1) < +\infty$ and $\mu(\eta) > 0$,*

$$d_{Kolm} \left(\mathbb{P}_\mu \left(\frac{1}{\sqrt{t}} \int_0^t f(X_s) ds \in \cdot \middle| \tau_\partial > t \right), \mathbb{Q}_{\eta \circ \mu} \left(\frac{1}{\sqrt{t}} \int_0^t f(X_s) ds \right) \right) \leq \frac{C \mu(\psi_1) / \mu(\eta)}{\sqrt{t}}. \quad (9)$$

In particular, (4) implies that $\sigma_f^2 < +\infty$ for any f bounded by 1, since, assuming without loss of generality that $\beta(f) = 0$, for all $k \geq 0$,

$$|\mathbb{E}_\beta^{\mathbb{Q}}(f(X_0)f(X_k))| = |\mathbb{E}_\beta^{\mathbb{Q}}(f(X_0)\mathbb{E}_{X_0}^{\mathbb{Q}}(f(X_k)))| \leq C\beta(\psi/\eta)e^{-\gamma k}. \quad (10)$$

Hence, by (9), there exists a Berry-Esseen theorem for the conditional probability measure $\mathbb{P}_\mu\left(\frac{1}{\sqrt{t}}\int_0^t f(X_s)ds \in \cdot \mid \tau_\partial > t\right)$ if and only if a Berry-Esseen theorem holds true for the Q -process. However, the technique of the moments, which is used in this paper, does not allow to prove a Berry-Esseen theorem for the Q -process satisfying Assumption 2. We refer then the reader to [23, 19, 5, 26] for Berry-Esseen-like theorems on discrete or continuous-time Markov processes. In particular, Theorem 1 and [23, Theorem 1.5] imply the following corollary.

Corollary 1. *Assume that the process $(X_t)_{t \geq 0}$ satisfies Assumption 1 (or equivalently Assumption 2) and that the Q -process is reversible and admits a spectral gap.*

Then there exists $C > 0$ such that, for all $f \in \mathbb{L}^\infty(\mathbb{1}_E)$ such that $\beta(f) = 0$ and $\sigma_f^2 > 0$, $t > 0$ and $\mu \ll \alpha$ such that $\mu(\psi_1) < +\infty$, $\mu(\eta) > 0$ and $\int_E \left(\frac{d\mu}{d\alpha}(x)\right)^2 \eta(x)\alpha(dx) < +\infty$,

$$\begin{aligned} d_{K olm} \left(\mathbb{P}_\mu \left(\frac{1}{\sqrt{t}} \int_0^t f(X_s) ds \in \cdot \mid \tau_\partial > t \right), \mathcal{N}(0, \sigma_f^2) \right) \\ \leq \frac{C}{\sqrt{t}} \left(\frac{\mu(\psi_1)}{\mu(\eta)} + \frac{1}{\mu(\eta)} \sqrt{\int_E \left(\frac{d\mu}{d\alpha}(x)\right)^2 \eta(x)\alpha(dx)} \frac{\|f\|_\infty}{\sigma_f} \right). \end{aligned}$$

This paper is only interested in processes conditioned not to be absorbed by absorbing states. Nevertheless, the following proofs can be adapted to general non-conservative semigroups satisfying Assumption 1. Some examples of such semigroups have been studied in [14, 2, 10, 30].

Theorem 1 will be proved at the last section. To prove it, we first need to show a central limit theorem for the Q -process satisfying (4). In particular, up to my knowledge, the papers dealing with central limit theorems for Markov processes require stronger hypotheses than (4). This central limit theorem allows then to conclude to a useful lemma, stated and proved in Section 3.

2 Central limit theorem for the Q -process

This first section aims to establish a central limit theorem for the Q -process. In the literature, central limit theorems for continuous-time Markov processes have been established in [20, 6, 23]. In particular, the papers [20, 6] made use of central limit theorems for martingales; the paper [23] used the Kato's theory applied to analytically perturbed operators.

In this paper, a central limit theorem will be proved for the Q -process studying the convergence of the moments of $\frac{1}{\sqrt{t}}\int_0^t f(X_s)ds$, for bounded functions f such that $\beta(f) = 0$ and $\sigma_f^2 > 0$. Up to my knowledge, this method to establish a central limit theorem for Markov processes is new and allows to weaken the needed assumptions on the ergodicity of the Q -process. In particular, we will show that the only hypothesis (4) allows to obtain a central limit theorem for the Q -process. However, this method is difficult to apply for discrete-time processes; we refer to [13, 12, 21, 16] for central limit theorems for discrete-time Markov chains.

In all what follows, for simplicity, we denote $\mathcal{B}_1(E)$ the set of the bounded by 1 measurable functions defined over E , and we define for all $x \in E'$

$$\psi(x) := \frac{\psi_1(x)}{\eta(x)}.$$

In this section, the following theorem will be proved.

Theorem 2. Under Assumption 1 (or equivalently Assumption 2), there exists two positive constants \tilde{C} and C_1 such that, for all $k \in \mathbb{Z}_+$, $\mu \in \mathcal{M}_1(E')$ such that $\mu(\psi) < +\infty$, $f \in \mathcal{B}_1(E')$ such that $\beta(f) = 0$ and $t > 0$,

$$\left| \mathbb{E}_\mu^{\mathbb{Q}} \left(\frac{1}{t^k} \left(\int_0^t f(X_s) ds \right)^{2k} \right) - \frac{(2k)!}{k!} \frac{\sigma_f^{2k}}{2^k} \right| \leq \tilde{C}^k C_1 \frac{k}{(k-1)!} \frac{\mu(\psi)}{t} \quad (11)$$

and

$$\lim_{t \rightarrow \infty} \mathbb{E}_\mu^{\mathbb{Q}} \left(\frac{1}{t^k \sqrt{t}} \left(\int_0^t f(X_s) ds \right)^{2k+1} \right) = 0.$$

In particular, for all $\mu \in \mathcal{M}_1(E')$ such that $\mu(\psi) < +\infty$ and $f \in \mathcal{B}_1(E')$ such that $\beta(f) = 0$ and $\sigma_f^2 > 0$,

$$\mathbb{Q}_\mu \left(\frac{1}{\sqrt{t}} \int_0^t f(X_s) ds \in \cdot \right) \xrightarrow[t \rightarrow \infty]{w} \mathcal{N}(0, \sigma_f^2).$$

Before proving Theorem 2, we need to prove the following lemmata.

Lemma 1. Denoting

$$\tilde{C} := \frac{2C\beta(\psi)}{\gamma} \vee \frac{2C}{\gamma} [C\gamma \vee 1 + \beta(\psi)],$$

where C and γ are the constants implied in (4), for all $f \in \mathcal{B}_1(E')$ such that $\beta(f) = 0$ and $\mu \in \mathcal{M}_1(E')$ and $s_2 \leq \dots \leq s_{2k}$,

$$\begin{aligned} & \left| \mathbb{E}_\mu^{\mathbb{Q}} \left(\left[\int_0^{s_2} f(X_{s_1}) ds_1 \right] f(X_{s_2}) \dots \left[\int_{s_{2k-2}}^{s_{2k}} f(X_{s_{2k-1}}) ds_{2k-1} \right] f(X_{s_{2k}}) \right) - \frac{\sigma_f^{2k}}{2^k} \right| \\ & \leq \tilde{C}^k \mu(\psi) \sum_{i=0}^{k-1} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})}, \quad (12) \end{aligned}$$

where $s_0 = 0$ by convention.

Proof. We prove it by induction on k . We begin by showing the case $k = 1$. For all $\mu \in \mathcal{M}_1(E')$ and $f \in \mathcal{B}_1(E')$ and $t \geq 0$,

$$\begin{aligned} \mathbb{E}_\mu^{\mathbb{Q}} \left(\left[\int_0^t f(X_s) ds \right] f(X_t) \right) &= \int_0^t \mathbb{E}_\mu^{\mathbb{Q}}(f(X_s) f(X_t)) ds \\ &= \int_0^t \mathbb{E}_{\mu_{Q_{t-s}}}^{\mathbb{Q}}(f(X_0) f(X_s)) ds, \quad (13) \end{aligned}$$

where we recall that $(Q_t)_{t \geq 0}$ is the semigroup for the Q -process. By (6), for all $x \in E'$ and $f \in \mathcal{B}_1(E')$ such that $\beta(f) = 0$,

$$|\mathbb{E}_x^{\mathbb{Q}}(f(X_0) f(X_s))| \leq \mathbb{E}_x^{\mathbb{Q}}(|\mathbb{E}_{X_0}^{\mathbb{Q}}(f(X_s))|) \leq C\psi(x) e^{-\gamma s}.$$

Hence, by (13), (4) and this last inequality, for all $t \geq 0$, $\mu \in \mathcal{M}_1(E')$ such that $\mu(\psi) < +\infty$ and $f \in \mathcal{B}_1(E')$ such that $\beta(f) = 0$,

$$\begin{aligned} \left| \mathbb{E}_\mu^{\mathbb{Q}} \left(\left[\int_0^t f(X_s) ds \right] f(X_t) \right) - \int_0^t \mathbb{E}_\beta^{\mathbb{Q}}(f(X_0) f(X_s)) ds \right| &\leq C\mu(\psi) \int_0^t e^{-\gamma(t-s)} \|\mathbb{E}^{\mathbb{Q}}(f(X_0) f(X_s))\|_{\mathbb{L}^\infty(\psi)} ds \\ &\leq C\mu(\psi) \int_0^t e^{-\gamma(t-s)} C e^{-\gamma s} ds \\ &\leq C^2 \mu(\psi) t e^{-\gamma t}. \quad (14) \end{aligned}$$

Moreover, since $\beta(f) = 0$, by (10), for all $t \geq 0$,

$$\left| \int_t^\infty \mathbb{E}_\beta^\mathbb{Q}(f(X_0)f(X_s))ds \right| \leq \int_t^\infty C\beta(\psi)e^{-\gamma s}ds \leq \frac{C\beta(\psi)}{\gamma}e^{-\gamma t}. \quad (15)$$

Hence, by the definition of $\tilde{C} > 0$ in the statement of the lemma,

$$\left| \mathbb{E}_\mu^\mathbb{Q} \left(\left[\int_0^t f(X_s)ds \right] f(X_t) \right) - \frac{\sigma_f^2}{2} \right| \leq \tilde{C}\mu(\psi)(t+1)e^{-\gamma t}.$$

This concludes the base case.

Let $k-1 \in \mathbb{N}$ be such that the hypothesis of induction is satisfied. Then, by the Markov property,

$$\begin{aligned} & \mathbb{E}_\mu^\mathbb{Q} \left(\int_0^{s_2} f(X_{s_1})ds_1 f(X_{s_2}) \dots \int_{s_{2k-2}}^{s_{2k}} f(X_{s_{2k-1}})ds_{2k-1} f(X_{s_{2k}}) \right) \\ &= \mathbb{E}_\mu^\mathbb{Q} \left(\int_0^{s_2} f(X_{s_1})ds_1 f(X_{s_2}) \mathbb{E}_{X_{s_2}}^\mathbb{Q} \left(\int_{s_2}^{s_4} f(X_{s_3-s_2})ds_3 f(X_{s_4-s_2}) \dots \int_{s_{2k-2}}^{s_{2k}} f(X_{s_{2k-1}-s_2})ds_{2k-1} f(X_{s_{2k}-s_2}) \right) \right) \\ &= \mathbb{E}_\mu^\mathbb{Q} \left(\int_0^{s_2} f(X_{s_1})ds_1 f(X_{s_2}) \mathbb{E}_{X_{s_2}}^\mathbb{Q} \left(\int_0^{s_4-s_2} f(X_{s_3})ds_3 f(X_{s_4-s_2}) \dots \int_{s_{2k-2}-s_2}^{s_{2k}-s_2} f(X_{s_{2k-1}})ds_{2k-1} f(X_{s_{2k}-s_2}) \right) \right). \end{aligned} \quad (16)$$

By hypothesis, for all $s_2 \leq s_4 \dots \leq s_{2k}$,

$$\begin{aligned} & \left| \mathbb{E}_{X_{s_2}}^\mathbb{Q} \left(\int_0^{s_4-s_2} f(X_{s_3-s_2})ds_3 f(X_{s_4-s_2}) \dots \int_{s_{2k-2}-s_2}^{s_{2k}-s_2} f(X_{s_{2k-1}})ds_{2k-1} f(X_{s_{2k}-s_2}) \right) - \frac{\sigma_f^{2k-2}}{2^{k-1}} \right| \\ & \leq \tilde{C}^{k-1}\psi(X_{s_2}) \sum_{i=1}^{k-1} s_2(s_{2(i+1)} - s_{2i} + 1)e^{-\gamma(s_{2(i+1)} - s_{2i})}. \end{aligned} \quad (17)$$

Moreover, since $\beta(f) = 0$, for all $\mu \in \mathcal{M}_1(E')$, for all $s_2 \geq 0$,

$$\left| \mathbb{E}_\mu^\mathbb{Q} \left(\int_0^{s_2} f(X_s)ds f(X_{s_2})\psi(X_{s_2}) \right) \right| \leq \frac{C}{\gamma}(1 + \beta(\psi))\mu(\psi), \quad (18)$$

where C is the constant implied in (4). As a matter of fact, for all $t \geq 0$, $\mu \in \mathcal{M}_1(E')$ and $f, g \in \mathbb{L}^\infty(\psi)$,

$$\int_0^t \mathbb{E}_\mu^\mathbb{Q} [f(X_s)g(X_t)] ds = \int_0^t \mathbb{E}_\mu^\mathbb{Q} [f(X_s)\mathbb{E}_{X_s}(g(X_{t-s}))] ds.$$

Thus, by (4), for all $t \geq 0$, $\mu \in \mathcal{M}_1(E')$, $f \in \mathcal{B}_1(E')$ and $g \in \mathbb{L}^\infty(\psi)$,

$$\begin{aligned} \left| \int_0^t \mathbb{E}_\mu^\mathbb{Q} [f(X_s)g(X_t)] ds - \int_0^t \mathbb{E}_\mu^\mathbb{Q} [f(X_s)] \beta(g) ds \right| & \leq C\|g\|_{\mathbb{L}^\infty(\psi)} \int_0^t \mu(\psi)e^{-\gamma(t-s)} ds \\ & \leq \frac{C}{\gamma}\|g\|_{\mathbb{L}^\infty(\psi)}\mu(\psi). \end{aligned}$$

Moreover, again by (4), for all $t \geq 0$, $\mu \in \mathcal{M}_1(E')$ and $f \in \mathcal{B}_1(E')$ such that $\beta(f) = 0$,

$$\left| \int_0^t \mathbb{E}_\mu^\mathbb{Q} [f(X_s)] ds \right| \leq C\mu(\psi) \int_0^t e^{-\gamma s} ds \leq \frac{C}{\gamma}\mu(\psi).$$

These two last inequalities applied to $g = f \times \psi$ imply (18).

Hence, by (16), (17) and (18),

$$\begin{aligned} & \left| \mathbb{E}_\mu^\mathbb{Q} \left(\int_0^{s_2} f(X_{s_1}) ds_1 f(X_{s_2}) \dots \int_{s_{2k-2}}^{s_{2k}} f(X_{2k-1}) ds_{2k-1} f(X_{s_{2k}}) \right) - \frac{\sigma_f^{2(k-1)}}{2^{k-1}} \mathbb{E}_\mu^\mathbb{Q} \left(\int_0^{s_2} f(X_{s_1}) ds_1 f(X_{s_2}) \right) \right| \\ & \leq \tilde{C}^{k-1} \sum_{i=1}^{k-1} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})} \mathbb{E}_\mu^\mathbb{Q} \left(\int_0^{s_2} f(X_s) ds f(X_{s_2}) \psi(X_{s_2}) \right) \\ & \leq \tilde{C}^{k-1} \frac{C}{\gamma} (1 + \beta(\psi)) \mu(\psi) \sum_{i=1}^{k-1} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})}. \end{aligned}$$

This and the case $k = 1$ conclude the induction. \square

We need also the following lemma.

Lemma 2. *For all $k \in \mathbb{N}$, there exists $C_k \in (0, +\infty)$ such that, for $t \geq 1$,*

$$\int_{0 \leq s_2 \dots \leq s_{2k} \leq t} \sum_{i=0}^{k-1} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})} ds_2 \dots ds_{2k} \leq C_k t^{k-1}. \quad (19)$$

Proof. We prove (19) by induction on k . The case $k = 1$ can easily be obtained by the reader for a given constant $C_1 > 0$. Now, assume that (19) holds true for $k - 1 \in \mathbb{N}$. For all $t \geq 0$,

$$\begin{aligned} & \sum_{i=0}^{k-1} \int_{0 \leq s_2 \dots \leq s_{2k} \leq t} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})} ds_2 \dots ds_{2k} \\ & = \int_{0 \leq s_2 \dots \leq s_{2k} \leq t} \sum_{i=0}^{k-2} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})} ds_2 \dots ds_{2k} \\ & \quad + \int_{0 \leq s_2 \dots \leq s_{2k} \leq t} (s_{2k} - s_{2k-1} + 1) e^{-\gamma(s_{2k} - s_{2k-1})} ds_2 \dots ds_{2k} \\ & = \int_0^t \left[\int_{0 \leq s_2 \dots \leq s_{2(k-1)} \leq s_{2k}} \sum_{i=0}^{k-2} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})} ds_2 \dots ds_{2(i-1)} \right] ds_{2k} \\ & \quad + \int_{0 \leq s_2 \dots \leq s_{2k} \leq t} (s_{2k} - s_{2(k-1)} + 1) e^{-\gamma(s_{2k} - s_{2(k-1)})} ds_2 \dots ds_{2k}. \quad (20) \end{aligned}$$

By hypothesis, for all $t \geq 0$,

$$\int_0^t \left[\int_{0 \leq s_2 \dots \leq s_{2(k-1)} \leq s_{2k}} \sum_{i=0}^{k-2} (s_{2(i+1)} - s_{2i} + 1) e^{-\gamma(s_{2(i+1)} - s_{2i})} ds_2 \dots ds_{2(i-1)} \right] ds_{2k} \leq \int_0^t C_{k-1} s_{2k}^{k-2} ds_{2k} = C_{k-1} \frac{t^{k-1}}{k-1}.$$

For all $t \geq 0$, the second term of (20) is equal to

$$\begin{aligned} & \int_{0 \leq s_{2(k-1)} \leq s_{2k} \leq t} (s_{2k} - s_{2(k-1)} + 1) e^{-\gamma(s_{2k} - s_{2(k-1)})} \left[\int_{0 \leq s_2 \dots \leq s_{2k-1}} ds_2 \dots ds_{2(k-2)} \right] ds_{2k-1} ds_{2k} \\ & = \int_{0 \leq r \leq s \leq t} (s - r + 1) e^{-\gamma(s-r)} \frac{r^{k-2}}{(k-2)!} dr ds \\ & = \int_0^t \left[\int_r^t (s - r + 1) e^{-\gamma(s-r)} ds \right] \frac{r^{k-2}}{(k-2)!} dr \\ & = \int_0^t \left(\int_0^{t-r} (u + 1) e^{-\gamma u} du \right) \frac{r^{k-2}}{(k-2)!} dr \leq \frac{C_1}{(k-1)!} t^{k-1}, \end{aligned}$$

where $C_1 < +\infty$ is exactly the same constant as for the case $k = 1$. Hence, (19) is proved with C_k satisfying the relation $C_k = \frac{C_{k-1}}{k-1} + \frac{C_1}{(k-1)!}$. By induction, for all $k \geq 2$,

$$C_k = \frac{C_1}{(k-1)!} + \frac{C_1}{(k-2)!}.$$

□

We can now prove Theorem 2.

Proof of Theorem 2. We begin by the convergence of the even moment. For all $\mu \in \mathcal{M}_1(E')$, $t \geq 0$, $f \in \mathcal{B}_1(E')$ and $k \in \mathbb{N}$,

$$\begin{aligned} & \mathbb{E}_\mu^{\mathbb{Q}} \left(\left(\int_0^t f(X_s) ds \right)^{2k} \right) \\ &= (2k)! \int_{0 \leq s_2 \leq \dots \leq s_{2k} \leq t} \int_0^{s_2} \int_{s_2}^{s_4} \dots \int_{s_{2k-2}}^{s_{2k}} \mathbb{E}_\mu^{\mathbb{Q}}(f(X_{s_1})f(X_{s_2}) \dots f(X_{s_{2k-1}})f(X_{s_{2k}})) ds_1 \dots ds_{2k}. \end{aligned} \quad (21)$$

Then, assuming moreover that $\beta(f) = 0$, by (21),(12) and (19),

$$\left| \mathbb{E}_\mu^{\mathbb{Q}} \left(\left(\int_0^t f(X_s) ds \right)^{2k} \right) - \frac{(2k)!}{k!} t^k \frac{\sigma_f^{2k}}{2^k} \right| \leq \tilde{C}^k \mu(\psi) \times C_k t^{k-1}, \quad (22)$$

which implies (11). Now, for all $\mu \in \mathcal{M}_1(E')$, $t \geq 0$, $k \in \mathbb{Z}_+$ and $f \in \mathcal{B}_1(E')$ such that $\beta(f) = 0$,

$$\begin{aligned} & \mathbb{E}_\mu^{\mathbb{Q}} \left(\left(\int_0^t f(X_s) ds \right)^{2k+1} \right) \\ &= (2k+1)! \int_0^t \mathbb{E}_\mu^{\mathbb{Q}} \left(f(X_s) \mathbb{E}_{X_s}^{\mathbb{Q}} \left[\int_{0 \leq s_2 \leq \dots \leq s_{2k+1} \leq t-s} f(X_{s_2}) \dots f(X_{s_{2k+1}}) ds_2 \dots ds_{2k+1} \right] \right) ds \\ &= (2k+1) \int_0^t \mathbb{E}_\mu^{\mathbb{Q}} \left(f(X_s) \mathbb{E}_{X_s}^{\mathbb{Q}} \left(\left(\int_0^{t-s} f(X_u) du \right)^{2k} \right) \right) ds \\ &= (2k+1) \int_0^t \mathbb{E}_\mu^{\mathbb{Q}} \left(f(X_{t-s}) \mathbb{E}_{X_{t-s}}^{\mathbb{Q}} \left(\left(\int_0^s f(X_u) du \right)^{2k} \right) \right) ds. \end{aligned}$$

By (22) and using that $\mathbb{E}_\mu^{\mathbb{Q}}[\psi(X_{t-s})] \leq (\beta(\psi) \|\eta\|_{\mathbb{L}^\infty(\psi_1)} + C) \mu(\psi)$ for all $\mu \in \mathcal{M}_1(E')$ and $s \leq t$ (this is a consequence of (4)), there exists $\hat{C} > 0$ such that

$$\left| \mathbb{E}_\mu^{\mathbb{Q}} \left(\left(\int_0^t f(X_s) ds \right)^{2k+1} \right) - \frac{(2k+1)!}{k!} \frac{\sigma_f^{2k}}{2^k} \int_0^t s^k \mathbb{E}_\mu^{\mathbb{Q}}(f(X_{t-s})) ds \right| \leq \tilde{C}^k \hat{C} \frac{k(2k+1)}{(k-1)!} \mu(\psi) \frac{t^k}{k}. \quad (23)$$

Since $\beta(f) = 0$, by (4), for all $\mu \in \mathcal{M}_1(E')$ and $s \leq t$,

$$|\mathbb{E}_\mu^{\mathbb{Q}}[f(X_{t-s})]| \leq C \mu(\psi) e^{-\gamma(t-s)}. \quad (24)$$

For all $t > 0$ and $k \in \mathbb{Z}_+$,

$$\frac{1}{t^{k+\frac{1}{2}}} \int_0^t s^k e^{-\gamma(t-s)} ds = \frac{e^{-\gamma t}}{t^{k+1/2}} \int_0^t s^k e^{\gamma s} ds \leq \frac{e^{-\gamma t}}{\sqrt{t}} \int_0^t e^{\gamma s} ds \leq \frac{1}{\gamma \sqrt{t}}. \quad (25)$$

We deduce from (23), (24) and (25) that there exists $\hat{C} > 0$ (different from the previous one) such that, for all $\mu \in \mathcal{M}_1(E')$ such that $\mu(\psi) < +\infty$ and $f \in \mathcal{B}_1(E')$ such that $\beta(f) = 0$,

$$\left| \frac{1}{t^{k+1/2}} \mathbb{E}_\mu^{\mathbb{Q}} \left(\left(\int_0^t f(X_s) ds \right)^{2k+1} \right) \right| \leq \tilde{C}^k \times \hat{C} \left[\frac{(2k+1)!}{2^k k!} + \frac{2k+1}{(k-1)!} \right] \frac{\mu(\psi)}{\sqrt{t}}. \quad (26)$$

The central limit theorem for the Q -process is deduced from Lévy's continuity theorem. \square

3 A useful lemma for Theorem 1

The aim of this section is to prove the following result, which can be seen as an improved central limit theorem for the Q -process.

Lemma 3. *Assume Assumption 1 (or equivalently Assumption 2). Then, for all $\mu \in \mathcal{M}_1(E')$ such that $\mu(\psi_1/\eta) < \infty$, $f \in \mathcal{B}_1(E')$ such that $\beta(f) = 0$ and $\sigma_f^2 > 0$ (as defined in (8)), and $\omega \in \mathbb{R}$,*

$$\lim_{t \rightarrow \infty} \sup_{g \in \mathbb{L}^\infty(\psi) : \|g\|_{\mathbb{L}^\infty(\psi)} \leq 1} \left| \mathbb{E}_\mu^{\mathbb{Q}} \left[e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} g(X_t) \right] - \beta(g) e^{-\frac{\sigma_f^2 \omega^2}{2}} \right| = 0, \quad (27)$$

where we recall that, for all $x \in E'$,

$$\psi(x) = \frac{\psi_1(x)}{\eta(x)}.$$

Proof of Lemma 3. For all $\mu \in \mathcal{M}_1(E')$, $f \in \mathcal{B}(E')$ such that $\beta(f) = 0$, $t \geq 0$, $k \in \mathbb{Z}_+$ and $g \in \mathbb{L}^\infty(\psi)$,

$$\begin{aligned} \mathbb{E}_\mu^{\mathbb{Q}} \left(\left(\int_0^t f(X_s) ds \right)^k g(X_t) \right) &= k \int_0^t \mathbb{E}_\mu^{\mathbb{Q}} \left(\left[\int_0^s f(X_u) du \right]^{k-1} f(X_s) g(X_t) \right) ds \\ &= k \int_0^t \mathbb{E}_\mu^{\mathbb{Q}} \left(\left[\int_0^s f(X_u) du \right]^{k-1} f(X_s) \mathbb{E}_{X_s}^{\mathbb{Q}}(g(X_{t-s})) \right) ds. \end{aligned}$$

Hence, for all $\mu \in \mathcal{M}_1(E')$, $t \geq 0$, $f \in \mathcal{B}_1(E')$ such that $\beta(f) = 0$, $k \in \mathbb{Z}_+$ and $g \in \mathbb{L}^\infty(\psi)$,

$$\begin{aligned} \mathbb{E}_\mu^{\mathbb{Q}} \left(\left(\int_0^t f(X_s) ds \right)^k g(X_t) \right) - \beta(g) \mathbb{E}_\mu^{\mathbb{Q}} \left(\left(\int_0^t f(X_s) ds \right)^k \right) \\ = k \int_0^t \mathbb{E}_\mu^{\mathbb{Q}} \left(\left(\int_0^s f(X_u) du \right)^{k-1} f(X_s) [\mathbb{E}_{X_s}^{\mathbb{Q}}(g(X_{t-s})) - \beta(g)] \right) ds. \end{aligned}$$

Thus, using that $e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} = \sum_{k=0}^{\infty} \frac{i^k \omega^k}{t^{k/2}} \left(\int_0^t f(X_s) ds \right)^k$ for all $t \geq 0$, $\omega \in \mathbb{R}$, and $f \in \mathcal{B}_1(E')$ such that $\beta(f) = 0$, then, using the above equality, for all $\mu \in \mathcal{M}_1(E')$ and

$g \in \mathbb{L}^\infty(\psi)$,

$$\begin{aligned}
& \mathbb{E}_\mu^\mathbb{Q} \left(e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} g(X_t) \right) - \beta(g) \mathbb{E}_\mu^\mathbb{Q} \left(e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} \right) \\
&= \sum_{k=0}^{\infty} \left(\frac{i\omega}{\sqrt{t}} \right)^k \frac{1}{k!} \left\{ \mathbb{E}_\mu^\mathbb{Q} \left(\left(\int_0^t f(X_s) ds \right)^k g(X_t) \right) - \beta(g) \mathbb{E}_\mu^\mathbb{Q} \left(\left(\int_0^t f(X_s) ds \right)^k \right) \right\} \\
&= \mathbb{E}_\mu^\mathbb{Q}(g(X_t)) - \beta(g) + \sum_{k=1}^{\infty} \left(\frac{i\omega}{\sqrt{t}} \right)^k \frac{1}{k!} \left\{ \mathbb{E}_\mu^\mathbb{Q} \left(\left(\int_0^t f(X_s) ds \right)^k g(X_t) \right) - \beta(g) \mathbb{E}_\mu^\mathbb{Q} \left(\left(\int_0^t f(X_s) ds \right)^k \right) \right\} \\
&= \mathbb{E}_\mu^\mathbb{Q}(g(X_t)) - \beta(g) + \frac{i\omega}{\sqrt{t}} \int_0^t \mathbb{E}_\mu^\mathbb{Q} \left(e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} f(X_s) [\mathbb{E}_{X_s}^\mathbb{Q}(g(X_{t-s})) - \beta(g)] \right) ds.
\end{aligned}$$

By (4) one has, for all $\mu \in \mathcal{M}_1(E')$, \mathbb{Q}_μ -almost surely and for all $s \leq t$ and $g \in \mathbb{L}^\infty(\psi)$,

$$|\mathbb{E}_{X_s}^\mathbb{Q}(g(X_{t-s})) - \beta(g)| \leq C \|g\|_{\mathbb{L}^\infty(\psi)} \frac{\psi_1(X_s)}{\eta(X_s)} e^{-\gamma(t-s)}.$$

Thus, for all $\mu \in \mathcal{M}_1(E')$, $t > 0$, $\omega \in \mathbb{R}$, $g \in \mathbb{L}^\infty(\psi)$ and $f \in \mathcal{B}_1(E')$ such that $\beta(f) = 0$,

$$\begin{aligned}
& \left| \mathbb{E}_\mu^\mathbb{Q} \left(e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} g(X_t) \right) - \beta(g) \mathbb{E}_\mu^\mathbb{Q} \left(e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} \right) \right| \\
& \leq C \mu(\psi) \|g\|_{\mathbb{L}^\infty(\psi)} e^{-\gamma t} + \frac{C \|g\|_{\mathbb{L}^\infty(\psi)} \omega}{\sqrt{t}} \int_0^t C e^{-\gamma(t-s)} \mathbb{E}_\mu^\mathbb{Q}(\psi(X_s)) ds.
\end{aligned}$$

This inequality and Theorem 2 implies (27) and concludes the proof. \square

4 Proof of Theorem 1

We can now prove Theorem 1. The first result presented in the statement of this theorem is a natural consequence of (9) and Theorem 2. We focus therefore on proving the inequality (9).

For all $\mu \in \mathcal{M}_1(E)$, $t > 0$ and $f \in \mathcal{B}_1(E)$, $g \in \mathbb{L}^\infty(\psi_1)$ and $k \in \mathbb{N}$,

$$\begin{aligned}
& \mathbb{E}_\mu \left(\left[\int_0^t f(X_s) ds \right]^k g(X_t) \middle| \tau_\partial > t \right) = k! \int_{0 \leq s_1 \leq \dots \leq s_k \leq t} \mathbb{E}_\mu(f(X_{s_1}) f(X_{s_2}) \dots f(X_{s_k}) g(X_t) | \tau_\partial > t) ds_1 \dots ds_k \\
&= k! \int_0^t \mathbb{E}_\mu \left(\left[\int_{0 \leq s_1 \leq \dots \leq s_{k-1} \leq s} f(X_{s_1}) \dots f(X_{s_{k-1}}) ds_1 \dots ds_{k-1} \right] f(X_s) g(X_t) \middle| \tau_\partial > t \right) ds \\
&= k \int_0^t \mathbb{E}_\mu \left(\left[\int_0^s f(X_u) du \right]^{k-1} f(X_s) g(X_t) \middle| \tau_\partial > t \right) ds \\
&= k \int_0^t \frac{1}{\mathbb{P}_\mu(\tau_\partial > t)} \mathbb{E}_\mu \left(\left[\int_0^s f(X_u) du \right]^{k-1} f(X_s) \mathbb{E}_{X_s}(g(X_{t-s}) \mathbb{1}_{\tau_\partial > t-s}) \mathbb{1}_{\tau_\partial > s} \right) ds.
\end{aligned}$$

For all $s \leq t$, $\mu \in \mathcal{M}_1(E)$, $g \in \mathbb{L}^\infty(\psi_1)$ and $x \in E$, denote

$$C_{\mu,g}(s, t, x) := \frac{\mu(\eta)}{e^{\lambda_0 s}} e^{\gamma(t-s)} \left\{ \frac{\mathbb{E}_x(g(X_{t-s}) \mathbb{1}_{\tau_\partial > t-s})}{\mathbb{P}_\mu(\tau_\partial > t)} - \frac{e^{\lambda_0 s} \eta(x) \alpha(g)}{\mu(\eta)} \right\}.$$

By triangular inequality, for all $s \leq t$ and $x \in E$,

$$|C_{\mu,g}(s, t, x)| \leq \frac{\mu(\eta)}{e^{\lambda_0 s}} e^{\gamma(t-s)} \left\{ \left| \frac{\mathbb{E}_x[g(X_{t-s}) \mathbb{1}_{\tau_\partial > t-s}]}{\mathbb{P}_\mu(\tau_\partial > t)} - \frac{e^{-\lambda_0(t-s)} \eta(x) \alpha(g)}{\mathbb{P}_\mu(\tau_\partial > t)} \right| + \left| \frac{e^{-\lambda_0(t-s)} \eta(x) \alpha(g)}{\mathbb{P}_\mu(\tau_\partial > t)} - \frac{e^{\lambda_0 s} \eta(x) \alpha(g)}{\mu(\eta)} \right| \right\}. \quad (28)$$

By (3),

$$\frac{\mu(\eta)}{e^{\lambda_0 s}} e^{\gamma(t-s)} \left| \frac{\mathbb{E}_x[g(X_{t-s}) \mathbb{1}_{\tau_\partial > t-s}]}{\mathbb{P}_\mu(\tau_\partial > t)} - \frac{e^{-\lambda_0(t-s)} \eta(x) \alpha(g)}{\mathbb{P}_\mu(\tau_\partial > t)} \right| \leq C \|g\|_{\mathbb{L}^\infty(\psi_1)} \psi_1(x) \mu(\eta) \frac{e^{-\lambda_0 t}}{\mathbb{P}_\mu(\tau_\partial > t)}.$$

Again by (3),

$$e^{\lambda_0 t} \mathbb{P}_\mu(\tau_\partial > t) \geq \mu(\eta) - C \mu(\psi_1) e^{-\gamma t}.$$

Hence, for all $t \geq \frac{1}{\gamma} \log \left(\frac{2C\mu(\psi_1)}{\mu(\eta)} \right)$,

$$\begin{aligned} \frac{\mu(\eta)}{e^{\lambda_0 t} \mathbb{P}_\mu(\tau_\partial > t)} &\leq \frac{1}{1 - C \frac{\mu(\psi_1)}{\mu(\eta)} e^{-\gamma t}} \\ &\leq 1 + 2C \frac{\mu(\psi_1)}{\mu(\eta)} e^{-\gamma t} \leq 2. \end{aligned}$$

For the second part of the right-hand side of the inequality (28),

$$\begin{aligned} \frac{\mu(\eta)}{e^{\lambda_0 s}} e^{\gamma(t-s)} \left| \frac{e^{-\lambda_0(t-s)} \eta(x) \alpha(g)}{\mathbb{P}_\mu(\tau_\partial > t)} - \frac{e^{\lambda_0 s} \eta(x) \alpha(g)}{\mu(\eta)} \right| &= \eta(x) \alpha(g) e^{\gamma(t-s)} \left| \frac{\mu(\eta)}{e^{\lambda_0 t} \mathbb{P}_\mu(\tau_\partial > t)} - 1 \right| \\ &\leq C \eta(x) \alpha(g) e^{-\gamma s} \frac{C \mu(\psi_1)}{e^{\lambda_0 t} \mathbb{P}_\mu(\tau_\partial > t)} \\ &\leq C \eta(x) \alpha(g) 2C \frac{\mu(\psi_1)}{\mu(\eta)}. \end{aligned}$$

Hence, these inequalities and (28) imply the existence of a constant $C' > 0$ such that, for all $s \leq t$ such that $t \geq \frac{1}{\gamma} \log \left(\frac{2C\mu(\psi_1)}{\mu(\eta)} \right)$ and $x \in E$,

$$|C_{\mu,g}(s, t, x)| \leq C' \|g\|_{\mathbb{L}^\infty(\psi_1)} \left[\psi_1(x) + \frac{\mu(\psi_1)}{\mu(\eta)} \eta(x) \right]. \quad (29)$$

Thus, for all $\mu \in \mathcal{M}_1(E)$, $f \in \mathcal{B}_1(E)$ such that $\beta(f) = 0$, $g \in \mathbb{L}^\infty(\psi_1)$, $k \in \mathbb{N}$ and $t > 0$,

$$\begin{aligned} \mathbb{E}_\mu \left(\left[\int_0^t f(X_s) ds \right]^k g(X_t) \Big| \tau_\partial > t \right) - \alpha(g) \mathbb{E}_{\eta \circ \mu}^{\mathbb{Q}} \left(\left[\int_0^t f(X_s) ds \right]^k \right) \\ = k \times e^{-\gamma t} \int_0^t e^{\gamma s} \mathbb{E}_{\eta \circ \mu}^{\mathbb{Q}} \left(\left(\int_0^s f(X_u) du \right)^{k-1} \frac{f(X_s) C_{\mu,g}(s, t, X_s)}{\eta(X_s)} \right) ds. \end{aligned}$$

By an inequality presented in [23, Section 3], for all $\mu \in \mathcal{M}_1(E)$, $t > 0$, $f \in \mathcal{B}_1(E)$ such that $\beta(f) = 0$ and $x \in \mathbb{R}$, and for $W > 0$,

$$\begin{aligned} \left| \mathbb{P}_\mu \left(\frac{1}{\sqrt{t}} \int_0^t f(X_s) ds \leq x \Big| \tau_\partial > t \right) - \mathbb{Q}_{\eta \circ \mu} \left(\frac{1}{\sqrt{t}} \int_0^t f(X_s) ds \leq x \right) \right| \\ \leq \frac{1}{\pi} \int_{-W}^W \frac{|\mathbb{E}_\mu(e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} \Big| \tau_\partial > t) - \mathbb{E}_{\eta \circ \mu}^{\mathbb{Q}}(e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds})|}{|\omega|} d\omega + \frac{24}{\pi \sqrt{\pi} W}. \quad (30) \end{aligned}$$

Similarly to the proof of Lemma 3, for all $t \geq \frac{1}{\gamma} \log \left(\frac{2C\mu(\psi_1)}{\mu(\eta)} \right)$, $\omega \in \mathbb{R}$, $\mu \in \mathcal{M}_1(E)$ and f such that $\beta(f) = 0$,

$$\begin{aligned} & \mathbb{E}_\mu \left(e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} \middle| \tau_\partial > t \right) - \mathbb{E}_{\eta \circ \mu}^\mathbb{Q} \left[e^{\frac{i\omega}{\sqrt{t}} \int_0^t f(X_s) ds} \right] \\ &= \sum_{k=1}^{\infty} \left(\frac{i\omega}{\sqrt{t}} \right)^k \frac{1}{k!} \left\{ \mathbb{E}_\mu \left(\left[\int_0^t f(X_s) ds \right]^k \middle| \tau_\partial > t \right) - \mathbb{E}_{\eta \circ \mu}^\mathbb{Q} \left(\left[\int_0^t f(X_s) ds \right]^k \right) \right\} \\ &= \sum_{k=1}^{\infty} \frac{i^k \omega^k}{t^{k/2}} \frac{1}{(k-1)!} e^{-\gamma t} \int_0^t e^{\gamma s} \mathbb{E}_{\eta \circ \mu}^\mathbb{Q} \left(\left(\int_0^s f(X_u) du \right)^{k-1} \frac{f(X_s) C_{\mu, \mathbb{1}_E}(s, t, X_s)}{\eta(X_s)} \right) ds \\ &= e^{-\gamma t} \int_0^t e^{\gamma s} \frac{i\omega}{\sqrt{t}} \mathbb{E}_{\eta \circ \mu}^\mathbb{Q} \left(e^{\frac{i\omega}{\sqrt{s}} \int_0^s f(X_u) du} \frac{f(X_s) C_{\mu, \mathbb{1}_E}(s, t, X_s)}{\eta(X_s)} \right) ds. \quad (31) \end{aligned}$$

By (29), the family of functions $(f(\cdot) C_{\mu, \mathbb{1}_E}(s, t, \cdot) / \eta(\cdot))_{t \geq \frac{1}{\gamma} \log \left(\frac{2C\mu(\psi_1)}{\mu(\eta)} \right), s \leq t}$ is uniformly upper-bounded in $\mathbb{L}^\infty(\psi)$, as soon as $\mu(\psi_1) < +\infty$ and $\mu(\eta) > 0$. Under these conditions, Theorem 3 implies that

$$\left| \mathbb{E}_{\eta \circ \mu}^\mathbb{Q} \left(e^{\frac{i\omega}{\sqrt{s}} \int_0^s f(X_u) du} \frac{f(X_s) C_{\mu, \mathbb{1}_E}(s, t, X_s)}{\eta(X_s)} \right) - \alpha(f \times C_{\mu, \mathbb{1}_E}(s, t, \cdot)) e^{-\frac{\sigma_f^2 \omega^2}{2}} \right|_{s \rightarrow \infty, t \geq s} \longrightarrow 0.$$

Moreover, by (29), one has for all $s \geq 0$,

$$\limsup_{t \rightarrow \infty} |\alpha(f \times C_{\mu, \mathbb{1}_E}(s, t, \cdot))| \leq C' \alpha(\psi_1) + C' \frac{\mu(\psi_1)}{\mu(\eta)}.$$

Thus, by triangular inequality,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \left| e^{-\gamma t} \int_0^t e^{\gamma s} \mathbb{E}_{\eta \circ \mu}^\mathbb{Q} \left(e^{\frac{i\omega}{\sqrt{s}} \int_0^s f(X_u) du} \frac{f(X_s) C_{\mu, \mathbb{1}_E}(s, t, X_s)}{\eta(X_s)} \right) ds \right| \\ & \leq \limsup_{t \rightarrow \infty} e^{-\gamma t} \int_0^t e^{\gamma s} \left| \alpha(f \times C_{\mu, \mathbb{1}_E}(s, t, \cdot)) e^{-\frac{\sigma_f^2 \omega^2}{2}} \right| ds \leq \frac{1}{\gamma} \left(C' \alpha(\psi_1) + C' \frac{\mu(\psi_1)}{\mu(\eta)} \right) e^{-\frac{\sigma_f^2 \omega^2}{2}}. \quad (32) \end{aligned}$$

Thus, using (30), (31) and (32),

$$\begin{aligned} & \left| \mathbb{P}_\mu \left(\frac{1}{\sqrt{t}} \int_0^t f(X_s) ds \leq x \middle| \tau_\partial > t \right) - \mathbb{Q}_{\eta \circ \mu} \left(\frac{1}{\sqrt{t}} \int_0^t f(X_s) ds \leq x \right) \right| \\ & \leq \frac{1}{\gamma \pi} \int_{-W}^W \frac{C' \alpha(\psi_1) + C' \frac{\mu(\psi_1)}{\mu(\eta)}}{\sqrt{t}} e^{-\frac{\sigma_f^2 \omega^2}{2}} d\omega + \frac{24}{\pi \sqrt{\pi W}}. \end{aligned}$$

This proves Theorem 1.

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