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Some remarks on Standard Gradient Models and Gradient Plasticity

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Abstract The standard gradient models, originally introduced in the works of Gurtin [16] and Fremond [13], have been intensively studied in the last decades for various applications in plasticity, damage mechanics and multi-phase analysis. In this paper, these models are revisited and discussed in relation with some classical descriptions of solids such as Plasticity and Visco-plasticity. The constitutive equations of such a model and the governing equations for a solid have been initially derived by these authors from an extended virtual work principle. Without this starting point, it is shown here that these equations appear as a generalized Biot equation for the solid and can be obtained directly from the global expression of the energy and dissipation potentials. This result gives the possibility to write the governing equations for standard gradient models of any order. These models also appear as a simple generalization of classical descriptions in solid mechanics. As an example, for a time-independent process such as incremental plasticity, our attention is focussed on the constitutive modeling and on the governing equations of the response. It is shown that an elastic regularization can be introduced to avoid some theoretical and numerical difficulties which are well known in the classical theory of rigid plasticity.

1 Introduction

The introduction of the gradients of the state variables such as the strain, the internal parameter and even the temperature in solid mechanics has been much discussed in the literature since the pioneering works of Mindlin and Toupin on second-gradient elasticity. In the last decades, many gradient theories have been proposed for the macroscopic modeling of materials and structures, cf. for example [1, 4, 13, 25, 16, 26, 30, 9, 29]. For example, in plasticity and damage mechanics, a possibility to avoid the existence of local zones of zero width and mesh-dependence problems is to introduce the gradient of the plastic strain or the damage parameter in order to penalize possible sharp localizations. The introduction of the gradient of these internal parameters can be done in different ways, at the level of the plastic or damage yield values as well as at the expression of the driving forces, cf. [21] and to the quoted references therein.

In particular, the standard gradient models, originally introduced in the works of Gurtin [16] and Fremond [13] have been intensively studied for various applications in plasticity, damage mechanics and multi-phase analysis. The constitutive equations of such a model and the governing equations for a solid have been initially derived by their authors from an extended version of the virtual work principle.

In this paper, these models are revisited and discussed formally. It is shown here that these governing equations can be obtained directly, without the *a priori* assumption of the extended virtual work principle, from the formalism of generalized standard materials i.e. from the expressions of the global energy and dissipation potentials of the solid. These equations appear as a generalization of Biot equation and thus as a straightforward generalization of classical models of elasticity, visco-plasticity and plasticity following the expressions of the energy and the dissipation potentials. In particular, this result shows that the standard gradient models of any order can be easily introduced and the associated governing equations are formally Biot equation.

It is also interesting to investigate if the gradient terms can be advantageously introduced in the energy potential or in the dissipation potential. As an example, for a time-independent process such as incremental plasticity and brittle damage, our attention is focussed on the constitutive modeling and on the governing equations of the response. In particular, in gradient plasticity, it is shown that an elastic regularization can be introduced to avoid some mathematical difficulty due to the presence of gradient terms in the expression of the dissipation potential, as in the example of classical rigid plasticity. In this case, a model with gradient terms in the dissipation can be approached by a model with gradient terms in the energy with some additional variables.

2 Standard Gradient Models

In the internal variable framework, the thermo-mechanical response of a solid V in a reference configuration is described by the fields of displacement \mathbf{u} , of internal parameter ϕ and of temperature \mathbf{T} . The internal parameter is a scalar or a tensor and represent physically hidden parameters such as micro-displacements or phase proportions or anelastic strains, etc. The displacement u and its gradient ∇u are naturally associated with the notion of force f and of stress σ . In the same spirit, the internal parameter ϕ and its gradient $\nabla\phi$ are associated with the internal force X and the internal stress Y .

For a standard gradient model, the governing equations for a solid can be given in the following way cf. [16, 13] in an isothermal transformation. The principal ingredients of the model are:

2.1 The assumption of Generalized Forces and Virtual Work Equation

It is first postulated that the variables (u, ϕ) satisfy an extended virtual work principle in the sense that:

$$P_i + P_j = P_e \quad \forall \quad \delta u, \delta\phi, \quad (1)$$

with

$$\begin{cases} P_i = \int_V (\sigma \cdot \nabla \delta u + X \cdot \delta\phi + Y \cdot \nabla \delta\phi) dV \\ P_j = \int_V \rho \ddot{u} \cdot \delta u dV \\ P_e = \int_V (f_{vu} \cdot \delta u + f_{v\phi} \cdot \delta\phi) dV + \int_{\partial V} (f_{su} \cdot \delta u + f_{s\phi} \cdot \delta\phi) da, \end{cases} \quad (2)$$

where (f_{vu}, f_{su}) and $(f_{v\phi}, f_{s\phi})$ are respectively external body and surface forces associated with the displacement and the internal parameter. This means that the *mechanical equilibrium* equations hold for the stress

$$\begin{cases} \nabla \cdot \sigma + f_{vu} = \rho \ddot{u} & \forall x \in V \\ \sigma \cdot n = f_{su} & \forall x \in \partial V \end{cases} \quad (3)$$

and the following *constitutive equilibrium* equations hold for the internal parameter after intergration by parts

$$\begin{cases} \nabla \cdot Y - X + f_{v\phi} = 0 & \forall x \in V \\ Y \cdot n = f_{s\phi} & \forall x \in \partial V. \end{cases} \quad (4)$$

These equations are easily understood when ϕ is a micro-displacement, X is then an internal volume force and Y is a micro-stress in the same spirit as stress σ .

2.2 *The assumption of Energy and Dissipation Potentials*

Standard gradient models also assume that there exist an energy potential W and a dissipation potential D per unit reference volume. Such an assumption, quite familiar in standard plasticity and viscoplasticity, means that the following equations hold:

$$\left\{ \begin{array}{l} W = W(\nabla u, \phi, \nabla \phi), \quad D = D(\nabla \dot{u}, \dot{\phi}, \nabla \dot{\phi}, \phi), \\ \sigma = \sigma_e + \sigma_d, \quad \sigma_e = W_{,\nabla u}, \quad \sigma_d = D_{,\nabla \dot{u}}, \\ X = X_e + X_d, \quad X_e = W_{,\phi}, \quad X_d = D_{,\dot{\phi}}, \\ Y = Y_e + Y_d, \quad Y_e = W_{,\nabla \phi}, \quad Y_d = D_{,\nabla \dot{\phi}}, \end{array} \right. \quad (5)$$

when the potentials W and D are smooth functions of its arguments and the dissipation potential is assumed to be state-dependent via the current value of ϕ . The relationships $X_d = D_{,\dot{\phi}}$, $Y_d = D_{,\nabla \dot{\phi}}$ describe a time-dependent behaviour of the materials and are commonly discussed in visco-elasticity, visco-plasticity, in phase change as in damage mechanics. The dissipation potential can be state-dependent via the current value of the variable ϕ .

The case of non-smooth dissipation potentials is also interesting in Solids Mechanics. For example, D is a positive homogeneous of degree 1 for time-independent process such as friction, plasticity, brittle fracture and brittle damage. In this case, the dissipation potential is convex but not differentiable and its derivative with respect to the rates in (5) must be understood in the sense of sub-gradient cf. [27, 14]. The resulting relationships between dissipative forces and fluxes in (5) will be written in the last section.

2.3 *Governing Equations*

The equations (3), (4), (5) are the governing equations of a standard gradient model. In terms of the two potentials, the governing equations for the fields of unknown \mathbf{u} , Φ are

$$\left\{ \begin{array}{l} \nabla \cdot (W_{,\nabla u} + D_{,\nabla \dot{u}}) + f_{vu} = \rho \ddot{u} \quad \forall x \in V, \\ (W_{,\nabla u} + D_{,\nabla \dot{u}}) \cdot n = f_{su} \quad \forall x \in \partial V, \\ \nabla \cdot (W_{,\nabla \phi} + D_{,\nabla \dot{\phi}}) - W_{,\phi} - D_{,\dot{\phi}} + f_{v\phi} = 0 \quad \forall x \in V, \\ (W_{,\nabla \phi} + D_{,\nabla \dot{\phi}}) \cdot n = f_{s\phi} \quad \forall x \in \partial V. \end{array} \right. \quad (6)$$

These equations describe the response of the solid from an initial position of state and velocity. The forces $f_{v\phi}$ and $f_{s\phi}$ appears as physical data. In this spirit, the condition $f_{v\phi} = 0$ and $f_{s\phi} = 0$ has been denoted as the **constitutive insulation condition** following a terminology due to Polizzotto [30]. The response of a solid under insulation condition has been discussed by several authors, cf. [12, 30, 9, 22].

3 Generalized Standard Formalism

It is clear however that the proposed extended virtual work principle must be interpreted as a particular assumption on the system behaviour, in other words a constitutive equation for the solid and not a law of nature. It might be interesting to investigate if, without this starting assumption, the governing equations could also be derived in a different way. This section is devoted to a different approach to obtain directly the governing equation (6).

The existence of the local energy and dissipation potentials W , D suggests a global approach for the solid based upon the notion of global energy and dissipation potentials. Indeed, the solid V admits as energy and dissipation potentials:¹

$$\mathbf{W}(\mathbf{U}) = \int_V W(\nabla u, \phi, \nabla \phi) dV \quad , \quad \mathbf{D}(\dot{\mathbf{U}}, \mathbf{U}) = \int_V D(\dot{\phi}, \nabla \dot{\phi}, \phi) dV, \quad (7)$$

where $\mathbf{U} = (\mathbf{u}, \Phi)$ denotes the fields of displacement and internal parameter.

It is shown now that the governing equations (6) can be also derived directly from the formalism of generalized standard materials [18]. This formalism states that the dissipative forces, obtained from the expression of the dissipation in an energy balance, are also derived from the dissipation potential.

¹ Bold face uppercase letters as Φ or \mathbf{u} refer to fields whereas normal letters ϕ and u refer to local values.

3.1 Dissipation Analysis

Under the applied forces

$$\mathbf{F} \cdot \delta \mathbf{U} = \int_V f_{vu} \cdot \delta u \, dV + \int_{\partial V} f_{su} \cdot \delta u \, da, \quad (8)$$

and insulation condition, the dissipation of the solid is by definition the unrecoverable part of the received energy per unit time

$$D_V = \mathbf{F} \cdot \dot{\mathbf{U}} - \frac{d}{dt}(\mathbf{W}(\mathbf{U}) + \mathbf{K}_t), \quad (9)$$

where $\mathbf{K}_t = \int_V \rho/2\dot{u}^2 \, dV$ denotes the kinetic energy. It follows that

$$D_V = \mathbf{F} \cdot \dot{\mathbf{U}} - \mathbf{W}(\mathbf{U})_{,\mathbf{U}} \cdot \dot{\mathbf{U}} - \mathbf{J} \cdot \dot{\mathbf{U}} \geq 0, \quad (10)$$

where \mathbf{J} denotes the field of inertia force

$$\mathbf{J} \cdot \delta \mathbf{U} = \int_V \rho \ddot{u} \cdot \delta u \, dV.$$

3.2 Generalized Standard Formalism

The dissipation D_V is a product of the force \mathbf{F}_d and flux $\dot{\mathbf{U}}$. For any field of fluxes $\delta \mathbf{U}$ defined on V , the power of the dissipative forces \mathbf{F}_d is

$$\mathbf{F}_d \cdot \delta \mathbf{U} = \mathbf{F} \cdot \delta \mathbf{U} - \mathbf{W}_{,\mathbf{U}} \cdot \delta \mathbf{U} - \int_V \rho \ddot{u} \delta u \, dV. \quad (11)$$

The generalized standard formalism consists of admitting that the dissipative force must also satisfy the expression

$$\mathbf{F}_d \cdot \delta \mathbf{U} = \mathbf{D}_{,\dot{\mathbf{U}}} \cdot \delta \mathbf{U} \quad \forall \delta \mathbf{U}, \quad (12)$$

if a dissipation potential $\mathbf{D}_{,\dot{\mathbf{U}}}$ also exists. Thus

$$\mathbf{W}_{,\mathbf{U}} \cdot \delta \mathbf{U} + \mathbf{D}_{,\dot{\mathbf{U}}} \cdot \delta \mathbf{U} = \mathbf{F} \cdot \delta \mathbf{U} + \mathbf{J} \cdot \delta \mathbf{U}. \quad (13)$$

It follows that the local equations (6) are recovered from a classical argument (Haar lemma in Variational Calculus)²

² For any tensor fields F and G , the variational condition $\int_V (F \cdot \delta \phi + G : \nabla \delta \phi) \, dV = 0$ for all $\delta \phi$, which can be written as $\int_V (F - \nabla \cdot G) \cdot \delta \phi \, dV + \int_{\partial V} (G \cdot n) \cdot \delta \phi \, da = 0$ for all $\delta \phi$, implies after Haar lemma that $F - \nabla \cdot G = 0$ in V and that $G \cdot n = 0$ on ∂V

In particular, this discussion shows that the generalized standard formalism includes the assumption of extended virtual work equation.

3.3 Extended Biot Equation

It follows that the governing equations and associated boundary conditions are the local expressions of a global Biot equation

$$\mathbf{W},_{\mathbf{U}} + \mathbf{D},_{\dot{\mathbf{U}}} = \mathbf{F} + \mathbf{J}. \quad (14)$$

In this spirit, the presence of higher gradients of the internal parameter can also be taken into account. For example, if the expression of the energy includes the second gradient $w(\nabla u, \phi, \nabla \phi, \nabla \nabla \phi)$, the same approach leads to the following body equations for ϕ

$$W,_{\phi} + D,_{\dot{\phi}} - \nabla \cdot (W,_{\nabla \phi} + D,_{\nabla \dot{\phi}}) + \nabla \cdot \nabla \cdot (W,_{\nabla \nabla \phi} + D,_{\nabla \nabla \dot{\phi}}) = f_{v\phi} \quad \forall \quad x \in V \quad (15)$$

and to appropriate boundary conditions.

Finally, for a standard gradient or higher-gradient model, the body equations for the displacement and the internal parameter of the solid submitted to a loading path $\mathbf{F}(\mathbf{t}) = (\mathbf{f}_{\mathbf{v}\mathbf{u}}(\mathbf{t}), \mathbf{f}_{\mathbf{s}\mathbf{u}}(\mathbf{t}))$ are given by an extended expression of Biot equation [2] for all $x \in V$:

$$\begin{cases} \frac{\delta W}{\delta u} + \frac{\delta D}{\delta \dot{u}} = f_{vu} - \rho \ddot{u}, \\ \frac{\delta W}{\delta \phi} + \frac{\delta D}{\delta \dot{\phi}} = f_{v\phi}, \end{cases} \quad (16)$$

with the following popular notation

$$\frac{\delta W}{\delta \phi} = W,_{\phi} - \nabla \cdot W,_{\nabla \phi} + \nabla \nabla \cdot W,_{\nabla \nabla \phi} - \dots \quad (17)$$

4 Time-Dependent Processes

The considered governing equations have been derived under the assumption of smoothness of the two potentials and are thus available only for time-dependent processes. Gradient models have been much considered in viscoplasticity as well as in damage mechanics, cf. [1, 4, 6, 8, 22, 17, 29, 23]. In particular, the gradient models of visco-elasticity have been often introduced in the study of different phenomena of diffusion and phase change following the phase field method, cf. for example [19]. Many discussions have been

devoted to the problem of strain localization and fracture, especially in the numerical computation of elastic-plastic solids. These works deal principally with the insulation case $f_{v\phi} = 0$ and $f_{s\phi} = 0$ because of the difficulty to define physically these actions. Some interesting discussions of the literature are reported here in order to obtain some examples on the physical nature of the internal parameters and on the practical interest of gradient models in the modeling of multi-physic phenomena in solids.

The modeling of damage of an elastic solid is an important subject in solid mechanics, cf. for example [28, 31]. In particular, the case of viscous damage is here reported because of its connection with the problem of strain localization and with the phase field method [19].

The internal parameter $0 \leq \phi \leq 1$ is here the damage proportion, $\phi = 0$ if no damage and $\phi = 1$ if full damage. For example, a simple viscous model of damage is obtained with the following expression of the energy and of the dissipation potentials

$$W(\nabla u, \phi, \nabla \phi) = (1 - \phi) w_{el}(\nabla u) + \frac{h}{2} \phi^2 + \frac{g}{2} |\nabla \phi|^2, \quad D(\dot{\phi}) = \frac{1}{2} \xi \dot{\phi}^2 + \frac{1}{2} \eta \nabla \dot{\phi}^2,$$

where w_{el} is the classical elastic energy, h , g and ξ , η are constants. The governing equations for the variation of the damage are:

$$(\xi I - \eta \Delta) \dot{\phi} = -h\phi + g\Delta\phi + w_{el}(\nabla u) \quad \forall x \in V \quad \text{and} \quad (\eta \dot{\phi} + g\phi)_{,n} = 0 \quad \forall x \in \partial V.$$

In this expression, the Laplacian operator at the left hand side is due to the presence of the gradient $\nabla \dot{\phi}$ in the dissipation potential and at the right hand side is due to the gradient $\nabla \phi$ in the energy. Both terms contribute to a non-local description of the constitutive equations.

In particular, the process of localization of the damage can be easily controlled by the values of the coefficients g and h . For example, if the solid is submitted to a very slow displacement control loading, its response at each load level is a stationary point of the energy functional since Biot equation leads in this case to the minimization of the total potential energy of the solid as in classical elasticity. With $h = \frac{G_c}{2\zeta}$, $g = G_c \zeta$, for vanishing ζ , the search for the response has a strong connection with the appearance of Griffith cracks of surface energy G_c . This gives an interesting method to detect the appearance of Griffith cracks in an elastic, brittle solid as the extension of damage zones, cf. Bourdin et al [3, 10]. Their interesting results show clearly the interests of gradient models in theoretical and numerical discussions.

5 Time-Independent Processes and Gradient Plasticity

For time-independent processes such as plasticity or brittle damage, the dissipation potential D is convex, positively homogeneous of degree 1 of the

rates $\dot{\phi}$

$$D(\mu\dot{\phi}, \mu\nabla\dot{\phi}, \phi) = \mu D(\dot{\phi}, \nabla\dot{\phi}, \phi) \quad \forall \quad \mu > 0. \quad (18)$$

Because of the loss of differentiability with respect to $(\dot{\phi}, \nabla\dot{\phi})$ when $(\dot{\phi}, \nabla\dot{\phi}) = (0, 0)$, the governing equations (6) must be specified.

5.1 Common models in Gradient Plasticity

The case of particular models admitting a gradient-independent dissipation potential

$$D = D(\dot{\phi}, \phi) \quad (19)$$

is first considered. Such a model will be denoted as a GW-model since the gradient is not introduced in the dissipation potential. The *force-flux* relationship is in this case:

$$X_d = \partial D(\dot{\phi}, \phi) \quad , \quad Y_d = 0, \quad (20)$$

in the sense of sub-gradient, cf. [27, 14, 28]. This means that the force X_d must belong to a convex domain of admissible forces $C = \partial D(0, \phi)$ and that the normality law is satisfied by the rate $\dot{\phi}$. There is no indetermination difficulty since from (4), $X_d = -X_e + \nabla \cdot Y_e$ is known at the present state.

For example, if $D(\dot{\phi}, \phi) = k(\phi)\|\dot{\phi}\|$, then the set of admissible forces is described by the plastic criterion $f(X_d, \phi) = \|X_d\| - k(\phi) \leq 0$ and the rate $\dot{\phi}$ must satisfy the normality law $\dot{\phi} = \lambda \frac{\partial f}{\partial X_d}$ with $\lambda \geq 0$, $\lambda f = 0$.

If gradient terms are included in the dissipation potential, i.e. $D = D(\dot{\phi}, \nabla\dot{\phi}, \phi)$ the model will be denoted as a GD-model and the force-flux relationship can be written as

$$X_d = \partial_{\dot{\phi}} D \quad , \quad Y_d = \partial_{\nabla\dot{\phi}} D. \quad (21)$$

This means that the force (X_d, Y_d) must belong to a convex set of admissible forces $C = \partial D_{(\dot{\phi}, \nabla\dot{\phi})}(0, 0, \phi)$ and that the normality law is satisfied by the rates $(\dot{\phi}, \nabla\dot{\phi})$.

For example, if

$$D = k(\phi)\|\dot{\phi}\| + \kappa(\phi)\|\nabla\dot{\phi}\|, \quad (22)$$

then the convex of admissible forces is given by two inequalities

$$f(X_d, \phi) = \|X_d\| - k(\phi) \leq 0 \quad , \quad \varphi(Y_d, \phi) = \|Y_d\| - \kappa(\phi) \leq 0, \quad (23)$$

and the rates $\dot{\phi}, \nabla\dot{\phi}$ must satisfy the normality law

$$\dot{\phi} = \lambda \frac{\partial f}{\partial X_d} \quad \text{with } \lambda \geq 0 \quad , \quad \lambda f = 0 \quad , \quad \nabla \dot{\phi} = \mu \frac{\partial \varphi}{\partial Y_d} \quad , \quad \mu \geq 0 \quad , \quad \mu \varphi = 0. \quad (24)$$

From (4), these forces must also satisfy the constitutive equilibrium equations ($X_e + X_d - \nabla \cdot (Y_d + Y_e) = 0$ and appropriate boundary conditions). However, these relationships do not determine (X_d, Y_d) when $(\dot{\phi}, \nabla \dot{\phi}) = (0, 0)$, even if the present state is known. This indetermination is the principal difficulty of GD-models.

The following case has also been considered in the literature, cf. for example [7, 8]:

$$D = k(\phi)(\|\dot{\phi}\|^2 + \ell^2 \|\nabla \dot{\phi}\|^2)^{1/2}, \quad (25)$$

and leads to a Mises-like plastic criterion and the normality law

$$\begin{cases} f = (\|X_d^p\|^2 + \frac{1}{\ell^2} \|Y_d^p\|^2)^{1/2} - k(\phi), \\ \dot{\epsilon}^p = \lambda \frac{\partial f}{\partial X_d}, \quad \nabla \dot{\epsilon}^p = \lambda \frac{\partial f}{\partial Y_d}, \\ f \leq 0, \quad \lambda \geq 0, \quad f\lambda = 0. \end{cases} \quad (26)$$

It is classical that the dissipation potential is obtained from the elastic domain by the maximum dissipation principle

$$D = D(\dot{\phi}, \nabla \dot{\phi}, \phi) = \max_{(X_d^*, Y_d^*) \in C_\phi} X_d^* \cdot \dot{\phi} + Y_d^* \cdot \nabla \dot{\phi}. \quad (27)$$

In small transformation, an interesting model of plasticity with isotropic-kinematic hardening consists of internal variable $\phi = (\epsilon^p, \gamma)$, of energy

$$W = w_e(\epsilon - \epsilon^p) + w_c(\epsilon^p) + w_i(\gamma) + w_g(\nabla \gamma)$$

and a Mises-like criterion of plasticity of the form

$$f(X_d^p, X_d^\gamma, Y_d^\gamma) = \|X_d^p\| + \frac{1}{\ell} \|Y_d^\gamma\| + X_d^\gamma - k \leq 0,$$

where k is a positive constant. The dissipation potential and the normality law are

$$D(\dot{\epsilon}^p, \dot{\gamma}, \nabla \dot{\gamma}) = \max_{f(X_d^{p*}, X_d^{\gamma*}, Y_d^{\gamma*}) \leq 0} X_d^{p*} \cdot \dot{\epsilon}^{p*} + Y_d^{\gamma*} \cdot \nabla \dot{\gamma} + X_d^{\gamma*} \dot{\gamma},$$

$$\begin{cases} \dot{\epsilon}^p = \lambda \frac{\partial f}{\partial X_d^p} = \lambda \frac{X_d^p}{\|X_d^p\|}, \\ \dot{\gamma} = \lambda \frac{\partial f}{\partial X_d^\gamma} = \lambda, \\ \nabla \dot{\gamma} = \lambda \frac{\partial f}{\partial Y_d^\gamma} = \lambda \frac{Y_d^\gamma}{\|Y_d^\gamma\|} \quad , \quad f \leq 0, \quad \lambda \geq 0, \quad f\lambda = 0. \end{cases}$$

γ is thus the equivalent plastic strain and the dissipation is $d = X_d^p \cdot \dot{\epsilon}^p + Y_d^\gamma \cdot \nabla \dot{\gamma} + X_d^\gamma \dot{\gamma} = k\lambda$.

At finite strain, several models of gradient plasticity have been proposed cf. Gurtin [17], using the classical multiplicative decomposition $\nabla u = F_e F_p$ where F_e and F_p are the elastic and plastic transformation gradients, the internal parameter is F_p :

$$W(\nabla u, F_p, \nabla F_p) = w_e(\nabla u, F_p) + w_c(F_p) + w_g(F_p, \text{Curl} F_p).$$

5.2 Governing Equations

Under the insulation condition, the governing equations, given by (1), (4), (21), are

$$\left\{ \begin{array}{l} \sigma = W_{,\nabla u}, \quad X_e = W_{,\phi}, \quad Y_e = W_{,\nabla \phi}, \\ X = X_e + X_d, \quad Y = Y_e + Y_d, \quad (X_d, Y_d) = \partial D(\dot{\phi}, \nabla \dot{\phi}, \phi), \\ P_i + P_j = P_e \quad \forall \delta u, \delta \phi, \\ P_i = \int_V (\sigma \cdot \nabla \delta u + X \cdot \delta \phi + Y \cdot \nabla \delta \phi) dV, \\ P_j = \int_V \rho \ddot{u} \cdot \delta u dV, \\ P_e = \int_V f_{vu} \cdot \delta u dV + \int_{\partial V} f_{su} \cdot \delta u da. \end{array} \right. \quad (28)$$

Because of the loss of differentiability of the dissipation potential, Biot equation (14) now takes the form of a variational inequality as in Classical Plasticity, cf. for example [5, 28, 11]:

Proposition 1 *The response of a solid $\mathbf{U}(t)$ under the loading $\mathbf{F}(t)$ is a solution of the evolutionary variational inequality*

$$\mathbf{W}_{,\mathbf{U}} \cdot (\delta \mathbf{U} - \dot{\mathbf{U}}) + \mathbf{D}(\delta \mathbf{U}, \mathbf{U}) - \mathbf{D}(\dot{\mathbf{U}}, \mathbf{U}) - (\mathbf{F} + \mathbf{J}) \cdot (\delta \mathbf{U} - \dot{\mathbf{U}}) \geq \mathbf{0} \quad \forall \delta \mathbf{U} \quad (29)$$

and vice versa.

Some mathematical results on the question of existence and uniqueness of a solution have also been given for GW-models, cf. Mainik & Mielke [23] in quasi-static transformation.

A discussion on existence and uniqueness of a solution has been recently given by Giacomini & Musesti [15] for the model (25).

5.3 Rigid plasticity and Regularization

It may be interesting to compare the advantages to include gradient terms in the energy or in the dissipation potentials. Some usefull remarks on this question can be found in the framework of rigid plasticity, cf. [20, 24] for example.

5.3.1 Recall on rigid plasticity

The classical theory of rigid plasticity gives the simplest example of GD-model. In this case, U is reduced to the displacement u and the energy and dissipation potentials are

$$W = 0 \quad , \quad D = k\|\nabla\dot{u}\|,$$

with the Mises-like criterion of plasticity. In a quasi-static transformation, the governing equations requires that the stress σ must satisfy the mechanical equilibrium, the plastic criterion $\|\sigma\| \leq k$ and the normality law for the rate $\nabla\dot{u}$. The difficulty to obtain an analytical or numerically approximated solution is well known and due principally to the lack of information on the stress field in the undeformed zone $\nabla\dot{u} = 0$. For both theoretical or numerical discussions, a regularization of the constitutive relation is often introduced in the classical literature. The technics of visco-plastic regularization or elastic-regularization have been currently adopted in practical applications.

5.3.2 Elastic Regularization

The elastic-regularization consists in replacing the rigid-plasticity model (which is a GD-model) by an elastic-plastic model (which is a GW-model). The energy and the dissipation are modified as

$$W^* = \frac{1}{2}(\nabla u - \epsilon^p) : E : (\nabla u - \epsilon^p) \quad , \quad D^* = k\|\epsilon^p\|,$$

with an additional independent variable ϵ^p . If E is strong enough, ϵ^p will be a good approximation of ∇u .

In the same spirit, for any GD-model, the elastic regularization consists to consider an associated GW-model by introducing an additional internal parameter β and an additional energy. For example, an additional term of the form $1/2r\|\beta - \nabla\phi\|^2$ can be included in the energy while $\nabla\dot{\phi}$ is replaced by $\dot{\beta}$ in the dissipation potential. This leads to a GW-model of internal parameters ϕ , β , of potentials w^* and D^* :

$$\begin{cases} w^*(\nabla u, \phi, \nabla \phi, \beta) = w(\nabla u, \phi, \nabla \phi) + \frac{1}{2}r\|\beta - \nabla \phi\|^2, \\ D^*(\dot{\phi}, \dot{\beta}, \phi) = D(\dot{\phi}, \dot{\beta}, \phi). \end{cases} \quad (30)$$

For the regularized model, the governing equations (28) lead to the same elastic domain C in the force space (X_d^ϕ, X_d^β) and the normality law for $(\dot{\phi}, \dot{\beta})$.

For example, by regularization, the GD-model (25) leads to the WD-model

$$\begin{cases} f = (\|X_d^\phi\|^2 + \frac{1}{\ell^2}\|X_d^\beta\|^2)^{1/2} - k(\phi), \\ \dot{\phi} = \lambda \frac{\partial f}{\partial X_d^\phi}, \dot{\beta} = \lambda \frac{\partial f}{\partial X_d^\beta}, \quad f \leq 0 \quad , \quad \lambda \geq 0 \quad , \quad f\lambda = 0, \\ X_d^\phi = -X_e^\phi + \nabla \cdot Y_e^\phi + r\Delta\phi - r\nabla \cdot \beta, \quad X_d^\beta = r(\nabla\gamma - \beta). \end{cases}$$

Thus β approaches $\nabla\gamma$ and X_d^β plays the role of Y_d^γ when the coefficient of rigidity r is high enough.

In the same spirit, the GD-model of isotropic and kinematic hardening with a Mises-like criterion of the form $f(X_d^p, X_d^\gamma, Y_d^\gamma) = \|X_d^p\| + \frac{1}{\ell}\|Y_d^\gamma\| + X_d^\gamma - k \leq 0$ leads to a regularized WD-model defined by

$$\begin{cases} f = \|X_d^p\| + X_d^\gamma + \frac{1}{\ell} \|X_d^\beta\| - k \leq 0, \\ \dot{\epsilon}^p = \lambda \frac{\partial f}{\partial X_d^p}, \quad \dot{\gamma} = \lambda \frac{\partial f}{\partial X_d^\gamma} = \lambda, \quad \dot{\beta} = \lambda \frac{\partial f}{\partial X_d^\beta}, \\ f \leq 0, \quad \lambda \geq 0, \quad f\lambda = 0, \\ X_d^p = \sigma - h\epsilon^p, \quad X_d^\gamma = -w'_i + g\Delta\gamma + \nabla \cdot r(\nabla\phi - \beta), \quad X_d^\beta = r(\nabla\gamma - \beta). \end{cases}$$

6 Conclusion

In this paper, the constitutive equations of standard gradient models are conveniently described from the expressions of the energy and the dissipation potentials. Our attention is focussed on the derivation of the governing equations as a generalized Biot equation, on the formalism of generalized standard materials and on time-independent processes such as incremental plasticity and brittle damage. In particular, in gradient plasticity the necessity of regularization of GD-models is underlined in order to avoid analytical and numerical difficulties in the practical applications.

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