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Exact boundary controllability of 1D semilinear wave equations through a constructive approach

Kuntal Bhandari ^{*} Jérôme Lemoine [†] Arnaud Münch [‡]

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Dedicated to the memory of Roland Glowinski

Abstract

The exact boundary controllability of the semilinear wave equation $y_{tt} - y_{xx} + f(y) = 0$, $x \in (0, 1)$ assuming that f is locally Lipschitz continuous and satisfies the growth condition $\limsup_{|r| \rightarrow \infty} |f(r)| / (|r| \ln^p |r|) \leq \beta$ for some β small enough and $p = 2$ has been obtained by Zuazua in 1993. The proof based on a non constructive fixed point arguments makes use of precise estimates of the observability constant for a linearized wave equation. Under the above asymptotic assumption with $p = 3/2$, by introducing a different fixed point application, we present a simpler proof of the exact boundary controllability which is not based on the cost of observability of the wave equation with respect to potentials. Then, assuming that f is locally Lipschitz continuous and satisfies the growth condition $\limsup_{|r| \rightarrow \infty} |f'(r)| / \ln^{3/2} |r| \leq \beta$ for some β small enough, we show that the above fixed point application is contracting yielding a constructive method to approximate the controls for the semilinear equation. Numerical experiments illustrate the results. The results can be extended to the multi-dimensional case and for nonlinearities involving the gradient of the solution.

AMS Classifications: 35L71, 93B05.

Keywords: Semilinear wave equation, Exact boundary controllability, Carleman estimates, Fixed point.

1 Introduction and main results

Let $\Omega := (0, 1)$ and let $T > 0$. We set $Q_T := \Omega \times (0, T)$. We consider the semilinear 1D wave equation

$$\begin{cases} y_{tt} - y_{xx} + f(y) = 0 & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(1, \cdot) = v & \text{in } (0, T), \\ (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \quad (1)$$

where $(u_0, u_1) \in \mathbf{V} := H_0^1(\Omega) \times L^2(\Omega)$ is a given initial state, $v \in H_0^1(0, T)$ is a control function and $f \in \mathcal{C}^0(\mathbb{R})$ is a nonlinear function such that $|f(r)| \leq C(1 + |r|) \ln^2(2 + |r|)$ for every $r \in \mathbb{R}$ and some $C > 0$. Then, (1) admits a unique weak solution in $\mathcal{C}^0([0, T]; H^1(\Omega)) \times \mathcal{C}^1([0, T]; L^2(\Omega))$ (see [CH80]).

The system (1) is said to be *exactly controllable* at time $T > 0$ if for any initial state $(u_0, u_1) \in \mathbf{V}$ and target data $(z_0, z_1) \in \mathbf{V}$, there exists a control function $v \in H_0^1(0, T)$ such that the associated solution to (1) satisfies $(y(\cdot, T), y_t(\cdot, T)) = (z_0, z_1)$. The controllability time $T > 0$ needs to be large enough in view of the finite speed of propagation of the solutions.

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Literature - The exact controllability for the linear wave equations is by now well-understood, see for instance the pioneer works by D. L. Russell, [Rus78], J.-L. Lions, [Lio88a, Lio88b], J. Lagnese & J.-L. Lions [LL88]; we also refer [BLR92].

The first work concerning the controllability of finite dimensional nonlinear wave equations has been done by Markus [Mar65] by the way of an implicit function theorem. Later on, this approach has been adapted for the local exact controllability of nonlinear wave equations by Chewning [Che76], Fattorini [Fat75]. Global exact distributed controllability for the semilinear wave equations in any space dimension has first been obtained by E. Zuazua [Zua88b, Zua88a, Zua90] assuming that the nonlinear functions are globally Lipschitz and asymptotically linear, i.e. assuming that $\limsup_{|r| \rightarrow \infty} |f(r)|/|r| < \infty$. For the boundary controllability case, this asymptotic assumption has been removed in [Zua91] in the framework of the HUM method introduced by Lions coupled with a fixed point argument.

Theorem 1. [Zua91, Theorem 2.1] *Assume that $T > 2$. Then, for every globally Lipschitz continuous function f such that $f' \in L^\infty(\mathbb{R})$ and $\gamma \in (0, 1)$, $\gamma \neq \frac{1}{2}$, the system (1) is exactly controllable in $H_0^\gamma(0, 1) \times H^{\gamma-1}(0, 1)$ with a control $v \in H_0^\gamma(0, T)$.*

Later on, this result (actually proved in a multidimensional situation) was recovered by I. Lasiecka and R. Triggiani, [LT91], using a global inversion theorem. The authors improved some regularity of their boundary control still assuming globally Lipschitz nonlinearity.

Then, in the framework of the distributed controllability with a control support $\omega \subset (0, 1)$, the assumption $f' \in L^\infty(\mathbb{R})$ has been relaxed by Zuazua.

Theorem 2. [Zua93, Theorem 1] *Let $\omega = (l_1, l_2)$ with $0 < l_1 < l_2 < 1$. Assume that $T > 2 \max\{l_1, 1 - l_2\}$, that f is locally Lipschitz continuous and satisfies*

$$(\mathbf{H}_1) \quad \limsup_{|r| \rightarrow \infty} \frac{|f(r)|}{|r| \ln_+^2 |r|} \leq \beta$$

for some β small enough depending only on ω and T . Then, for any $(u_0, u_1), (z_0, z_1) \in \mathbf{V}$, the system

$$\begin{cases} y_{tt} - y_{xx} + f(y) = v \mathbf{1}_\omega & \text{in } Q_T, \\ y(0, \cdot) = y(1, \cdot) = 0 & \text{in } (0, T), \\ (y(0, \cdot), y_t(0, \cdot)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \quad (2)$$

is exactly controllable with control in $L^2(\omega \times (0, T))$: there exists $v \in L^2(\omega \times (0, T))$ such that $(y(\cdot, T), y_t(\cdot, T)) = (z_0, z_1)$.

Here and in the sequel, we note

$$\ln_+ |r| = \begin{cases} 0 & \text{if } |r| \leq 1 \\ \ln |r| & \text{else.} \end{cases}$$

Moreover, it is proved in [Zua93, Theorem 2] that, if f behaves like $-s \ln^p(|s|)$ with $p > 2$ as $|s| \rightarrow +\infty$, then the system is not exactly controllable in any time $T > 0$, due to an uncontrollable blow-up phenomenon. Theorem 1 has been slightly improved in [CKL02], weakening the condition (\mathbf{H}_1) into

$$\limsup_{|r| \rightarrow +\infty} \left| \int_0^r f(r') dr' \right| \left(|r| \prod_{k=1}^{+\infty} \ln^{[k]}(e_k + r^2) \right)^{-2} < +\infty$$

where $\ln^{[k]}$ denotes the k^{th} iterate of \ln and $e_k > 0$ is such that $\ln^{[k]}(e_k) = 1$. This growth condition is optimal since the solution of (2) may blow up whenever f grows faster at infinity and has the bad sign. The multi-dimensional case in which Ω is a bounded domain of \mathbb{R}^d , $d > 1$, with a $\mathcal{C}^{1,1}$ boundary has been addressed in [LZ00]; assuming that the support ω of the control function is a neighborhood of $\partial\Omega$ and that $T > \text{diam}(\Omega \setminus \omega)$, the exact controllability of (2) is proved under the

growth condition $\limsup_{|r| \rightarrow +\infty} \frac{|f(r)|}{|r| \ln_+^{1/2} |r|} < \beta$ for some β small enough. For control domains ω satisfying the classical multiplier assumption (see [Lio88b]), exact controllability has been proved in [Zha00] assuming that f is globally Lipschitz continuous. We also mention [CT06] where a positive boundary controllability result is proved for steady-state initial and final data and for T large enough by a quasi-static deformation approach. We also mention the work [DLZ03] by B. Dehman, G. Lebeau and E. Zuazua which is concerned with the controllability and stabilizability of some subcritical semilinear wave equations in $\Omega' \subset \mathbb{R}^3$. Assuming that the nonlinearity $f \in C^3(\mathbb{R})$ satisfies

$$f(0) = 0, \quad rf(r) \geq 0, \quad |f^{(j)}(r)| \leq C(1 + |r|)^{p-j}, \quad j = 1, 2, 3; \quad 1 \leq p < 5$$

the exact internal controllability of the semilinear wave equations at time $T := T(u_0, u_1) > 0$ that depends on the size of the initial data $(u_0, u_1) \in H_0^1(\Omega') \times L^2(\Omega')$. See also [DL09] achieving the same result in a uniform time under smallness assumption on the initial data. The sign condition has been weakened in [JL14] to an asymptotic sign assumption leading to a semi-global internal controllability result in the sense that the target data is prescribed in a precise subset of $H_0^1(\Omega') \times L^2(\Omega')$.

The above results devoted to internal controllability, notably Theorem 2, can be employed together with the domain extension method to get indirectly boundary controllability results for system (1) of interest in the present work.

The proof of Theorem 2 is based on a Leray Schauder fixed point argument applies to the operator $\Lambda : L^\infty(Q_T) \rightarrow L^\infty(Q_T)$, where $y := \Lambda(z)$ is a controlled solution with the control function v of the linear boundary value problem

$$\begin{cases} y_{tt} - y_{xx} + \widehat{f}(z) y = -f(0) + v1_\omega & \text{in } Q_T, \\ y(0, \cdot) = y(1, \cdot) = 0 & \text{in } (0, T), \\ (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \quad \widehat{f}(r) := \begin{cases} \frac{f(r) - f(0)}{r} & \text{if } r \neq 0 \\ f'(0) & \text{if } r = 0 \end{cases} \quad (3)$$

satisfying $(y(\cdot, T), y_t(\cdot, T)) = (z_0, z_1)$. The stability of the operator Λ in $L^\infty(Q_T)$ is based on a precise estimate of the cost of the control v in term of the potential \widehat{f} and data $(u_0, u_1), (z_0, z_1)$.

Objective - The general goal addressed in this work is the approximation of the controllability problem associated with (1), that is to construct an explicit sequence $(y_k, v_k)_{k \in \mathbb{N}}$ converging strongly toward a control-state pair solution (y, v) for (1). Although almost sharp with respect to the nonlinearity, Theorem 2 is not constructive as it does not provide any convergent sequences $(y_k)_{k \in \mathbb{N}}$ to a fixed point of Λ , i.e. to a controlled solution y of (2). This is due to the fact that the operator Λ is not contracting in general.

Assuming slightly stronger assumptions on f , a constructive convergent sequence has been proposed by the third author and E. Trélat in [MT22] using a least-squares approach coupled with a Newton type linearization.

Theorem 3. [MT22, Theorem 2.3] *Let $\omega = (l_1, l_2)$ with $0 < l_1 < l_2 < 1$. Assume that $T > 2 \max\{l_1, 1 - l_2\}$ and that $f \in C^1(\mathbb{R})$ satisfies*

$$(\mathbf{H}'_1) \quad \exists \alpha > 0, \text{ s.t. } |f'(r)| \leq \alpha + \beta \ln_+^2 |r|, \quad \forall r \in \mathbb{R}$$

for some $\beta > 0$ small enough depending only on ω and T and

$$(\overline{\mathbf{H}}_p) \quad \exists p \in (0, 1] \text{ such that } \sup_{\substack{a, b \in \mathbb{R} \\ a \neq b}} \frac{|f'(a) - f'(b)|}{|a - b|^p} < +\infty.$$

Then, for any $(u_0, u_1), (z_0, z_1) \in \mathbf{V}$, one can construct a sequence $(y_k, v_k)_{k \in \mathbb{N}}$ converging strongly to a controlled pair for (2) satisfying $(y(\cdot, T), y_t(\cdot, T)) = (z_0, z_1)$. Moreover, after a finite number of iterations, the convergence is of order at least $1 + p$.

The hypothesis on f are stronger here than in Theorem 1: it should be noted however that the function $f(r) = a + br + \beta r \ln(1 + |r|)^2$, $a, b \in \mathbb{R}$ which is somehow the limit case in (\mathbf{H}_1) satisfies (\mathbf{H}'_1) and $(\overline{\mathbf{H}}_1)$. On the other hand, Theorem 3 is constructive, contrary to Theorem 1. The construction makes appear the operator $\Lambda_1 : L^\infty(Q_T) \rightarrow L^\infty(Q_T)$ where $y := \Lambda_1(z)$ is a controlled solution with the control function v of the linear boundary value problem

$$\begin{cases} y_{tt} - y_{xx} + f'(z)y = v1_\omega + f'(z)z - f(z) & \text{in } Q_T, \\ y(0, \cdot) = y(1, \cdot) = 0 & \text{in } (0, T), \\ (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega. \end{cases} \quad (4)$$

Theorem 3 is extended to a multidimensional case, i.e. $\Omega \subset \mathbb{R}^d$ with $d \leq 3$ in [BLM21] under the same condition on f except that the exponent 2 in (\mathbf{H}'_1) is replaced by an exponent 1/2.

Main result of the present work - In this paper, we prove the following result, directly in the framework of the boundary controllability.

Theorem 4. *Assume $T > 2$. Let $s > 0$ large enough.*

- *There exists $\beta^* > 0$ such that if $f \in \mathcal{C}^0(\mathbb{R})$ satisfies*

$$(\mathbf{H}_2) \quad \exists \alpha_1, \alpha_2 > 0, \text{ s.t. } |f(r)| \leq \alpha_1 + |r| \left(\alpha_2 + \beta^* \ln_+^{3/2} |r| \right), \quad \forall r \in \mathbb{R}$$

then system (1) is exactly controllable at time T for initial data in \mathbf{V} with controls in $H_0^1(0, T)$.

- *There exists $\beta^* > 0$ such that if $f \in \mathcal{C}^1(\mathbb{R})$ satisfies*

$$(\mathbf{H}'_2) \quad \exists \alpha > 0, \text{ s.t. } |f'(r)| \leq \alpha + \beta^* \ln_+^{3/2} |r|, \quad \forall r \in \mathbb{R}$$

then, for any initial state (u_0, u_1) and final state (z_0, z_1) in \mathbf{V} , one can construct a sequence $(y_k, v_k)_{k \in \mathbb{N}^}$ that converges strongly to a controlled pair (y, v) in $(\mathcal{C}^0([0, T]; H^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega))) \times H_0^1(0, T)$ for the system (1). Moreover, the convergence of (y_k, v_k) holds at least with a linear rate for the norm $\|\rho(s) \cdot\|_{L^2(Q_T)} + \|\rho_1(s) \cdot\|_{L^2(0, T)}$ where ρ, ρ_1 are defined in (8) and s is chosen sufficiently large depending on $\|(u_0, u_1)\|_{\mathbf{V}}$ and $\|(z_0, z_1)\|_{\mathbf{V}}$.*

To our knowledge, this result is the first proposing a constructive approximation of boundary controls for the semilinear wave equation without the assumption that f is globally Lipschitz. Under smallness assumptions on the data, we mention the recent works [CCCR21] and [NZWF19].

Concerning the first part of the theorem, if we compare with Theorem 2 (leading indirectly to boundary controllability result by the extension method), the assumption on the asymptotic behavior of f is slightly stronger with an exponent $\frac{3}{2}$ instead of 2. This is due to the fact the method in [Zua93] based on explicit computation (using the d'Alembert formula) is genuinely one dimensional, while the present work is based on Carleman estimates valid in any space dimension. On the other hand, this first part relaxed the regularity assumption to $f \in \mathcal{C}^0(\mathbb{R})$ instead of f locally Lipschitz continuous. Moreover, this first part of Theorem 4 differs from Theorem 2 on the functional spaces as it is based on a different fixed point application leading to a simpler proof. In particular, it is not based on the analysis of the cost of observability of the wave equation with potential. Concerning the second part of the theorem, it relaxes the Hölder assumption $(\overline{\mathbf{H}}_p)$ on f' but still leads to a constructive method. As we shall see, this is related to an appropriate choice of the parameter s related to the norm of the initial condition. Again, to our knowledge, this is the first result leading to a convergent approximation of boundary controls for superlinear nonlinearities without smallness assumption notably on the initial condition and target (contrary to the recent works [CCCR21] and [NZWF19]).

Theorem 4 is obtained by adapting the recent work [ELM21] devoted to a semilinear heat equation. We introduce the following linearized controllability problem: for \hat{y} in a suitable class $\mathcal{C}_R(s)$ depending

on a free parameter $s \geq 1$, find the control v such that the solution y of

$$\begin{cases} y_{tt} - y_{xx} = -f(\hat{y}) & \text{in } Q_T, \\ y(0, \cdot) = 0, y(1, \cdot) = v & \text{in } (0, T), \\ (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \quad (5)$$

satisfies $(y(\cdot, T), y_t(\cdot, T)) = (z_0, z_1)$ in Ω , and (y, v) corresponds to the minimizer of a functional J_s depending on s and involving Carleman weight functions (see Remark 3).

This will define an operator $\Lambda_s : \hat{y} \mapsto y$ from some suitable class $\mathcal{C}_R(s)$ into itself, on which we can use fixed point theorems for s sufficiently large depending on $\|u_0\|_{L^\infty(\Omega)}$, namely Schauder fixed point theorem for the first item of Theorem 4, and Banach-Picard fixed point theorem for the second item, allowing to exhibit a simple sequence of convergent approximations of the control and controlled trajectory. The analysis of the fixed point operator is based on Carleman estimates as they allow to get precise estimates on the control and controlled trajectories in term of the parameter s . Choosing the Carleman parameter large allows to limit the influence of lower order terms and get suitable contracting properties. Such tricks have already been used in the context of inverse problems reformulated through a least-squares functional in [BDBE13] and [KKNS17].

With respect to the heat equation considered in [ELM21, LM22], the Carleman weights are not singular with respect to the time variable, avoiding technicalities. On the other hand, the regularity issue is more delicate for the hyperbolic case which does not enjoys regularizing property. This is *a fortiori* true for boundary control : precisely, in order to get L^∞ estimate for the controlled trajectories solution of (1), the boundary control v needs to be more regular than $L^2(0, T)$. Hopefully, it turns out that the optimal state-control pair for the functional J_s (see Remark 3) involving L^2 norms enjoys suitable regularity property as soon as the initial and final data belongs to \mathbf{V} and satisfy compatibility condition at $x = 0$ and $x = 1$. This point is crucial in our analysis.

Outline - The paper is organized as follows. In Section 2, we derive a controllability result for the linear wave equation with precise estimates in term of the right hand side, the initial data and the Carleman parameter s large enough. In particular, we prove that the optimal control for the $L^2(0, T)$ norm belongs actually to $H^1(0, T)$: this result stated in Theorem 7 is proven in Appendix A. Then, in Section 3, we prove, for any time $T > 2$ the uniform null controllability of (1) assuming that f is continuous and satisfies the condition (\mathbf{H}_2) . Then in Section 3.5, assuming that f' is continuous and satisfies the condition (\mathbf{H}'_1) , we show that the operator Λ_s is contracting, yielding the convergence of the Picard iterates $y_{k+1} = \Lambda_s(y_k)$. Section 5 illustrates the result with some numerical experiments while Section 6 concludes with some remarks.

Notations. In this article, C denote generic constants depending on Ω and T , which may change from line to line, but are independent of the Carleman parameter s .

2 Controllability results for the linear wave equation

This section is devoted to a controllability result for a linear wave equation with a right hand side $B \in L^2(Q_T)$ and initial data $(u_0, u_1) \in \mathbf{V}$. Precisely, for any $(z_0, z_1) \in \mathbf{V}$ and $T > 0$ large enough, we are interested by the existence of a control function $v \in H^1_0(0, T)$ such that the solution y of

$$\begin{cases} y_{tt} - y_{xx} = B & \text{in } Q_T, \\ y(0, \cdot) = 0, y(1, \cdot) = v & \text{in } (0, T), \\ (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \quad (6)$$

satisfies $(y(\cdot, T), y_t(\cdot, T)) = (z_0, z_1)$. Though this linear control is by now standard, we aim to get precise weighted estimate of a state-control pair in a given functional space in the data, which will

crucial to handle the nonlinear system (1). We employ Carleman estimates as fundamental tool (see [BY17]).

2.1 A global Carleman estimate

For any $\beta \in (0, 1)$ and $x_0 < 0$, we define the auxiliary function

$$\psi(x, t) = |x - x_0|^2 - \beta \left(t - \frac{T}{2} \right)^2 + M_0 \quad \text{in } Q_T, \quad (7)$$

where $M_0 > 0$ is chosen in such a way that ψ is strictly positive. Then, for any $\lambda > 0$, we define $\phi(x, t) = e^{\lambda\psi(x, t)}$. For all $s \geq s_0$, let us now define the following weight functions

$$\rho(s; x, t) := e^{-s\phi(x, t)}, \quad \rho_1(s; t) = \rho(s; 1, t), \quad \forall (x, t) \in Q_T. \quad (8)$$

Remark that

$$e^{-cs} \leq \rho(s; x, t) \leq e^{-s}, \quad e^{-cs} \leq \rho_1(s; t) \leq e^{-s} \quad \text{in } Q_T$$

with $c := \|\phi\|_{L^\infty(Q_T)}$, that $\rho^{-1}, \rho \in \mathcal{C}^\infty(\overline{Q_T})$ and that $\rho_1, \rho_1^{-1} \in \mathcal{C}^\infty([0, T])$. In short, we shall write $\rho(s)$ and $\rho_1(s)$ to denote the above weight functions.

Then, for any $\delta > 0$ such that $T - 2\delta > 2 \sup_{\overline{\Omega}} |x - x_0|$, we introduce a cut-off function $\eta \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfying the following properties:

$$\begin{cases} 0 \leq \eta(t) \leq 1 & \text{in } (\delta, T - \delta), \\ \eta(t) = 1 & \text{in } [2\delta, T - 2\delta], \\ \eta(t) = 0 & \text{in } (-\infty, \delta] \cup [T - \delta, +\infty). \end{cases} \quad (9)$$

Let

$$P := \{w \in \mathcal{C}^0([0, T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega)), w_{tt} - w_{xx} \in L^2(Q_T)\}.$$

Recall that $w_x(1, \cdot) \in L^2(0, T)$ for every $w \in P$ (see [Lio88a, Theorem 4.1]).

The controllability property for the linear system (6) is based on the following Carleman estimate with boundary observation at $x = 1$.

Theorem 5. *Assume $T > 2$. There exists $s_0 > 0$, $\lambda > 0$ and $C > 0$, such that for any $s \geq s_0$, we have the following Carleman inequality*

$$\begin{aligned} & s \int_{Q_T} \rho^{-2}(s) (|w_t|^2 + |w_x|^2) + s^3 \int_{Q_T} \rho^{-2}(s) |w|^2 \\ & \quad + s \int_{\Omega} \rho^{-2}(s; \cdot, 0) (|w_t(\cdot, 0)|^2 + |w_x(\cdot, 0)|^2) + s^3 \int_{\Omega} \rho^{-2}(s; \cdot, 0) |w(\cdot, 0)|^2 \\ & \quad + s \int_{\Omega} \rho^{-2}(s; \cdot, T) (|w_t(\cdot, T)|^2 + |w_x(\cdot, T)|^2) dx + s^3 \int_{\Omega} \rho^{-2}(s; \cdot, T) |w(\cdot, T)|^2 \\ & \leq C \int_{Q_T} \rho^{-2}(s) |w_{tt} - w_{xx}|^2 + Cs \int_0^T \eta^2(t) \rho_1^{-2}(s) |w_x(1, \cdot)|^2 \quad (10) \end{aligned}$$

for every $w \in P$.

Proof. We refer to [CFCM13, Lemma 2.3] using [BDBE13, Remark 2.9 and Theorem 2.5]. Remark that the occurrence of the terms at $t = T$ on the left hand side are due to the fact that $\rho(\cdot, t) = \rho(\cdot, T - t)$ and the reversibility of the wave operator. □

2.2 Application to controllability

In a standard way, Theorem 5 allows to deduce some controllability results for the system (6). For any $s \geq s_0$, we define the bilinear form

$$(w, z)_{P,s} := \int_{Q_T} \rho^{-2}(s) LwLz + s \int_0^T \eta^2(t) \rho_1^{-2}(s) w_x(1, t) z_x(1, t), \quad (11)$$

for any $w, z \in P$. Here and in what follows, we use the notation $Lw := w_{tt} - w_{xx}$. It is easily seen that (11) defines a scalar product in P and if P_s denotes P endowed with this scalar product, then P_s is an Hilbert space.

We can state the main result of this section, devoted without loss of generality to the null controllability case, for which $(z_0, z_1) = (0, 0)$ in Ω .

Theorem 6. *Assume that $T > 2$ and let $\eta \in C^\infty(\mathbb{R})$ be a cut-off function satisfying (9). For $s \geq s_0$, $B \in L^2(Q_T)$ and $(u_0, u_1) \in \mathbf{V}$, there exists unique $w_s \in P_s$, depending only on B, u_0, u_1 such that*

$$(w_s, z)_{P,s} = \int_{\Omega} u_1 z(\cdot, 0) dx - \int_{\Omega} u_0 z_t(\cdot, 0) dx + \int_{Q_T} Bz, \quad \forall z \in P_s. \quad (12)$$

Then $v_s(t) = s\eta^2(t)\rho_1^{-2}(s)(w_s)_x(1, t)$ is a control function for (6) where $y_s = \rho^{-2}(s)Lw_s$ is the associated controlled trajectory, that is $y_s(x, T) = (y_s)_t(x, T) = 0$ for all $x \in \Omega$ and the operator defined by

$$\Lambda_s^0 : (B, u_0, u_1) \mapsto y_s \quad (13)$$

is linear, continuous from $L^2(Q_T) \times H_0^1(\Omega) \times L^2(\Omega)$ to $L^2(Q_T)$.

Moreover, we have the following estimates for y_s and v_s for some constant $C > 0$ independent of s :

$$\begin{aligned} & \|\rho(s)y_s\|_{L^2(Q_T)} + s^{-1/2} \left\| \frac{\rho_1(s)}{\eta} v_s \right\|_{L^2(\delta, T-\delta)} \\ & \leq C \left(s^{-3/2} \|\rho(s)B\|_{L^2(Q_T)} + s^{-1/2} e^{-s} \|u_0\|_{L^2(\Omega)} + s^{-3/2} e^{-s} \|u_1\|_{L^2(\Omega)} \right). \end{aligned} \quad (14)$$

Before going to the proof, we make the following remarks.

Remark 1. *It is well-known that the boundary controllability of (6) with $L^2(0, T)$ controls holds true with initial data (u_0, u_1) only in $L^2(\Omega) \times H^{-1}(\Omega)$. We start directly with $(u_0, u_1) \in \mathbf{V}$ since the application of some fixed point theorem to deal with the semilinear case shall require regularity on the state-control pair (see Section 2.3).*

Remark 2. *In the framework of exact controllability with no vanishing target $(z_0, z_1) \in \mathbf{V}$, the right hand side of estimate (14) contains the extra quantities*

$$s^{-1/2} e^{-s} \|z_0\|_{L^2(\Omega)} + s^{-3/2} e^{-s} \|z_1\|_{L^2(\Omega)}.$$

The point here to be noted is that the coefficients (powers of s or exponentials associated with s) in front of the norms of u_0, u_1 and z_0, z_1 are the same, and this would hold for any subsequent estimates.

This is why, there is no loss of generality to choose $(z_0, z_1) = (0, 0)$ which will make the computations shorter and simpler.

Proof. We first ensure the solvability of the variational equation (12). Since $(\cdot, \cdot)_{P,s}$ is a scalar product on P_s , we only need to check that the right hand side of (12) is a linear continuous form on P_s .

- For all $z \in P_s$: since $\rho(s)B \in L^2(Q_T)$, we have

$$\left| \int_{Q_T} Bz \right| \leq \left(\int_{Q_T} |\rho(s)B|^2 \right)^{1/2} \left(\int_{Q_T} |\rho^{-1}(s)z|^2 \right)^{1/2}.$$

Now, since z enjoys the Carleman inequality (10), one has $\|\rho^{-1}(s)z\|_{L^2(Q_T)} \leq Cs^{-3/2}\|z\|_{P_s}$ (recall the definition of the inner product (11) on P_s). Thus, we have

$$\left| \int_{Q_T} Bz \right| \leq Cs^{-3/2}\|\rho(s)B\|_{L^2(Q_T)}\|z\|_{P_s}.$$

- Next, we observe that

$$\begin{aligned} \left| \int_{\Omega} u_0 z_t(\cdot, 0) \, dx \right| &\leq \|\rho(s; \cdot, 0)u_0\|_{L^2(\Omega)} \|\rho^{-1}(s; \cdot, 0)z_t(\cdot, 0)\|_{L^2(\Omega)} \\ &\leq Cs^{-1/2}e^{-s}\|u_0\|_{L^2(\Omega)}\|z\|_{P_s}, \end{aligned}$$

using the Carleman inequality (10) and that $|\rho(s; x, 0)| = |e^{-s\phi(x, 0)}| \leq e^{-s}$ (since $\phi \geq 1$).

- Similar, we get

$$\begin{aligned} \left| \int_{\Omega} u_1 z(\cdot, 0) \, dx \right| &\leq \|\rho(s; \cdot, 0)u_1\|_{L^2(\Omega)} \|\rho^{-1}(s; \cdot, 0)z(\cdot, 0)\|_{L^2(\Omega)} \\ &\leq Cs^{-3/2}e^{-s}\|u_1\|_{L^2(\Omega)}\|z\|_{P_s}. \end{aligned}$$

Combining the above three items, the right hand side of (12) corresponds to a linear functional on P_s . The Riesz representation theorem, implies the existence of a unique $w_s \in P_s$ satisfying the formulation (12) which additionally satisfies

$$\|w_s\|_{P_s} \leq C \left(s^{-3/2}\|\rho(s)B\|_{L^2(Q_T)} + s^{-1/2}e^{-s}\|u_0\|_{L^2(\Omega)} + s^{-3/2}e^{-s}\|u_1\|_{L^2(\Omega)} \right), \quad (15)$$

where the constant $C > 0$ is independent of $s \geq s_0$.

Then, set $y_s = \rho^{-2}(s)Lw_s$ and $v_s = s\eta^2\rho^{-2}(s)(w_s)_x(1, \cdot)$. From the equality (12), the pair (y_s, v_s) satisfies

$$\int_{Q_T} y_s Lz \, dx dt + \int_0^T v_s z_x(1, \cdot) \, dt = \int_{\Omega} u_1 z(\cdot, 0) \, dx - \int_{\Omega} u_0 z_t(\cdot, 0) \, dx + \int_{Q_T} Bz, \quad \forall z \in P_s,$$

meaning that $y_s \in L^2(Q_T)$ is a solution to the linear system (6) associated with the function $v_s \in L^2(0, T)$ in the sense of transposition. By uniqueness, y_s indeed solves (6) in a weak sense. Eventually, using the estimate (15) for w_s , we get that $\rho(s)y_s = \rho^{-1}(s)Lw_s \in L^2(Q_T)$ and $s^{-1/2}\rho_1(s)v_s = s^{1/2}\eta^2\rho_1^{-1}(s)(w_s)_x(1, \cdot) \in L^2(0, T)$ and deduce the weighted estimate (14). \square

Remark 3. *The functions y_s and v_s introduced by Theorem 6 can be characterized as the unique minimizer of the following functional*

$$J_s(y, v) = s \int_{Q_T} \rho^2(s)y^2 + \int_0^T \eta^{-2}\rho_1^2(s)v^2 \quad (16)$$

over the set $\{(y, v) \in L^2(Q_T) \times L^2(0, T) \text{ solution of (6) with } y(\cdot, T) = y_t(\cdot, T) = 0 \text{ in } \Omega\}$. We refer to [CFCM13, Section 2] for the details.

Remark 4. The controlled state $y_s = \rho^{-2}Lw_s$ satisfies

$$\begin{cases} Ly_s = B & \text{in } Q_T, \\ y_s(0, \cdot) = 0, \quad y_s(1, \cdot) = s\eta^2\rho_1^{-2}(s)(w_s)_x(1, \cdot) & \text{in } (0, T), \\ (y_s(\cdot, 0), (y_s)_t(\cdot, 0)) = (u_0, u_1), & \text{in } \Omega, \end{cases} \quad (17)$$

implying (by standard regularity results for the wave equation) that $y_s \in C^0([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$. On the other hand, the function w_s uniquely satisfies the equation

$$\begin{cases} Lw_s = \rho^2 y_s & \text{in } Q_T, \\ w_s(0, \cdot) = w_s(1, \cdot) = 0 & \text{in } (0, T), \end{cases} \quad (18)$$

implying that $(w_s(\cdot, 0), \partial_t w_s(\cdot, 0)) \in \mathbf{V}$ (see estimate (10)) and $w_s \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$.

2.3 Estimate for the state-control pair in $C^0([0, T]; H^1(\Omega)) \times H^1(0, T)$

In this section, we prove that the state-control pair (y_s, v_s) given by Theorem 6 enjoys additional regularity property, under the assumption $(u_0, u_1) \in \mathbf{V}$ and the introduction of the cut-off function η with respect to the time variable. In particular, we obtain that $v_s \in H_0^1(0, T)$ and $y_s \in L^\infty(Q_T)$. This gain is crucial for the analysis of the semilinear case.

Theorem 7. Let any $(u_0, u_1) \in \mathbf{V}$ and $B \in L^2(Q_T)$ be given. Then, the solution (y_s, v_s) of (6) defined in Theorem 6 satisfies $v_s \in H^1(0, T)$, $y_s \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ and the following estimate :

$$\begin{aligned} \|\rho(s)(y_s)_t\|_{L^2(Q_T)} + s^{-1/2}\|\rho_1(s)(v_s)_t\|_{L^2(0, T)} &\leq C \left(s^{-1/2}\|\rho(s)B\|_{L^2(Q_T)} + s^{-1/2}e^{-s}\|u_1\|_{L^2(\Omega)} \right. \\ &\quad \left. + s^{1/2}e^{-s}\|u_0\|_{L^2(\Omega)} + s^{-1/2}e^{-s}\|(u_0)_x\|_{L^2(\Omega)} \right). \end{aligned} \quad (19)$$

We refer to [EZ10, Theorem 5.4] where a similar gain of regularity is proved in the slightly simpler case of control of minimal $L^2(0, T)$ -norm, i.e. J_s in (16) is replaced by $J(y, v) = \|v\|_{L^2(0, T)}^2$. We also refer to [DL09] for internal control by introducing a cut-off function in space. The proof of Theorem 7 is long and requires several steps. It is done in Appendix A.

Let us prescribe the following regularity estimate for the controlled trajectory y_s .

Lemma 1. Let us recall the controlled trajectory y_s and the control v_s for the linear system (6), defined by Theorem 6. Then, y_s satisfies the following bound

$$\begin{aligned} \|y_s\|_{C^0([0, T]; H^1(\Omega))} + \|(y_s)_t\|_{C^0([0, T]; L^2(\Omega))} \\ \leq C \left(\|B\|_{L^2(Q_T)} + e^{cs}\|\rho(s)B\|_{L^2(Q_T)} + e^{(c-1)s}\|(u_0)_x\|_{L^2(\Omega)} \right. \\ \left. + se^{(c-1)s}\|u_0\|_{L^2(\Omega)} + s^{1/2}e^{(c-1)s}\|u_1\|_{L^2(\Omega)} \right), \end{aligned} \quad (20)$$

where $C > 0$ is a constant that does not depend on $s \geq s_0$, and $c = \|\phi\|_{L^\infty(Q_T)}$.

Proof. It is well-known that for given data $(u_0, u_1) \in \mathbf{V}$ and $B \in L^2(Q_T)$, we have

$$\begin{aligned} \|y_s\|_{C^0([0, T]; H^1(\Omega))} + \|(y_s)_t\|_{C^0([0, T]; L^2(\Omega))} \\ \leq C \left(\|B\|_{L^2(Q_T)} + \|(v_s)_t\|_{L^2(0, T)} + \|(u_0)_x\|_{L^2(\Omega)} + \|u_1\|_{L^2(\Omega)} \right). \end{aligned} \quad (21)$$

Then, using the estimate (19) from Theorem 7, it is not difficult to obtain (20). \square

3 Controllability result for the semilinear problem with $f \in \mathcal{C}^0(\mathbb{R})$: a Schauder fixed point argument

For any $s \geq s_0$ and $R > 0$, we introduce the class $\mathcal{C}_R(s)$, defined as the closed convex subset of $L^\infty(Q_T)$

$$\mathcal{C}_R(s) := \left\{ \hat{y} \in L^\infty(Q_T) : \|\hat{y}\|_{L^\infty(Q_T)} \leq R, \|\rho(s)\hat{y}\|_{L^2(Q_T)} \leq R^{1/2} \right\} \quad (22)$$

and assume that the nonlinear function $f \in \mathcal{C}^0(\mathbb{R})$ in (1) satisfies the growth assumption **(H₂)** for some β^* positive precisely chosen later.

Then, for $T > 2$, $s \geq s_0$ (to be fixed later) and for all $\hat{y} \in \mathcal{C}_R(s)$, we first solve the linearized boundary control problem, given by

$$\begin{cases} y_{tt} - y_{xx} = -f(\hat{y}) & \text{in } Q_T, \\ y(0, \cdot) = 0, \quad y(1, \cdot) = v & \text{in } (0, T), \\ (y(\cdot, 0), y_t(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega \end{cases} \quad (23)$$

with v such that $(y(\cdot, T), y_t(\cdot, T)) = (0, 0)$ in Ω . The existence of a controlled trajectory $y \in L^\infty(Q_T)$ is guaranteed by Theorem 7 with the source term $-f(\hat{y}) \in L^2(Q_T)$. Now, our aim is to prove that there exists a fixed point of the following operator

$$\Lambda_s : L^\infty(Q_T) \mapsto L^\infty(Q_T), \quad \Lambda_s(\hat{y}) = y. \quad (24)$$

Note that, $\Lambda_s(\hat{y}) = \Lambda_s^0(-f(\hat{y}), u_0, u_1)$, as per the definition (13).

Claim: We are going to show that

1. for $\beta^* > 0$ small enough, there exist $R > 0$ large enough and $s \geq s_0$ such that $\mathcal{C}_R(s)$ is stable under the map Λ_s ; see Section 3.2;
2. $\Lambda_s(\mathcal{C}_R(s))$ is relatively compact subset of $\mathcal{C}_R(s)$ for the norm $\|\cdot\|_{L^\infty(Q_T)}$; see Section 3.3;
3. Λ_s is a continuous map in $\mathcal{C}_R(s)$ for the topology induced by the norm $\|\cdot\|_{L^\infty(Q_T)}$; see Section 3.4.

Accordingly, by the Schauder fixed point theorem, there will exist a fixed point of Λ_s , denote by y , which will be the controlled trajectory for our semilinear problem (1).

3.1 Estimate of $\|\Lambda_s(\hat{y})\|_{L^\infty(Q_T)}$

We begin with the following lemma.

Lemma 2. *Assume $T > 2$ and $f \in \mathcal{C}^0(\mathbb{R})$ satisfies **(H₂)**. For any $s \geq s_0$ and $\hat{y} \in L^\infty(Q_T)$, the quantity $f(\hat{y})$ satisfies the following estimates:*

$$\begin{aligned} \|f(\hat{y})\|_{L^2(Q_T)} &\leq \alpha_1 T + \left(\alpha_2 + \beta^* \ln_+^{3/2} \|\hat{y}\|_{L^\infty(Q_T)} \right) e^{cs} \|\rho(s)\hat{y}\|_{L^2(Q_T)}, \\ \|\rho(s)f(\hat{y})\|_{L^2(Q_T)} &\leq \alpha_1 T e^{-s} + \left(\alpha_2 + \beta^* \ln_+^{3/2} \|\hat{y}\|_{L^\infty(Q_T)} \right) \|\rho(s)\hat{y}\|_{L^2(Q_T)}, \\ \|f(\hat{y})\|_{L^\infty(Q_T)} &\leq \alpha_1 T + \left(\alpha_2 + \beta^* \ln_+^{3/2} \|\hat{y}\|_{L^\infty(Q_T)} \right) \|\hat{y}\|_{L^\infty(Q_T)}, \end{aligned}$$

with $c = \|\phi\|_{L^\infty(Q_T)}$.

Proof. The proof of above lemma follows from the growth assumption (\mathbf{H}_2) on f . Observe that

$$\begin{aligned} \left(\int_{Q_T} |f(\widehat{y})|^2 \right)^{1/2} &\leq \alpha_1 T + \|\rho^{-1}(s)\|_\infty \left(\int_{Q_T} |\rho(s)\widehat{y}|^2 (\alpha_2 + \beta^* \ln_+^{3/2} |\widehat{y}|)^2 \right)^{1/2} \\ &\leq \alpha_1 T + (\alpha_2 + \beta^* \ln_+^{3/2} \|\widehat{y}\|_{L^\infty(Q_T)}) e^{cs} \|\rho(s)\widehat{y}\|_{L^2(Q_T)}, \end{aligned}$$

where we have used that $\|\rho^{-1}(s)\|_{L^\infty(Q_T)} = \|e^{s\phi}\|_{L^\infty(Q_T)} \leq e^{cs}$. The other estimates are obtained in a similar way. \square

Proposition 1. *Under the assumptions of Lemma 2, for $s \geq s_0$ and for all $\widehat{y} \in L^\infty(Q_T)$, the solution $y = \Lambda_s(\widehat{y})$ to the linearized system (23) satisfies the following estimates:*

$$\begin{aligned} \|\rho(s)y\|_{L^2(Q_T)} + s^{-1/2} \|\rho_1(s)v\|_{L^2(0,T)} &\leq C s^{-3/2} \left(\alpha_2 + \beta^* \ln_+^{3/2} \|\widehat{y}\|_{L^\infty(Q_T)} \right) \|\rho(s)\widehat{y}\|_{L^2(Q_T)} \\ &\quad + C s^{-3/2} \alpha_1 T e^{-s} + C s^{-1/2} e^{-s} \|u_0\|_{L^2(\Omega)} + C s^{-3/2} e^{-s} \|u_1\|_{L^2(\Omega)}, \end{aligned} \quad (25)$$

$$\begin{aligned} \|\rho(s)y_t\|_{L^2(Q_T)} + s^{-1/2} \|\rho_1(s)v_t\|_{L^2(0,T)} &\leq C s^{-1/2} \left(\alpha_2 + \beta^* \ln_+^{3/2} \|\widehat{y}\|_{L^\infty(Q_T)} \right) \|\rho(s)\widehat{y}\|_{L^2(Q_T)} \\ &\quad + C s^{-1/2} \alpha_1 T e^{-s} + C s^{-1/2} e^{-s} \|(u_0)_x\|_{L^2(\Omega)} + C s^{1/2} e^{-s} \|u_0\|_{L^2(\Omega)} + C e^{-s} \|u_1\|_{L^2(\Omega)}. \end{aligned} \quad (26)$$

Moreover, $y \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ and

$$\begin{aligned} \|y\|_{L^\infty(Q_T)} &\leq C \alpha_1 T + C \alpha_1 T e^{(c-1)s} \\ &\quad + C \left(\alpha_2 + \beta^* \ln_+^{3/2} \|\widehat{y}\|_{L^\infty(Q_T)} \right) e^{cs} \|\rho(s)\widehat{y}\|_{L^2(Q_T)} + C e^{(c-1)s} \|(u_0)_x\|_{L^2(\Omega)} \\ &\quad + C s e^{(c-1)s} \|u_0\|_{L^2(\Omega)} + C s^{1/2} e^{(c-1)s} \|u_1\|_{L^2(\Omega)}. \end{aligned} \quad (27)$$

Proof. Put $B = -f(\widehat{y})$ in the linear model (6). Then, the proof is followed as a consequence of Theorem 6, Theorem 7, Lemma 1 and Lemma 2. \square

3.2 Stability of the class $\mathcal{C}_R(s)$ for suitable choices of parameters

We express the result in term of the following lemma. We hereby recall the set $\mathcal{C}_R(s)$ defined in (22).

Lemma 3. *Under the assumptions of Lemma 2, if β^* in (\mathbf{H}_2) is small enough, there exists an s and $R > 0$ large enough, such that we have*

$$\Lambda_s(\mathcal{C}_R(s)) \subset \mathcal{C}_R(s). \quad (28)$$

where $\mathcal{C}_R(s)$ is the class given in (22).

Proof. We start with any $\widehat{y} \in \mathcal{C}_R(s)$ for $s \geq s_0 \geq 1$ and we look for the bounds of the solution $y = \Lambda_s(\widehat{y})$ (to (23)) with respect to the associated norms. Since $\widehat{y} \in \mathcal{C}_R(s)$, one has $\|\rho(s)\widehat{y}\|_{L^2(Q_T)} \leq R^{1/2}$ and $\|\widehat{y}\|_{L^\infty(Q_T)} \leq R$. Therefore, the estimate (25) yields

$$\begin{aligned} \|\rho(s)y\|_{L^2(Q_T)} &\leq C s^{-3/2} \left(\alpha_2 + \beta^* \ln_+^{3/2} R \right) R^{1/2} + C s^{-3/2} \alpha_1 T e^{-s} \\ &\quad + C s^{-1/2} e^{-s} \|u_0\|_{L^2(\Omega)} + C s^{-3/2} e^{-s} \|u_1\|_{L^2(\Omega)}. \end{aligned} \quad (29)$$

Similarly, estimate (27) implies

$$\begin{aligned} \|y\|_{L^\infty(Q_T)} &\leq C \alpha_1 T + C \alpha_1 T e^{(c-1)s} + C \left(\alpha_2 + \beta^* \ln_+^{3/2} R \right) e^{cs} R^{1/2} \\ &\quad + C e^{(c-1)s} \|(u_0)_x\|_{L^2(\Omega)} + C s e^{(c-1)s} \|u_0\|_{L^2(\Omega)} + C s^{1/2} e^{(c-1)s} \|u_1\|_{L^2(\Omega)}. \end{aligned} \quad (30)$$

We now fix the parameter s in term of R as follows :

$$s = \frac{1}{32c} \ln_+ R, \quad \text{with } c = \|\phi\|_{L^\infty(Q_T)} > 1, \quad (31)$$

where $R > 0$ is chosen large enough to ensure $s \geq s_0 \geq 1$. With this choice of s , the solution $y = \Lambda_s(\hat{y})$ satisfies, in view of (29) and the fact that \hat{y} belongs to $\mathcal{C}_R(s)$,

$$\begin{aligned} \|\rho(s)y\|_{L^2(Q_T)} &\leq \frac{C(32c)^{3/2}}{\ln_+^{3/2} R} (\alpha_2 + \beta^* \ln_+^{3/2} R) R^{1/2} + \frac{C\alpha_1 T (32c)^{3/2}}{R^{1/32c} \ln_+^{3/2} R} \\ &\quad + \frac{C\sqrt{32c}}{R^{1/32c} \ln_+^{1/2} R} \|u_0\|_{L^2(\Omega)} + \frac{C(32c)^{3/2}}{R^{1/32c} \ln_+^{3/2} R} \|u_1\|_{L^2(\Omega)} \end{aligned} \quad (32)$$

Thus, if $\beta^* > 0$ is small enough such that

$$C(32c)^{3/2} \beta^* < 1/4, \quad (33)$$

it can be guaranteed for large enough $R > 0$ that

$$\begin{cases} \frac{C(32c)^{3/2}}{\ln_+^{3/2} R} (\alpha_2 + \beta^* \ln_+^{3/2} R) R^{1/2} + \frac{C\alpha_1 T (32c)^{3/2}}{R^{1/32c} \ln_+^{3/2} R} \leq \frac{1}{3} R^{1/2}, \\ \frac{C\sqrt{32c}}{R^{1/32c} \ln_+^{1/2} R} \|u_0\|_{L^2(\Omega)} \leq \frac{1}{3} R^{1/2}, \\ \frac{C(32c)^{3/2}}{R^{1/32c} \ln_+^{3/2} R} \|u_1\|_{L^2(\Omega)} \leq \frac{1}{3} R^{1/2} \end{cases} \quad (34)$$

involving, in view of 32 that $\|\rho(s)y\|_{L^2(Q_T)} \leq R^{1/2}$.

Similarly, in view of (30) and the fact that \hat{y} belongs to $\mathcal{C}_R(s)$, we infer that

$$\begin{aligned} \|y\|_{L^\infty(Q_T)} &\leq C\alpha_1 T + C\alpha_1 T R^{(\frac{1}{32} - \frac{1}{32c})} + C(\alpha_2 + \beta^* \ln_+^{3/2} R) R^{1/32} R^{1/2} + \\ &+ \frac{C}{32c} (\ln_+ R) R^{(\frac{1}{32} - \frac{1}{32c})} \|u_0\|_{L^2(\Omega)} + C R^{(\frac{1}{32} - \frac{1}{32c})} \|(u_0)_x\|_{L^2(\Omega)} + \frac{C}{\sqrt{32c}} (\ln_+^{1/2} R) R^{(\frac{1}{32} - \frac{1}{32c})} \|u_1\|_{L^2(\Omega)}. \end{aligned} \quad (35)$$

Taking $\beta^* > 0$ as before and R large enough, we infer that (recall that $c = \|\phi\|_{L^\infty(Q_T)} > 1$ so that $0 < \frac{1}{32} - \frac{1}{32c} < 1$)

$$\begin{cases} C(\alpha_2 + \beta^* \ln_+^{3/2} R) R^{1/32} R^{1/2} \leq R/5, \\ \frac{C}{32c} (\ln_+ R) R^{(\frac{1}{32} - \frac{1}{32c})} \|u_0\|_{L^2(\Omega)} \leq R/5, \\ C R^{(\frac{1}{32} - \frac{1}{32c})} \|(u_0)_x\|_{L^2(\Omega)} \leq R/5, \\ \frac{C}{\sqrt{32c}} (\ln_+^{1/2} R) R^{(\frac{1}{32} - \frac{1}{32c})} \|u_1\|_{L^2(\Omega)} \leq R/5, \\ C\alpha_1 T + C\alpha_1 T R^{(\frac{1}{32} - \frac{1}{32c})} \leq R/5 \end{cases} \quad (36)$$

implying from (35) that $\|y\|_{L^\infty(Q_T)} \leq R$. It follows that $y = \Lambda_s(\hat{y}) \in \mathcal{C}_R(s)$. This concludes the proof. \square

Remark 5. *The smallness condition on β^* is explicit:*

$$\beta^* < \frac{1}{4C(32c)^{3/2}}, \quad (37)$$

where C is the constant appearing in Proposition 1.

Remark 6. *Provided we impose the relation (31), the above proof shows that $\mathcal{C}_R(s)$ is stable for Λ_s for any $R \geq R_0$ (equivalently $s \geq s_0$) for a suitably large R_0 (equivalently s_0). With the above choices, in view of (34)-(36), the lower bound R_0 depends on $\|(u_0, u_1)\|_{\mathbf{V}}$ as a power of $\|(u_0, u_1)\|_{\mathbf{V}}$, so that the lower bound s_0 can be chosen as depending logarithmically on $\|(u_0, u_1)\|_{\mathbf{V}}$. Note also that there is no upper bound for R so that the parameter s (appearing notably in the definition of the weights) can be taken arbitrarily large.*

3.3 Relative compactness of the set $\Lambda_s(\mathcal{C}_R(s))$

Proposition 2. *Under the assumptions of Lemma 3, $\Lambda_s(\mathcal{C}_R(s))$ is a relatively compact subset of $\mathcal{C}_R(s)$ for the $L^\infty(Q_T)$ norm.*

Proof. Let $(y_n)_{n \in \mathbb{N}}$ be a bounded sequence in $\Lambda_s(\mathcal{C}_R(s))$. We prove that there exists a subsequence $(y_{n_k})_{k \in \mathbb{N}}$ of $(y_n)_{n \in \mathbb{N}}$ that converges strongly to some $y \in \mathcal{C}_R(s)$ with respect to the $L^\infty(Q_T)$ norm.

Thanks to Lemma 3, $(y_n)_{n \in \mathbb{N}}$ is a bounded sequence in $\mathcal{C}_R(s)$ and so, there exists a subsequence $y_{n_k} \in \mathcal{C}_R(s)$ and $y \in \mathcal{C}_R(s)$ such that

$$y_{n_k} \rightharpoonup^* y \text{ weakly}^* \text{ in } L^\infty(Q_T), \text{ as } k \rightarrow +\infty. \quad (38)$$

Now, since $(y_{n_k})_{k \in \mathbb{N}} \subset \Lambda_s(\mathcal{C}_R(s))$, there is a sequence $(\widehat{y}_{n_k})_{k \in \mathbb{N}} \subset \mathcal{C}_R(s)$ such that $y_{n_k} = \Lambda_s(\widehat{y}_{n_k})$, $\forall k \in \mathbb{N}$. More precisely, there exists a sequence $(v_{n_k})_{k \in \mathbb{N}} \in H_0^1(0, T)$ such that, for all $k \in \mathbb{N}$, y_{n_k} satisfies

$$\begin{cases} (y_{n_k})_{tt} - (y_{n_k})_{xx} = -f(\widehat{y}_{n_k}) & \text{in } Q_T, \\ y_{n_k}(0, t) = 0, \quad y_{n_k}(1, t) = v_{n_k}(t) & \text{in } (0, T), \\ (y_{n_k}(\cdot, 0), (y_{n_k})_t(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega. \end{cases}$$

Moreover, for some $C_1 > 0$, we have

$$\|y_{n_k}\|_{C^0([0, T]; H_0^1(\Omega))} \leq C_1 R, \quad \|(y_{n_k})_t\|_{C^0([0, T]; L^2(\Omega))} \leq C_1 R,$$

thanks to their estimates in (20) for $B = -f(\widehat{y}_{n_k})$ and the analysis in Lemma 3. Since the embedding $\{y \in L^\infty(0, T; H_0^1(\Omega)) \mid y_t \in L^\infty(0, T; L^2(\Omega))\} \hookrightarrow C^0([0, T]; C^0(\overline{\Omega}))$ is compact (see [Sim87, Corollary 8 p. 90 and Lemma 12 p. 91]), this ensures the strong convergence of $(y_{n_k})_{k \in \mathbb{N}}$ in $C^0(\overline{Q_T})$ as $k \rightarrow +\infty$. \square

3.4 Continuity of the map Λ_s in $\mathcal{C}_R(s)$

We prove the following result.

Proposition 3. *Under the assumptions and result of Lemma 3, the map $\Lambda_s : \mathcal{C}_R(s) \rightarrow \mathcal{C}_R(s)$ is continuous with respect to the $L^\infty(Q_T)$ norm.*

Proof. Let $(\widehat{y}_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{C}_R(s)$ such that $\widehat{y}_n \rightarrow \widehat{y}$ as $n \rightarrow +\infty$ w.r.t. the $L^\infty(Q_T)$ norm for some $\widehat{y} \in \mathcal{C}_R(s)$.

Let $y_n = \Lambda_s(\widehat{y}_n)$ and prove that $y_n \rightarrow y := \Lambda_s(\widehat{y})$ as $n \rightarrow +\infty$ w.r.t. the same norm. Since $f \in C^0(\mathbb{R})$, f is uniformly continuous in $[-R, R]$ implying that

$$f(\widehat{y}_n) \rightarrow f(\widehat{y}) \text{ in } L^\infty(Q_T), \text{ as } n \rightarrow +\infty. \quad (39)$$

and thus $f(\widehat{y}_n) \rightarrow f(\widehat{y})$ in $L^2(Q_T)$ as $n \rightarrow +\infty$.

Now (as mentioned in Theorem 6), $\Lambda_s(\widehat{y}_n) = \Lambda_s^0(-f(\widehat{y}_n), u_0, u_1)$ is linear continuous map from $L^2(Q_T) \times H_0^1(\Omega) \times L^2(\Omega)$ to $L^2(Q_T)$. Consequently, $y_n \rightarrow y$ as $n \rightarrow +\infty$. \square

3.5 Proof of the first item of Theorem 4

Taking β^* small enough (see (33)) so that Lemma 3 applies, with s and R given by (31), we can apply Schauder fixed point theorem to Λ_s on $\mathcal{C}_R(s)$: there exists $y_s \in \mathcal{C}_R(s) \subset L^\infty(Q_T)$ such that $y_s = \Lambda_s(y_s)$. By construction of Λ_s , there exists a function $v \in H_0^1(0, T)$ such that y_s is the solution of the null controllability problem (23) with $\hat{y} = y_s$: it follows that this element y_s is a controlled solution of the semilinear wave equation (1).

4 Construction of control by Banach fixed point approach with $f \in \mathcal{C}^1(\mathbb{R})$: proof of the second item of Theorem 4

In this section, we assume that f is locally Lipschitz continuous and that f' satisfies (\mathbf{H}'_2) with β^* small as before. Remark that condition (\mathbf{H}'_2) implies the condition (\mathbf{H}_2) used in the previous section.

We endow the convex set $\mathcal{C}_R(s)$ with the distance d defined by $d(y, z) = \|\rho(s)(y - z)\|_{L^2(Q_T)}$. We easily check that $(\mathcal{C}_R(s), d)$ is a complete space. In the next proposition, we prove that the operator $\Lambda_s : \mathcal{C}_R(s) \rightarrow \mathcal{C}_R(s)$ is a contracting mapping leading to constructive method to find its fixed point.

Proposition 4. *Assume that f satisfies (\mathbf{H}'_2) with β^* satisfying (37), R and s as chosen in Lemma 3. Then, for any $\hat{y}_1, \hat{y}_2 \in \mathcal{C}_R(s)$,*

$$d(\Lambda_s(\hat{y}_2), \Lambda_s(\hat{y}_1)) \leq \frac{1}{2}d(\hat{y}_2, \hat{y}_1). \quad (40)$$

In particular, Λ_s is a contraction mapping from $\mathcal{C}_R(s)$ into itself.

Proof. Let $\hat{y}_1, \hat{y}_2 \in \mathcal{C}_R(s)$. From (14), we get that

$$\|\rho(s)(\Lambda_s(\hat{y}_2) - \Lambda_s(\hat{y}_1))\|_{L^2(Q_T)} \leq C s^{-3/2} \|\rho(s)(f(\hat{y}_2) - f(\hat{y}_1))\|_{L^2(Q_T)}.$$

Then, we can use (\mathbf{H}'_2) to deduce

$$\|\rho(s)(\Lambda_s(\hat{y}_2) - \Lambda_s(\hat{y}_1))\|_{L^2(Q_T)} \leq C(32c)^{3/2} \left(\frac{\alpha}{\ln_+^{3/2} R} + \beta^* \right) \|\rho(s)(\hat{y}_2 - \hat{y}_1)\|_{L^2(Q_T)}, \quad (41)$$

since s is given by (31). Since $C(32c)^{3/2}\beta^* \leq 1/4$, the result follows as soon as R is large enough. \square

Theorem 8. *Let $(u_0, u_1) \in \mathbf{V}$. Assume that f is locally Lipschitz continuous and satisfies (\mathbf{H}'_2) with β^* satisfying (37), s and R as chosen in Lemma 3. Then, for any $y_0 \in \mathcal{C}_R(s)$, the sequence $(y_k)_{k \in \mathbb{N}^*} \subset \mathcal{C}_R(s)$ given by*

$$y_{k+1} = \Lambda_s(y_k), \quad k \geq 0,$$

(where Λ_s is defined by (24)) together with the associated sequence of controls $(v_k)_{k \in \mathbb{N}^} \subset H_0^1(0, T)$ strongly converges strongly in $L^2(Q_T) \times L^2(0, T)$ to a controlled solution for (1). Moreover, the convergence is at least linear with respect to the distance d .*

Proof. The convergence of the sequence $(y_k)_{k \in \mathbb{N}}$ toward $y = \Lambda_s(y) \in \mathcal{C}_R(s)$ with linear rate follows from the contraction property of Λ_s :

$$\begin{aligned} \|\rho(s)(y - y_k)\|_{L^2(Q_T)} &= \|\rho(s)(\Lambda_s(y) - \Lambda_s(y_{k-1}))\|_{L^2(Q_T)} \\ &\leq \frac{1}{2^k} \|\rho(s)(y - y_0)\|_{L^2(Q_T)} \leq \frac{1}{2^k} (R^{1/2} + e^{-s} \|y_0\|_{L^2(Q_T)}). \end{aligned}$$

Let now $v \in H_0^1(0, T)$ be associated with y so that $y - y_k$ satisfies, for every $k \in \mathbb{N}^*$

$$\begin{cases} (y - y_k)_{tt} - (y - y_k)_{xx} = -(f(y) - f(y_{k-1})) & \text{in } Q_T, \\ (y - y_k)(0, \cdot) = 0, \quad (y - y_k)(1, \cdot) = (v - v_k) & \text{in } (0, T), \\ ((y - y_k)(\cdot, 0), (y - y_k)_t(\cdot, 0)) = (0, 0) & \text{in } \Omega, \\ ((y - y_k)(\cdot, T), (y - y_k)_t(\cdot, T)) = (0, 0) & \text{in } \Omega. \end{cases} \quad (42)$$

Estimate (14) then implies (recall $s = \frac{1}{32c} \ln_+ R$)

$$\begin{aligned} \|\rho_1(s)(v - v_k)\|_{L^2(0,T)} &\leq C s^{-1} \|\rho(s)(f(y) - f(y_{k-1}))\|_{L^2(Q_T)} \\ &\leq C \frac{32c}{\ln_+ R} (\alpha + \beta^* \ln_+^{3/2} R) \|\rho(s)(y - y_{k-1})\|_{L^2(Q_T)}. \end{aligned}$$

and therefore the convergence is at a linear rate of the sequence $(v_k)_{k \in \mathbb{N}^*}$ toward an exact control for (1). \square

Remark 7. *It can be observed from (41) that the constant appearing in front of $\|\rho(s)(\widehat{y}_2 - \widehat{y}_1)\|_{L^2(Q_T)}$ is getting smaller as R (consequently s) getting larger. In particular, if f satisfies*

$$\lim_{|r| \rightarrow +\infty} \frac{|f'(r)|}{\ln_+^{3/2} |r|} = 0, \quad (43)$$

then, for any given $\epsilon > 0$ (however small), the map Λ_s is ϵ -contractive for large enough $s \geq s_0$. In other words, the speed of convergence of the sequence $(y_k)_{k \geq 1}$ introduced by Theorem 8 increases with s .

As a corollary of the previous result and the classical Banach-Picard's fixed point theorem, the contraction property of the operator Λ_s for β^* small enough given in (33) and s and R given by (31) allows to define a convergent sequence $(y_k, v_k)_{k \in \mathbb{N}}$ to a controlled pair for (1) and prove the following precise version of the second item of Theorem 4.

5 Numerical illustrations

We present some numerical illustrations of the convergence result given by Theorem 8 and emphasize the influence of the parameter s . More precisely, for s large enough, we compute the sequence $(y_k, v_k)_{k \in \mathbb{N}}$ solution to

$$\begin{cases} y_{k,tt} - y_{k,xx} = -f(y_{k-1}) & \text{in } Q_T, \\ y_k(0, \cdot) = 0, \quad y_k(1, \cdot) = v_k & \text{in } (0, T), \\ (y_k(\cdot, 0), (y_k)_t(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega, \\ (y_k(\cdot, T), (y_k)_t(\cdot, T)) = (0, 0) & \text{in } \Omega, \end{cases} \quad (44)$$

obtained through the variational formulation (12) with the source term $B = -f(y_{k-1})$. We first sketch the algorithm and then discuss some numerical experiments obtained with the software FreeFem++ (see [Hec12]).

5.1 Construction of the sequence $(y_k, v_k)_{k \geq 1}$

Starting with some suitable initial guess $y_0 \in \mathcal{C}_R(s)$, we can obtain the solution y_k to (44) with a control v_k based on Theorem 6. Assume that the value of the Carleman parameter s satisfies Lemma 3. Then, for each $k \geq 1$, we define the unique solution $w_k \in P_s$ (see Theorem 6) of

$$(w_k, z)_{P_s} = \int_{\Omega} u_1 z(\cdot, 0) dx - \int_{\Omega} u_0 z_t(\cdot, 0) dx - \int_{Q_T} f(y_{k-1}) z dx dt \quad \forall z \in P_s, \quad (45)$$

then we set $y_k = \rho^{-2}(s) L w_k$ in Q_T and $v_k = s \eta^2 \rho_1^{-2}(s) (w_k)_x(1, \cdot)$ in $(0, T)$.

The numerical approximation of the variational formulation (45) has been addressed in [CM15, CFCM13] and more recently in [BFMO21]. A conformal finite dimensional approximations, say $P_{s,h}$ of P_s , leads to a strong convergent approximation $w_{k,h}$ of w_k for the P_s norm as the discretization parameter h goes to 0. Then, from $w_{k,h}$, we can define the approximated controlled solution

$y_{k,h} := \rho^{-2}(s)Lw_{k,h}$ and $v_{k,h} := s\eta^2\rho_1^{-2}(s)(w_{k,h})_x(1, \cdot)$. In our semilinear setting, we shall employ an equivalent but different formulation, more appropriate for numerical purposes. First, in order to avoid the possible numerical blow up of the weights for s large, we introduce a change of variable. Second, in order to avoid second differentiation in order to compute y_k from the definition $y_k = \rho^{-2}(s)(w_{k,tt} - w_{k,xx})$, we incorporate directly the controlled state solution in the formulation. Therefore, we introduce the variables

$$m_k = \rho_1^{-1}(s)w_k, \quad p_k = \rho^{-1}(s)Lw_k \quad \text{in } Q_T \quad (46)$$

so that $p_k = \rho^{-1}(s)L(\rho_1(s)m_k)$ and $y_k = \rho^{-1}(s)p_k$ and then replace the formulation (45) by the equivalent and well-posed following mixed formulation: find $(m_k, p_k, \lambda_k) \in \rho^{-1}(s)P_s \times L^2(Q_T) \times L^2(Q_T)$ solution of

$$\left\{ \begin{array}{l} \int_{Q_T} p_k \bar{p} \, dxdt + s \int_0^T \eta^2(t)(m_k)_x(1, t) \bar{m}_x(1, t) dt + \int_{Q_T} \lambda_k (\bar{p} - \rho^{-1}(s)L(\rho_1(s)\bar{m})) \, dxdt \\ = \int_0^1 u_1 \rho_1(s, 0) \bar{m}(\cdot, 0) \, dx - \int_0^1 u_0(x) [\rho_1(0; s) \bar{m}_t(0, x) + (\partial_t \rho_1)(0; s) \bar{m}(0, x)] \, dx \\ - \int_{Q_T} f(\rho^{-1}(s)p_{k-1}) \rho_1(s) \bar{m} \, dxdt, \\ \int_{Q_T} \bar{\lambda} (p_k - \rho^{-1}(s)L(\rho_1(s)m_k)) \, dxdt = 0, \end{array} \right. \quad (47)$$

for all $(\bar{m}, \bar{p}, \bar{\lambda}) \in \rho^{-1}(s)P_s \times L^2(Q_T) \times L^2(Q_T)$. The variable λ_k stands as a Lagrange multiplier for the constraint $p_k - \rho^{-1}(s)L^*(\rho_1(s)m_k) = 0$ in Q_T . We check the following inequality

$$\begin{aligned} \rho^{-1}(s)L^*(\rho_1(s)m_k) &= \rho^{-1}(s)\rho_1(s)Lm_k + \rho^{-1}(s)\partial_{tt}\rho_1(s)m_k + 2\rho^{-1}(s)\partial_t\rho_1(s)(m_k)_t \\ &= A_1 Lm_k + A_2 m_k + A_3 (m_k)_t, \end{aligned}$$

with

$$\left\{ \begin{array}{l} \partial_t \rho_1(s; t) = s\lambda\beta(2t - T)\phi(1, t)\rho_1(s; t), \\ \partial_{tt}\rho_1(s; t) = 2s\lambda\beta\phi(1, t)\rho_1(s; t) - s\lambda^2\beta^2(2t - T)^2\phi(1, t)\rho_1(s; t) + s^2\lambda^2\beta^2(2t - T)^2\phi^2(1, t)\rho_1(s; t), \\ A_1 = \rho^{-1}(s)\rho_1(s), \quad A_2 = \rho^{-1}(s)\partial_{tt}\rho_1(s), \quad A_3 = 2\rho^{-1}(s)\partial_t\rho_1(s) \end{array} \right.$$

and we observe that the functions A_i do not contains exponential with positive arguments. For instance, we get

$$\rho^{-1}(s)\rho_1(s) = e^{-s(\phi(1,t) - \phi(x,t))},$$

and recall that $\phi(1, t) - \phi(x, t) \geq 0$ in Q_T . Eventually, from the solution (m_k, p_k, λ_k) , the controlled pair (y_k, v_k) can be retrieved using the formula

$$y_k = \rho^{-1}(s)p_k, \quad v_k = s\eta^2\rho_1^{-1}(s)(m_k)_x(1, \cdot). \quad (48)$$

The sequence $(y_k, v_k)_{k \geq 1}$ is initialized with $(y_0, v_0) = (0, 0)$ so that the iteration (y_1, v_1) is the solution to the linear system (44) with the right hand side $B = -f(y_0) = -f(0)$. We perform the iterations until the following criterion is fulfilled

$$\frac{\|\rho(s)y_{k+1} - \rho(s)y_k\|_{L^2(Q_T)}}{\|\rho(s)y_k\|_{L^2(Q_T)}} \leq 10^{-6}. \quad (49)$$

We denote by k^* the smallest integer k such that (49) holds.

Eventually, concerning the approximation of the formulation (47), we use a conformal space-time finite element method (as addressed in [CM15]). We introduce a regular triangulation \mathcal{T}_h of

Q_T such that $\overline{Q}_T = \bigcup_{K \in \mathcal{T}_h} K$. We assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family, where the index h is such that $h = \max_{K \in \mathcal{T}_h} \text{diam}(K)$. We then approximate of the variables p_k and λ_k in the space $P_h := \{p_h \in \mathcal{C}^0(\overline{Q}_T) : p_h|_K \in \mathbb{P}_1(K), \forall K \in \mathcal{T}_h\} \subset L^2(Q_T)$, where $\mathbb{P}_1(K)$ denotes the space of affine functions both in x and t . On the other hand, the variable m_k is approximated with the space $V_h := \{v_h \in \mathcal{C}^1(\overline{Q}_T) : v_h|_K \in \mathbb{P}(K), \forall K \in \mathcal{T}_h\} \subset \rho_1^{-1}(s)P_s$, where $\mathbb{P}(K)$ denotes the composite Hsieh-Clough-Tocher \mathcal{C}^1 element defined for triangles. We refer to [Cia02, page 356] and [BH81] where the implementation has been discussed. We refer to [BFMO21] for the numerical analysis of the formulation (47).

5.2 Experiments

In what follows, we take $T = 2.5$ and $(u_0(x), u_1(x)) = c_{u_0}(\sin(\pi x), 0)$ with $x \in \Omega = (0, 1)$ parametrized by the real c_{u_0} . Then we define the various weight functions appearing in the Carleman inequality (10) in Section 2.1 as follows: we take

$$\psi(x, t) = (x + 0.02)^2 - 0.9(t - T/2)^2 + 2, \quad \phi(x, t) = e^{\frac{1}{2}\psi(x, t)} \quad \text{in } Q_T,$$

so that $\psi \geq 0.5$ in Q_T . The weights ρ and ρ_1 are then defined by (8). The cut-off function η is chosen as follows:

$$\eta(t) = e^{-\frac{1}{(t+10^{-6})(T-t+10^{-6})}}, \quad \forall t \in (0, T).$$

Eventually, we employ a regular space-time mesh composed of 25600 triangles and 13041 vertices corresponding to the discretization parameter $h \approx 1.25 \times 10^{-2}$.

5.2.1 Nonlinear functions with growth $r \ln^{3/2}(2 + |r|)$

In the mixed formulation (47), let us first consider the semilinear function

$$f(r) = c_f r(\alpha_2 + \beta^* \ln^{3/2}(2 + |r|)), \quad \forall r \in \mathbb{R} \quad (50)$$

with $\alpha_2 = \beta^* = 1$ and some $c_f \in \mathbb{R}^*$, so that $f(0) = 0$. We check that f satisfies (\mathbf{H}_2) and (\mathbf{H}'_2) . In this case, the source term in (47) can be rigorously written as

$$\rho_1(s)f(\rho^{-1}(s)p_{k-1}) = c_f \rho_1(s)\rho^{-1}(s)p_{k-1} \left(\alpha_2 + \beta^* \ln^{3/2}(2 + |y_{k-1}|) \right). \quad (51)$$

I. Experiments for fixed (c_f, c_{u_0}) w.r.t. the parameter s . Let us make the following experiments given by Table 1, 2, for some fixed parameters c_f (associated with the nonlinear function) and c_{u_0} (associated with the initial data). For some large parameters $c_f = 5$ and $c_{u_0} = 20$, it has been checked that the value $s = 1$ is not large enough to imply the Banach contraction property (ensuring the convergence of the algorithm w.r.t. the criterion (49)). Then, by choosing $s \geq 2$, we recover the required convergence criterion (49). We provide the results in Table 1.

Figure 1 depicts the evolution of the relative error $\frac{\|\rho^{(s)}y_{k+1} - \rho^{(s)}y_k\|_{L^2(Q_T)}}{\|\rho^{(s)}y_k\|_{L^2(Q_T)}}$ w.r.t. to the iteration number k for $s \in \{1, 2, 3, 4, 5\}$. In agreement with Remark 7, we observe that the decay of the error is amplified with larger values of s .

Table 2 reports experiments in the unfavorable situation for which $c_f < 0$. We checked that for $c_f = -5$ ($c_{u_0} = 20$ as previous), the convergence is observed from $s = 3$. It is noticeable that the L^2 norms of the solutions and the associated controls are relatively larger compare to the case of positive c_f given by Table 1.

s	$\ y_{k^*}\ _{L^2(Q_T)}$	$\ \rho(s)y_{k^*}\ _{L^2(Q_T)}$	$\ v_{k^*}\ _{L^2(0,T)}$	$\ \rho_1(s)v_{k^*}\ _{L^2(0,T)}$	$\ v_{k^*}\ _{H_0^1(0,T)}$	$\ v_{k^*}\ _{L^\infty(0,T)}$	k^*
2	59.756	1.996×10^{-1}	215.164	2.801×10^{-2}	798.404	313.009	20
3	58.542	3.922×10^{-2}	210.267	3.725×10^{-4}	781.826	306.162	16
4	50.894	8.256×10^{-3}	179.039	1.06×10^{-5}	675.977	262.196	13
5	43.345	1.796×10^{-3}	148.817	5.624×10^{-7}	575.226	219.946	12
6	37.920	3.991×10^{-4}	130.515	3.096×10^{-8}	523.09	196.314	11
7	37.669	9.014×10^{-5}	144.458	1.709×10^{-9}	610.474	224.476	10
8	49.551	2.063×10^{-5}	207.905	9.607×10^{-11}	874.74	318.437	9

Table 1: $c_{u_0} = 20$; $c_f = 5$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of (y_{k^*}, v_{k^*}) w.r.t. s .

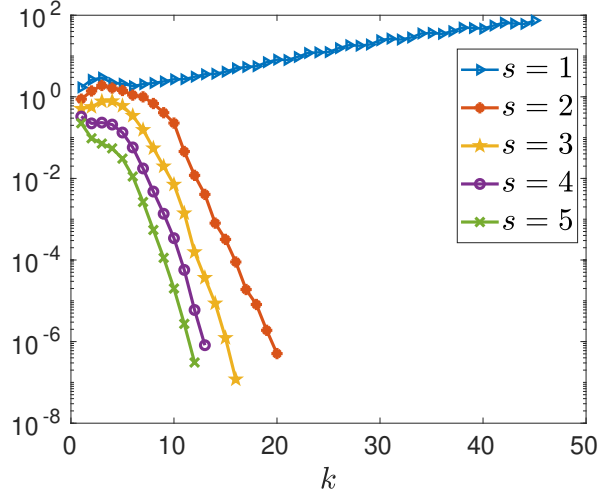


Figure 1: Evolution of the relative error $\frac{\|\rho(s)y_{k+1} - \rho(s)y_k\|_{L^2(Q_T)}}{\|\rho(s)y_k\|_{L^2(Q_T)}}$ w.r.t. iterations k for $(c_f, c_{u_0}) = (5, 20)$.

s	$\ y_{k^*}\ _{L^2(Q_T)}$	$\ \rho(s)y_{k^*}\ _{L^2(Q_T)}$	$\ v_{k^*}\ _{L^2(0,T)}$	$\ \rho_1(s)v_{k^*}\ _{L^2(0,T)}$	$\ v_{k^*}\ _{H_0^1(0,T)}$	$\ v_{k^*}\ _{L^\infty(0,T)}$	k^*
3	3846.94	6.768×10^{-1}	3905.78	1.298×10^{-1}	57298.8	4576.07	32
4	2621.55	3.154×10^{-2}	2888.41	4.064×10^{-3}	35910.6	2919.18	23
5	2120.91	2.711×10^{-3}	2393.94	1.851×10^{-4}	29619.4	2373.23	20
6	1842.65	4.677×10^{-4}	2088.93	9.851×10^{-6}	26358.5	2093.53	17
7	1666.94	9.993×10^{-5}	1910.54	5.657×10^{-7}	24108.5	1902.18	15
8	1539.51	2.231×10^{-5}	1680.7	3.396×10^{-8}	22176.5	1815.9	14

Table 2: $c_{u_0} = 20$; $c_f = -5$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of (y_{k^*}, v_{k^*}) w.r.t. s .

II. Experiments for fixed (s, c_{u_0}) w.r.t. c_f . Hereafter, for fixed Carleman parameter $s = 3$, we consider several values of c_f to look the number of iterations for which the pair the solution y_{k^*} satisfies the criterion (49); see Table 3.

We observed that for these parameters, i.e., $s = 3$, $c_{u_0} = 20$, the algorithm fails to converge when c_f is large, which indeed verifies our theoretical results, since the constant c_f should be small enough to apply the fixed point arguments. In fact, for large absolute values of c_f , the algorithm still may converge provided we start with larger values of the Carleman parameter s , which is in accordance with Lemma 3 and Theorem 8. For instance, if $c_f = 8$, then $s = 3$ does not give the convergence result whereas by choosing $s = 4$, we recover the required convergence criterion (49).

c_f	$\ y_{k^*}\ _{L^2(Q_T)}$	$\ \rho(s)y_{k^*}\ _{L^2(Q_T)}$	$\ v_{k^*}\ _{L^2(0,T)}$	$\ \rho_1(s)v_{k^*}\ _{L^2(0,T)}$	$\ v_{k^*}\ _{L^\infty(0,T)}$	k^*
-5	3846.94	6.768×10^{-1}	3905.78	1.298×10^{-1}	4576.07	32
-4	600.956	1.243×10^{-1}	558.073	2.074×10^{-2}	662.127	21
-3	107.61	5.635×10^{-2}	81.8475	4.419×10^{-3}	132.963	14
-2	21.07	4.807×10^{-2}	19.084	6.158×10^{-4}	20.4579	10
-1	9.876	4.528×10^{-2}	7.147	1.246×10^{-4}	8.432	7
0	11.804	4.352×10^{-2}	11.331	1.945×10^{-4}	13.791	1
1	15.491	4.223×10^{-2}	24.214	1.457×10^{-4}	30.46	7
2	16.79	4.123×10^{-2}	32.562	2.003×10^{-4}	43.481	9
3	16.529	4.042×10^{-2}	31.947	2.880×10^{-4}	38.572	10
4	28.201	3.977×10^{-2}	80.496	3.114×10^{-4}	115.364	13
5	58.542	3.922×10^{-2}	210.267	3.725×10^{-4}	306.162	16
6	94.037	3.875×10^{-2}	376.436	6.475×10^{-4}	564.801	19
7	113.327	3.835×10^{-2}	482.024	1.037×10^{-3}	753.926	25

Table 3: $c_{u_0} = 20$; $s = 3$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of (y_{k^*}, v_{k^*}) w.r.t. c_f .

III. Experiments for fixed (s, c_f) w.r.t. the parameter c_{u_0} . We now fix the parameters s and c_f and then vary the size of the initial data u_0 in terms of the parameter c_{u_0} . We give some results in Table 4 and Table 5 for $(s, c_f) = (3, -2)$ and $(s, c_f) = (3, 2)$ respectively. One can observe that for large c_{u_0} also, the algorithm converges. The quantity $C(y, v)$ defined by (following the estimates in (6) or Proposition 1)

$$C(y, v) = \frac{\|\rho(s)y\|_{L^2(Q_T)} + s^{-1/2}\|\rho_1(s)v\|_{L^2(0,T)}}{s^{-3/2}\|\rho(s)f(y)\|_{L^2(Q_T)} + s^{-1/2}e^{-s}\|u_0\|_{L^2(\Omega)}} \quad (52)$$

is uniform with respect to the quantity c_{u_0} in agreement with our theoretical results.

c_{u_0}	$\ y_{k^*}\ _{L^2(Q_T)}$	$\ \rho(s)y_{k^*}\ _{L^2(Q_T)}$	$\ v_{k^*}\ _{L^2(0,T)}$	$\ \rho_1(s)v_{k^*}\ _{L^2(0,T)}$	$\ v_{k^*}\ _{L^\infty(0,T)}$	$C(y_{k^*}, v_{k^*})$	k^*
10	6.303	2.326×10^{-2}	2.667	5.574×10^{-5}	3.0515	9.55×10^{-2}	8
50	168.803	1.317×10^{-1}	173.604	6.733×10^{-3}	199.033	9.19×10^{-2}	12
100	780.659	3.143×10^{-1}	607.61	3.152×10^{-2}	957.58	9.01×10^{-2}	14
200	3588.67	9.19×10^{-1}	4084.4	1.352×10^{-1}	4256.47	8.47×10^{-2}	16
500	26475.4	5.158	26106.3	8.975×10^{-1}	29134.9	7.33×10^{-2}	19
1000	113305	20.98	92618.1	3.7697	123829	6.39×10^{-2}	21
2000	477431	86.6439	494352	15.881	513495	5.54×10^{-2}	23
5000	3.188×10^6	563.405	3.493×10^6	106.615	3.606×10^6	4.62×10^{-2}	28
10000	1.31×10^7	2306.73	1.19×10^7	450.263	1.719×10^7	4.08×10^{-2}	31
15000	2.973×10^7	5256.44	2.531×10^7	1048.23	4.124×10^7	3.81×10^{-2}	33
20000	5.305×10^7	9424.13	4.619×10^7	1910.76	7.554×10^7	3.64×10^{-2}	35

Table 4: $(s, c_f) = (3, -2)$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of (y_{k^*}, v_{k^*}) w.r.t. c_{u_0} .

IV. Evolution of the controlled solutions. In this paragraph, we present some figures of the controlled solutions and the associated controls for our semilinear system. We fix $c_f = -3$ and $c_{u_0} = 10$. Figures 2 provides the controlled solutions y_{k^*} for $s = 1, 5$ and 9 respectively. The corresponding optimal control is given in Figure 3-Left for $s \in \{1, 3, 5, 9\}$. The evolution of the $L^2(\Omega)$ norm w.r.t. $t \in (0, T)$ is depicted in fig. 3-Right. Figures 4 and 5 are concerned with the case $c_f = 3$, leading to control-state pairs with lower norms.

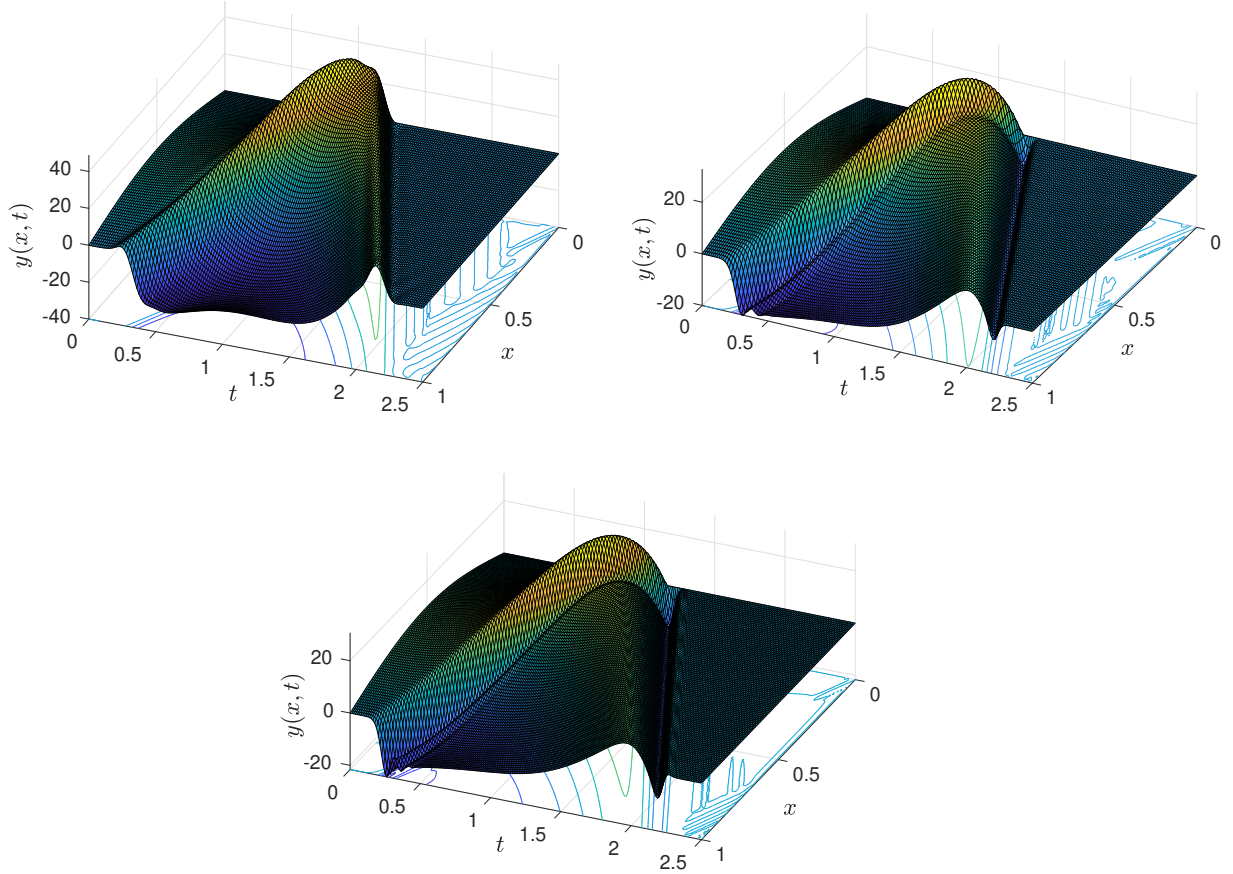


Figure 2: Controlled solution y_{k^*} for $c_f = -3$, $c_{u_0} = 10$ and $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; $s \in \{1, 5, 9\}$.

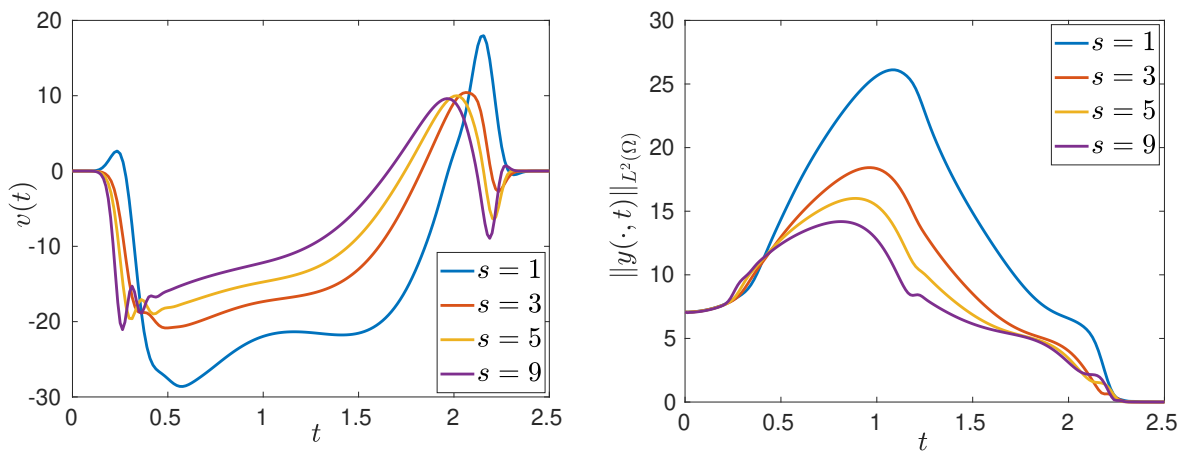


Figure 3: $c_f = -3$, $c_{u_0} = 10$ and $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Left: Control v_{k^*} w.r.t. s ; Right: Evolution of $\|y_{k^*}(\cdot, t)\|_{L^2(\Omega)}$ w.r.t. time t .

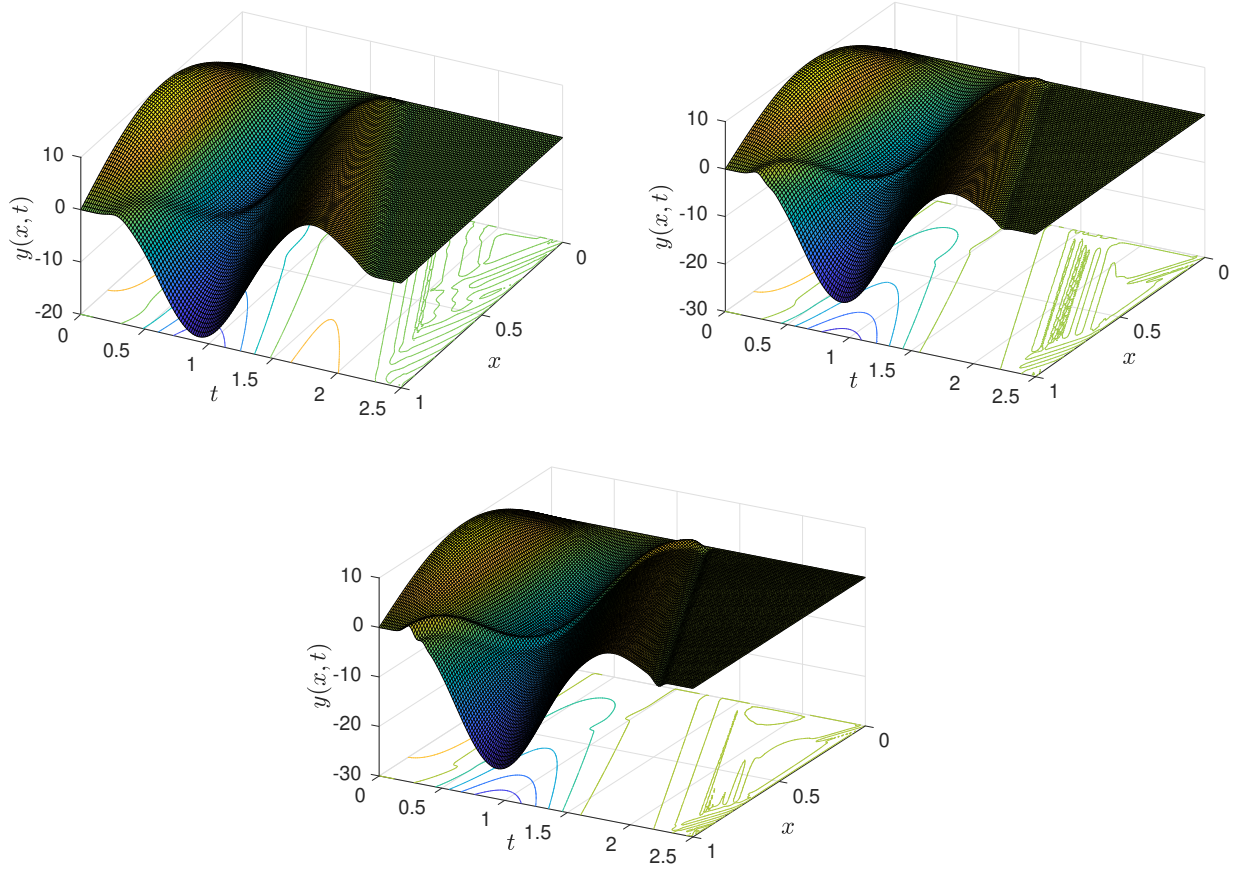


Figure 4: Controlled solution y_{k^*} for $c_f = 3$, $c_{u_0} = 10$ and $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; $s \in \{1, 5, 9\}$.

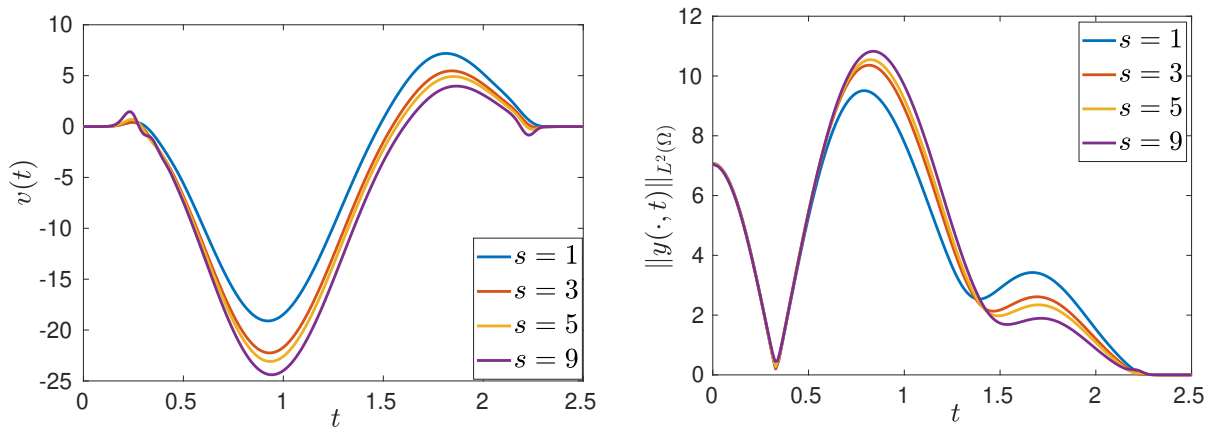


Figure 5: $c_f = 3$, $c_{u_0} = 10$ and $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Left: Control v_{k^*} w.r.t. t ; Right: Evolution of $\|y_{k^*}(\cdot, t)\|_{L^2(\Omega)}$ w.r.t. t .

c_{u_0}	$\ y_{k^*}\ _{L^2(Q_T)}$	$\ \rho(s)y_{k^*}\ _{L^2(Q_T)}$	$\ v_{k^*}\ _{L^2(0,T)}$	$\ \rho_1(s)v_{k^*}\ _{L^2(0,T)}$	$\ v_{k^*}\ _{L^\infty(0,T)}$	$C(y_{k^*}, v_{k^*})$	k^*
10	8.302	2.084×10^{-2}	15.062	7.91×10^{-5}	19.481	8.75×10^{-2}	8
50	40.582	1.015×10^{-1}	77.126	6.83×10^{-4}	100.52	7.66×10^{-2}	10
100	99.183	2.005×10^{-1}	223.618	1.517×10^{-3}	302.591	7.18×10^{-2}	11
200	331.403	3.963×10^{-1}	974.631	3.04×10^{-3}	1367.74	6.72×10^{-2}	13
500	1476.63	9.765×10^{-1}	5043.43	9.201×10^{-3}	7091.56	6.16×10^{-2}	14
1000	3667.51	1.933	13216	2.736×10^{-2}	18969.3	5.79×10^{-2}	16
2000	7261.36	3.830	26737.1	7.799×10^{-2}	39969.8	5.45×10^{-2}	17
5000	17146.3	9.467	62553.3	2.722×10^{-1}	80179.7	5.05×10^{-2}	19
10000	76904.8	18.79	314884	7.301×10^{-1}	481311	4.78×10^{-2}	21
15000	182361	28.073	775890	1.378	1.1725×10^6	4.65×10^{-2}	22
20000	317709	37.334	1.379×10^6	2.197	2.0743×10^6	4.56×10^{-2}	23

Table 5: $(s, c_f) = (3, 2)$; $f(r) = c_f r(1 + \ln^{3/2}(2 + |r|))$; Norms of (y_{k^*}, v_{k^*}) w.r.t. c_{u_0} .

5.3 Nonlinear functions with growth $r|\cos(r^2)|\ln^{3/2}(2 + |r|)$

Let us consider the following nonlinear function

$$f(r) = c_f r |\cos(r^2)| (\alpha_2 + \beta^* \ln^{3/2}(2 + |r|)), \quad \forall r \in \mathbb{R} \quad (53)$$

with some $c_f \in \mathbb{R}^*$ and $\alpha_2 = \beta^* = 1$. It satisfies the assumption (\mathbf{H}_2) but not (\mathbf{H}'_2) . We check that for small values of c_f , the algorithm converges for $s = 1$. For instance, in Table 6 we give some experiments for $c_f = -1$, $c_{u_0} = 20$ and $s \in \{1, 2, 3, 4, 5\}$. On the other hand, with $c_f = -2$, the method fails to converge for $s \in \{1, 2, 3, 4, 5\}$, meaning that the contraction property is lost. The convergence for s larger than 6, see Table 7. Moreover, we see that the number of iterations k^* to fulfill the convergence criterion (49) is very high compare to the case of $c_f = -1$.

s	$\ y_{k^*}\ _{L^2(Q_T)}$	$\ \rho(s)y_{k^*}\ _{L^2(Q_T)}$	$\ v_{k^*}\ _{L^2(0,T)}$	$\ \rho_1(s)v_{k^*}\ _{L^2(0,T)}$	$\ v_{k^*}\ _{H^1_0(0,T)}$	$\ v_{k^*}\ _{L^\infty(0,T)}$	k^*
1	10.415	1.311	7.893	1.635×10^{-1}	33.08	7.976	24
2	10.212	2.2974×10^{-1}	7.929	4.541×10^{-3}	33.398	8.195	20
3	9.987	4.457×10^{-2}	8.002	1.759×10^{-4}	34.3	8.627	18
4	9.799	9.128×10^{-3}	8.085	8.968×10^{-6}	35.616	8.975	14
5	9.657	1.94×10^{-3}	8.196	4.972×10^{-7}	37.011	9.326	14

Table 6: $c_{u_0} = 20$; $c_f = -1$; Norms of (y_{k^*}, v_{k^*}) w.r.t. s ; when $f = c_f r |\cos(r^2)|(1 + \ln^{3/2}(2 + |r|))$.

s	$\ y_{k^*}\ _{L^2(Q_T)}$	$\ \rho(s)y_{k^*}\ _{L^2(Q_T)}$	$\ v_{k^*}\ _{L^2(0,T)}$	$\ \rho_1(s)v_{k^*}\ _{L^2(0,T)}$	$\ v_{k^*}\ _{H^1_0(0,T)}$	$\ v_{k^*}\ _{L^\infty(0,T)}$	k^*
6	10.657	4.275×10^{-4}	6.459	1.014×10^{-8}	24.197	7.473	87
7	10.494	9.516×10^{-5}	6.464	6.434×10^{-10}	25.258	7.641	118
8	10.519	2.155×10^{-5}	6.466	4.577×10^{-11}	26.736	7.465	90
9	10.641	4.946×10^{-6}	6.548	3.664×10^{-12}	29.271	7.774	45

Table 7: $c_{u_0} = 20$; $c_f = -2$; Norms of (y_{k^*}, v_{k^*}) w.r.t. s ; when $f = c_f r |\cos(r^2)|(1 + \ln^{3/2}(2 + |r|))$.

5.4 Nonlinear functions with growth $r \ln^p(2 + |r|)$ for $p \geq 2$

In this section, we first consider the following form of the nonlinear function:

$$f(r) = c_f r (\alpha_2 + \beta^* \ln^2(2 + |r|)) \quad \forall r \in \mathbb{R} \quad (54)$$

which satisfies (\mathbf{H}_1) but not (\mathbf{H}_2) nor (\mathbf{H}'_2) . For $c_f = 4$ and $c_{u_0} = 10$, we have checked that the algorithm does not converge for the Carleman parameters $s = 1, 2$. For the experiments, we need at least $s > 2$ to fulfill the convergence criterion (49). We present some results in Table 8.

But, as soon as we increase the L^2 -norm of the initial data u_0 , the result is getting worse, even if we keep the value of $c_f = 4$. For instance, considering $c_{u_0} = 20$ is giving the convergence for $s > 4$, see Table 9. In other words, the algorithm does not really fit w.r.t. large values of norms for the initial data.

s	$\ y_{k^*}\ _{L^2(Q_T)}$	$\ \rho(s)y_{k^*}\ _{L^2(Q_T)}$	$\ v_{k^*}\ _{L^2(0,T)}$	$\ \rho_1(s)v_{k^*}\ _{L^2(0,T)}$	$\ v_{k^*}\ _{H^1_0(0,T)}$	$\ v_{k^*}\ _{L^\infty(0,T)}$	k^*
3	18.16	1.981×10^{-2}	61.872	1.687×10^{-4}	247.795	93.918	17
4	14.998	4.168×10^{-3}	47.839	5.926×10^{-6}	196.934	72.472	14
5	12.562	9.055×10^{-4}	37.038	2.862×10^{-7}	158.616	55.242	13
6	11.791	2.0098×10^{-4}	38.389	1.472×10^{-8}	176.911	59.853	11
7	12.7496	4.535×10^{-5}	46.642	7.7768×10^{-10}	221.245	75.804	10

Table 8: $c_{u_0} = 10$; $c_f = 4$; $f(r) = c_f r(1 + \ln^2(2 + |r|))$; Norms of (y_{k^*}, v_{k^*}) w.r.t. s .

s	$\ y_{k^*}\ _{L^2(Q_T)}$	$\ \rho(s)y_{k^*}\ _{L^2(Q_T)}$	$\ v_{k^*}\ _{L^2(0,T)}$	$\ \rho_1(s)v_{k^*}\ _{L^2(0,T)}$	$\ v_{k^*}\ _{H^1_0(0,T)}$	$\ v_{k^*}\ _{L^\infty(0,T)}$	k^*
5	131.602	1.774×10^{-3}	616.056	4.783×10^{-7}	2668.36	1004.82	36
6	119.542	3.95×10^{-4}	559.19	2.594×10^{-8}	2470.46	933.931	25

Table 9: $c_{u_0} = 20$; $c_f = 4$; $f(r) = c_f r(1 + \ln^2(2 + |r|))$; Norms of (y_{k^*}, v_{k^*}) w.r.t. s .

We also perform some experiments for $s = 3$, $c_{u_0} = 10$ to see how the algorithm behaves with respect to different values of c_f . In Table 10, we see that the algorithm converges for the values of $c_f \in [-3, 4]$. On the other hand, for the same quantities $(s, c_{u_0}) = (3, 10)$, we have the divergence of our method when the nonlinear parameter $c_f \leq -4$ or $c_f \geq 5$.

c_f	$\ y_{k^*}\ _{L^2(Q_T)}$	$\ \rho(s)y_{k^*}\ _{L^2(Q_T)}$	$\ v_{k^*}\ _{L^2(0,T)}$	$\ \rho_1(s)v_{k^*}\ _{L^2(0,T)}$	$\ v_{k^*}\ _{L^\infty(0,T)}$	$C(y_{k^*}, v_{k^*})$	k^*
-3	433.899	8.057×10^{-2}	473.795	1.517×10^{-2}	503.043	4.122×10^{-2}	36
-2	16.512	2.446×10^{-2}	19.926	6.372×10^{-4}	20.044	9.318×10^{-2}	12
-1	5.0298	2.271×10^{-2}	3.587	5.838×10^{-5}	4.268	9.85×10^{-2}	7
0	5.902	2.176×10^{-2}	5.6657	9.726×10^{-5}	6.895	1.073×10^{-1}	1
1	7.819	2.108×10^{-2}	12.589	7.245×10^{-5}	15.975	9.249×10^{-2}	7
2	8.536	2.056×10^{-2}	17.4798	1.067×10^{-4}	23.876	8.187×10^{-2}	9
3	8.425	2.014×10^{-2}	17.138	1.519×10^{-4}	20.768	7.375×10^{-2}	11
4	18.16	1.981×10^{-2}	61.872	1.687×10^{-4}	93.918	6.724×10^{-2}	17

Table 10: $c_{u_0} = 10$; $s = 3$; $f(r) = c_f r(1 + \ln^2(2 + |r|))$; Norms of (y_{k^*}, v_{k^*}) w.r.t. c_f .

Next, we make some experiments for the nonlinearities f that behave like $r \ln^p |r|$ at infinity when $p > 2$ and therefore does not satisfy (\mathbf{H}_1) . Below, we consider the nonlinear function

$$f_p(r) = c_f r(1 + \ln^p(2 + |r|)), \quad \text{for } p > 2, \quad \forall r \in \mathbb{R}. \quad (55)$$

We refer to Table 11 for some results.

We have observed that by choosing $p \geq 2.4$, the algorithm does not converge anymore. More precisely, the norms of the solutions and controls are blowing up as number of iterations are increasing, which is in accordance with the result in Theorem 2.

	$\ y_{k^*}\ _{L^2(Q_T)}$	$\ \rho(s)y_{k^*}\ _{L^2(Q_T)}$	$\ v_{k^*}\ _{L^2(0,T)}$	$\ \rho_1(s)v_{k^*}\ _{L^2(0,T)}$	$\ v_{k^*}\ _{L^\infty(0,T)}$	$C(y_{k^*}, v_{k^*})$	k^*
$p = 2$	16.512	2.446×10^{-2}	19.926	6.372×10^{-4}	20.044	9.318×10^{-2}	12
$p = 2.05$	20.581	2.472×10^{-2}	23.047	8.334×10^{-4}	25.093	9.293×10^{-2}	12
$p = 2.1$	26.176	2.508×10^{-2}	24.076	1.099×10^{-3}	32.173	9.237×10^{-2}	13
$p = 2.15$	34.134	2.564×10^{-2}	26.570	1.468×10^{-3}	44.653	9.077×10^{-2}	15
$p = 2.2$	47.681	2.662×10^{-2}	49.532	2.011×10^{-3}	63.213	8.617×10^{-2}	17
$p = 2.25$	76.081	2.883×10^{-2}	84.956	2.938×10^{-3}	95.777	7.511×10^{-2}	20
$p = 2.3$	136.668	3.542×10^{-2}	121.539	5.045×10^{-3}	156.929	5.612×10^{-2}	26

Table 11: $c_f = -2$, $s = 3$, $c_{u_0} = 10$; Norms of (y_{k^*}, v_{k^*}) w.r.t. f_p ; f_p given by (55).

6 Concluding remarks

By introducing a functional in the Carleman setting different than in the seminal paper of Zuazua in 1993, we have derived, under similar assumptions, a somehow simpler proof of the boundary controllability of a semilinear wave equation of the form $y_{tt} - y_{xx} + f(y) = 0$. Moreover, assuming an additional growth assumption on f' , we have constructed a sequence of state-control pairs, solution of a linear boundary controllability problem, converging pointwise and with a linear rate to a solution of the semilinear equation. As in the recent work [ELM21] devoted to the distributed controllability for a semilinear heat equation, the analysis emphasizes the role of the Carleman weights parameterized by the real s . Numerical experiments illustrates that the speed of convergence of the sequence is amplified as the Carleman parameter s is larger.

Our analysis is based on a simple fixed point strategy which consists to see the nonlinear term as a source term. It would be interesting to analyze whether or not the fixed point operator introduced by Zuazua in 1993, involving a potential, is, after reformulation in a functional Carleman setting, contracting for s large enough.

The fixed point argument employed here requires uniform bounds of the controlled trajectories for a linear wave equation: this is achieved by assuming the initial data in $H_0^1(\Omega) \times L^2(\Omega)$ and by imposing that the control satisfies at the initial and final time some compatibility conditions with the solution: this leads to boundary controls in $H_0^1(0, T)$ and then, in our one dimensional situation, to trajectories in $L^\infty(Q_T)$. Assuming more regularity on the initial conditions, we may extend our results for multi-dimensional situations and for nonlinearities depending on the gradient of the solution. This will be addressed in future works.

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A Appendix: Proof of Theorem 7

In what follows, in order to simplify the notations, we shall just write $\rho = \rho(t)$ and $\rho_1 = \rho_1(t)$ instead of $\rho = \rho(s; x, t)$ and $\rho_1 = \rho_1(s; t)$.

Preliminary to the proof and following [EZ10], for all $f \in \mathcal{C}^0(\mathbb{R}; E)$ (where E is a Banach space) and any $\tau > 0$, we define $\delta_\tau f := f(t + \frac{\tau}{2}) - f(t - \frac{\tau}{2})$ and

$$\mathcal{T}_\tau f := \frac{1}{\tau} \delta_\tau \left(\frac{\delta_\tau f}{\tau} \right) = \frac{f(t + \tau) - 2f(t) + f(t - \tau)}{\tau^2}.$$

Let now $w_s \in P_s$ and the solution $y_s \in L^2(Q_T)$ be given by Theorem 6. Then, z defined by $z = \mathcal{T}_\tau w_s$ belongs to P_s , where w_s as well as y_s can be extended uniquely on $(-\infty, 0)$ and $(T, +\infty)$.

Indeed, in the interval $(-\infty, 0)$ the solution y_s satisfies the following set of equations

$$\begin{cases} Ly_s = 0 & \text{in } \Omega \times (-\infty, 0), \\ y_s(0, t) = y_s(1, t) = 0 & \text{for } t \in (-\infty, 0), \\ (y_s(\cdot, 0), \partial_t y_s(\cdot, 0)) = (u_0, u_1) & \text{in } \Omega, \end{cases} \quad (56)$$

where the source term $B \in L^2(Q_T)$ is assumed to be extendable by 0 outside $(0, T)$. Recall that the boundary condition $y_s(1, t) = 0$ holds outside $(0, T)$ since $\eta = 0$ (appearing in the formula of v_s) vanishes outside $(\delta, T - \delta)$.

Similarly, in $(T, +\infty)$ we can define the solution y_s uniquely, and $y_s(t) = 0$ for all $t \geq T$. It follows that the solution y_s satisfies $y_s \in \mathcal{C}^0(\mathbb{R}; L^2(\Omega)) \cap \mathcal{C}^1(\mathbb{R}; H^{-1}(\Omega))$ and $y_s \in \mathcal{C}^0((-\infty, \delta]; H^1(\Omega)) \cap \mathcal{C}^1((-\infty, \delta]; L^2(\Omega))$ and $y_s \in \mathcal{C}^0([T - \delta, +\infty); H^1(\Omega)) \cap \mathcal{C}^1([T - \delta, +\infty); L^2(\Omega))$ (see [LLT86]). We extend as well the weights ρ and ρ_1 in $\Omega \times \mathbb{R}$ so that it preserves smoothness and positivity properties.

This ensures the extension of the solution w_s which satisfies the following set of equations in \mathbb{R}

$$\begin{cases} Lw_s = \rho^2 y_s & \text{in } \Omega \times \mathbb{R}, \\ w_s(0, t) = w_s(1, t) = 0, & \text{in } \mathbb{R}. \end{cases} \quad (57)$$

Moreover, it can be seen that $Lw_s = 0$ in $[T, +\infty)$, since y_s is a controlled solution to (56).

We now proceed to the proof of Theorem 7, done in three steps.

Step 1 : We suppose first that $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$ and $B \in \mathcal{D}(0, T; L^2(\Omega))$ and prove that $v_s \in H^1(0, T)$ and $(y_s)_t \in L^2(Q_T)$.

We start by considering the variational formulation (12) by choosing $z = \mathcal{T}_\tau w_s$ as test function. Since $w_s \in \mathcal{C}^0(\mathbb{R}; H_0^1(\Omega)) \cap \mathcal{C}^1(\mathbb{R}; L^2(\Omega))$ solves (57), it is clear that $\mathcal{T}_\tau w_s \in \mathcal{C}^0([0, T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega))$, $(\mathcal{T}_\tau w_s)_x \in L^2(0, T)$. With this z , the formulation reads

$$\begin{aligned} & \int_{Q_T} \rho^{-2} Lw_s L\mathcal{T}_\tau w_s \, dxdt + s \int_0^T \eta^2(t) \rho_1^{-2}(w_s)_x(1, t) \mathcal{T}_\tau (w_s)_x(1, t) dt \\ & = \int_\Omega u_1(x) \mathcal{T}_\tau w_s(x, 0) \, dx - \int_\Omega u_0(x) \mathcal{T}_\tau (w_s)_t(x, 0) \, dx + \int_{Q_T} B \mathcal{T}_\tau w_s \, dxdt. \end{aligned} \quad (58)$$

Sub-step 1. Let us start with the first integral in the left hand side of (58). We have

$$\begin{aligned} & \int_{Q_T} \rho^{-2}(t) Lw_s(t) L\mathcal{T}_\tau w_s(t) \, dxdt \\ & = \frac{1}{\tau} \int_{Q_T} \rho^{-2}(t) Lw_s(t) \frac{Lw_s(t + \tau) - Lw_s(t)}{\tau} \, dxdt \\ & \quad - \frac{1}{\tau} \int_{Q_T} \rho^{-2}(t) Lw_s(t) \frac{Lw_s(t) - Lw_s(t - \tau)}{\tau} \, dxdt \\ & = \frac{1}{\tau} \int_{Q_T} \rho^{-2}(t) Lw_s(t) \frac{Lw_s(t + \tau) - Lw_s(t)}{\tau} \, dxdt \\ & \quad - \frac{1}{\tau} \int_{-\tau}^{T-\tau} \int_\Omega \rho^{-2}(t + \tau) Lw_s(t + \tau) \frac{Lw_s(t + \tau) - Lw_s(t)}{\tau} \, dxdt \\ & = \int_{Q_T} \left(\frac{\rho^{-2}(t) Lw_s(t) - \rho^{-2}(t + \tau) Lw_s(t + \tau)}{\tau} \right) \left(\frac{Lw_s(t + \tau) - Lw_s(t)}{\tau} \right) \, dxdt \\ & \quad - \frac{1}{\tau} \int_{-\tau}^0 \int_\Omega \rho^{-2}(t + \tau) Lw_s(t + \tau) \left(\frac{Lw_s(t + \tau) - Lw_s(t)}{\tau} \right) \, dxdt \end{aligned}$$

since $Lw_s = \rho^2 y_s = 0$ on $[T, +\infty)$. Now, observe that

$$\begin{aligned}
& \int_{Q_T} \left(\frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau} \right) \left(\frac{Lw_s(t+\tau) - Lw_s(t)}{\tau} \right) dxdt \\
&= \int_{Q_T} \rho^2(t) \left(\frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau} \right) \rho^{-2}(t) \left(\frac{Lw_s(t+\tau) - Lw_s(t)}{\tau} \right) dxdt \\
&= - \int_{Q_T} \rho^2(t) \left| \frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau} \right|^2 dxdt \\
&\quad + \int_{Q_T} \rho^2(t) \left(\frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau} \right) \left(\frac{\rho^{-2}(t) - \rho^{-2}(t+\tau)}{\tau} \right) Lw_s(t+\tau) dxdt.
\end{aligned} \tag{59}$$

The equality (59) then reads

$$\begin{aligned}
& \int_{Q_T} \rho^{-2}Lw_s(t)L\mathcal{T}_\tau w_s(t) dxdt \\
&= - \int_{Q_T} \rho^2(t) \left| \frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau} \right|^2 dxdt \\
&\quad + \int_{Q_T} \rho^2(t) \left(\frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau} \right) \left(\frac{\rho^{-2}(t) - \rho^{-2}(t+\tau)}{\tau} \right) Lw_s(t+\tau) dxdt \\
&\quad - \frac{1}{\tau} \int_{-\tau}^0 \int_{\Omega} \rho^{-2}(t+\tau)Lw_s(t+\tau) \left(\frac{Lw_s(t+\tau) - Lw_s(t)}{\tau} \right) dxdt.
\end{aligned} \tag{60}$$

Next, we shall look into the second term in the left hand side of (58). First, recall the smooth function η given by (9) satisfies $\eta = 0$ in $(-\infty, \delta] \cup [T - \delta, +\infty)$ (with $\delta > 0$ given in (9)). Then, in a similar way that have lead to (59), we have assuming $\tau \leq \delta$:

$$\begin{aligned}
& \int_0^T \eta^2(t)\rho_1^{-2}(t)(w_s)_x(1,t)\mathcal{T}_\tau(w_s)_x(1,t)dt \\
&= \int_0^T \left(\frac{\eta^2(t)\rho_1^{-2}(t)(w_s)_x(1,t) - \eta^2(t+\tau)\rho_1^{-2}(t+\tau)(w_s)_x(1,t+\tau)}{\tau} \right) \\
&\quad \times \left(\frac{(w_s)_x(1,t+\tau) - (w_s)_x(1,t)}{\tau} \right) dt.
\end{aligned} \tag{61}$$

Then, using the identity

$$ad - bc = \frac{(a-c)(b+d) - (a+c)(b-d)}{2}, \quad \forall(a, b, c, d) \in \mathbb{R}^4 \tag{62}$$

with $a = \eta^2(t)\rho_1^{-2}(t)$, $b = (w_s)_x(1, t + \tau)$, $c = \eta^2(t + \tau)\rho_1^{-2}(t + \tau)$ and $d = (w_s)_x(1, t)$, we obtain from (61)

$$\begin{aligned}
& \int_0^T \eta^2(t)\rho_1^{-2}(t)(w_s)_x(1,t)\mathcal{T}_\tau(w_s)_x(1,t)dt \\
&= \int_0^T \frac{(\eta^2(t)\rho_1^{-2}(t) - \eta^2(t+\tau)\rho_1^{-2}(t+\tau))((w_s)_x(1,t) + (w_s)_x(1,t+\tau))}{2\tau} \\
&\quad \times \left(\frac{(w_s)_x(1,t+\tau) - (w_s)_x(1,t)}{\tau} \right) dt \\
&\quad - \int_0^T \frac{(\eta^2(t)\rho_1^{-2}(t) + \eta^2(t+\tau)\rho_1^{-2}(t+\tau))}{2} \left| \frac{(w_s)_x(1,t+\tau) - (w_s)_x(1,t)}{\tau} \right|^2 dt.
\end{aligned} \tag{63}$$

Now, using (60) and (63) in the formulation (58), we have

$$\begin{aligned}
& \int_{Q_T} \rho^2(t) \left| \frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau} \right|^2 dxdt \\
& + s \int_0^T \frac{(\eta^2(t)\rho_1^{-2}(t) + \eta^2(t+\tau)\rho_1^{-2}(t+\tau))}{2} \left| \frac{(w_s)_x(1, t+\tau) - (w_s)_x(1, t)}{\tau} \right|^2 dt \\
= & \int_{Q_T} \rho^2(t) \left(\frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau} \right) \left(\frac{\rho^{-2}(t) - \rho^{-2}(t+\tau)}{\tau} \right) Lw_s(t+\tau) dxdt \\
& - \frac{1}{\tau} \int_{-\tau}^0 \int_{\Omega} \rho^{-2}(t+\tau)Lw_s(t+\tau) \left(\frac{Lw_s(t+\tau) - Lw_s(t)}{\tau} \right) dxdt \\
& + s \int_0^T \frac{(\eta^2(t)\rho_1^{-2}(t) - \eta^2(t+\tau)\rho_1^{-2}(t+\tau)) ((w_s)_x(1, t) + (w_s)_x(1, t+\tau))}{2\tau} \\
& \quad \times \left(\frac{(w_s)_x(1, t+\tau) - (w_s)_x(1, t)}{\tau} \right) dt \\
& - \int_{Q_T} B\mathcal{T}_\tau w_s dxdt - \int_{\Omega} u_1 \mathcal{T}_\tau w_s(\cdot, 0) dx + \int_{\Omega} u_0 \mathcal{T}_\tau (w_s)_t(\cdot, 0) dx \\
:= & I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \tag{64}
\end{aligned}$$

Sub-step 2. In this step, we obtain precise estimates for the terms I_1 and I_3 and then an estimate of the left hand side of (64).

(i) *Estimate of I_1 .* Young's inequality leads to

$$\begin{aligned}
|I_1| \leq & \frac{1}{2} \int_{Q_T} \rho^2(t) \left| \frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau} \right|^2 dxdt \\
& + \frac{1}{2} \int_{Q_T} \rho^2(t) \left| \left(\frac{\rho^{-2}(t) - \rho^{-2}(t+\tau)}{\tau} \right) Lw_s(t+\tau) \right|^2 dxdt. \tag{65}
\end{aligned}$$

(ii) *Estimate of I_3 .* We have

$$\begin{aligned}
|I_3| = & s \left| \int_0^T \frac{(\eta(t)\rho_1^{-1}(t) - \eta(t+\tau)\rho_1^{-1}(t+\tau)) ((w_s)_x(1, t) + (w_s)_x(1, t+\tau))}{2\tau} \right. \\
& \quad \left. \times \frac{(\eta(t)\rho_1^{-1}(t) + \eta(t+\tau)\rho_1^{-1}(t+\tau)) ((w_s)_x(1, t+\tau) - (w_s)_x(1, t))}{\tau} dt \right| \\
\leq & 2s \int_0^T \left| \frac{(\eta(t)\rho_1^{-1}(t) - \eta(t+\tau)\rho_1^{-1}(t+\tau)) ((w_s)_x(1, t) + (w_s)_x(1, t+\tau))}{2\tau} \right|^2 dt \\
& + \frac{s}{8} \int_0^T |\eta(t)\rho_1^{-1}(t) + \eta(t+\tau)\rho_1^{-1}(t+\tau)|^2 \left| \frac{(w_s)_x(1, t+\tau) - (w_s)_x(1, t)}{\tau} \right|^2 dt \\
\leq & 2s \int_0^T \left| \frac{(\eta(t)\rho_1^{-1}(t) - \eta(t+\tau)\rho_1^{-1}(t+\tau)) ((w_s)_x(1, t) + (w_s)_x(1, t+\tau))}{2\tau} \right|^2 dt \\
& + \frac{s}{2} \int_0^T \frac{(\eta^2(t)\rho_1^{-2}(t) + \eta^2(t+\tau)\rho_1^{-2}(t+\tau))}{2} \left| \frac{(w_s)_x(1, t+\tau) - (w_s)_x(1, t)}{\tau} \right|^2 dt. \tag{66}
\end{aligned}$$

(iii) A first estimate of the left hand side of (64). The previous estimates and (64) give

$$\begin{aligned}
& \frac{1}{2} \int_{Q_T} \rho^2(t) \left| \frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau} \right|^2 dxdt \\
& + \frac{s}{2} \int_0^T \frac{(\eta^2(t)\rho_1^{-2}(t) + \eta^2(t+\tau)\rho_1^{-2}(t+\tau))}{2} \left| \frac{(w_s)_x(1,t) - (w_s)_x(1,t+\tau)}{\tau} \right|^2 dt \\
& \leq \frac{1}{2} \int_{Q_T} \rho^2(t) \left| \left(\frac{\rho^{-2}(t) - \rho^{-2}(t+\tau)}{\tau} \right) Lw_s(t+\tau) \right|^2 dxdt \\
& + \frac{1}{\tau} \left| \int_{-\tau}^0 \int_{\Omega} \rho^{-2}(t+\tau)Lw_s(t+\tau) \left(\frac{Lw_s(t+\tau) - Lw_s(t)}{\tau} \right) dxdt \right| \\
& + 2s \int_0^T \left| \frac{(\eta(t)\rho_1^{-1}(t) - \eta(t+\tau)\rho_1^{-1}(t+\tau))((w_s)_x(1,t) + (w_s)_x(1,t+\tau))}{2\tau} \right|^2 dt \\
& + \left| \int_{Q_T} B\mathcal{T}_\tau w_s dxdt \right| + \left| \int_{\Omega} u_0 \mathcal{T}_\tau (w_s)_t(\cdot, 0) dx \right| + \left| \int_{\Omega} u_1 \mathcal{T}_\tau w_s(\cdot, 0) dx \right| \\
& := J_1 + J_2 + J_3 + J_4 + J_5 + J_6.
\end{aligned} \tag{67}$$

Sub-step 3 : We prove that the left hand side of (67) is bounded uniformly with respect to $\tau \in [0, \delta]$.

(i) J_1 is bounded. Since $\rho^{-2} \in \mathcal{C}^\infty(\overline{Q_T})$, $(\rho^{-1})_t = -2s\lambda\beta(t - \frac{T}{2})\phi\rho^{-1}$ and $Lw_s \in \mathcal{C}^0(\mathbb{R}; L^2(\Omega))$:

$$\int_{Q_T} \rho^2(t) \left| \left(\frac{\rho^{-2}(t) - \rho^{-2}(t+\tau)}{\tau} \right) Lw_s(t+\tau) \right|^2 dxdt \rightarrow 4s^2\lambda^2\beta^2 \int_{Q_T} \left(t - \frac{T}{2} \right)^2 \phi^2(t)\rho^2(t)y_s^2 dxdt$$

as $\tau \rightarrow 0$ and thus J_1 is bounded.

(ii) J_2 is bounded. Since $\rho^{-2}Lw_s = y_s \in \mathcal{C}^0(\mathbb{R}; L^2(\Omega))$, $\rho^{-2} \in \mathcal{C}^\infty(\overline{Q_T})$ and $Lw_s = \rho^2 y_s \in \mathcal{C}^1((-\infty, \delta]; L^2(\Omega))$ we have, as $\tau \rightarrow 0$

$$\frac{1}{\tau} \left| \int_{-\tau}^0 \int_{\Omega} \rho^{-2}(t+\tau)Lw_s(t+\tau) \left(\frac{Lw_s(t+\tau) - Lw_s(t)}{\tau} \right) dxdt \right| \rightarrow \left| \int_{\Omega} y_s(0)(\rho^2 y_s)_t(0) \right|$$

and thus J_2 is bounded.

(iii) J_3 is bounded. Since $(w_s)_x(1, \cdot) \in L^2(0, T)$ and $\eta\rho_1^{-1} \in \mathcal{C}^1(\mathbb{R})$ we have

$$\begin{aligned}
& 2s \int_0^T \left| \frac{(\eta(t)\rho_1^{-1}(t) - \eta(t+\tau)\rho_1^{-1}(t+\tau))((w_s)_x(1,t) + (w_s)_x(1,t+\tau))}{2\tau} \right|^2 dt \\
& \rightarrow 2s \int_0^T |(\eta\rho_1^{-1})_t(t)(w_s)_x(1,t)|^2 dt
\end{aligned}$$

as $\tau \rightarrow 0$ and thus J_3 is bounded.

(iv) J_4 is bounded. For τ small enough, since $B \in \mathcal{D}(\mathbb{R}; L^2(\Omega))$ and $w_s \in \mathcal{C}^1(\mathbb{R}; L^2(\Omega))$, we have

$$\left| \int_{Q_T} B\mathcal{T}_\tau w_s dxdt \right| = \left| \int_{-\tau}^{T+\tau} \int_{\Omega} \mathcal{T}_\tau B w_s dxdt \right| \rightarrow \left| \int_{Q_T} B_{tt} w_s dxdt \right| \tag{68}$$

as $\tau \rightarrow 0$ and thus J_4 is bounded.

(v) J_5 is bounded. We have $Lw_s = \rho^2 y_s \in \mathcal{C}^0(\mathbb{R}; L^2(\Omega))$ and $w_s \in \mathcal{C}^0(\mathbb{R}; H_0^1(\Omega))$, thus $(w_s)_{tt} = Lw_s + (w_s)_{xx} \in \mathcal{C}(\mathbb{R}; H^{-1}(\Omega))$. We then have, for all $t \in \mathbb{R}$:

$$(w_s)_t(t) - (w_s)_t(0) = \int_0^t Lw_s(\xi) d\xi + \int_0^t (w_s)_{xx}(\xi) d\xi.$$

This yields

$$\begin{aligned}
\mathcal{T}_\tau(w_s)_t(0) &= \frac{(w_s)_t(\tau) - 2(w_s)_t(0) + (w_s)_t(-\tau)}{\tau^2} \\
&= \frac{1}{\tau^2} \left(\int_0^\tau Lw_s(\xi) \, d\xi + \int_0^{-\tau} Lw_s(\xi) \, d\xi + \int_0^\tau (w_s)_{xx}(\xi) \, d\xi + \int_0^{-\tau} (w_s)_{xx}(\xi) \, d\xi \right) \\
&= \frac{2}{\tau} \int_0^\tau \frac{\xi}{\tau} \left(\frac{Lw_s(\xi) - Lw_s(-\xi)}{2\xi} \right) \, d\xi + \frac{2}{\tau} \int_0^\tau \frac{\xi}{\tau} \left(\frac{(w_s)_{xx}(\xi) - (w_s)_{xx}(-\xi)}{2\xi} \right) \, d\xi.
\end{aligned}$$

Now, since $Lw_s = \rho^2 y_s \in \mathcal{C}^1((-\infty, \delta]; L^2(\Omega))$, we write that

$$\begin{aligned}
&\int_\Omega u_0 \frac{2}{\tau} \int_0^\tau \frac{\xi}{\tau} \left(\frac{Lw_s(\xi) - Lw_s(-\xi)}{2\xi} \right) \, d\xi \, dx \\
&\rightarrow \int_\Omega u_0 (\rho^2 y_s)_t(0) \, dx = -2s\lambda\beta T \int_\Omega \phi(0)\rho^2(0)u_0^2 \, dx + \int_\Omega \rho^2(0)u_0u_1 \, dx \quad \text{as } \tau \rightarrow 0. \tag{69}
\end{aligned}$$

On the other hand, since $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $w_s \in \mathcal{C}^0(\mathbb{R}; H_0^1(\Omega))$:

$$\begin{aligned}
&\frac{2}{\tau} \int_0^\tau \left\langle \frac{\xi}{\tau} \left(\frac{(w_s)_{xx}(\xi) - (w_s)_{xx}(-\xi)}{2\xi} \right), u_0 \right\rangle_{H^{-1}, H_0^1} \, d\xi \\
&= \frac{2}{\tau} \int_0^\tau \frac{\xi}{\tau} \int_\Omega (u_0)_{xx} \frac{(w_s)_{xx}(\xi) - (w_s)_{xx}(-\xi)}{2\xi} \, dx \, d\xi \\
&\rightarrow \int_\Omega (u_0)_{xx} (w_s)_t(0) \, dx \quad \text{as } \tau \rightarrow 0 \tag{70}
\end{aligned}$$

since moreover $w_s \in \mathcal{C}^1(\mathbb{R}; L^2(\Omega))$. Thus

$$\int_\Omega u_0 \mathcal{T}_\tau(w_s)_t(\cdot, 0) \, dx \rightarrow -2s\lambda\beta T \int_\Omega \phi(0)\rho^2(0)u_0^2 + \int_\Omega \rho^2(0)u_0u_1 + \int_\Omega (u_0)_{xx} (w_s)_t(0) \, dx \tag{71}$$

as $\tau \rightarrow 0$ and thus J_5 is bounded.

(vi) J_6 is bounded. We have $Lw_s = \rho^2 y_s \in \mathcal{C}^0(\mathbb{R}; L^2(\Omega))$ and $w_s \in \mathcal{C}^1(\mathbb{R}; H_0^1(\Omega))$, thus $(w_s)_{tt} = \rho^2 y_s + (w_s)_{xx} \in \mathcal{C}^0(\mathbb{R}; H^{-1}(\Omega))$. Therefore

$$\mathcal{T}_\tau w(0) = \frac{w(\tau) - 2w(0) + w(-\tau)}{\tau^2} \rightarrow (w_s)_{tt}(0) = (w_s)_{xx}(0) + \rho^2(0)u_0 \text{ in } H^{-1}(\Omega)$$

as $\tau \rightarrow 0$ and thus

$$\begin{aligned}
&\int_\Omega u_1 \mathcal{T}_\tau w(\cdot, 0) \, dx \rightarrow \langle (w_s)_{tt}(0), u_1 \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \\
&= \langle (w_s)_{xx}(0), u_1 \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \int_\Omega \rho^2(0)u_1u_0 \, dx \\
&= - \int_\Omega (w_s)_x(\cdot, 0)(u_1)_x \, dx + \int_\Omega \rho^2(0)u_1u_0 \, dx. \tag{72}
\end{aligned}$$

as $\tau \rightarrow 0$. $J_6 = \left| \int_\Omega u_1 \mathcal{T}_\tau w(\cdot, 0) \, dx \right|$ is therefore bounded.

(vii) Then we can conclude, from (67), that the terms

$$\int_{Q_\tau} \rho^2(t) \left| \frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau} \right|^2 \, dx \, dt$$

and

$$\int_0^T \frac{(\eta^2(t)\rho_1^{-2}(t) + \eta^2(t+\tau)\rho_1^{-2}(t+\tau))}{2} \left| \frac{(w_s)_x(1, t) - (w_s)_x(1, t+\tau)}{\tau} \right|^2 \, dt$$

are bounded. Remark that this implies that the two terms $\int_{Q_T} \rho^{-2}(t) |L(\frac{\delta_\tau w_s}{\tau})|^2 dxdt$ and $\int_0^T \eta^2(t) \rho_1^{-2}(t) |(\frac{\delta_\tau w_s}{\tau})_x(1, t)|^2 dt$ are bounded; indeed,

$$\begin{aligned} & \int_0^T \eta^2(t) \rho_1^{-2}(t) |(\frac{\delta_\tau w_s}{\tau})_x(1, t)|^2 dt \\ & \leq 2 \int_0^T \frac{(\eta^2(t) \rho_1^{-2}(t) + \eta^2(t + \tau) \rho_1^{-2}(t + \tau))}{2} \left| \frac{(w_s)_x(1, t) - (w_s)_x(1, t + \tau)}{\tau} \right|^2 dt. \end{aligned}$$

We also have

$$\begin{aligned} \int_{Q_T} \rho^{-2}(t) |L(\frac{\delta_\tau w_s}{\tau})|^2 dxdt & \leq 2 \int_{Q_T} \rho^2(t) \left| \frac{\rho^{-2}(t) Lw_s(t) - \rho^{-2}(t + \tau) Lw_s(t + \tau)}{\tau} \right|^2 dxdt \\ & \quad + 2 \int_{Q_T} \rho^2(t) \left| \frac{\rho^{-2}(t) - \rho^{-2}(t + \tau)}{\tau} Lw_s(t + \tau) \right|^2 dxdt \end{aligned}$$

and each term of the right hand side is bounded.

Sub-step 4. In this step, we prove that $v_s \in H^1(0, T)$ and $y_s \in \mathcal{C}^0([0, T]; H^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega))$.

Since $\frac{\delta_\tau w_s}{\tau} \in \mathcal{C}^0([0, T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega))$ and satisfies $(\frac{\delta_\tau w_s}{\tau})_x(1, \cdot) \in L^2(0, T)$ then the Carleman estimates (10) gives

$$\begin{aligned} & s \int_{Q_T} \rho^{-2}(t) \left(|(\frac{\delta_\tau w_s}{\tau})_t|^2 + |(\frac{\delta_\tau w_s}{\tau})_x|^2 \right) dxdt + s^3 \int_{Q_T} \rho^{-2}(t) |\frac{\delta_\tau w_s}{\tau}|^2 dxdt \\ & \quad + s \int_\Omega \rho^{-2}(0) \left(|(\frac{\delta_\tau w_s}{\tau})_t(x, 0)|^2 + |(\frac{\delta_\tau w_s}{\tau})_x(x, 0)|^2 \right) dx + s^3 \int_\Omega \rho^{-2}(0) |\frac{\delta_\tau w_s}{\tau}(x, 0)|^2 dx \\ & \leq C \int_{Q_T} \rho^{-2}(t) |L(\frac{\delta_\tau w_s}{\tau})|^2 dxdt + Cs \int_0^T \eta^2(t) \rho_1^{-2}(t) |(\frac{\delta_\tau w_s}{\tau})_x(1, t)|^2 dt. \quad (73) \end{aligned}$$

Therefore, since the right hand side is bounded, $(\frac{\delta_\tau w_s}{\tau})_t$ and $(\frac{\delta_\tau w_s}{\tau})_x$ are bounded in $L^2(Q_T)$ and thus $(w_s)_{tt} \in L^2(Q_T)$ and $(w_s)_t \in L^2(0, T; H_0^1(\Omega))$. Moreover, $\frac{\delta_\tau w_s}{\tau}(\cdot, 0)$ is bounded in $H_0^1(\Omega)$ thus $(w_s)_t(\cdot, 0) \in H_0^1(\Omega)$. We also have $L(\frac{\delta_\tau w_s}{\tau})$ bounded in $L^2(Q_T)$ so $L(w_s)_t \in L^2(Q_T)$. Thus $(w_s)_t$ satisfies

$$\begin{cases} L(w_s)_t \in L^2(Q_T), \\ (w_s)_t(0, t) = (w_s)_t(1, t) = 0, \quad t \in (0, T) \\ ((w_s)_t(0), (w_s)_{tt}(0)) \in H_0^1(\Omega) \times L^2(\Omega) \end{cases}$$

and thus $(w_s)_t \in \mathcal{C}^0([0, T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega))$ and $(w_s)_{tx}(1, \cdot) \in L^2(0, T)$. Therefore from the definition of v_s , $v_s \in H^1(0, T)$ while from the equation satisfied by (y_s, v_s) (see (17)), $y_s \in \mathcal{C}^0([0, T]; H^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega))$.

Remark 8. We then have $w_s \in \mathcal{C}^1([0, T]; H_0^1(\Omega)) \cap \mathcal{C}^2([0, T]; L^2(\Omega))$ and from the equation satisfied by w_s , since $Lw_s \in \mathcal{C}^1([0, T]; L^2(\Omega))$ we deduce that $(w_s)_{xx} = (w_s)_{tt} - Lw_s \in \mathcal{C}^0([0, T]; L^2(\Omega))$ and thus that $(w_s)_{xx}(\cdot, 0) \in L^2(\Omega)$.

Step 2 : In this step, we give estimates on $(v_s)_t$ and $(y_s)_t$.

First of all, since $(w_s)_t \in \mathcal{C}^0([0, T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0, T]; L^2(\Omega))$, $L(w_s)_t \in L^2(Q_T)$ and $(w_s)_{tx}(1, \cdot) \in L^2(0, T)$, we can write the Carleman estimate (10) for $(w_s)_t$ leading to

$$\begin{aligned} & s \int_{Q_T} \rho^{-2}(t) (|(w_s)_{tt}|^2 + |(w_s)_{tx}|^2) dxdt + s^3 \int_{Q_T} \rho^{-2}(t) |(w_s)_t|^2 dxdt \\ & \quad + s \int_\Omega \rho^{-2}(0) (|(w_s)_{tt}(x, 0)|^2 + |(w_s)_{tx}(x, 0)|^2) dx + s^3 \int_\Omega \rho^{-2}(0) |(w_s)_t(x, 0)|^2 dx \\ & \leq C \int_{Q_T} \rho^{-2}(t) |(Lw_s)_t|^2 dxdt + Cs \int_0^T \eta^2(t) \rho_1^{-2}(t) |(w_s)_{tx}(1, t)|^2 dt. \quad (74) \end{aligned}$$

Sub-step 1 : In this step, we pass to the limit when $\tau \rightarrow 0$ in equation (64). We have, since $y_s = \rho^{-2}Lw_s \in \mathcal{C}^1(\mathbb{R}; L^2(\Omega))$:

$$\int_{Q_T} \rho^2(t) \left| \frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau} \right|^2 dxdt \rightarrow \int_{Q_T} \rho^2(t) |(y_s)_t|^2 dxdt$$

as $\tau \rightarrow 0$ and since $(w_s)_{tx}(1, \cdot) \in L^2(-\delta, T + \delta)$ and $\eta\rho_1^{-1} \in \mathcal{C}(\mathbb{R})$:

$$\begin{aligned} \int_0^T \frac{(\eta^2(t)\rho_1^{-2}(t) + \eta^2(t+\tau)\rho_1^{-2}(t+\tau))}{2} \left| \frac{(w_s)_x(1, t+\tau) - (w_s)_x(1, t)}{\tau} \right|^2 dt \\ \rightarrow \int_0^T \eta^2(t)\rho_1^{-2}(t) |(w_s)_{tx}(1, t)|^2 dt \end{aligned}$$

as $\tau \rightarrow 0$. Since $y_s = \rho^{-2}Lw_s \in \mathcal{C}^1(\mathbb{R}; L^2(\Omega))$, $Lw_s \in \mathcal{C}^1(\mathbb{R}; L^2(\Omega))$ and $(\rho^{-1})_t = -2s\lambda\beta(t - \frac{T}{2})\phi\rho^{-1}$ in Q_T , we infer that

$$\begin{aligned} \int_{Q_T} \rho^2(t) \left(\frac{\rho^{-2}(t)Lw_s(t) - \rho^{-2}(t+\tau)Lw_s(t+\tau)}{\tau} \right) \left(\frac{\rho^{-2}(t) - \rho^{-2}(t+\tau)}{\tau} \right) Lw_s(t+\tau) dxdt \\ \rightarrow -2s\lambda\beta \int_{Q_T} (t - \frac{T}{2})\phi(t)\rho^2(t)(y_s)_t y_s dxdt \end{aligned}$$

and

$$\frac{1}{\tau} \int_{-\tau}^0 \int_{\Omega} \rho^{-2}(t+\tau)Lw_s(t+\tau) \left(\frac{Lw_s(t+\tau) - Lw_s(t)}{\tau} \right) dxdt \rightarrow \int_{\Omega} y_s(0)(\rho^2 y_s)_t(0) dx$$

as $\tau \rightarrow 0$.

Similarly, since $w_s \in \mathcal{C}^2(\mathbb{R}; L^2(\Omega))$ and $(w_s)_{tx}(1, \cdot) \in L^2(-\delta, T + \delta)$,

$$\begin{aligned} \int_0^T \frac{(\eta^2(t)\rho_1^{-2}(t) - \eta^2(t+\tau)\rho_1^{-2}(t+\tau))}{2\tau} ((w_s)_x(1, t) + (w_s)_x(1, t+\tau)) \\ \times \left(\frac{(w_s)_x(1, t+\tau) - (w_s)_x(1, t)}{\tau} \right) dt \\ = \int_0^{T-\delta} \frac{(\eta^2(t)\rho_1^{-2}(t) - \eta^2(t+\tau)\rho_1^{-2}(t+\tau))}{2\tau} ((w_s)_x(1, t) + (w_s)_x(1, t+\tau)) \\ \times \left(\frac{(w_s)_x(1, t+\tau) - (w_s)_x(1, t)}{\tau} \right) dt \\ \rightarrow \int_0^T (\eta^2\rho_1^{-2})_t (w_s)_x(1, t)(w_s)_{tx}(1, t) dt, \end{aligned}$$

and

$$\int_{Q_T} B\mathcal{T}_\tau w_s dxdt \rightarrow \int_{Q_T} B(w_s)_{tt} dxdt.$$

as $\tau \rightarrow 0$. Since $(w_s)_{tt}(\cdot, 0) \in L^2(\Omega)$, the convergence (72) reads

$$\int_{\Omega} u_1 \mathcal{T}_\tau w_s(\cdot, 0) dx \rightarrow \int_{\Omega} (w_s)_{tt}(\cdot, 0) u_1 dx.$$

Similarly, since $(w_s)_t(\cdot, 0) \in H_0^1(\Omega)$, (71) reads

$$\int_{\Omega} u_0 \mathcal{T}_\tau (w_s)_t(\cdot, 0) dx \rightarrow -2s\lambda\beta T \int_{\Omega} \phi(0)\rho^2(0)u_0^2 dx + \int_{\Omega} \rho^2(0)u_0 u_1 dx - \int_{\Omega} (u_0)_x (w_s)_{tx}(\cdot, 0) dx.$$

We conclude that the limit with respect to $\tau \rightarrow 0$ in (64) leads to the following equality

$$\begin{aligned}
& \int_{Q_T} \rho^2(t) |(y_s)_t|^2 \, dx dt + s \int_0^T \eta^2(t) \rho_1^{-2}(t) |(w_s)_{tx}(1, t)|^2 \, dt \\
&= -2s\lambda\beta \int_{Q_T} \left(t - \frac{T}{2}\right) \phi(t) \rho^2(t) (y_s)_t y_s \, dx dt - \int_{\Omega} y_s(0) (\rho^2 y_s)_t(0) \\
&\quad + s \int_0^T (\eta^2 \rho_1^{-2})_t(t) (w_s)_x(1, t) (w_s)_{tx}(1, t) \, dt - \int_{Q_T} B(w_s)_{tt} \, dx dt - \int_{\Omega} (w_s)_{tt}(\cdot, 0) u_1 \, dx \\
&\quad - 2s\lambda\beta T \int_{\Omega} \phi(0) \rho^2(0) u_0^2 \, dx + \int_{\Omega} \rho^2(0) u_1 u_0 \, dx - \int_{\Omega} (u_0)_x (w_s)_{tx}(\cdot, 0) \, dx \\
&:= K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7 + K_8
\end{aligned} \tag{75}$$

Sub-step 2 : In this step, we estimate each term $K_i, i = 1, \dots, 8$.

(i) We get that, there exists $C > 0$ only depending on T such that

$$|K_1| \leq \frac{1}{8} \int_{Q_T} \rho^2(t) |(y_s)_t|^2 \, dx dt + Cs^2 \int_{Q_T} \rho^2(t) |y_s|^2 \, dx dt.$$

(ii) Similarly, recalling that $y_s(\cdot, 0) = u_0$ and $(y_s)_t(\cdot, 0) = u_1$, there exists $C > 0$ such that

$$|K_2| \leq Cs \|\rho(0) u_0\|_{L^2(\Omega)}^2 + \|\rho(0) u_0\|_{L^2(\Omega)} \|\rho(0) u_1\|_{L^2(\Omega)}.$$

(iii) Using that $(\rho_1^{-1})_t = -2s\lambda\beta(t - \frac{T}{2})\phi\rho_1^{-1}$, we obtain

$$\begin{aligned}
|K_3| &= \left| s \int_0^T (\eta^2 \rho_1^{-2})_t (w_s)_x(1, t) (w_s)_{tx}(1, t) \, dt \right| \\
&\leq 2 \left(s \int_0^T |(\eta \rho_1^{-1})_t|^2 |(w_s)_x(1, t)|^2 \, dt \right)^{1/2} \left(s \int_0^T \eta^2(t) \rho_1^{-2}(t) |(w_s)_{tx}(1, t)|^2 \, dt \right)^{1/2} \\
&\leq C \left(s^3 \int_0^T \eta^2(t) \rho_1^{-2}(t) |(w_s)_x(1, t)|^2 \, dt + s \int_0^T \rho_1^{-2}(t) |(w_s)_x(1, t)|^2 \, dt \right)^{1/2} \times \\
&\quad \left(s \int_0^T \eta^2(t) \rho_1^{-2}(t) |(w_s)_{tx}(1, t)|^2 \, dt \right)^{1/2} \\
&\leq C \left(s \int_{\delta}^{T-\delta} \frac{\rho_1^2(t)}{\eta^2(t)} v_s^2 \, dt + s \int_0^T \rho_1^{-2}(t) |(w_s)_x(1, t)|^2 \, dt \right) + \frac{s}{8} \int_0^T \eta^2(t) \rho_1^{-2}(t) |(w_s)_{tx}(1, t)|^2 \, dt.
\end{aligned}$$

We now estimate the term $\int_0^T \rho_1^{-2}(t) |(w_s)_x(1, t)|^2 \, dt$ appearing in the previous inequality: proceeding as in [Lio88a, Lemma 3.7] with $q(x, t) = x\rho^{-2}(x, t)$ such that $q(0, t) = 0$ and $q(1, t) = \rho_1^{-2}(t)$, we get the equality

$$\frac{1}{2} \int_0^T \rho_1^{-2}(t) |(w_s)_x(1, t)|^2 \, dt = 2 \int_{Q_T} x \rho^{-1} \rho_t^{-1} w_x w_t + \frac{1}{2} \int_{Q_T} (\rho^{-2} - 2x \rho^{-1} \rho_x^{-1}) (w_x^2 + w_t^2) + \int_{\Omega} [x \rho^{-2} w_t w_x]_0^T.$$

Writing that $|\rho^{-1} \rho_x^{-1}| \leq Cs\rho^{-2}$ and $|\rho^{-1} \rho_t^{-1}| \leq Cs\rho^{-2}$, we obtain (since $s \geq 1$)

$$\frac{1}{2} \int_0^T \rho_1^{-2}(t) |(w_s)_x(1, t)|^2 \, dt \leq Cs \int_{Q_T} \rho^{-2} (w_x^2 + w_t^2) + Cs \int_{\Omega} \left(\rho^{-2}(0) (w_t^2 + w_x^2)(0) + \rho^{-2}(T) (w_t^2 + w_x^2)(T) \right)$$

leading, using the Carleman estimate (10), to

$$s \int_0^T \rho_1^{-2}(t) |(w_s)_x(1, t)|^2 \, dt \leq Cs \left(\|\rho y_s\|_{L^2(Q_T)}^2 + s^{-1} \left\| \frac{\rho_1}{\eta} v_s \right\|_{L^2(\delta, T-\delta)}^2 \right). \tag{76}$$

Thus,

$$|K_3| \leq C s \left(\|\rho y_s\|_{L^2(Q_T)}^2 + \|\frac{\rho_1}{\eta} v_s\|_{L^2(\delta, T-\delta)}^2 \right) + \frac{s}{8} \int_0^T \eta^2(t) \rho_1^{-2}(w_s)_{tx}(1, t) dt.$$

(iv) Using the Carleman estimate (74) we have

$$\begin{aligned} |K_4| &= \left| \int_{Q_T} B(w_s)_{tt} dx dt \right| \leq \left(s^{-1} \int_{Q_T} \rho^2 B^2 \right)^{1/2} \left(s \int_{Q_T} \rho^{-2} |(w_s)_{tt}|^2 \right)^{1/2} \\ &\leq C \left(s^{-1} \int_{Q_T} \rho^2 B^2 \right)^{1/2} \left(\int_{Q_T} \rho^{-2} |(\rho^2 y_s)_t|^2 dx dt + s \int_0^T \eta^2(t) \rho_1^{-2} |(w_s)_{tx}(1, t)|^2 dt \right)^{1/2} \\ &\leq C \left(s^{-1} \int_{Q_T} \rho^2 B^2 \right)^{1/2} \left(s^2 \int_{Q_T} \rho^2 |y_s|^2 dx dt + \int_{Q_T} \rho^2 |(y_s)_t|^2 dx dt + s \int_0^T \eta^2(t) \rho_1^{-2} |(w_s)_{tx}(1, t)|^2 dt \right)^{1/2} \\ &\leq C \left(s^{-1} \int_{Q_T} \rho^2 B^2 + s^2 \int_{Q_T} \rho^2 |y_s|^2 dx dt \right) + \frac{1}{8} \int_{Q_T} \rho^2 |(y_s)_t|^2 dx dt + \frac{s}{8} \int_0^T \eta^2(t) \rho_1^{-2} |(w_s)_{tx}(1, t)|^2 dt. \end{aligned}$$

(v) Similarly, using again the Carleman estimate (74) we have

$$\begin{aligned} |K_5| &= \left| \int_{\Omega} (w_s)_{tt}(\cdot, 0) u_1 dx \right| \leq \|\rho(0) u_1\|_{L^2(\Omega)} \|\rho^{-1}(0) (w_s)_{tt}(\cdot, 0)\|_{L^2(\Omega)} \\ &\leq C s^{-1/2} \|\rho(0) u_1\|_{L^2(\Omega)} \left(\int_{Q_T} \rho^{-2}(t) |(\rho^2 y_s)_t|^2 dx dt + s \int_0^T \eta^2(t) \rho_1^{-2} |(w_s)_{tx}(1, t)|^2 dt \right)^{1/2} \\ &\leq C s^{-1/2} \|\rho(0) u_1\|_{L^2(\Omega)} \left(s^2 \int_{Q_T} \rho^2 |y_s|^2 dx dt + \int_{Q_T} \rho^2 |(y_s)_t|^2 dx dt + s \int_0^T \eta^2(t) \rho_1^{-2} |(w_s)_{tx}(1, t)|^2 dt \right)^{1/2} \\ &\leq C \left(s^{-1} \|\rho(0) u_1\|_{L^2(\Omega)}^2 + s^2 \int_{Q_T} \rho^2 |y_s|^2 dx dt \right) + \frac{1}{8} \int_{Q_T} \rho^2 |(y_s)_t|^2 dx dt + \frac{s}{8} \int_0^T \eta^2(t) \rho_1^{-2} |(w_s)_{tx}(1, t)|^2 dt. \end{aligned}$$

(vi) Simpler, we get

$$|K_6| \leq C s \|\rho(0) u_0\|_{L^2(\Omega)}^2$$

(vii) and

$$|K_7| \leq \|\rho(0) u_0\|_{L^2(\Omega)} \|\rho(0) u_1\|_{L^2(\Omega)}.$$

(viii) Eventually, (74) leads to

$$\begin{aligned} |K_8| &= \left| \int_{\Omega} (u_0)_x (w_s)_{tx}(\cdot, 0) \right| \leq \|\rho(0) (u_0)_x\|_{L^2(\Omega)} \|\rho^{-1}(0) (w_s)_{tx}(\cdot, 0)\|_{L^2(\Omega)} \\ &\leq C s^{-1/2} \|\rho(0) (u_0)_x\|_{L^2(\Omega)} \left(\int_{Q_T} \rho^{-2}(t) |(\rho^2 y_s)_t|^2 dx dt + C s \int_0^T \eta^2(t) \rho_1^{-2}(t) |(w_s)_{tx}(1, t)|^2 dt \right)^{1/2} \\ &\leq C s^{-1/2} \|\rho(0) (u_0)_x\|_{L^2(\Omega)} \left(s^2 \int_{Q_T} \rho^2(t) |y_s|^2 dx dt + \int_{Q_T} \rho^2(t) |(y_s)_t|^2 dx dt \right. \\ &\quad \left. + s \int_0^T \eta^2(t) \rho_1^{-2}(t) |(w_s)_{tx}(1, t)|^2 dt \right)^{1/2} \\ &\leq C \left(s^{-1} \|\rho(0) (u_0)_x\|_{L^2(\Omega)}^2 + s^2 \int_{Q_T} \rho^2(t) |y_s|^2 dx dt \right) + \frac{1}{8} \int_{Q_T} \rho^2(t) |(y_s)_t|^2 dx dt \\ &\quad + \frac{s}{8} \int_0^T \eta^2(t) \rho_1^{-2}(t) |(w_s)_{tx}(1, t)|^2 dt. \end{aligned}$$

Sub-step 3 : In this step, we give estimates on $(v_s)_t$ and $(y_s)_t$. Collecting the previous estimates, we get from (75)

$$\begin{aligned} & \int_{Q_T} \rho^2(t) |(y_s)_t|^2 \, dxdt + s \int_0^T \eta^2(t) \rho_1^{-2}(t) |(w_s)_{tx}(1, t)|^2 \, dt \\ & \leq C \left(s^2 \|\rho y_s\|_{L^2(Q_T)}^2 + s \left\| \frac{\rho_1}{\eta} v \right\|_{L^2(\delta, T-\delta)}^2 + s^{-1} \int_{Q_T} \rho^2 B^2 \right. \\ & \quad \left. + s \|\rho(0)u_0\|_{L^2(\Omega)}^2 + s^{-1} \|\rho(0)(u_0)_x\|_{L^2(\Omega)}^2 + s^{-1} \|\rho(0)u_1\|_{L^2(\Omega)}^2 \right). \end{aligned} \quad (77)$$

We have

$$\begin{aligned} & s^{-1} \int_0^T \rho_1^2(t) |(v_s)_t|^2 \, dt \\ & \leq C \left(s \int_0^T \eta^2(t) \rho_1^{-2}(t) |(w_s)_{tx}(1, t)|^2 \, dt + s \int_\delta^{T-\delta} \frac{\rho_1^2(t)}{\eta^2(t)} v_s^2(t) \, dt + s \int_0^T \rho_1^{-2}(t) |(w_s)_x(1, t)|^2 \, dt \right) \end{aligned}$$

thus using the estimates (76) and (14), (77) implies for $s \geq s_0 \geq 1$ that

$$\begin{aligned} & \int_{Q_T} \rho^2(t) |(y_s)_t|^2 \, dxdt + s^{-1} \int_0^T \rho_1^2(t) |(v_s)_t|^2 \, dt \\ & \leq C \left(s^{-1} \|\rho B\|_{L^2(Q_T)}^2 + s e^{-2s} \|u_0\|_{L^2(\Omega)}^2 + s^{-1} e^{-2s} \|(u_0)_x\|_{L^2(\Omega)}^2 + s^{-1} e^{-2s} \|u_1\|_{L^2(\Omega)}^2 \right) \end{aligned} \quad (78)$$

which gives the announced estimate (19) in the case of regular data.

Step 3 : Case where $B \in L^2(Q_T)$ and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. We proceed by density: there exist $(u_0^n)_{n \in \mathbb{N}} \in H^2(\Omega) \cap H_0^1(\Omega)$, $(u_1^n)_{n \in \mathbb{N}} \in H_0^1(\Omega)$ and $(B^n)_{n \in \mathbb{N}} \in \mathcal{D}(0, T; L^2(\Omega))$ such that $u_0^n \rightarrow u_0$ in $H_0^1(\Omega)$, $u_1^n \rightarrow u_1$ in $L^2(\Omega)$ and $B^n \rightarrow B$ in $L^2(Q_T)$ as $n \rightarrow \infty$.

Let (y_s^n, v_s^n) be the solution of (6) given in Theorem 6 associated to (u_0^n, u_1^n, B^n) . Then, by linearity, we have for all $(n, m) \in \mathbb{N}^2$, from (14)

$$\begin{aligned} & \|\rho(y_s^n - y_s^m)\|_{L^2(Q_T)} + s^{-1/2} \|\rho_1(v_s^n - v_s^m)\|_{L^2(0, T)} \\ & \leq \|\rho(y_s^n - y_s^m)\|_{L^2(Q_T)} + s^{-1/2} \left\| \frac{\rho_1}{\eta} (v_s^n - v_s^m) \right\|_{L^2(\delta, T-\delta)} \\ & \leq C \left(s^{-3/2} \|\rho(B^n - B^m)\|_{L^2(Q_T)} + s^{-1/2} e^{-s} \|u_0^n - u_0^m\|_{L^2(\Omega)} + s^{-3/2} e^{-s} \|u_1^n - u_1^m\|_{L^2(\Omega)} \right), \end{aligned}$$

while from (19)

$$\begin{aligned} & \|\rho(y_s^n - y_s^m)_t\|_{L^2(Q_T)} + s^{-1/2} \|\rho_1(v_s^n - v_s^m)_t\|_{L^2(0, T)} \\ & \leq \left(s^{-1/2} \|\rho(B^n - B^m)\|_{L^2(Q_T)} + s^{1/2} e^{-s} \|u_0^n - u_0^m\|_{L^2(\Omega)} \right. \\ & \quad \left. + s^{-1/2} e^{-s} \|(u_0^n - u_0^m)_x\|_{L^2(\Omega)} + s^{-1/2} e^{-s} \|u_1^n - u_1^m\|_{L^2(\Omega)} \right) \end{aligned}$$

and from (21).

$$\begin{aligned} & \|(y_s^n - y_s^m)_t\|_{L^\infty(0, T; L^2(\Omega))} + \|(y_s^n - y_s^m)_x\|_{L^\infty(0, T; L^2(\Omega))} \leq C \left(\|B^n - B^m\|_{L^2(Q_T)} + \|u_0^n - u_0^m\|_{L^2(\Omega)} \right. \\ & \quad \left. + \|(u_1^n - u_1^m)_x\|_{L^2(\Omega)} + \|v_s^n - v_s^m\|_{H^1(0, T)} \right). \end{aligned}$$

Therefore $v_s^n \rightarrow v_s$ in $H^1(0, T)$ and $y_s^n \rightarrow y_s \in C^0([0, T]; H^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ and, passing to the limit in the equation (6) satisfied by (y_s^n, v_s^n) , we obtain that (y_s, v_s) solves (6). Moreover, passing to the limit in the estimate (19) satisfied by (y_s^n, v_s^n) , we deduce that (y_s, v_s) also satisfies (19). Using (10), we easily check that (y_s, v_s) satisfies $v_s = s\eta^2 \rho_1^{-2} (w_s)_x(1, \cdot)$ and $y_s = \rho^{-2} L w_s$ where $w_s \in P_s$ is the unique solution of (12). The proof of Theorem 7 is complete.

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