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Growth-fragmentation model for a population presenting heterogeneity in growth rate: Malthus parameter and long-time behavior

Anaïs Rat^{*,†} Magali Tournus[†]

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Abstract

The goal of the present paper is to explore the long-time behavior of the growth-fragmentation equation formulated in the case of equal mitosis and variability in growth rate, under fairly general assumptions on the coefficients. The first results concern the monotonicity of the Malthus parameter with respect to the coefficients. Existence of a solution to the associated eigenproblem is then stated in the case of a finite set of growth rates thanks to Kreĭn-Rutman theorem and a series of estimates on moments. Afterwards, adapting the classical general relative entropy (GRE) method enables us to ensure uniqueness of the eigenelements and derive the long-time asymptotics of the Cauchy problem. We prove convergence towards the steady state including in the case of individual exponential growth known to exhibit oscillations at large times in absence of variability. A few numerical simulations are eventually performed in the case of linear growth rate to illustrate our monotonicity results and the fact that variability, providing enough mixing in the heterogeneous population, is sufficient to re-establish asynchronicity.

Keywords: Growth-fragmentation equation, structured population, heterogeneity in growth rate, eigenproblem, Malthus parameter, long-time behavior, general relative entropy, Kreĭn-Rutman.

MSC 2010: 35B40, 35Q92, 45C05, 45K05, 47A75, 92D25.

1 INTRODUCTION

Among the class of structured population models, *growth-fragmentation equations* have raised throughout the last decades a wide literature. Their first formulation traces back to 1967 by three independent groups of biophysicists, Bell & Anderson [5], Sinko & Streifer [51], and Fredrickson, Ramkrishna & Tsuchiya [25]. Their need was to add/substitute size, or possibly several other physiological variables [25], to the age structuring variable of the already known renewal equation in order to account and learn from the finer structure of the data made available by technical progress (in particular electronic *Coulter counters*

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allowing for accurate cell counting and sizing [1, 30]). Yet, mathematical tools were insufficient to tackle the problem other than numerically and we have to wait more than a decade, until the 1980s and the development of semi-group theory, for the first theoretical results [18, 31, 36]. Since then, the equation has been extensively studied under weaker assumptions or variant reformulations (linear to non-linear version, symmetric to asymmetric or general fragmentation, etc.) thanks to new tools from both deterministic and probabilistic approaches. Such interest is likely due to the multiple domains of application as well as the mathematical complexity and richness of questions deriving from apparently simple formulations (well-posedness, long-time asymptotic behavior, coefficient estimation through inverse problem, for the most classical). For further information on growth-fragmentation equations we refer to the book [40] of Metz & Diekmann (1986) or the review papers [2, 32] of Arino (1995) and Gyllenberg (2007).

1.1 The classical growth-fragmentation model

We consider a population of cells that grow in size according to some *deterministic growth rate* τ and divide with a certain size-dependent *probability per unit of time* γ . We assume constant environment so that *coefficients* τ and γ only depend on the individual variables –usually called *structuring variables*, they fully characterize individual evolution.

In this subsection, the only structuring variable is the *size* x . We call it size although x can be volume, length, dry mass, protein content, etc. as long as it is conserved through division. Finally, we assume *equal mitosis* considering that dividing cells split into two cells of equal size. We obtain the classical *growth-fragmentation equation*

$$\begin{cases} \frac{\partial}{\partial t} n(t, x) + \frac{\partial}{\partial x} (\tau(x)n(t, x)) + \gamma(x)n(t, x) = 4\gamma(2x)n(t, 2x), \\ \tau(0)n(t, 0) = 0, \quad n(0, x) = n^{in}(x), \end{cases} \quad (\mathbf{S}_t)$$

where $n(t, x)$ stands for the density of cells of size $x > 0$ at time $t \geq 0$. The 4 in factor of the right-hand term (which stands for new-born cells) is the product of two factors 2 that arise from getting *two* cells with birth size in $(x, x + dx)$ out of the division of one cell with size in the interval $(2x, 2x + 2dx)$ *twice* as big.

Although non-conservative, this *population balance equation* derives from the combination of different conservation laws: integrating (\mathbf{S}_t) against 1 and x respectively, one gets that

- the *total number of cells* is left unchanged by growth but increased by fragmentation

$$\frac{d}{dt} \int_0^\infty n(t, x) dx = \int_0^\infty \gamma(x)n(t, x) dx,$$

- and conversely, fragmentation conserves the *total mass*, besides increased by growth

$$\frac{d}{dt} \int_0^\infty xn(t, x) dx = \int_0^\infty \tau(x)n(t, x) dx.$$

A remarkable feature of many reproducing populations observed before crowding or resource limitation can occur, therefore expected to be captured by the model, is exponential growth coupled to asynchronicity –or *asynchronous exponential growth (A.E.G.)* [53]. *Asynchrony* is the property of a system to forget the shape of its initial distribution at large times and asymptotically stabilize in the sense that the proportion of individuals in a given *cohort* –a sub-population sharing same traits– becomes constant as time progresses.

Biologists also talk about *desynchronization effect* [13] since no matter how synchronized, i.e. narrow distributed, the initial population is (Dirac distribution included in the case of *clonal* populations), it progressively desynchronizes as it aligns to some stable distribution. Mathematically, A.E.G. corresponds to the existence of a *stationary profile* N , independent of the initial state, and positive constants C , λ such that:

$$n(t, x) \underset{t \rightarrow +\infty}{\sim} C e^{\lambda t} N(x),$$

where the only memory of the initial state is a weighted average contained in C . The asymptotic exponential growth rate λ is called *Malthus parameter* or sometimes *fitness*.

We refer to Mischler and Scher's article [45] for a study and review of the long-time behavior. Good balance between growth and fragmentation rates is required to ensure A.E.G: if fragmentation dominates growth around zero the total mass goes to a Dirac in zero [21] and conversely, it dilutes to infinity if growth dominates fragmentation for large sizes, see [24] for examples of such non-existence of steady profile.

Another situation where A.E.G fails, more surprising although already anticipated in 1967 by Bell & Anderson [5], is the case of ideal bacterial growth: equal mitosis and linear growth rate $\tau(x) = vx$ for some positive v (or any growth rate s.t. $2\tau(x) = \tau(2x)$). In this setting indeed the equation lacks dissipativity, thus losing its regularizing effect and keeping stronger memory of the initial state: we still have exponential growth but convergence in shape towards a time-periodic limit preserving the (otherwise vanishing) singular part of measure solution. For instance, starting from a Dirac mass in x , the distribution at any larger time t is a sum of Dirac masses supported on a subset of $\{xe^{vt}2^{-n}\}_{n \in \mathbb{N}}$ [27]. An intuitive explanation is that under individual exponential growth, cells with same size, no matter when they respectively divide, give birth to cells of same size. The property of having same size is thus passed on through generations together with properties of the initial population [5, 18]. See [31] for the first proof of convergence (1988), most notably improved in [6, 10, 20], or [27] for the latest results and a nice review.

In his 1995 survey [2], Arino lists the modifications of this limit case that have been made to re-establish A.E.G. With no surprise, it concerns one of the two conditions – equal mitosis or $\tau(2x) = 2\tau(x)$ – and somehow consists in adding variability: allowing for

- different sizes at birth, that is *asymmetric/unequal division*, one daughter cell inheriting a fraction $r \neq \frac{1}{2}$ of her mother's mass, the other one the remaining $1 - r$, see Heijmans (1984) [36] for the first proof,
- different growth rates, subdividing the population into *quiescent* non-growing (tumor) cells and *proliferating* growing cells [33],

or, he suggests, even both combined which has only been studied very recently by Cloez, de Saporta and Roget [15].

The first option has been well studied, especially through the larger model of *general fragmentation* that accounts for stochastic mother/daughter size ratios at division [7, 45], but few studies have focused on other forms of variability. In the present paper, motivated by biological evidence (see [28, 37] and the numerous references therein), we investigate the second option: variability in growth rate within the size-structured setting, more suited to cell division than the age-structured one (at least for E. Coli as indicates [50]).

1.2 A model accounting for inter-cell variability in growth rate

Up to our knowledge, no theoretical study on the asymptotics of a size structured model encompassing more than the two ways of growing mentioned above is available. A model allowing for a continuous set of growth rates was formulated through piecewise deterministic Markov branching tree by Doumic, Hoffmann, Krell & Robert [22] in order to estimate the division rate more accurately. The model was then used by Olivier [47] to quantify the variation of the Malthus parameter with respect to variability in the aging and growth rates of age and size-structured populations, respectively. Theoretical results are obtained in a deterministic framework for the age-structured model, but again, no theoretical study is carried out in the size-structured case, she relies on the stochastic approach of Doumic et al. to get numerical results later detailed in Subsection 2.2.2.

To account for such inter-cell variability one needs to describe cells, by their size still, but also by a new structuring variable: an individual *feature* or *trait* v , attributed at birth and conserved all life long, that determines cells' own growth rates $\tau(v, \cdot)$. It remains to define a rule for the transmission of this feature over generations. Denoting by \mathcal{V} the *set of admissible features*, we introduce a kernel κ , the *variability kernel*, supported on $\mathcal{V} \times \mathcal{V}$ such that $\kappa(v, v')dv'$ is the probability of a cell of feature v to give birth to cells of feature in $[v', v' + dv']$; in particular it must verify

$$\int_{\mathcal{V}} \kappa(v, v') dv' = 1, \quad \forall v \in \mathcal{V}.$$

Within this new setting, the density $n(t, v, x)$ of cells of feature $v \in \mathcal{V}$ and size $x > 0$ at time $t \geq 0$ evolves as

$$\begin{cases} \frac{\partial}{\partial t} n(t, v, x) + \frac{\partial}{\partial x} (\tau(v, x)n(t, v, x)) = \mathcal{F}(n)(t, v, x), \\ \tau(v', 0)n(t, v', 0) = 0, \quad n(0, v, x) = n^{in}(v, x), \end{cases} \quad (\mathbf{S}_t + \mathbf{V})$$

with the fragmentation operator \mathcal{F} acting on a function f through

$$\mathcal{F}f(v, x) := -\gamma(v, x)n(t, v, x) + 4 \int_{\mathcal{V}} \gamma(v', 2x)n(t, v', 2x)\kappa(v', v) dv'.$$

Let us clarify why the division rate depends on v although variability is only assumed on growth. When assuming division to be only triggered by size, we actually aim at defining a *division rate per unit of size*, say β , function of the size only such that the probability $\beta(x)dx$, to divide before reaching size $x + dx$ when having reached size x , is common to all cells. A straightforward dimensional analysis however indicates that the division rate γ of the equation is a *rate per unit of time*. In absence of variability in growth rate, $\gamma(x)dt$ is thus the probability for any cell reaching size x to divide in at most dt . Relating γ and β is then easy given cells' growth rate: any cell of size x at time t , therefore instantaneously growing at speed $\tau(x)$, will take a time $dt = \frac{dx}{\tau(x)}$ at most to grow of dx at most, figure 1. This imposes the natural relation $\gamma(x) := \tau(x)\beta(x)$, as described for example in [19, 34].

Returning to the model with variability, γ should depends on v as well but only through τ , as a consequence of the relation $\gamma := \tau\beta$ ensuring $\gamma(v, x)dt = \beta(x)dx$. So one should not be mistaken: only growth is subjected to variability, division keeps being triggered by size only, in the sense that $\beta = \beta(x)$. Note that this modeling assumption is made for the size/size-increment equation (or equivalent) in both biological [52] and mathematical [23, 26, 35] papers, and mimicked by Oliver [47] for the size-structured model with variability.

It is clear that model (\mathbf{S}_t) can be formally obtained from model $(\mathbf{S}_t + \mathbf{V})$ by

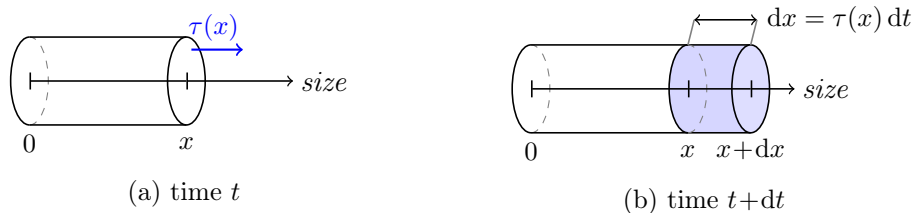


Figure 1: Scheme of the growth mechanism of a cell with growth rate $\tau = \tau(x)$.

- choosing an initial state composed of cells of a unique given feature \bar{v} ,
- and imposing to every new-born cell to have feature \bar{v} by setting $\kappa(v, \cdot) = \delta_{\bar{v}}$ for all v ,

after which the support of the distribution of features among cells, initially reduced to $v = \bar{v}$, will conserve through time since there is no transport in feature. This way, we have $n(t, dv, x) = \tilde{n}(t, x)\delta_{\bar{v}}(dv)$ and finding a solution \tilde{n} to model (\mathbf{S}_t) is equivalent to finding a solution n to this “reset” model $(\mathbf{S}_t + \mathbf{V})$.

Studying the long-time asymptotic behavior of $(\mathbf{S}_t + \mathbf{V})$ requires to study its eigenproblem, whose solution can be seen as a stationary solution to the problem on $ne^{\lambda t}$ (solution to $(\mathbf{S}_t + \mathbf{V})$ rescaled by the exponential rate inherent to its unrestricted growth):

$$\begin{cases} \frac{\partial}{\partial x}(\tau N) + \lambda N = \mathcal{F}N, & \tau(\cdot, 0)N(\cdot, 0) = 0, & \int_{\mathcal{V}} \int_0^{\infty} N = 1, & N \geq 0, \\ -\tau \frac{\partial}{\partial x} \phi + \lambda \phi = \mathcal{F}^* \phi, & \int_{\mathcal{V}} \int_0^{\infty} N \phi = 1, & \phi \geq 0, \end{cases} \quad (\mathbf{S} + \mathbf{V})$$

with λ the Malthus parameter previously mentioned and \mathcal{F}^* the dual operator of \mathcal{F} by

$$\mathcal{F}^* h(v, x) := -\gamma(v, x)h(v, x) + 2\gamma(v, x) \int_{\mathcal{V}} \kappa(v, v')h\left(v', \frac{x}{2}\right) dv'. \quad (1)$$

Note that the direct equation is defined in the weak sense: *weak solutions* to the direct problem of $(\mathbf{S} + \mathbf{V})$ are functions $N \in L^1(\mathcal{V} \times \mathbb{R}_+)$ such that for all $\varphi \in \mathcal{C}_c^\infty(\mathcal{V} \times [0, +\infty))$

$$\begin{aligned} \int_{\mathcal{V}} \int_0^{\infty} N(v, x) \left(-\tau(v, x) \frac{\partial}{\partial x} \varphi(v, x) + (\lambda + \gamma(v, x)) \varphi(v, x) \right) dx dv \\ = 4 \int_{\mathcal{V}} \int_0^{\infty} \left(\int_{\mathcal{V}} \gamma(v', 2x) N(v', 2x) \kappa(v', v) dv' \right) \varphi(v, x) dv dx = 0. \end{aligned}$$

As for the adjoint equation, it should hold in $L^1_{loc}(0, +\infty)$ for ϕ in $W^{1, \infty}(0, +\infty)$.

1.3 Malthus parameter

Characterizing the variation of the Malthus parameter λ with respect to (w.r.t.) a variation in the coefficients of the problem is a question of both biological and mathematical interest. In the context of adaptive dynamics, it arises as soon as one seeks to question the optimality of the coefficients in the Darwinian sense of maximizing the growth speed of the population –for example to explain observations [42] or optimize therapeutic strategy like cancer [8, 14] or antibiotic treatments [3]. Such concerns also come to mind for anyone that wishes to gain insight into the mathematical model, through a deeper understanding of the interplay between the coefficients and the eigenvalue λ solution [11]. Though, relatively few mathematical studies have addressed it (see, Subsection 2.2 for a brief review).

What interests us in the following, is the influence of monotonous variations of the growth or division rate. A natural question is whether or not the resulting variation of λ is monotonous. One could think for example, that increasing the division rate should increase the instantaneous number of newborn cells and in turn, the exponential growth rate λ of the overall population. No matter how intuitive this can be it is not necessarily true as proved in [11].

In this article, Calvez, Doumic and Gabriel, multiply by a factor α the growth rate (or the division rate but the problem is proved equivalent) and study the limit of the corresponding solution λ_α when α goes to zero or infinity. At the limit, the problem becomes pure fragmentation, or pure transport respectively: the mass tends to a Dirac in zero (or goes to infinity resp.) so, at the end, the only quantity that matters to the asymptotic exponential growth speed of the population is the value of the division rate at infinitely small (large resp.) sizes. This formal reasoning is summarized by their [11, Theorem 1]: $\lim_{\alpha \rightarrow L} \lambda_\alpha = \lim_{x \rightarrow L} \gamma(x)$, for L equal to 0 or $+\infty$, which especially implies the possible non-monotonicity of the Malthus parameter (for example if γ , non-monotonous, cancels in zero and infinity).

The question then, which still remains open, is to find necessary and sufficient conditions on the coefficients for the Malthus parameter to depend in a monotonous way on them.

1.4 Outline of the paper

Our objective is to bring insight in the effect of variability between individuals on the dynamic of the whole population: can such cell-level heterogeneity be responsible for any population-level strategy? in other words, is the population with variability growing faster? for which families of growth rates, which transmission laws? are, among others, related biological questions that we would like to answer. From a mathematical point of view, how does variability impacts the asymptotic behavior of the system? does it introduce enough mixing between individuals to recover the A.E.G. property in the case of individual exponential growth? The main results of the present paper are the following:

- Theorem 1 (no variability) tells us that the modeling assumption $\gamma := \tau\beta$ is sufficient to ensure monotonicity of the Malthus parameter with respect to the growth rate,
- Theorem 2 (variability) provides bounds for the Malthus parameter λ solution to $(\mathbf{S}+\mathbf{V})$. The lower (upper) bound is the Malthus parameter λ_1 (λ_2 resp.) associated with the problem with no variability and any growth rate $\tau_1 \leq \tau(v, \cdot)$ (resp. $\tau(v, \cdot) \leq \tau_2$),
- Theorem 3 states existence of eigenelements (λ, N, ϕ) solution to $(\mathbf{S}+\mathbf{V})$,
- Proposition 2 ensures that $(\mathbf{S}_t+\mathbf{V})$ verifies a GRE inequality,
- from which we get uniqueness of eigenelements (Proposition 5) and long-time convergence of renormalized solutions to $(\mathbf{S}_t+\mathbf{V})$ to the steady state N in $L^1(\phi)$ -norm (Theorem 4). Proofs strongly rely on the GRE principle and the localization of the support of the eigenvectors (Proposition 1). An interesting consequence is that providing sufficient mixing in feature, encoded in a condition of irreducibility on the variability kernel κ , convergence to a steady profile holds in the special case of linear growth rate.

The paper is organized as follows. Section 2 exposes our set of assumptions and main results. Proofs are given all along the section, except the longer proof of Theorem 3 that is postponed to Section 4. Section 3 is dedicated to the illustration of the long-time convergence in the case of linear growth rate, through theoretical study of a simple case and numerical results. Computations also provide us with estimates of λ supporting Theorem 2.

2 MAIN RESULTS

All along the paper, we adopt the following notation:

$$\mathcal{S} = \mathcal{V} \times (0, +\infty),$$

introduce the space of functions L^p -integrable on a neighborhood of zero

$$L_0^p := \{f : \exists a > 0, f \in L^p(0, a)\}, \quad 1 \leq p \leq +\infty,$$

and the set of non-negative functions with at most polynomial growth or decay at infinity:

$$\mathcal{P}_\infty := \left\{ f \geq 0, f \in L_{loc}^\infty(0, +\infty) : \exists \nu, \omega \geq 0, \limsup_{x \rightarrow +\infty} \{x^{-\nu} f(x)\} < \infty, \liminf_{x \rightarrow +\infty} \{x^\omega f(x)\} > 0 \right\}.$$

2.1 Assumptions

We formulate the following hypotheses:

(HV $_{\mathcal{Q}}$) The *set of features* \mathcal{V} is finite: $\exists M \in \mathbb{N}^* : \mathcal{V} = \{v_1, v_2, \dots, v_M\}$.

To lighten notations we introduce $\mathcal{I} := \{1, \dots, M\}$ and use indifferently the notations $f(v_i, \cdot)$ or f_i , for any i in \mathcal{I} , whether one considers f as a map on \mathcal{V} to some functional space X or as a vector in X^M . In particular we write equivalently $\ell^1(\mathcal{I}; L^1(0, +\infty))$ and $L^1(\mathcal{S})$.

(H τ) The *growth rate* τ satisfies: there exist $\nu_0, \omega_0 \geq 0$, s.t. for all v in \mathcal{V} ,

(H τ_{pos}) $\forall K \subset (0, +\infty)$ compact, $\exists m_K > 0 : \tau(v, \cdot) \geq m_K$ a.e. on K

(H τ_0) $\frac{x^{\nu_0}}{\tau(v, \cdot)} \in L_0^1$ and $x^{\omega_0} \tau(v, \cdot) \in L_0^\infty$

(H τ_∞) $\tau(v, \cdot) \in \mathcal{P}_\infty$

(H γ) The *division rate per unit of time* γ satisfies: for all v in \mathcal{V} , $\gamma(v, \cdot) \in \mathcal{P}_\infty$.

(H γ - τ) There exists a function β , depending only on the size variable and satisfying **(H β)**, such that $\gamma(v, x) = \beta(x)\tau(v, x)$, $\forall v \in \mathcal{V}$, a.e. $x \in (0, +\infty)$.

(H β) The fragmentation/growth ratio β , or *division rate per unit of size*, satisfies

(H β_{supp}) β is supported on $[b, +\infty)$ for some $b \geq 0$

(H β_0) $\beta \in L_0^1$

(H β_∞) $\lim_{x \rightarrow +\infty} x\beta(x) = +\infty$

(H κ) Under **(HV $_{\mathcal{Q}}$)**, the *variability kernel* $\kappa : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}_+$ can be seen as a square matrix of size M , in which case $\kappa = (\kappa_{ij}) \in \mathcal{M}_M(\mathbb{R}_+)$ satisfies

(H κ_{prob}) κ is a *stochastic (or probability or transition) matrix*:

$$\kappa_{ij} \geq 0, \quad \forall (i, j) \in \mathcal{I}^2, \quad \sum_{j=1}^M \kappa_{ij} = 1, \quad \forall i \in \mathcal{I},$$

(H κ_{irr}) κ is *irreducible*: $\forall (i, j) \in \mathcal{I}^2, \exists m \in \mathbb{N} : \kappa_{ij}^{(m)} := (\kappa^m)_{ij} > 0$.

(H $\tau_{heterogeneity}$) $\forall (v', v) \in \text{supp}(\kappa) : v' \neq v, 2\tau(v, x) \neq \tau(v', 2x)$ a.e. $x \in (\frac{b}{2}, +\infty)$,

A few remarks on these assumptions:

- *Set of features.* Taking a finite set of features $(\mathbf{H}\mathcal{V}_{\mathcal{G}})$ is sufficient to model most biological situations where variability manifests as clear physiological differences between distinct subpopulations (see e.g. [15] which accounts for the difference in growth rates between the *old pole* and *new pole cells*, or [3] which studies *persistent* cells, resistant to antibiotic treatments but having reduced growth rate contrary to the rest of the population). Likewise, one can resume from the continuous vision of variability, of the type of [22, 47]

$$(\mathbf{H}\mathcal{V}_{\mathcal{C}}) \quad \mathcal{V} \text{ is a compact interval of } (0, +\infty),$$

to the discrete one, simply by partitioning \mathcal{V} in characteristic intervals \mathcal{V}_i and considering any cell with feature in \mathcal{V}_i as a cell with feature \bar{v}_i , a certain average on \mathcal{V}_i .

- *Growth and fragmentation rates.* Assumption $(\mathbf{H}\beta_0)$ together with $(\mathbf{H}\tau_{pos})$, $(\mathbf{H}\gamma-\tau)$ and $(\mathbf{H}\gamma)$ implies that β is $L^1_{loc}(\mathbb{R}_+)$. Hypotheses $(\mathbf{H}\tau)$, $(\mathbf{H}\gamma)$ and $(\mathbf{H}\beta)$ altogether is nothing but having, for any v in \mathcal{V} , that $\tau(v, \cdot)$, $\gamma(v, \cdot)$ and their ratio satisfy the assumptions made by Doumic and Gabriel in [24] (to ensure existence of a solution to (\mathbf{S})) with the additional requirement that γ is $L^\infty(\mathbb{R}_+^*)$. Under $(\mathbf{H}\mathcal{V}_{\mathcal{G}})$, this straightforward adaptation of their assumptions is sufficient to state existence but we prove it under the additional modeling assumption $(\mathbf{H}\gamma-\tau)$ which brings small simplifications.

The additional assumption $(\mathbf{H}\tau_{heterogeneity})$ is crucial to characterize the functions canceling the dissipation of entropy, needed to establish uniqueness of eigenelements and convergence. It can be interpreted as follows: every time a mother of feature v gives birth to cells with same features $v' \neq v$, the daughters cannot gain together the exact mass that their mother would have gained if it had not divided; or equivalently from a macroscopic point of view: the mass that would have grown the fraction $\kappa(v', v)$ of all dividing cells of feature v' cannot be the total mass gained by their daughters born with feature $v \neq v'$.

- *Variability kernel.* For $m > 0$, $\kappa_{ij}^{(m)}$ represents the probability for a cell descending from m generations of a cell with feature v_i to have feature v_j . Therefore $(\mathbf{H}\kappa_{irr})$ means that any cell has a non-zero probability to transmit any feature in a finite number of division (M at most). Besides, κ can be associated to the directed graph having $\{v_1, \dots, v_M\}$ as vertices and the pairs (v_i, v_j) such that $\kappa_{ij} > 0$ as oriented edges. Then, $(\mathbf{H}\kappa_{irr})$ is equivalent to the graph of κ being *strongly connected* (every vertex is reachable from every other vertex).

The first assumption on κ easily translates to the continuous setting $(\mathbf{H}\mathcal{V}_{\mathcal{C}})$ (this is κ being a *Markov* or *probability kernel*¹ as assumed in [22, 47]). It is however more complicated to generalize $(\mathbf{H}\kappa_{irr})$, crucial to establish the positivity result of Proposition 1, to the continuous setting (see [41] for a generalization). Additionally, irreducibility alone could not be appropriate to extend our results to larger \mathcal{V} : for countable infinite sets \mathcal{V} , we might additionally need κ *recurrent* [46]; as for continuous sets, irreducibility could be less relevant as suggest the recent articles [4, 16] whose more probabilistic approach via Harris's theorem proves particularly suited to obtain existence of a steady state and long-time convergence at exponential rate in the setting of measure solutions.

¹ $\kappa(v, \cdot)$ is a probability measure for every v (i.e. $v \mapsto \int_{\mathcal{V}} \varphi(v')\kappa(v, dv')$, for $\varphi \in \mathcal{C}(\mathcal{V})$, is Lebesgue measurable and $\int_{\mathcal{V}} \kappa(v, dv') = 1$, $v \in \mathcal{V}$) and $v \mapsto \kappa(v, A)$ is a measurable function for all $A \in \mathcal{B}(\mathcal{V})$.

2.2 Variation of the Malthus parameter with respect to coefficients

2.2.1 In absence of variability. When there is no variability in growth rate, the eigenproblem associated to the Cauchy problem (\mathbf{S}_t) writes as follows: for $x > 0$,

$$\begin{cases} \frac{d}{dx}(\tau(x)N(x)) + (\lambda + \gamma(x))N(x) = 4\gamma(2x)N(2x), & \tau(0)N(0) = 0, & \int N = 1, & N \geq 0, \\ -\tau(x)\frac{d}{dx}\phi(x) + (\lambda + \gamma(x))\phi(x) = 2\gamma(x)\phi\left(\frac{x}{2}\right), & \int N\phi = 1, & \phi \geq 0, \end{cases} \quad (\mathbf{S})$$

and the only coefficients of the problem are τ and γ . In the general fragmentation model, note that another coefficient is the fragmentation kernel (that is $\frac{1}{2}(\delta_{x=ry} + \delta_{x=(1-r)y})$ in the case of *general mitosis*, with $r = \frac{1}{2}$ in our case of equal mitosis).

Among existing studies on the variation of λ w.r.t variations of the coefficients, Michel [42] focuses on the influence of the asymmetry in birth size between daughters (given by the parameter r just mentioned) from two different points of view –either by differentiation of λ with respect to the asymmetry parameter r or by a min-max principle providing a handful expression of λ . Monotonicity of λ w.r.t r is proved in the case of constant growth rate and compactly supported division rate that satisfies conditions proved sufficient for the adjoint vector to be concave or convex. In particular, depending on the form of the division rate, symmetric division can be detrimental to the growth of the overall population.

A corpus of articles by Clairambault et al. investigates the influence on λ of the time periodic dependence of coefficients (division and death rates) induced by the circadian rhythm. However, they rather consider the renewal (age-structured) equation [29] or a system of renewal equations accounting for different phases of the cell cycle [8, 14].

Campillo, Champagnat and Fritsch [12] provide sufficient conditions on τ , γ and γ/τ for the monotonicity of λ w.r.t. monotonous variations of the coefficients. The result is derived in a probabilistic framework by coupling techniques and eventually extended to the deterministic framework thanks to a relation between the survival probability of the stochastic model and the Malthus parameter solution to the deterministic problem.

With only deterministic tools and straightforward arguments, we propose another set of conditions.

Lemma 1. *Set $\mathcal{V} := \{v_1, v_2\}$ and take τ, γ satisfying $(\mathbf{H}\tau)$, $(\mathbf{H}\gamma)$ such that $\frac{\tau(v_i, \cdot)}{\gamma(v_i, \cdot)}$ satisfies $(\mathbf{H}\beta)$ for $i \in \{1, 2\}$. Denote by (λ_i, N_i, ϕ_i) , $\lambda_i > 0$, the weak solution to the eigenproblem (\mathbf{S}) with coefficients $\tau_i := \tau(v_i, \cdot)$ and $\gamma_i := \gamma(v_i, \cdot)$. Assume the following*

$$i) \tau_1 \leq \tau_2, \quad ii) \left(\frac{\gamma_2}{\tau_2} - \frac{\gamma_1}{\tau_1}\right) \left(\frac{1}{2}\phi_i - \phi_i\left(\frac{\cdot}{2}\right)\right) \leq 0 \quad i = 1 \text{ or } 2, \quad \text{a.e. on } (0, +\infty),$$

with notation $\phi\left(\frac{\cdot}{2}\right) : x \mapsto \phi\left(\frac{x}{2}\right)$. Then we have

$$\lambda_1 \leq \lambda_2.$$

Proof. The proof relies on the duality relation between eigenvectors. Existence and uniqueness of positive eigenelements is guaranteed by [24, Theorem 1]. To simplify notations we introduce the bracket $\langle f, g \rangle := \int_0^\infty fg$. Without loss of generality, assume that *ii)* holds for $i = 2$ (otherwise we can still invert indices 1 and 2, or equivalently inverse all the

inequalities). Using equation **(S)** on N_1 and ϕ_2 successively, we have:

$$\begin{aligned}\lambda_1 \langle N_1, \phi_2 \rangle &= \left\langle -\frac{d}{dx}(\tau_1 N_1) - \gamma_1 N_1 + 4\gamma_1(2\cdot)N_1(2\cdot), \phi_2 \right\rangle, \\ &= \left\langle N_1, \tau_1 \frac{d}{dx} \phi_2 - \gamma_1 \phi_2 + 2\gamma_1 \phi_2\left(\frac{\cdot}{2}\right) \right\rangle, \\ &= \left\langle N_1, \frac{\tau_1}{\tau_2} \left(\lambda_2 \phi_2 + \gamma_2 \phi_2 - 2\gamma_2 \phi_2\left(\frac{\cdot}{2}\right) \right) - \gamma_1 \phi_2 + 2\gamma_1 \phi_2\left(\frac{\cdot}{2}\right) \right\rangle, \\ &= \lambda_2 \left\langle N_1, \frac{\tau_1}{\tau_2} \phi_2 \right\rangle + \left\langle \tau_1 N_1, \left(\frac{\gamma_2}{\tau_2} - \frac{\gamma_1}{\tau_1} \right) \left(\phi_2 - 2\phi_2\left(\frac{\cdot}{2}\right) \right) \right\rangle,\end{aligned}$$

leading, thanks to *i*), to the following inequality

$$(\lambda_1 - \lambda_2) \langle N_1, \phi_2 \rangle \leq \left\langle \tau_1 N_1, \left(\frac{\gamma_2}{\tau_2} - \frac{\gamma_1}{\tau_1} \right) \left(\phi_2 - 2\phi_2\left(\frac{\cdot}{2}\right) \right) \right\rangle,$$

which makes clear that $\lambda_1 \leq \lambda_2$ as soon as *ii*) is verified. \square

In particular when **(H γ - τ)** is satisfied, *ii*) is an equality and Lemma 1 directly brings monotonicity of the Malthus parameter with respect to the growth rate as stated below.

Theorem 1 (Monotony of the Malthus parameter). *Let us take notations of Lemma 1 and make the same assumptions on τ and γ . Assume in addition that **(H γ - τ)** holds, then*

$$\tau_1 \leq \tau_2 \text{ a.e. on } (0, +\infty) \implies \lambda_1 \leq \lambda_2,$$

and similarly with strict inequalities.

It should be noticed that another set of assumptions can be derived, namely:

$$i) \gamma_1 \leq \gamma_2, \quad ii) \left(\frac{\gamma_2}{\tau_2} - \frac{\gamma_1}{\tau_1} \right) \frac{d}{dx} \phi_2 \leq 0 \quad i = 1 \text{ or } 2, \quad \text{a.e. on } (0, +\infty),$$

since the third equality in the proof can be replaced by

$$\lambda_1 \langle N_1, \phi_2 \rangle = \lambda_2 \left\langle N_1, \frac{\gamma_1}{\gamma_2} \phi_2 \right\rangle + \left\langle N_1, \left(\frac{\tau_1}{\tau_2} - \frac{\gamma_1}{\gamma_2} \right) \tau_2 \frac{d}{dx} \phi_2 \right\rangle,$$

A few remarks on Lemma 1 before continuing on the problem with variability.

- Lemma 1 and Theorem 1 remain valid for general fragmentation kernels verifying the general assumptions of [24].

- Assumption *ii*) depends on the adjoint eigenvector which makes it hardly interpretable. Although not very satisfying, we kept it for several reasons. First, it needs to be satisfied for only one *i*, therefore in practice one can obtain monotonicity between a problem with explicit solution (λ, N, ϕ) and another problem, with no explicit solution but lower (or greater) growth rate and greater (lower resp.) fragmentation/growth ratio wherever $\phi\left(\frac{\cdot}{2}\right) \leq \frac{1}{2}\phi$ –or simply lower (or greater) growth rate when $\phi\left(\frac{\cdot}{2}\right) = \frac{1}{2}\phi$ (which is true for most of the known solutions, see e.g. the explicit solutions given in [24] for uniform fragmentation, or in [6, 49] for mitosis).

Second, it is interesting to see that we retrieve conditions similar to those of [12], but through a different and much quicker approach allowing for more general coefficients. Assuming $\tau_1 \leq \tau_2$ and $\frac{\gamma_2}{\tau_2} \leq \frac{\gamma_1}{\tau_1}$ as in [12], we require $\phi_i\left(\frac{\cdot}{2}\right) \leq \frac{1}{2}\phi_i$ where they ask for γ to be monotonous not only in the v variable but also in the x variable.

- For our lemma to be interpretable in terms of coefficients only, the problem comes down to finding assumptions on the coefficients allowing to characterize the sign of $\frac{d}{dx} \phi$ (or $\phi\left(\frac{\cdot}{2}\right) - \frac{1}{2}\phi$, implied because of **(S)**) in the spirit of [42] that derives conditions on γ for ϕ to be concave or convex.

2.2.2 In presence of variability. When it comes to the question of the impact of variability on the Malthus parameter even less studies are available although attracting a lot of attention from biologists, see [39] for a recent review in the biological community.

Among the most notable studies, Cloez, de Saporta and Roget [15] add to the morphological asymmetry between daughters studied by Michel [42] a physiological asymmetry, attributing different growth rates to daughters depending on whether or not they have bigger size. In particular, in the case τ linear (and $\lim_{x \rightarrow L} \gamma(x) = L$, for $L = 0$ or $+\infty$) they explicitly compute the partial derivatives of λ w.r.t. the asymmetry parameters and find that in presence of morphological asymmetry, physiological asymmetry maximizes λ if largest cell growth faster.

Rather than two different growth rates in the whole population, Olivier [47] allows for a continuous set of growth rates but equal mitosis, this is model $(\mathbf{S}_t + \mathbf{V})$ under $(\mathbf{H}\mathcal{V}_\ell)$. Her numerical results indicate that variability in growth rate do not favor the global growth of the population –more precisely she exhibits, under specific coefficients, monotonicity of the Malthus parameter w.r.t. the coefficient of variation (CV) of growth rates– in short, w.r.t a parameter α characterizing the level of variability in the sense that the set of features $\mathcal{V}_\alpha = \alpha\mathcal{V} + (1 - \alpha)\bar{v}$ is more or less spread out depending on α .

Relying on the ideas used to treat the case with no variability we can compare the eigenvalue of problem $(\mathbf{S} + \mathbf{V})$ to eigenvalues solution to the problem with no variability.

Theorem 2 (Monotonicity of the Malthus parameter with variability). *Assume $(\mathbf{H}\kappa_{prob})$ and consider (λ, N, ϕ) , with $\lambda > 0$, a weak solution to $(\mathbf{S} + \mathbf{V})$ with coefficients τ and γ satisfying $(\mathbf{H}\gamma - \tau)$ so that $\beta := \frac{\gamma}{\tau}$ is only function of x . Let τ_1 and τ_2 satisfy $(\mathbf{H}\tau)$ be s.t.*

$$\tau_1 \leq \tau(v, \cdot) \leq \tau_2 \quad \text{a.e. on } (0, +\infty), \quad \forall v \in \mathcal{V},$$

and define $\gamma_i := \beta\tau_i$, for $i = 1, 2$. Assume that γ_i and β satisfy $(\mathbf{H}\gamma)$ and $(\mathbf{H}\beta)$, and denote by (λ_i, N_i, ϕ_i) , $\lambda_i > 0$, the weak solution to the eigenproblem (\mathbf{S}) with no variability and coefficients τ_i and γ_i . Then we have (and similarly with strict inequalities):

$$\lambda_1 \leq \lambda \leq \lambda_2.$$

Proof. Proceeding similarly to the case with no variability and but with duality bracket $\langle f, g \rangle = \iint_{\mathcal{S}} fg$, and using $(\mathbf{H}\kappa_{prob})$, we find:

$$\lambda \langle N, \phi_i \rangle = \lambda_i \left\langle N, \frac{\tau}{\tau_i} \phi_i \right\rangle + \left\langle \tau N, \left(\frac{\gamma_i}{\tau_i} - \frac{\gamma}{\tau} \right) \left(\phi_i - 2\phi_i\left(\frac{\cdot}{2}\right) \right) \right\rangle, \quad i \in \{1, 2\}, \quad (2)$$

From $(\mathbf{H}\gamma - \tau)$ and the definition of the γ_i , the second term of the right-hand side cancels and the inequality on λ is a direct consequence of the inequality on τ . \square

In particular, a population where $(\mathbf{H}\gamma - \tau)$ holds and where features vary from v_{min} to v_{max} between individuals asymptotically grows faster, respectively slower, than the population where all cells have feature v_{min} or v_{max} , respectively.

Note that Theorem 2, which assume existence of a solution to $(\mathbf{S} + \mathbf{V})$, holds true in the continuous formulation $(\mathbf{H}\mathcal{V}_\ell)$. Besides, a finer condition of the type of Lemma 1 can be formulated from (2).

2.3 Eigenvalue problem

In the following, we elaborate on previously established results adapted to our case to prove existence of a solution to $(\mathbf{S}+\mathbf{V})$ and GRE related results (subsections 2.4 and 2.5).

Theorem 3 (Existence of eigenelements). *Under assumptions $(\mathbf{H}\mathcal{V}_{\mathcal{D}})$, $(\mathbf{H}\tau)$, $(\mathbf{H}\gamma)$, $(\mathbf{H}\gamma-\tau)$, $(\mathbf{H}\beta)$ and $(\mathbf{H}\kappa_{prob})$, there exists a weak solution (λ, N, ϕ) to the eigenproblem $(\mathbf{S}+\mathbf{V})$ with $\lambda > 0$, and for every $i \in \mathcal{I}$ we have, for ν_0 defined by $(\mathbf{H}\tau_0)$:*

$$x^\alpha \tau_i N_i \in L^p(\mathbb{R}_+), \quad \forall \alpha > -\nu_0, \quad 1 \leq p \leq +\infty, \quad x^\alpha \tau_i N_i \in W^{1,1}(\mathbb{R}_+), \quad \forall \alpha \geq 0, \\ \exists k > 0 : \frac{\phi_i}{1+x^k} \in L^\infty(\mathbb{R}_+), \quad \tau_i \frac{\partial}{\partial x} \phi_i \in L_{loc}^\infty(\mathbb{R}_+),$$

The proof of is postponed to Section 4. It follows the proof of [24, Theorem 1] in which Doumic and Gabriel state, under similar assumptions on the coefficients, the existence of eigenelements to the problem with no variability but general fragmentation. As in [24], we can also derive a priori results on the localization of the support of the eigenvectors.

Proposition 1 (Positivity). *Consider N and ϕ eigenvectors solution to $(\mathbf{S}+\mathbf{V})$ with $\lambda > 0$. Assume $(\mathbf{H}\mathcal{V}_{\mathcal{D}})$, $(\mathbf{H}\tau_{pos})$, $(\mathbf{H}\gamma-\tau)$, $(\mathbf{H}\beta_{supp})$ and $(\mathbf{H}\kappa_{irr})$ satisfied. Then, for all $i \in \mathcal{I}$*

$$\text{supp } N_i = \left[\frac{b}{2}, +\infty\right), \quad \tau_i(x)N_i(x) > 0 \quad \text{a.e. } x \in \left(\frac{b}{2}, +\infty\right), \\ \phi_i(x) > 0 \quad \text{a.e. } x > 0,$$

with b defined by $(\mathbf{H}\beta_{supp})$. Additionally, for any $j \in \mathcal{I}$ such that $\frac{1}{\tau_j} \in L_0^1$ we have

$$\phi_j(0) > 0.$$

Note that the support of the eigenvectors being of the form $V \times X$, for V and X subsets of \mathcal{V} and $[0, +\infty)$, respectively, is a consequence of the support of τ_i and γ_i being independent of i (see $(\mathbf{H}\tau_{pos})$ and $(\mathbf{H}\gamma-\tau)$). The support of the direct eigenvector N is quite natural considering the following remarks.

- *Along the x -axis.* Under equal mitosis the smallest size a new-born cell can have, regardless its feature, is $\frac{b}{2}$ since b is the smallest size at which division can occur. Besides if $b > 0$, any cell of feature v_i and size $x < b$ (and $x > 0$ otherwise it stays with size $x = 0$ from the boundary condition $\tau_i(0)n_i(t, 0) = 0$) reaches size b in finite time (namely $\int_x^b \frac{1}{\tau_i}$) so all individuals will have at least size $\frac{b}{2}$ after some time (or keep size 0). Likewise, the probability $e^{-\int_s^x \beta(y)dy}$ to survive from size s up to size x without dividing is positive for all x so we expect arbitrarily large cells to be found in the population at large times.
- *Along the v -axis.* Because of the irreducibility condition $(\mathbf{H}\kappa_{irr})$ any cell can transmit any feature after a finite number of divisions, so the population should occupy the whole set of features (and stay in it) in finite time.

As for the adjoint vector, the integrability condition on some τ_j for $\phi_j(0) > 0$ can be explained interpreting the equation on ψ (see $(\mathbf{S}_t^*+\mathbf{V})$ later) as a backward equation (where in a sense cells are shrinking and aggregating over time). Then, we expect to find cells of feature v_j reaching size 0 at large times, meaning $\phi_j(0) > 0$, if and only if the time $\int_0^x \frac{1}{\tau_j} dx$ to reach size 0 from size $x > 0$ (in absence of ‘‘aggregation’’) is finite –that is $\frac{1}{\tau_j} \in L_0^1$.

Proof. We introduce for all i in \mathcal{I} , the sets

$$I_{\leftarrow}(i) := \{j \in \mathcal{I} : \kappa_{ji} > 0\}, \quad I_{\rightarrow}(i) := \{j \in \mathcal{I} : \kappa_{ij} > 0\}, \quad (3)$$

of the indices of all the features of cells that can have a daughter or a mother, respectively, with feature v_i . Note, that for every $i \in \mathcal{I}$, none of them is empty thanks to $(\mathbf{H}\kappa_{irr})$.

Direct eigenvector. Integrating on $[0, x]$ the equation of $(\mathbf{S}+\mathbf{V})$ satisfied by $N \geq 0$ brings: for all $i \in \mathcal{I}$, for x in $(0, +\infty)$,

$$\tau_i(x)N_i(x) \leq 2 \int_0^{2x} \beta(s) \sum_{j \in \mathcal{I}} (\tau_j(s)N_j(s)\kappa_{ji}) ds, \quad (4)$$

which makes clear that $\tau N \geq 0$ cancels on $\mathcal{V} \times [0, \frac{b}{2}]$ since β cancels on $[0, b]$ (see $(\mathbf{H}\beta_{\text{supp}})$). It remains to prove that τN is positive elsewhere. For some positive x_0 we define

$$F_i(x) := \tau_i(x)N_i(x) e^{\Lambda_i(x)}, \quad \Lambda_i(x) := \int_{x_0}^x \frac{\lambda + \gamma_i(s)}{\tau_i(s)} ds, \quad \forall i \in \mathcal{I}, \quad \text{a.e. } x \in [x_0, +\infty),$$

and obtain, through equation $(\mathbf{S}+\mathbf{V})$ again, that for all i and for x in $[x_0, +\infty)$

$$\frac{d}{dx} F_i(x) = 4 \sum_{j \in \mathcal{I}} (\gamma_j(2x)N_j(2x)\kappa_{ji}) e^{\Lambda_i(x)} \geq 0.$$

This tells us for all i in \mathcal{I} , that if $\tau_i(\cdot)N_i(\cdot)$ becomes positive at some $x_1 \geq x_0$, then it stays positive on $[x_1, +\infty)$. Let thus introduce the smallest size of individuals with feature v_i at equilibrium

$$\ell_i := \text{ess inf}_{(0, \infty)} \{x : \tau_i(x)N_i(x) > 0\}, \quad i \in \mathcal{I},$$

and prove that all the ℓ_i , $i \in \mathcal{I}$, are equal to $\frac{b}{2}$. We already know that they are greater than $\frac{b}{2}$ so it only remains to prove that ℓ_i is lower than $\frac{b}{2}$ for every i .

For all $i \in \mathcal{I}$, given that $x \mapsto \tau_i(x)N_i(x)$ cancels on $[0, \ell_i]$ and because τ_i can be bounded from below by a positive constant on any compact sets of $(0, +\infty)$ from $(\mathbf{H}\tau_{\text{pos}})$, N_i cancels on $(0, \ell_i]$. Integrating the direct equation of $(\mathbf{S}+\mathbf{V})$ on $(0, \ell_i)$ hence brings:

$$0 = \sum_{j \in \mathcal{I}} \int_0^{\ell_i} \gamma_j(2x)N_j(2x)\kappa_{ji} dx = \frac{1}{2} \sum_{j \in I_{\leftarrow}(i)} \int_{\max(b, \ell_j)}^{2\ell_i} \beta(x)\tau_j(x)N_j(x)\kappa_{ji} dx \quad \forall i \in \mathcal{I}.$$

with *positive integrand* at the right-hand side which tells us that necessarily:

$$\forall i \in \mathcal{I}, \quad 2\ell_i \leq \max(b, \ell_j), \quad \forall j \in I_{\leftarrow}(i). \quad (5)$$

Now consider any i_1, i_2 in \mathcal{I} . Assumption $(\mathbf{H}\kappa_{irr})$ indicates that there exists $m = m_{i_1 i_2} \leq M$ such that $\kappa_{i_1 i_2}^m > 0$, meaning that we can find a path connecting v_{i_1} to v_{i_2} in the directed graph associated to κ , namely that there is $(v_{p_1}, \dots, v_{p_m}) \in \mathcal{V}^m$ such that:

$$\begin{cases} v_{p_1} = v_{i_1}, & v_{p_m} = v_{i_2}, \\ \kappa(v_{p_i}, v_{p_{i+1}}) = \kappa_{p_i p_{i+1}} > 0, & \text{i.e. } p_{i+1} \in I(p_i), \quad \forall i \in \{1, \dots, m-1\}. \end{cases}$$

Applying (5) iteratively to the p_i , for i from 1 to m , we obtain: $\forall i \in \{1, \dots, m-1\}$

$$\ell_{i_1} \leq \frac{1}{2} \max\left(b, \frac{\ell_{p_i}}{2^i}\right) \leq \frac{1}{2} \max\left(b, \frac{\ell_{p_{i+1}}}{2^{i+1}}\right) \leq \frac{1}{2} \max\left(b, \frac{\ell_{i_2}}{2^m}\right). \quad (6)$$

The inequality tells us that as soon as v_{i_1} and v_{i_2} communicate (v_{i_1} leads to v_{i_2} and v_{i_2} to v_{i_1} in the graph of κ), then ℓ_{i_1} and ℓ_{i_2} are both either infinite or inferior or equal to $\frac{b}{2}$

–it suffices to apply (6) to (i_2, i_1) and (i_2, i_1) , distinguishing between cases $\ell_{i_2} \leq 2^m b$ or $\ell_{i_2} > 2^m b$, and ℓ_{i_1} lower or greater than $2^{m_{i_2 i_1}} b$, respectively. However the irreducibility condition $(\mathbf{H}\kappa_{irr})$ implies that all the features communicate, so we deduce that all the ℓ_i , $i \in \mathcal{I}$, are either infinite –this would imply that all the N_i are null-functions on \mathbb{R}_+ which is immediately excluded by the normalization condition on N – or lower or equal to $\frac{b}{2}$.

Adjoint eigenvector. Following the same idea we define $G_i := \phi_i e^{-\Lambda_i}$ on $(x_0, +\infty)$, for x_0 positive, and compute for all $i \in \mathcal{I}$ and for $x \in (x_0, +\infty)$:

$$\frac{d}{dx} G_i(x) = -2\beta(x) \sum_{j \in \mathcal{I}} \kappa_{ij} \phi_j \left(\frac{x}{2} \right) e^{-\Lambda_i(x)} \leq 0, \quad (7)$$

so, as soon as ϕ_i vanishes it remains null, plus G_i is constant on $(0, b]$ for every $i \in \mathcal{I}$.

We prove that for all $i \in \mathcal{I}$, ϕ_i never reaches zero. Again, we introduce:

$$k_i := \operatorname{ess\,sup}_{[0, +\infty]} \{x : \phi_i(x) > 0\}, \quad i \in \mathcal{I}.$$

By definition, ϕ_i is zero on $[k_i, +\infty)$ for any i in \mathcal{I} , therefore so are G_i and $\frac{d}{dx} G_i$. Thus integrating (7) on $[k_i, +\infty)$ yields after straightforward computations, similar to those done for the direct eigenvector, that:

$$\sum_{j \in I_{\rightarrow}(i)} \kappa_{ij} \int_{\max(\frac{b}{2}, \frac{k_i}{2})}^{k_j} \beta(2x) \phi_j(x) e^{-\Lambda_i(2x)} dx = 0, \quad i \in \mathcal{I},$$

with *positive integrand* which leads to:

$$\forall i \in \mathcal{I}, \quad k_j \leq \max\left(\frac{b}{2}, \frac{k_i}{2}\right), \quad \forall j \in I_{\rightarrow}(i). \quad (8)$$

Same arguments based on (8) and $(\mathbf{H}\kappa_{irr})$ hold and allow us to deduce that the k_i , $i \in \mathcal{I}$, are either all infinite or all inferior or equal to $\frac{b}{2}$. The first option is contradicted by the normalization condition on ϕ . As for the second one, it proves impossible unless all k_i are zero. Indeed if $0 < k_i \leq \frac{b}{2}$ for some $i \in \mathcal{I}$, we take $x_0 \in (0, k_i)$ and get G_i constant on $[x_0, k_i]$ (constant on $[x_0, b]$) equal to $G_i(k_i) = 0$ which contradicts the definition of k_i .

Assume now that $\frac{1}{\tau_j} \in L_0^1$ for some $j \in \mathcal{I}$. Then we can take $x_0 = 0$ in the definition of G_i and $\phi_j(0) = 0$ would imply that ϕ_j is zero everywhere, i.e. $k_i = +\infty$ which we said is impossible, so $\phi_j(0) > 0$. \square

A few remarks on the proof:

- This proof also works in the case where the graph associated to κ is not strongly connected but decomposes in strongly connected components partitioned, say in $(\mathcal{V}_1, \dots, \mathcal{V}_p)$ associated to a partition $(\mathcal{I}_1, \dots, \mathcal{I}_p)$ of \mathcal{I} . In this case, inequality (6) implies that the property of $i \mapsto \ell_i$ to be either infinite or upper bounded by $\frac{b}{2}$ is valid on each of the \mathcal{I}_i , and the normalization condition on N then yields $\ell \leq \frac{b}{2}$ on at least one \mathcal{I}_i , but ℓ possibly infinite elsewhere (meaning N null outside $\mathcal{V}_i \times [\frac{b}{2}, +\infty)$).
- Likewise, equation (4) tells us that if there was a feature v_i that somehow could not be transmitted over generations, this is $I_{\leftarrow}(i) = \emptyset$, then $\tau_i N_i$ would be zero on \mathbb{R}_+ .

2.4 General Relative Entropy

The GRE principle (see [43,44,48]) holds in the variability setting and can be stated under assumption $(\mathbf{H}\mathcal{V}_\varphi)$ of \mathcal{V} continuous, so we first keep integral notation for the v variable. Likewise, we start with a general abstract setting which does not involve the eigenelements yet but a solution to the dual equation given, for \mathcal{F}^* defined by (1), as:

$$\frac{\partial}{\partial t}\psi(t, v, x) + \tau(v, x)\frac{\partial}{\partial x}\psi(t, v, x) = -\mathcal{F}^*\psi(t, v, x), \quad (\mathbf{S}_t^* + \mathbf{V})$$

We adopt the compact notation $u_t(v, x) := u(t, v, x)$.

Definition 1 (GRE). *We define the general relative entropy of n with respect to p by*

$$E^H[n|p](t) := \iint_S \psi_t(v, x)p_t(v, x)H\left(\frac{n_t(v, x)}{p_t(v, x)}\right) dv dx,$$

with $H: \mathbb{R} \rightarrow \mathbb{R}$ a differentiable function, ψ weak solution to $(\mathbf{S}_t^* + \mathbf{V})$ and n and p such that the integral is defined.

Proposition 2 (GRE principle). *Assume $(\mathbf{H}\gamma\text{-}\tau)$, $(\mathbf{H}\beta_{\text{supp}})$ and $(\mathbf{H}\kappa_{\text{prob}})$ are satisfied and n and p are weak solutions to $(\mathbf{S}_t + \mathbf{V})$, and ψ to $(\mathbf{S}_t^* + \mathbf{V})$, with $p \geq 0$ and ψ such that $\text{supp}(\psi_t) \cap \mathcal{V} \times [\frac{b}{2}, +\infty) \subset \text{supp}(p_t)$ for all positive t . Then, the entropy dissipation $D^H[n|p]$ of n w.r.t. p , defined by*

$$\frac{d}{dt}E^H[n|p](t) := -D^H[n|p](t),$$

can be expressed, for any locally differentiable $H: \mathbb{R} \rightarrow \mathbb{R}$, as

$$\begin{aligned} D^H[n|p](t) &= -4 \iint_S \psi_t(v, x) \int_{\mathcal{V}} \kappa(v', v)\gamma(v', 2x)p_t(v', 2x) \\ &\quad \times \left(H'\left(\frac{n_t(v, x)}{p_t(v, x)}\right) \left[\frac{n_t(v', 2x)}{p_t(v', 2x)} - \frac{n_t(v, x)}{p_t(v, x)} \right] - \left[H\left(\frac{n_t(v', 2x)}{p_t(v', 2x)}\right) - H\left(\frac{n_t(v, x)}{p_t(v, x)}\right) \right] \right) dv' dv dx. \end{aligned}$$

In particular, for H convex and $\psi \geq 0$ the general relative entropy is decreasing since $D^H[n|p](t) \geq 0$, and for H strictly convex and ψ positive we have

$$D^H[n|p](t) = 0 \iff \frac{n_t(v', 2x)}{p_t(v', 2x)} = \frac{n_t(v, x)}{p_t(v, x)}, \quad (v', v, x) \in \text{supp}(\kappa(v', v)\gamma(v', 2x)).$$

Proof. The standard computation, see e.g. [48, p.93], is not affected by the additional variable v , we only need to make sure quantities are well defined since p here is not necessarily positive everywhere –we rely on $\text{supp}(\gamma) := \mathcal{V} \times [b, +\infty)$ from $(\mathbf{H}\gamma\text{-}\tau)$ and $(\mathbf{H}\beta_{\text{supp}})$.

Using equation $(\mathbf{S}_t + \mathbf{V})$ verified by both n and p we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{n_t(v, x)}{p_t(v, x)}\right) + \tau(v, x)\frac{\partial}{\partial x}\left(\frac{n_t(v, x)}{p_t(v, x)}\right) &= \left[\frac{1}{p_t^2}\left(p_t\left(\frac{\partial}{\partial t}n_t + \frac{\partial}{\partial x}(\tau n_t)\right) - n_t\left(\frac{\partial}{\partial t}p_t + \frac{\partial}{\partial x}(\tau p_t)\right)\right) \right](v, x), \\ &= 4 \int_{\mathcal{V}} \frac{p_t(v', 2x)}{p_t(v, x)} \kappa(v', v)\gamma(v', 2x) \left[\frac{n_t(v', 2x)}{p_t(v', 2x)} - \frac{n_t(v, x)}{p_t(v, x)} \right] dv', \end{aligned} \quad (9)$$

Multiplying by $H'\left(\frac{n_t(v, x)}{p_t(v, x)}\right)$, we find

$$\begin{aligned} \frac{\partial}{\partial t}H\left(\frac{n_t}{p_t}\right) + \tau\frac{\partial}{\partial x}H\left(\frac{n_t}{p_t}\right) &= 4 \int_{\mathcal{V}} \frac{p_t(v', 2x)}{p_t(v, x)} \kappa(v', v)\gamma(v', 2x)H'\left(\frac{n_t(v, x)}{p_t(v, x)}\right) \left[\frac{n_t(v', 2x)}{p_t(v', 2x)} - \frac{n_t(v, x)}{p_t(v, x)} \right] dv', \\ &:= A(v, x). \end{aligned} \quad (10)$$

Besides, using the equations on ψ and p we have

$$\frac{\partial}{\partial t}(\psi_t p_t) + \frac{\partial}{\partial x}(\tau \psi_t p_t) = \left(\frac{\partial}{\partial t} \psi_t + \tau \frac{\partial}{\partial x} \psi_t\right) p_t + \left(\frac{\partial}{\partial t} p_t + \frac{\partial}{\partial x}(\tau p_t)\right) \psi_t = -\mathcal{F}^*(\psi_t) p_t + \mathcal{F}(p_t) \psi_t,$$

and thus

$$\frac{\partial}{\partial t} \left(\psi_t p_t H\left(\frac{n_t}{p_t}\right) \right) + \frac{\partial}{\partial x} \left(\tau \psi_t p_t H\left(\frac{n_t}{p_t}\right) \right) = \left(\mathcal{F}(p_t) \psi_t - \mathcal{F}^*(\psi_t) p_t \right) H\left(\frac{n_t}{p_t}\right) + \psi_t p_t A.$$

It then remains to integrate for (v, x) in \mathcal{S} to get an expression of the entropy dissipation. The x -derivative term cancels and we obtain

$$\frac{d}{dt} \iint_{\mathcal{S}} \psi_t p_t H\left(\frac{n_t}{p_t}\right) dv dx = \underbrace{\iint_{\mathcal{S}} \left(\mathcal{F}(p_t) \psi_t - \mathcal{F}^*(\psi_t) p_t \right) H\left(\frac{n_t}{p_t}\right) dv dx}_{:=I(t)} + \iint_{\mathcal{S}} \psi_t p_t A dv dx. \quad (11)$$

that is well define from the assumption on the support of ψ , p and the expression of the support of γ . The first part $I(t)$ of the dissipation is handled through a change of variables in its second term in order to factorize by $\mathcal{F}(p_t)(v, x) \psi_t(v, x)$:

$$I(t) = 4 \iint_{\mathcal{S}} \int_{\mathcal{V}} \kappa(v', v) \psi_t(v, x) \gamma(v', 2x) p_t(v', 2x) \left[H\left(\frac{n_t(v, x)}{p_t(v, x)}\right) - H\left(\frac{n_t(v', 2x)}{p_t(v', 2x)}\right) \right] dv' dv dx,$$

which, combined with (10) and (11), brings the expected formulation.

The inequality in the case H convex (strictly convex) follows from the non-negativity (positivity) of ϕ and p , γ and κ , and a classical inequality of convexity. \square

As direct consequence of Proposition 2 we have the following classical properties.

Proposition 3. *Assume hypothesis of Theorem 3 satisfied and let (λ, N, ϕ) be the solution to $(\mathbf{S}+\mathbf{V})$. Assume n is a solution to $(\mathbf{S}_t+\mathbf{V})$. Then, setting $\tilde{n} := n e^{-\lambda t}$ we have: $\forall t \geq 0$*

i) (Conservation law). $\iint_{\mathcal{S}} \tilde{n}_t(v, x) \phi(v, x) dv dx = \iint_{\mathcal{S}} n^{in}(v, x) \phi(v, x) dv dx,$

ii) (Contraction principle). $\iint_{\mathcal{S}} |\tilde{n}_t(v, x)| \phi(v, x) dv dx \leq \iint_{\mathcal{S}} |n^{in}(v, x)| \phi(v, x) dv dx.$

More precisely, $s \mapsto (\iint_{\mathcal{S}} |\tilde{n}_s| \phi)$ is a decreasing function of time,

iii) (Maximum Principle) $|n^{in}(v, x)| \leq CN(v, x) \implies |\tilde{n}_t(v, x)| \leq CN(v, x).$

Proof. From Proposition 1, ϕ is positive on \mathcal{S} and $\text{supp}(N) = \mathcal{V} \times [\frac{b}{2}, +\infty)$. Proposition 2 thus applies to $p_t := N e^{\lambda t}$ and $\psi_t := \phi e^{-\lambda t}$, respective solutions to $(\mathbf{S}_t+\mathbf{V})$ and $(\mathbf{S}_t^*+\mathbf{V})$, and for well chosen convex differentiable entropy functions we retrieve:

i) the conservation law, after taking $H(u) = u$ and $-H$,

ii) the contraction principle from choosing $H(u) = |u|$,

iii) and the maximum principle from $H(u) = (|u| - C)_+^2$. Indeed, $|n^{in}| \leq CN$ then implies $H(\frac{n^{in}}{N}) = 0$. Therefore, $E^H[n^{in}|N](0) = 0$ and because the entropy decays (from $D^H[\tilde{n}|N](t) = D^H[n|N e^{\lambda t}](t) \geq 0$) we have that for all times t , $E^H[\tilde{n}|N](t) = \iint_{\mathcal{S}} \phi N H(\frac{\tilde{n}_t}{N})$ is 0, and deduce the expected result. \square

Before going further we establish two useful results, also derived from the GRE and later needed to establish convergence.

Proposition 4 (BV-regularity). *Let assume hypothesis of Theorem 3 are satisfied and consider (λ, N, ϕ) the solution to $(\mathbf{S}+\mathbf{V})$. Assume that $n \in \mathcal{C}(\mathbb{R}_+, L^1(\mathcal{S}; \phi \, d\text{v}dx))$ is a weak solution to $(\mathbf{S}_t+\mathbf{V})$ with initial data satisfying*

$$|n^{in}| \leq CN, \quad \frac{\partial}{\partial x}(\tau n^{in}) \in L^1(\mathcal{S}; \phi \, d\text{v}dx),$$

then, the renormalized solution $\tilde{n} := n e^{-\lambda t}$ satisfies: $\forall t \geq 0, i \in \mathcal{I}$,

$$\int_0^\infty \left| \frac{\partial}{\partial t} \tilde{n}_i(t, x) \right| \phi_i(x) \, dx \leq C(n^{in}), \quad \int_0^\infty \left| \frac{\partial}{\partial x} \tilde{n}_i(t, x) \right| \phi_i(x) \, dx \leq C(n^{in}).$$

Proof. Time derivative. The renormalization of $(\mathbf{S}_t+\mathbf{V})$, satisfied by \tilde{n} , writes as:

$$\frac{\partial}{\partial t} \tilde{n}_i(t, x) + \frac{\partial}{\partial x}(\tau_i(x) \tilde{n}_i(t, x)) + (\lambda + \gamma_i(x)) \tilde{n}_i(t, x) = 4 \sum_{j \in \mathcal{I}} \gamma_j(2x) \tilde{n}_j(t, 2x) \kappa_{ji}.$$

Differentiating it in time yields to the same equation verified by $q = \frac{\partial}{\partial t} \tilde{n}$, to which we can apply the contraction principle of Proposition 3 and derive:

$$I_t := \sum_{i \in \mathcal{I}} \int_0^\infty |q_i(t, x)| \phi_i(x) \, dx \leq \sum_{i \in \mathcal{I}} \int_0^\infty |q_i(0, x)| \phi_i(x) \, dx.$$

Evaluating at $t = 0$ the equation verified by \tilde{n} , we have

$$q_i(0, x) = -\frac{\partial}{\partial x}(\tau_i(x) n_i^{in}(x)) - (\lambda + \gamma_i(x)) n_i^{in}(x) + 4 \sum_{j \in \mathcal{I}} \gamma_j(2x) n_j^{in}(2x) \kappa_{ji},$$

whose absolute value can be controlled, bounding $|n^{in}|$ by CN and then replacing $\sum_j \gamma_j(2x) N_j(2x) \kappa_{ji}$ by the other terms of the equation on N , to obtain:

$$I_t \leq \sum_{i \in \mathcal{I}} \int_0^\infty \phi_i \left(\left| \frac{\partial}{\partial x}(\tau_i n_i^{in}) \right| + C \left| \frac{\partial}{\partial x}(\tau_i N_i) \right| \right) + 2\lambda C + 2C \sum_{i \in \mathcal{I}} \int_0^\infty \gamma_i N_i \phi_i.$$

The term in n^{in} is controlled by assumption, the other ones thanks to the estimates on (N, ϕ) provided by Theorem 3, using in particular that

$$\sum_{i \in \mathcal{I}} \int_0^\infty \gamma_i(x) N_i(x) \phi_i(x) \, dx \leq \left\| \frac{\phi}{1+x^k} \right\|_{L^\infty(\mathcal{S})} \left\| (1+x^k) \gamma N \right\|_{L^1(\mathcal{S})}.$$

Size derivative. We use the result just obtained for the time derivative. \square

Lemma 2 (GRE minimizers). *Assume n, p, ψ and H satisfy assumptions of Proposition 2, with ψ positive and H strictly convex. Assume $(\mathbf{H}\mathcal{V}_\emptyset)$, $(\mathbf{H}\kappa_{irr})$, $(\mathbf{H}\tau_{heterogeneity})$, $(\mathbf{H}\gamma-\tau)$ and $(\mathbf{H}\beta_{\text{supp}})$ satisfied. Then*

$$D^H[n|p] \equiv 0 \quad \iff \quad \exists C \in \mathbb{R} : \frac{n_i(t, x)}{p_i(t, x)} = C, \quad \forall t > 0, \quad \forall i \in \mathcal{I}, \quad x \in \left(\frac{b}{2}, +\infty\right).$$

Proof. Before starting, we introduce I_- and I_+ as defined by (3) and formulate the “support of $(v', v, x) \mapsto \kappa(v', v) \gamma(v', 2x)$ ” with discrete *index* notations:

$$\Delta := \overline{\left\{ (j, i, x) \in \mathcal{I}^2 \times \mathbb{R}_+ : \kappa_{ji} \gamma_j(2x) > 0 \right\}},$$

which can be simplified, thanks to the assumption on the support of γ , into

$$\Delta = \left\{ (j, i) \in \mathcal{I}^2 : j \in I_{\leftarrow}(i) \right\} \times \left[\frac{b}{2}, +\infty \right).$$

Also, note that because $I_{\leftarrow}(i)$ is not empty for every $i \in \mathcal{I}$ thanks to $(\mathbf{H}\kappa_{irr})$ we have

$$\left\{ (i, x) : (j, i, x) \in \Delta, j \in \mathcal{I} \right\} = \mathcal{I} \times \left[\frac{b}{2}, +\infty \right). \quad (12)$$

Now, from Proposition 2 we rather prove that the second assertion is equivalent to

$$\frac{n_i(t, x)}{p_i(t, x)} = \frac{n_j(t, 2x)}{p_j(t, 2x)}, \quad \forall t > 0, \quad (j, i, x) \in \Delta. \quad (13)$$

From now on we set $u_i := n_i/p_i$ for all $i \in \mathcal{I}$ to simplify notations. It is clear that $u_i = C$ on $(0, +\infty) \times \left[\frac{b}{2}, +\infty \right)$ for all $i \in \mathcal{I}$, implies (13). Let us prove the converse.

Recalling (9), we use (13) (which holds for any $i \in \mathcal{I}$, $x \in \left[\frac{b}{2}, +\infty \right)$ from (12)) and get:

$$\frac{\partial}{\partial t} u_i(t, x) + \tau_i(x) \frac{\partial}{\partial x} u_i(t, x) = 0, \quad t > 0, \quad (i, x) \in \mathcal{I} \times \left[\frac{b}{2}, +\infty \right). \quad (14)$$

On the one hand, (13) gives: for $t > 0$, $(j, i, x) \in \Delta$

$$\frac{\partial}{\partial t} (u_i(t, x)) + \tau_i(x) \frac{\partial}{\partial x} (u_i(t, x)) = \frac{\partial}{\partial t} (u_j(t, 2x)) + \tau_i(x) \frac{\partial}{\partial x} (u_j(t, 2x)),$$

which is

$$\frac{\partial}{\partial t} u_i(t, x) + \tau_i(x) \frac{\partial}{\partial x} u_i(t, x) = \frac{\partial}{\partial t} u_j(t, 2x) + 2\tau_i(x) \frac{\partial}{\partial x} u_j(t, 2x),$$

that, combined with (14), eventually brings

$$\frac{\partial}{\partial t} u_j(t, 2x) + 2\tau_i(x) \frac{\partial}{\partial x} u_j(t, 2x) = 0, \quad t > 0, \quad (j, i, x) \in \Delta. \quad (15)$$

On the other hand, (14) being true for all $i \in \mathcal{I}$ and $x \in \left[\frac{b}{2}, +\infty \right)$, we have a fortiori

$$\frac{\partial}{\partial t} u_j(t, 2x) + \tau_j(2x) \frac{\partial}{\partial x} u_j(t, 2x) = 0, \quad t > 0, \quad (j, x) \in \mathcal{I} \times \left[\frac{b}{4}, +\infty \right). \quad (16)$$

Taking the difference between equations (15) and (16) we find

$$(2\tau_i(x) - \tau_j(2x)) \frac{\partial}{\partial x} u_j(t, 2x) = 0, \quad t > 0, \quad (j, i, x) \in \Delta.$$

Assumption $(\mathbf{H}\tau_{heterogeneity})$ ensures that the first factor does not cancel on Δ , thus

$$\frac{\partial}{\partial x} u_i(t, x) = 2 \frac{\partial}{\partial x} u_j(t, 2x) = 0, \quad t > 0, \quad (j, i, x) \in \Delta.$$

From (12), this means that for all $i \in \mathcal{I}$, u_i is constant w.r.t. the size variable on the whole set $\left(\frac{b}{2}, +\infty \right)$, which, injected in (14), yields that it is also constant in the time variable:

$$\frac{\partial}{\partial t} u_i(t, x) = 0, \quad \frac{\partial}{\partial t} u_i(t, x) = 0, \quad t > 0, \quad (i, x) \in \mathcal{I} \times \left(\frac{b}{2}, +\infty \right).$$

Therefore $u_i(t, x) = C_i$ for $t > 0$, and $(i, x) \in \mathcal{I} \times \left(\frac{b}{2}, +\infty \right)$, and we get by (13) that:

$$C_i = u_i(t, x) = u_j(t, 2x) = C_j \quad \forall (j, i, x) \in \Delta,$$

which yields that for all $i \in \mathcal{I}$, we have $C_j = C_i$ as soon as $j \in I_{\leftarrow}(i)$. But then for all $j \in I_{\leftarrow}(i)$, the map $k \mapsto C_k$ is constant on $I_{\leftarrow}(j)$, and so on, so that it is constant on sets of indices corresponding to any strongly connected component of the graph of κ . We conclude thanks to $(\mathbf{H}\kappa_{irr})$: the graph of κ is strongly connected, thus $k \mapsto C_k$ constant on \mathcal{I} . \square

Another interesting consequence of the GRE principle is the uniqueness of the direct eigenvector solution to $(\mathbf{S}+\mathbf{V})$. Uniqueness for the whole eigenproblem can also be derived, see Proposition 5 below whose proof is left to Section 4.

Proposition 5 (Uniqueness of eigenelements). *Assume assumptions of Theorem 3 are satisfied and that, in addition, $(\mathbf{H}\kappa_{irr})$ holds. Then, there exists a unique solution to the eigenproblem $(\mathbf{S}+\mathbf{V})$.*

2.5 Long-time asymptotic behavior

After solving the eigenproblem, a natural wish is to characterize the asymptotic behavior of a solution to the Cauchy problem $(\mathbf{S}_t + \mathbf{V})$ for which we expect A.E.G.:

$$n(t, v, x) \underset{t \rightarrow +\infty}{\sim} C e^{\lambda t} N(v, x).$$

The GRE principle then proves very useful to get information on the evolution of the distance between n and $p = e^{\lambda t} N(v, x)$ in well-chosen L^1 -norm (ponderated by ϕ by taking $\psi = e^{-\lambda t} \phi$ in the GRE inequality) since it indicates that the entropy of n w.r.t. p decreases in time. However we need, first to be sure that $(\mathbf{S}_t + \mathbf{V})$ admits a weak solution in the suited space $\mathcal{C}(\mathbb{R}_+, L^1(\mathcal{S}; \phi \, dv dx))$ for any initial condition allowing us to use the maximum principle of Proposition 3-iii).

To state convergence we thus assume existence of a solution in $\mathcal{C}(\mathbb{R}_+, L^1(\mathcal{S}; \phi \, dv dx))$ to $(\mathbf{S}_t + \mathbf{V})$ for some initial data satisfying $|n^{in}| \leq CN$. When \mathcal{V} is finite $(\mathbf{H}\mathcal{V}_{\mathcal{D}})$, a straightforward adaptation of [7], however guarantees the existence of a weak solution to the Cauchy problem $(\mathbf{S}_t + \mathbf{V})$ in $\mathcal{C}(\mathbb{R}_+, L^1(\mathcal{S}; \phi \, dv dx))$ with initial condition $n^{in} \in L^1(\mathcal{S}; \phi)$, providing stronger assumptions on the coefficients. Introducing

$$\mathcal{P}_{x_0, \omega_1} := \{f : \exists K_1 \geq K_0 > 0, \omega_0 \in (0, \omega_1], K_0 x^{\omega_0} \mathbf{1}_{x \geq x_0} \leq f(x) \leq K_1 \max(1, x^{\omega_1}), x > 0\},$$

these assumptions state as follows: for all $v \in \mathcal{V}$:

- $\tau(v, \cdot) : (0, +\infty) \rightarrow (0, +\infty)$ is \mathcal{C}^1 and belong to $\mathcal{P}_{1,1}$
- $\gamma(v, \cdot) : (0, +\infty) \rightarrow [0, +\infty)$ is continuous with connected support (our $(\mathbf{H}\beta_{\text{supp}})$) and belongs to $\mathcal{P}_{x_0, \omega}$ for some positive x_0, ω ,
- either $\frac{1}{\tau(v, \cdot)}$ or $\frac{\gamma(v, \cdot)}{\tau(v, \cdot)}$ and $\frac{x^{\nu_0}}{\tau(v, \cdot)}$, for $\nu_0 \geq 0$, in to L^1_0 (this is $(\mathbf{H}\beta_0)$ and $(\mathbf{H}\tau_0)$ resp.),

Existence of a weak solution to $(\mathbf{S}_t + \mathbf{V})$ is besides derived in [22], as the expectation of the empirical measure of the underlying Markov process over smooth test functions. The result is stated for the continuous compact setting $(\mathbf{H}\mathcal{V}_{\mathcal{C}})$ without the modeling assumption $(\mathbf{H}\gamma\text{-}\tau)$ but continuous division rate γ satisfying $\gamma(0) = 0$ and $\int^\infty x^{-1} \gamma = \infty$ (the analogous of $(\mathbf{H}\beta_\infty)$) and linear growth rate $\tau : (v, x) \mapsto vx$.

Theorem 4 (Convergence to equilibrium). *Assume $(\mathbf{H}\mathcal{V}_{\mathcal{D}})$, $(\mathbf{H}\tau)$, $(\mathbf{H}\gamma)$, $(\mathbf{H}\beta)$ and $(\mathbf{H}\kappa)$. Take (λ, N, ϕ) the solution to $(\mathbf{S} + \mathbf{V})$, and assume $n \in \mathcal{C}(\mathbb{R}_+, L^1(\mathcal{S}; \phi \, dv dx))$ is a solution to $(\mathbf{S}_t + \mathbf{V})$ with initial condition n^{in} verifying $|n^{in}| \leq CN$, then we have*

$$\lim_{t \rightarrow \infty} \sum_{i \in \mathcal{I}} \int_0^\infty |n_i(t, x) e^{-\lambda t} - \rho N_i(x)| \phi_i(x) \, dx = 0, \quad \rho := \sum_{i \in \mathcal{I}} \int_0^\infty n_i^{in}(x) \phi_i(x) \, dx.$$

Proof. We follow the now classical proof given in Perthame's book [48, p.98]. Note that one can always regularize the initial data so that it satisfies the assumptions of Proposition 4, then contraction principle ensures that the convergence of the regularized problem implies converge of the initial problem. Therefore, it is enough to prove converge for n^{in} satisfying the assumptions of Proposition 4.

First step. For all $t \geq 0$, all $i \in \mathcal{I}$ and for $x > 0$, we set

$$h_i(t, x) := n_i(t, x) e^{-\lambda t} - \rho N_i(x)$$

and notice that $h e^{\lambda t}$ satisfies the growth-fragmentation $(\mathbf{S}_t + \mathbf{V})$ (linear and satisfied by both n and $N e^{\lambda t}$) with regularized initial data. Thus Proposition 4 applies and ensures

$$\int_0^\infty \left| \frac{\partial}{\partial t} h_i(t, x) \right| \phi_i(x) dx \leq C(n^{in}), \quad \int_0^\infty \left| \frac{\partial}{\partial x} h_i(t, x) \right| \phi_i(x) dx \leq C(n^{in}). \quad (17)$$

Likewise, Proposition 3 applies and brings first (Maximum principle and conservation law)

$$|h| \leq CN, \quad \sum_{i \in \mathcal{I}} \int_0^\infty h_i(t, x) \phi_i(x) dx = \sum_{i \in \mathcal{I}} \int_0^\infty (n_i^{in}(x) - \rho N_i(x)) \phi_i(x) dx = 0, \quad (18)$$

and second (contraction principle) that

$$R(t) := \sum_{i \in \mathcal{I}} \int_0^\infty |n_i(t, x) e^{-\lambda t} - \rho N_i(x)| \phi_i(x) dx$$

decreases towards some non-negative constant, say L , as t tend to infinity. We thus want to prove that L is zero.

Second step. Instead of considering the convergence of R as t tends to infinity, we introduce the family of functions $(h^{(k)})_{k \in \mathbb{N}}$, defined for all $t \geq 0$ and for $x > 0$ by

$$h^{(k)}(t, x) := h(t + k, x), \quad k \in \mathbb{N},$$

and rather consider the $L^1([0, T] \times \mathbb{R}_+; \phi_i dx)$ -convergence of the sequences $(h_i^{(k)})_{k \geq 0}$, for every $i \in \mathcal{I}$. Thanks to (17) the sequences of $h_i^{(k)}$, for $i \in \mathcal{I}$, are uniformly bounded in BV (with weight ϕ_i). We can thus extract (after M successive diagonal extractions) a subsequence from $(h^{(k)})_{k \in \mathbb{N}}$, denoted identically, that converges for all $X > 0$ and each i th component strongly in $L^1([0, T] \times [0, X]; \phi_i dx)$. The maximum principle ($|h^k| \leq CN$) however ensure that the convergence is strong on $[0, T] \times \mathbb{R}_+$. Denote by g the limit; it satisfies the renormalization by $e^{-\lambda t}$ of problem $(\mathbf{S}_t + \mathbf{V})$ and verifies

$$|g_i(t, x)| \leq C_0 N_i(x), \quad \forall i \in \mathcal{I}.$$

In particular the support of g is included in the support of N , that is $\mathcal{V} \times [\frac{b}{2}, +\infty)$ from Proposition 1, so that $g \equiv N \equiv 0$ on $\mathcal{V} \times [0, \frac{b}{2})$.

Third step. We can now work on proving that g is zero. The GRE principle stated in Proposition 2 applies to $n := h e^{\lambda t}$, $p := N e^{\lambda t}$, $\psi := \phi e^{-\lambda t}$ and $H(u) := u^2$ convex, bringing

$$\begin{aligned} -4 \sum_{i \in \mathcal{I}} \int_0^\infty \phi_i(x) \left(\sum_{j \in \mathcal{I}} \kappa_{ji} \gamma_j(2x) N_j(2x) \left| \frac{h_i(t, x)}{N_i(x)} - \frac{h_j(t, 2x)}{N_j(2x)} \right|^2 \right) dx \\ = \frac{d}{dt} \sum_{i \in \mathcal{I}} \int_0^\infty \frac{\phi_i(x)}{N_i(x)} |h_i(t, x)|^2 dx < +\infty. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^\infty \sum_{i, j \in \mathcal{I}} \kappa_{ji} \phi_i(x) \gamma_j(2x) N_j(2x) \left| \frac{h_i^{(k)}(t, x)}{N_i(x)} - \frac{h_j^{(k)}(t, 2x)}{N_j(2x)} \right|^2 dx \\ = \int_k^\infty \sum_{i, j \in \mathcal{I}} \kappa_{ji} \phi_i(x) \gamma_j(2x) N_j(2x) \left| \frac{h_i(t, x)}{N_i(x)} - \frac{h_j(t, 2x)}{N_j(2x)} \right|^2 dx \xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

Passing to the strong limit the $h^{(k)}$ in the first integral we get:

$$\int_0^\infty \sum_{i, j \in \mathcal{I}} \kappa_{ji} \phi_i(x) \gamma_j(2x) N_j(2x) \left| \frac{g_i(t, x)}{N_i(x)} - \frac{g_j(t, 2x)}{N_j(2x)} \right|^2 dx = 0.$$

Fourth step. We can finally apply Lemma 2 and get that $g_i(t, x) = CN_i(v, x)$ for all $i \in \mathcal{I}$, and a.e. $x \in (\frac{b}{2}, +\infty)$ (and thus a.e. on \mathbb{R}_+ from the last remark of step 2). To conclude we use the normalization condition on ϕ and the conservation law (18) also valid on g after passing to the limit in k : $C = C \sum_i \int_{\mathcal{S}} N_i \phi_i = \sum_i \int_{\mathcal{S}} g_i(t) \phi_i = 0$ and thus $L = 0$. \square

3 ILLUSTRATION OF THE RESULTS

3.1 Comparison of our result with existing literature

The main message of Theorem 4 is that heterogeneity in the growth rate, providing enough mixing in feature, enables convergence towards a stationary profile. We provide here an example that sheds light on how our result fits in the current knowledge.

Let us consider the case where $\mathcal{V} = \{v_1, v_2\}$ with $v_1 < v_2$, take $\tau(v, x) := vx$, $\gamma : (v, x) \mapsto \tau(v, x)\beta(x)$ with β such that $(\mathbf{H}\gamma)$ and $(\mathbf{H}\beta)$ are satisfied, and define the two following variability kernels –a reducible one, κ^{red} standing for a population with absolutely no mixing in feature, and an irreducible one, κ^{irr} for a population with mixing:

$$\kappa^{red} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \kappa^{irr} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In the non-mixing case, the Cauchy problem $(\mathbf{S}_t + \mathbf{V})$ is a system of two growth fragmentation equations that are not coupled

$$\begin{cases} \frac{\partial}{\partial t} n_1(t, x) + v_1 \frac{\partial}{\partial x} (x n_1(t, x)) + \gamma_1(x) n_1(t, x) = 4\gamma_1(2x) u_1(t, 2x), \\ \frac{\partial}{\partial t} n_2(t, x) + v_2 \frac{\partial}{\partial x} (x n_2(t, x)) + \gamma_2(x) n_2(t, x) = 4\gamma_2(2x) u_2(t, 2x), \\ n_1(0, x) = n_1^{in}(x), \quad n_2(0, x) = n_2^{in}(x), \end{cases} \quad (19)$$

whose associated eigenproblem, on the contrary, is coupled but only through the eigenvalue λ^{red} . Taken independently, each equation is associated with a first positive eigenvalue associated with eigenelements (λ_i, N_i, ϕ_i) , see [24]. Still, neither $e^{-\lambda_1 t} u_1(t, x)$ nor $e^{-\lambda_2 t} u_2(t, x)$ converge towards a stationary profile as t goes to infinity. Instead, we know from [6] that they exhibit a periodic behavior for large times.

Besides, since Assumptions $(\mathbf{H}\mathcal{V}_{\mathcal{D}})$, $(\mathbf{H}\tau)$, $(\mathbf{H}\gamma)$, $(\mathbf{H}\gamma-\tau)$, $(\mathbf{H}\beta)$, $(\mathbf{H}\kappa_{prob})$ are satisfied, Theorem 3 applies and guarantees the existence of eigenelements $(\lambda^{red}, N^{red}, \phi^{red})$ associated to system (19), with

$$0 < \lambda_1 \leq \lambda^{red} \leq \lambda_2, \quad N_i^{red} \geq 0, \quad \phi_i^{red} \geq 0, \quad i \in \{1, 2\},$$

where the first inequalities are ensured by Theorem 2. However, the irreducibility condition $(\mathbf{H}\kappa_{irr})$ is not satisfied by κ^{red} , so uniqueness is not ensured and Proposition 1 does not apply to guarantee that N_1^{red} and N_2^{red} are non-zero. Adapting the proof of Proposition 1 to each of the strongly connected components of the graph of κ , simply reduced to $\{v_1\}$ and $\{v_2\}$, shows that for each $i \in \{1, 2\}$, N_i^{red} is either zero everywhere or positive on $(\frac{b}{2}, +\infty)$. The normalization condition

$$\sum_{i \in \{1, 2\}} \int_0^\infty N_i^{red}(x) dx = 1$$

implies that at least for one $i \in \{1, 2\}$, N_i^{red} is non-zero. Both N_1^{red} and N_2^{red} cannot be non-zero, otherwise Lemma 2 would work (in equality (13), we need the existence of

$i \neq j$ such that $N_i^{red} \neq 0$ and $N_j^{red} \neq 0$) and we could use it to prove the long-time convergence of system (19) towards a stationary state (fourth step of Theorem 4). That would contradict [6]. The only possibility is that $N_1^{red} \equiv 0$ and $N_2^{red}(x) > 0$ for $x \geq \frac{b}{2}$ or the opposite and then $\lambda = \lambda_2$ or $\lambda = \lambda_1$ respectively.

In the mixing case, the equations of system $(\mathbf{S}_t + \mathbf{V})$ are coupled through their source term:

$$\begin{cases} \frac{\partial}{\partial t} n_1(t, x) + v_1 \frac{\partial}{\partial x} (x n_1(t, x)) + \gamma_1(x) n_1(t, x) = 4\gamma_2(2x) u_2(t, 2x), \\ \frac{\partial}{\partial t} n_2(t, x) + v_2 \frac{\partial}{\partial x} (x n_2(t, x)) + \gamma_2(x) n_2(t, x) = 4\gamma_1(2x) u_1(t, 2x), \\ n_1(0, x) = n_1^{in}(x), \quad n_2(0, x) = n_2^{in}(x), \end{cases}$$

and the irreducibility condition $(\mathbf{H}\kappa_{irr})$, missing in the non-mixing case, is now satisfied. We can thus apply successively Theorem 3, Proposition 1 and Theorem 4 to get

$$\sum_{i=1}^2 \int_{\mathbb{R}^+} |n_i(t, x) e^{-\lambda^{irr} t} - N_i^{irr}(x) \phi_i^{irr}(x)| \phi_i^{irr}(x) dX \xrightarrow[t \rightarrow \infty]{} 0.$$

for some $\lambda^{irr} > 0$, and for $i \in \{1, 2\}$, $N_i^{irr} > 0$ over $(\frac{b}{2}, +\infty)$ and $\phi_i^{irr} > 0$ over $(0, +\infty)$.

These two simple cases illustrate that the existence result (Theorem 3) holds for every probability matrix κ , in particular reducible ones. The irreducibility condition on κ comes into play to characterize the functions canceling the dissipation of entropy (Lemma 2), which then proves crucial to establish uniqueness of the steady state and convergence towards it.

3.2 Numerical illustration

Similarly to the previous subsection, we focus here on the special case of linear growth rates to illustrate the convergence result of Theorem 4. We numerically approximate and compare the long-time asymptotics in presence and absence of mixing in feature.

We choose $M = 3$ different features, namely $\mathcal{V} = \{1, 2, 5\}$, and approximate on the grid

$$\mathcal{S}_N := \mathcal{V} \times \{x_0, \dots, x_{2N}\}, \quad x_m := 2^{\frac{-m-N}{k}}, \quad m \in \{0, \dots, 2N\}, \quad k = 100, \quad N = 200$$

the time-evolution of the following initial data (taken identical for all features)

$$n^{in} : (v, x) \mapsto C x^a e^{-bx^2}, \quad a = 22, \quad b = 60, \quad C \text{ s.t. } \|n^{in}\|_{L^1(\mathcal{V} \times (0, x_{2N}))} = 1,$$

under the law given by the Cauchy problem $(\mathbf{S}_t + \mathbf{V})$ with coefficients

$$\tau : (v, x) \mapsto vx, \quad \gamma : (v, x) \mapsto x^2 \tau(v, x)$$

and variability kernel κ^{red} or κ^{irr} , for the non-mixing and mixing case respectively:

$$\kappa^{red} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \kappa^{irr} := \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.5 & 0.4 & 0.1 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}.$$

The numerical scheme used to obtain the following curves is an adaptation of the scheme developed in [6] to capture the oscillations that appear as time progresses in the case of equal mitosis, linear growth rate and no variability. Our scheme is identical for $M = 1$,

consisting in a splitting with upwind scheme of CFL = 1 on a geometrical grid to avoid diffusivity, and is simply extended to $M > 1$ with consequence that CFL is still 1 for the fastest individuals, those with feature $v_{max} = \max(\mathcal{V})$, but is $\frac{v_i}{v_{max}}\text{CFL} < 1$ for the other individuals, with feature v_i . For this reason we expect these cohorts to present a little bit of numerical dissipativity.

We see on Figure 4a that in absence of mixing the fastest subpopulation quickly dominates the others but does not stabilize and keeps oscillating instead. However, as soon as mixing is introduced (Figure 4b), no subpopulation is overwhelming the others anymore and all stabilize to a steady distribution.

We besides approximate the Malthus parameter λ solution to the associated eigenproblem, by three different ways to control as much as possible the approximation error:

$$\lambda \approx \frac{\frac{d}{dt}(\iint_{\mathcal{S}} n_t)}{\iint_{\mathcal{S}} n_t} := \lambda_n(t), \quad \lambda = \frac{\iint_{\mathcal{S}} \tau N}{\iint_{\mathcal{S}} x N} \approx \frac{\iint_{\mathcal{S}} \tau n_t}{\iint_{\mathcal{S}} x n_t} := \lambda_\tau(t), \quad \lambda = \frac{\iint_{\mathcal{S}} \gamma N}{\iint_{\mathcal{S}} n_t} \approx \frac{\iint_{\mathcal{S}} \gamma n_t}{\iint_{\mathcal{S}} n_t} := \lambda_\gamma(t).$$

The first is the instantaneous exponential rate of growth of the population, the other two correspond to the approximation by $n_t \approx e^{\lambda t} N$ of analytical expressions of λ from the equation at equilibrium. The time evolution of these estimates as well as the evolution of the log of the total number of individuals $\ln(\iint_{\mathcal{S}} n_t)$, are plotted on Figure 2 and Figure 3 respectively.

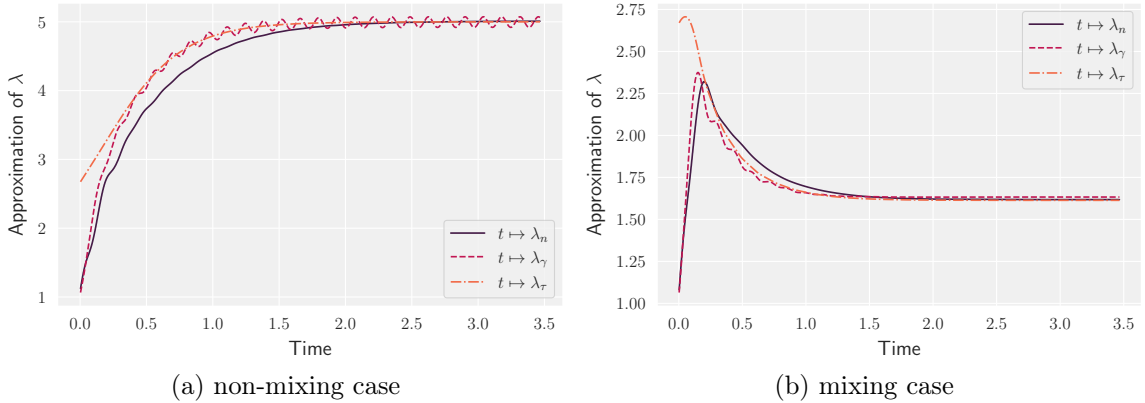


Figure 2: Time evolution of estimates of λ , obtained for coefficients $\tau(v, x) = vx$, $\gamma(v, x) = x^2\tau(v, v)$ and the two variability kernels κ^{red} (left) and κ^{irr} (right). For any $t > 0$, $\lambda_n(t)$ is obtained by linear regression of $s \mapsto \ln(\iint_{\mathcal{S}} n(s, v, x) dv dx)$ on the time interval $[\frac{t}{2}, t]$.

After, performing a linear regression of $\ln(\iint_{\mathcal{S}} n_t)$ and an average on the last times of the computation, we obtained the following estimations of the Malthus parameter:

	λ_n	λ_τ	λ_γ
Non-mixing	4.9959	4.9999	5.0136
Mixing	1.6136	1.6105	1.6393

Table 1: Estimations of the Malthus parameters through different methods.

As expected, we retrieve the Malthus parameter $\lambda^{red} \approx \lambda_{max} = v_{max}$ that corresponds to a population where all cells have same feature v_{max} in the case with no mixing, and $\lambda^{irr} \in (v_{min}, v_{max})$, v_{min} and v_{max} the extremum values of \mathcal{V} , in the case of mixing.

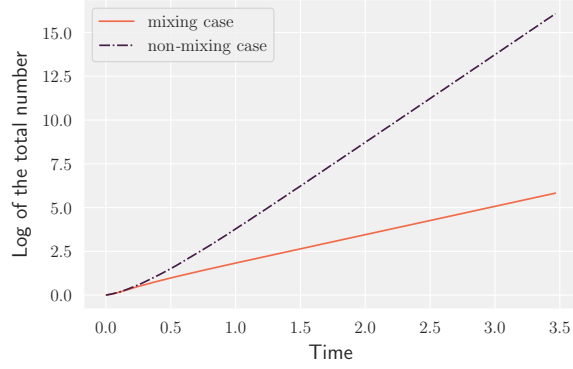


Figure 3: Time evolution of the log of the total number, $t \mapsto \ln \left(\iint_{\mathcal{S}} n(t, v, x) \, dv dx \right)$, for coefficients $\tau(v, x) = vx$, $\gamma(v, x) = x^2 \tau(v, v)$ and the two variability kernels κ^{red} (dashed line) and κ^{irr} .

A deeper study of the variation of the Malthus parameter with respect to variation in κ or \mathcal{V} would benefit to the understanding of mixing mechanisms. In agreement with Olivier's numerical results [47], it should be noticed that in the case of a *homogeneous kernel* (such that $\kappa_{ij} = \frac{1}{M}$ for all $i, j \in \{1, \dots, M\}$) and the setting described above for the other parameters, the estimated Malthus parameter (around 2.1), is lower than the average on the features (2.7) –that is the average on the Malthus parameters of the subpopulations grown independently.

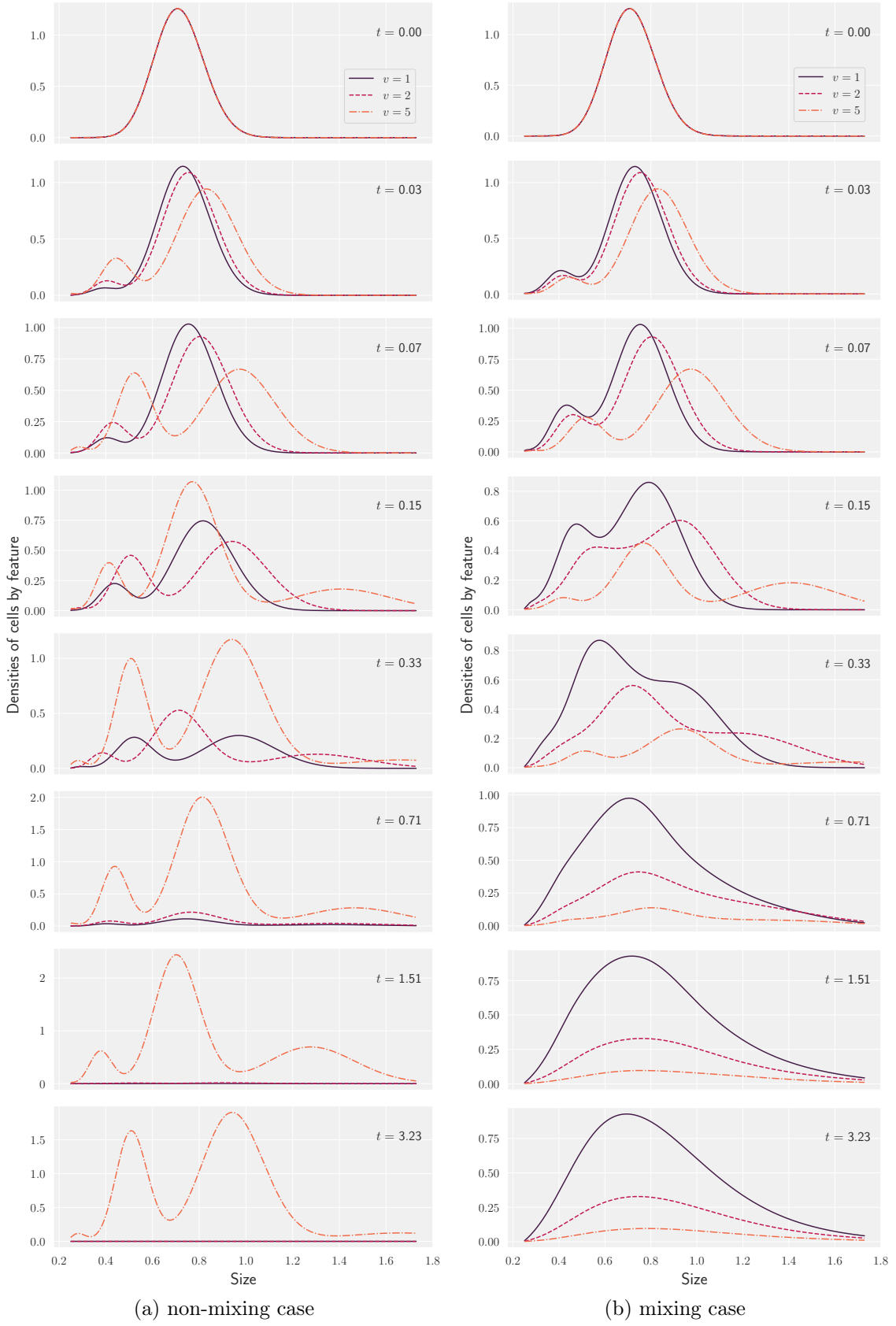


Figure 4: Time evolution of the size distribution per feature, for coefficients $\tau(v, x) = vx$, $\gamma(v, x) = x^2\tau(v, v)$ and the two variability kernel κ^{red} (left) and κ^{irr} (right).

4 PROOFS

4.1 Proof of Theorem 3

We adapt the existence proof of [24, Theorem 1] for the setting with no variability to our case. It basically consists in obtaining uniform estimates on a truncated problem, whose existence is proved through regularization and Kreĭn-Rutman theorem, as follows:

- i) we “truncate” problem $(\mathbf{S}+\mathbf{V})$ to work with a nicer “truncated problem”, $(\mathbf{S}+\mathbf{V}_{\eta,\delta})$ defined on the compact set $\mathcal{V} \times (0, R)$ and having positive boundary condition,
- ii) regularize the truncated problem through convolution by a mollifier sequence $(\rho_\varepsilon)_{\varepsilon>0}$ so that Kreĭn-Rutman can apply and provide existence of eigenelements $(\lambda^\varepsilon, N^\varepsilon, \phi^\varepsilon)$,
- iii) derive estimates on these eigenelements to be able to pass to the limit $\varepsilon \rightarrow 0$ and obtain positive eigenelements $(\lambda^{\eta,\delta}, N^{\eta,\delta}, \phi^{\eta,\delta})$ solution to the truncated problem,
- iv) then bound $(\lambda^{\eta,\delta})_{\eta,\delta>0}$ in $(0, +\infty)$ to get a $\lambda > 0$ limit when $\eta, \delta \rightarrow 0$ (and $R \rightarrow \infty$),
- v) derive estimates on the moments of $\tau^\eta N^{\eta,\delta}$, uniform in η and δ , to be able to pass to the limit the direct problem of $(\mathbf{S}+\mathbf{V})$, $\delta \rightarrow 0$ first, and then $\eta \rightarrow 0$,
- vi) and similarly with the adjoint problem by uniform estimates on $\phi^{\eta,\delta}$.

Proof. Step i). In order to get compactness and positivity, both required to apply Kreĭn-Rutman theorem, we first consider the problem on a bounded domain

$$\mathcal{S}_R := \mathcal{V} \times [0, R], \quad R > 0,$$

and second, endow the direct problem with a positive boundary condition δ and set the growth rate τ to $\eta > 0$ around $x = 0$:

$$\tau^\eta := \begin{cases} \eta & \text{on } \mathcal{V} \times [0, \eta], \\ \tau & \text{elsewhere,} \end{cases}$$

to make it bounded from below on \mathcal{S}_R by the positive constant

$$\mu = \mu(\eta, R) := \min_{i \in \mathcal{I}} \left(\operatorname{ess\,inf}_{(0,R)} \tau_i^\eta \right) > 0. \quad (20)$$

Following $(\mathbf{H}\gamma\text{-}\tau)$, we also define on \mathbb{R}_+ the functions $\gamma_i^\eta := \beta\tau_i^\eta$, for i in \mathcal{I} .

The truncation of the adjoint problem simply follows as the adjoint eigenproblem of the truncated direct eigenproblem endowed with null boundary condition at $x = R$, and finally the whole “truncated” problem states as follow: $\forall i \in \mathcal{I}$, a.e. $x \in (0, R)$,

$$\left\{ \begin{array}{l} \frac{\partial}{\partial x} [\tau_i^\eta(x) N_i^{\eta,\delta}(x)] + (\lambda^{\eta,\delta} + \gamma_i^\eta(x)) N_i^{\eta,\delta}(x) = 4 \sum_{j \in \mathcal{I}} \left(\gamma_j^\eta(2x) N_j^{\eta,\delta}(2x) \kappa_{ji} \right) \mathbf{1}_{[0,R]}(2x), \\ \tau_i^\eta(0) N_i^{\eta,\delta}(0) = \delta, \quad \sum_{j \in \mathcal{I}} \left(\int_0^R N_j^{\eta,\delta}(s) ds \right) = 1, \quad N_i^{\eta,\delta}(x) > 0, \\ -\tau_i^\eta(x) \frac{\partial}{\partial x} \phi_i^{\eta,\delta}(x) + (\lambda^{\eta,\delta} + \gamma_i^\eta(x)) \phi_i^{\eta,\delta}(x) = 2\gamma_i^\eta(x) \sum_{j \in \mathcal{I}} \left(\phi_j^{\eta,\delta}\left(\frac{x}{2}\right) \kappa_{ij} \right) + \delta \phi_i^{\eta,\delta}(0), \\ \phi_i^{\eta,\delta}(R) = 0, \quad \sum_{j \in \mathcal{I}} \left(\int_0^R N_j^{\eta,\delta}(s) \phi_j^{\eta,\delta}(s) ds \right) = 1, \quad \phi_i^{\eta,\delta}(x) \geq 0, \end{array} \right. \quad (\mathbf{S}+\mathbf{V}_{\eta,\delta})$$

From now on, δ and η being to be brought to zero, we choose them lower than 1.

Step ii). Besides compactness and positivity, one needs to work in a space whose positive cone has non-empty interior to apply the strong form of Kreĭn-Rutman theorem [17, 48]. In particular, L^p spaces are not appropriate but the space of continuous functions is. Let us thus regularize $(\mathbf{S} + \mathbf{V}_{\eta, \delta})$ in order to work in $\mathcal{C}([0, R])^M$ endowed with the norm in $\ell^\infty(\mathcal{I}; \mathcal{C}([0, R]))$ or in equivalently in $\ell^1(\mathcal{I}; \mathcal{C}([0, R]))$:

$$\|n\|_{\ell^\infty} := \max_{i \in \mathcal{I}} \|n_i\|_\infty, \quad \|n\|_{\ell^1} := \sum_{i \in \mathcal{I}} \|n_i\|_\infty, \quad (21)$$

where $\|\cdot\|_\infty$ refers to the supremum norm in $\mathcal{C}([0, R])$. We define, for $(\rho_\varepsilon)_{\varepsilon > 0}$ a sequence of mollifiers and $*$ the convolution product²:

$$\beta^\varepsilon := \rho_\varepsilon * \beta, \quad \mathbf{1}_{[0, R]}^\varepsilon := \rho_\varepsilon * \mathbf{1}_{[0, R]}, \quad \tau_i^\varepsilon := \rho_\varepsilon * \tau_i^\eta, \quad \gamma_i^\varepsilon := \beta^\varepsilon \tau_i^\varepsilon, \quad \forall i \in \mathcal{I},$$

omitting η to lighten notations. The sequence is chosen with $\text{supp}(\rho_\varepsilon) \subset [-\varepsilon, 0]$, for all positive ε , such that $\mathbf{1}_{[0, R]}^\varepsilon$ cancels on $(R, +\infty)$ (and is 1 on $[0, R - \varepsilon]$), which gives sense to the regularized truncated equation $(\mathbf{S} + \mathbf{V}_\varepsilon)$ defined only for $x \leq R$. We also mention

$$\beta^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \beta \quad L^1(0, R), \quad \tau_i^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \tau_i^\eta \quad L^1(0, R), \quad \forall i \in \mathcal{I}, \quad (22)$$

from the properties of the convolution with mollifiers [9], as well as the inequalities:

$$\|\beta^\varepsilon\|_{L^1(0, R)} \leq 2 \|\beta\|_{L^1(0, R)} \quad \varepsilon \text{ small enough}, \quad \mu \leq \tau_i^\varepsilon \leq \|\tau^\eta\|_{L^\infty(S_R)}, \quad \forall i \in \mathcal{I}. \quad (23)$$

In particular $(\tau^\varepsilon)_{\varepsilon > 0}$, bounded in $L^\infty([0, R])$, converges in $L^\infty([0, R])$ for the weak* topology (towards τ^η) so that we can also prove L^1 -convergence for $(\gamma^\varepsilon)_{\varepsilon > 0}$: $\forall i \in \mathcal{I}$,

$$\|\gamma_i^\varepsilon - \gamma_i\|_{L^1(0, R)} \leq \|\tau^\eta\|_{L^\infty(S_R)} \|\beta^\varepsilon - \beta\|_{L^1(0, R)} + \|\beta(\tau_i^\varepsilon - \tau_i^\eta)\|_{L^1(0, R)} \xrightarrow{\varepsilon \rightarrow 0} 0 \quad (24)$$

Conditions are now gathered to use Kreĭn-Rutman theorem and derive the existence of a solution $(\lambda^\varepsilon, N^\varepsilon, \phi^\varepsilon)$ to the following regularized problem (written omitting η and δ , again for simplicity). The proof is left in Appendix A.1.

Theorem 5. *Under assumptions $(\mathbf{H}\tau)$ $(\mathbf{H}\gamma)$ $(\mathbf{H}\beta)$ and $(\mathbf{H}\kappa_{prob})$, for $\delta MR < \mu$ and $\varepsilon > 0$, there exists a unique solution $(\lambda^\varepsilon, N^\varepsilon, \phi^\varepsilon) \in \mathbb{R} \times \mathcal{C}^1([0, R])^M \times \mathcal{C}^1([0, R])^M$ to the regularized eigenproblem: $\forall i \in \mathcal{I}, \forall x \in [0, R]$,*

$$\left\{ \begin{array}{l} \frac{\partial}{\partial x} [\tau_i^\varepsilon(x) N_i^\varepsilon(x)] + (\lambda^\varepsilon + \gamma_i^\varepsilon(x)) N_i^\varepsilon(x) = 4 \sum_{j \in \mathcal{I}} (\gamma_j^\varepsilon(2x) N_j^\varepsilon(2x) \kappa_{ji}) \mathbf{1}_{[0, R]}^\varepsilon(2x), \\ \tau_i^\varepsilon N_i^\varepsilon(0) = \delta, \quad \sum_{j \in \mathcal{I}} \left(\int_0^R N_j^\varepsilon(s) ds \right) = 1, \quad N_i^\varepsilon(x) > 0, \\ -\tau_i^\varepsilon(x) \frac{\partial}{\partial x} \phi_i^\varepsilon(x) + (\lambda^\varepsilon + \gamma_i^\varepsilon(x)) \phi_i^\varepsilon(x) = 2\gamma_i^\varepsilon(x) \sum_{j \in \mathcal{I}} \phi_j^\varepsilon\left(\frac{x}{2}\right) \kappa_{ij} + \delta \phi_i^\varepsilon(0), \\ \phi_i^\varepsilon(R) = 0, \quad \sum_{j \in \mathcal{I}} \left(\int_0^R N_j^\varepsilon(s) \phi_j^\varepsilon(s) ds \right) = 1, \quad \phi_i^\varepsilon(x) > 0. \end{array} \right. \quad (\mathbf{S} + \mathbf{V}_\varepsilon)$$

²For any $L^1_{loc}(\mathbb{R})$ function f , the convolution product $*$ is defined for all φ in $\mathcal{C}_c(\mathbb{R})$ (the space of continuous compactly functions on \mathbb{R}), as $f * \varphi : x \mapsto \int_0^\infty f(s) \varphi(s - x) ds$. For f only defined on \mathbb{R}_+ , $f * \varphi$ makes sense as the truncation to \mathbb{R}_+ of the convolution between f extended to \mathbb{R} by zero and φ in $\mathcal{C}_c(\mathbb{R}_+)$.

From now on, we arbitrarily impose on the truncation parameters to satisfy

$$2\delta MR = \mu \quad (25)$$

so that the condition $\delta MR < \mu$ holds. As a result, it should be noted that for η fixed and δ tending toward zero, R tend to infinity. This is not completely direct since $\mu = \mu(\eta, R)$ does not only depend on η . To be convinced, one can notice that $(\mathbf{H}\tau_\infty)$ ensures:

$$\exists \omega \geq 0, C > 0, A > \eta \quad : \quad \min_{i \in \mathcal{I}} \left(\operatorname{ess\,inf}_{(A,R)} \tau_i^\eta \right) \geq \frac{C}{R^\omega}, \quad \forall R > A,$$

while $(\mathbf{H}\tau_{pos})$ provides $\tau = \tau^\eta$ with a lower bound on $\mathcal{V} \times [\eta, A]$, say $m_{[\eta, A]}$. At the end, together with condition (25) we have $2\delta MR = \mu > \min(\eta, m_{[\eta, A]}, \frac{C}{R^\omega})$ making things clear.

It follows that R can be seen not as a parameter anymore but as functions of parameters η and δ , and then μ as well from (20): $R = R(\eta, \delta)$, $\mu = \mu(\eta, \delta)$.

Step iii). To find a solution to problem $(\mathbf{S} + \mathbf{V}_{\eta, \delta})$ it remains to bring ε to zero in the weak formulation of $(\mathbf{S} + \mathbf{V}_\varepsilon)$. To do so, we rely on the following estimates, derived in Appendix A.1 right after the proof of Theorem 5.

Lemma 3. *For ε, δ and η fixed positive, there exists constants $\lambda_{up}, N_{low}, N_{up}$, and ϕ_{up} , depending on η and δ but independent of ε , such that: $\forall i \in \mathcal{I}, \forall x \in [0, R]$,*

$$0 < \lambda^\varepsilon \leq \lambda_{up}, \quad 0 < N_{low} \leq N_i^\varepsilon(x) \leq N_{up}, \quad 0 \leq \phi_i^\varepsilon(x) \leq \phi_{up}.$$

Eigenvalue. From $(\lambda_\varepsilon)_\varepsilon$ bounded in $(0, +\infty)$ we extract a subsequence, still denoted $(\lambda_\varepsilon)_\varepsilon$ for simplicity, that converges to some non-negative $\lambda^{\eta, \delta}$.

Eigenvectors. Similarly, since for all i in \mathcal{I} families $(N_i^\varepsilon)_{\varepsilon > 0}$ and $(\phi_i^\varepsilon)_{\varepsilon > 0}$ are bounded in $L^\infty(0, R)$ we can extract from $(N^\varepsilon)_{\varepsilon > 0}$ and $(\phi^\varepsilon)_{\varepsilon > 0}$ (through M successive extractions) subsequences, denoted the same, that converge component-wise weakly* in $L^\infty(0, R)$:

$$\int_0^R N_i^\varepsilon \varphi \xrightarrow{\varepsilon \rightarrow 0} \int_0^R N_i \varphi, \quad \int_0^R \phi_i^\varepsilon \varphi \xrightarrow{\varepsilon \rightarrow 0} \int_0^R \phi_i \varphi, \quad \forall \varphi \in L^1(0, R), \quad \forall i \in \mathcal{I}. \quad (26)$$

Let us check that (λ, N, ϕ) is (weak) solution to $(\mathbf{S} + \mathbf{V}_{\eta, \delta})$:

1. From the bound from below on $(N^\varepsilon)_{\varepsilon > 0}$ and $(\phi^\varepsilon)_{\varepsilon > 0}$ we have that N positive, bounded from below by N_{low} , and ϕ non-negative.
2. We prove that N satisfies the direct equation of $(\mathbf{S} + \mathbf{V}_{\eta, \delta})$. As classical solution N^ε is weak solution to $(\mathbf{S} + \mathbf{V}_\varepsilon)$: $\forall \varphi \in \mathcal{C}_c^\infty([0, R])^M$,

$$\begin{aligned} \sum_{i \in \mathcal{I}} \int_0^R N_i^\varepsilon(x) \left(-\tau_i^\varepsilon(x) \frac{\partial}{\partial x} \varphi_i(x) + (\lambda^\varepsilon + \gamma_i^\varepsilon(x)) \varphi_i(x) \right) dx - \delta \sum_{i \in \mathcal{I}} \varphi_i(0) \\ = 2 \sum_{i \in \mathcal{I}} \int_0^R \gamma_i^\varepsilon(x) N_i^\varepsilon(x) \mathbb{1}_{[0, R]}^\varepsilon(x) \left(\sum_{j \in \mathcal{I}} \kappa_{ij} \varphi_j \left(\frac{x}{2} \right) \right) dx, \end{aligned} \quad (27)$$

and we show that the limit $\varepsilon \rightarrow 0$ exists and is nothing but the weak formulation of the direct problem of $(\mathbf{S} + \mathbf{V}_{\eta, \delta})$. For the first term we have:

$$\begin{aligned} \left| \sum_{i \in \mathcal{I}} \int_0^R \left(\tau_i^\varepsilon N_i^\varepsilon - \tau_i^\eta N_i \right) \frac{\partial}{\partial x} \varphi_i \right| \leq \sum_{i \in \mathcal{I}} \left(\|\tau_i^\varepsilon - \tau_i^\eta\|_{L^1(0, R)} \right) \left\| \frac{\partial}{\partial x} \varphi \right\|_{\ell^\infty} N_{up} \\ + \left| \sum_{i \in \mathcal{I}} \int_0^R \tau_i^\eta (N_i^\varepsilon - N_i) \frac{\partial}{\partial x} \varphi_i \right| \end{aligned}$$

with $\tau_i^\eta \frac{\partial}{\partial x} \varphi_i \in L^1(0, R)$, making clear, with (22) and (26), that right-hand side tends towards zero as ε does. The convergence also holds for the other terms of the left-hand side of (27) since $(\lambda^\varepsilon)_{\varepsilon>0}$ converges towards $\lambda^{\eta,\delta}$ and $(\gamma^\varepsilon)_{\varepsilon>0}$ towards γ^η . As for the right-hand term, we perform the same kind of computations relying on $(\mathbf{H}\kappa_{prob})$:

$$\begin{aligned} & \left| \sum_{i \in \mathcal{I}} \int_0^R \left(\gamma_i^\varepsilon(x) N_i^\varepsilon(x) \mathbf{1}_{[0,R]}^\varepsilon(x) - \gamma_i^\eta(x) N_i(x) \right) \sum_{j \in \mathcal{I}} \kappa_{ij} \varphi_j \left(\frac{x}{2} \right) dx \right| \\ & \leq \|\varphi\|_{\ell^\infty} N_{up} \sum_{i \in \mathcal{I}} \left(\|\gamma_i^\varepsilon - \gamma_i^\eta\|_{L^1(0,R)} + \|\gamma_i^\eta\|_{L^1(R-\varepsilon,R)} \right) + \sum_{i \in \mathcal{I}} \int_0^R (N_i^\varepsilon - N_i) \tilde{\varphi}_i^\eta, \end{aligned}$$

with $\tilde{\varphi}_i^\eta := \gamma_i^\eta \sum_j \kappa_{ij} \varphi_j(\frac{\cdot}{2}) \in L^1(0, R)$, and again all terms tend to zero when ε does thanks to (22), $\gamma_i^\eta \in L^1_{loc}(\mathbb{R}_+^*)$ and (26).

3. Similarly one can prove that ϕ satisfies the adjoint equation of $(\mathbf{S}+\mathbf{V}_{\eta,\delta})$.
4. Plus, testing for all i , $\varphi \equiv 1$ against N_i^ε in (26), and summing for i in \mathcal{I} , brings the expected normalization condition on N .
As for ϕ , we have $1 = \sum_i \int N_i^\varepsilon \phi_i^\varepsilon \leq N_{up} \sum_i \int \phi_i^\varepsilon$ so ϕ is non-zero and satisfies the normalization condition up to renormalization.

Step iv). *Limit as $\delta, \eta \rightarrow 0$ for $\lambda^{\eta,\delta}$.* We compare $\lambda^{\eta,\delta}$ with the *positive* eigenvalues $\lambda_{min}^{\eta,\delta}$ and $\lambda_{max}^{\eta,\delta}$ solution to the reformulation of $(\mathbf{S}+\mathbf{V}_{\eta,\delta})$ in absence of variability, when all cells have feature $v_{min} := \min(\mathcal{V})$ or $v_{max} := \max(\mathcal{V})$, respectively. The existence and positivity of such eigenvalues is ensured by the proof of [24, Theorem 1] in the mitosis case. Mimicking the proof of Theorem 2 (working on the truncated domain \mathcal{S}_R with a slightly different formulation in δ and η does not change computations) we get:

$$0 < \lambda_{min}^{\eta,\delta} \leq \lambda^{\eta,\delta} \leq \lambda_{max}^{\eta,\delta}, \quad \eta, \delta > 0. \quad (28)$$

The proof of [24, Theorem 1] also provides positive $\lambda_{min}^\eta, \lambda_{max}^\eta, \lambda_{min}$, and λ_{max} , such that

$$\lambda_{min}^{\eta,\delta} \xrightarrow{\delta \rightarrow 0} \lambda_{min}^\eta, \quad \lambda_{max}^{\eta,\delta} \xrightarrow{\delta \rightarrow 0} \lambda_{max}^\eta, \quad \eta > 0, \quad \lambda_{min}^\eta \xrightarrow{\eta \rightarrow 0} \lambda_{min}, \quad \lambda_{max}^\eta \xrightarrow{\eta \rightarrow 0} \lambda_{max}.$$

Passing to the limit $\delta \rightarrow 0$ and $\eta \rightarrow 0$ successively in (28) yields to the existence of subsequences $(\lambda^{\eta,\delta})_{\eta,\delta>0}$ and $(\lambda^\eta)_{\eta>0}$, denoted the same, and λ such that:

$$\lambda^{\eta,\delta} \xrightarrow{\delta \rightarrow 0} \lambda^\eta > 0, \quad \eta > 0, \quad \lambda^\eta \xrightarrow{\eta \rightarrow 0} \lambda > 0,$$

Step v). *Limit as $\delta \rightarrow 0$ for $N^{\eta,\delta}$.* Let us fix $\eta > 0$ and recall that δ brought to zero brings R to infinity. To pass to the limit $\delta \rightarrow 0$, we bound the moments of $\tau^\eta N^{\eta,\delta}$ in $\ell^1(\mathcal{I}; W^{1,1}([0, R]))$, uniformly in δ . We start bounding $(x^\alpha \gamma^\eta N^{\eta,\delta})_{\delta>0}$, for all $\alpha \geq 0$:

- first for $\alpha \geq m := \max(2, \omega_0 + 1)$, with ω_0 defined by $(\mathbf{H}\tau_0)$. Multiplying the direct equation in $(\mathbf{S}+\mathbf{V}_{\eta,\delta})$ by x^α and integrating on \mathcal{S}_R brings with a change of variables:

$$\begin{aligned} & \sum_{i \in \mathcal{I}} R^\alpha \tau_i^\eta(R) N_i^{\eta,\delta}(R) - \alpha \sum_{i \in \mathcal{I}} \int_0^R x^{\alpha-1} \tau_i^\eta(x) N_i^{\eta,\delta}(x) dx + \lambda^{\eta,\delta} \sum_{i \in \mathcal{I}} \int_0^R x^\alpha N_i^{\eta,\delta}(x) dx \\ & + \sum_{i \in \mathcal{I}} \int_0^R x^\alpha \gamma_i^\eta(x) N_i^{\eta,\delta}(x) dx = \frac{1}{2^{\alpha-1}} \sum_{i \in \mathcal{I}} \int_0^R x^\alpha \gamma_i^\eta(x) N_i^{\eta,\delta}(x) dx, \end{aligned} \quad (29)$$

hence the inequality

$$\left(1 - \frac{1}{2^{\alpha-1}}\right) \sum_{i \in \mathcal{I}} \int_0^R x^\alpha \gamma_i^\eta(x) N_i^{\eta, \delta}(x) dx \leq \alpha \sum_{i \in \mathcal{I}} \int_0^R x^{\alpha-1} \tau_i^\eta(x) N_i^{\eta, \delta}(x) dx.$$

According to assumption $(\mathbf{H}\beta_\infty)$ there exists $A_\alpha \geq \eta$ such that

$$\tau_i^\eta(x) = \tau_i(x) \leq \frac{x}{\alpha 2^\alpha} \gamma_i^\eta(x), \quad x \geq A_\alpha, \quad \forall i \in \mathcal{I}.$$

By definition $x \mapsto x^{\omega_0} \tau_i(x)$ is essentially bounded from above on a neighborhood of 0 and so is τ^η . Combined with assumption $(\mathbf{H}\tau_0)$, we obtain that for η small enough

$$\|x^{\omega_0} \tau_i^\eta\|_{L^\infty(0, A_\alpha)} \leq \|x^{\omega_0} \tau_i\|_{L^\infty(0, A_\alpha)} < +\infty, \quad \forall i \in \mathcal{I}.$$

From all these considerations, it follows that for all i in \mathcal{I} :

$$\begin{aligned} \int_0^R x^{\alpha-1} \tau_i^\eta(x) N_i^{\eta, \delta}(x) dx &\leq A_\alpha^{\alpha-1-\omega_0} \int_0^{A_\alpha} x^{\omega_0} \tau_i^\eta(x) N_i^{\eta, \delta}(x) dx + \int_{A_\alpha}^R x^{\alpha-1} \tau_i^\eta(x) N_i^{\eta, \delta}(x) dx, \\ &\leq A_\alpha^{\alpha-1-\omega_0} \|x^{\omega_0} \tau_i\|_{L^\infty(0, A_\alpha)} \int_0^{A_\alpha} N_i^{\eta, \delta}(x) dx + \frac{1}{\alpha 2^\alpha} \int_{A_\alpha}^R x^\alpha \gamma_i^\eta(x) N_i^{\eta, \delta}(x) dx, \end{aligned}$$

We conclude using the normalization condition on $N^{\eta, \delta}$: for all $\eta > 0$ small and $\alpha \geq m$

$$\sum_{i \in \mathcal{I}} \int_0^R x^\alpha \gamma_i^\eta(x) N_i^{\eta, \delta}(x) dx \leq \alpha \left(1 - \frac{3}{2^\alpha}\right)^{-1} A_\alpha^{\alpha-1-\omega_0} \|x^{\omega_0} \tau\|_{\ell^1(\mathcal{I}; L^\infty(0, A_\alpha))} := B_\alpha. \quad (30)$$

- Then for $0 \leq \alpha < m$. To extend estimates to smaller α we make sure there is no problem around $x = 0$ focusing on bounding $(\sum_i \tau_i^\eta N_i^{\eta, \delta})_{\delta > 0}$ essentially around zero so the moments of $\sum_i \gamma_i^\eta N_i^{\eta, \delta} = \beta \sum_i \tau_i^\eta N_i^{\eta, \delta}$ can be bounded as well around 0. Let fix ρ in $(0, \frac{1}{2})$ and define $x_\rho > 0$ (lower than R for R big) as the unique point such that

$$\int_0^{x_\rho} \beta(x) dx := \rho, \quad (31)$$

which is well defined since β is non-negative integrable around zero from $(\mathbf{H}\beta_0)$. Integrating $(\mathbf{S} + \mathbf{V}_{\eta, \delta})$ on sizes lower than any $x \in (0, x_\rho)$ and traits, gives:

$$\begin{aligned} \sum_{i \in \mathcal{I}} \tau_i^\eta(x) N_i^{\eta, \delta}(x) &\leq \delta M + 2 \sum_{i \in \mathcal{I}} \int_0^{2x} \gamma_i^\eta(s) N_i^{\eta, \delta}(s) \mathbf{1}_{[0, R]}(s) ds, \\ &\leq \delta M + 2 \left\| \sum_{i \in \mathcal{I}} \tau_i^\eta N_i^{\eta, \delta} \right\|_{L^\infty(0, x_\rho)} \int_0^{x_\rho} \beta(s) ds + \frac{2}{x_\rho^m} \sum_{i \in \mathcal{I}} \int_{x_\rho}^R s^m \gamma_i^\eta(s) N_i^{\eta, \delta}(s) ds. \end{aligned}$$

Remembering that δ has been taken inferior to 1, it follows that

$$\left\| \sum_{i \in \mathcal{I}} \tau_i^\eta N_i^{\eta, \delta} \right\|_{L^\infty(0, x_\rho)} \leq \frac{1}{1-2\rho} \left(M + \frac{2}{x_\rho^m} B_m \right) := D_0.$$

Now let us go back to $(\sum_i \int_0^R x^\alpha \gamma_i^\eta N_i^{\eta, \delta})_{\delta, \eta}$ for $0 \leq \alpha < m$. We have

$$\begin{aligned} \sum_{i \in \mathcal{I}} \int_0^R x^\alpha \gamma_i^\eta(x) N_i^{\eta, \delta}(x) dx &\leq D_0 \int_0^{x_\rho} x^\alpha \beta(s) ds + x_\rho^{\alpha-m} \sum_{i \in \mathcal{I}} \int_{x_\rho}^R x^m \gamma_i^\eta(x) N_i^{\eta, \delta}(x) dx \\ &\leq D_0 \rho x_\rho^\alpha + x_\rho^{\alpha-m} := B_\alpha. \end{aligned} \quad (32)$$

At the end, combining (30) and (32) brings

$$\forall \alpha \geq 0, \exists B_\alpha : \sum_{i \in \mathcal{I}} \int_0^R x^\alpha \gamma_i^\eta(x) N_i^{\eta, \delta}(x) dx \leq B_\alpha, \quad \delta, \eta > 0. \quad (33)$$

Finally, we can control $(x^\alpha \sum_i \tau_i^\eta N_i^\eta)_\delta$, with $\alpha > -1$, in $L^1(0, R)$. Using, from the definition of γ^η and $(\mathbf{H}\beta_\infty)$, the fact that there exists $\hat{x} > \eta$ such that:

$$\tau_i^\eta(x) \leq x \gamma_i^\eta(x), \quad \text{a.e. } x \geq \hat{x}, \quad \forall i \in \mathcal{I},$$

we have that for $R > \hat{x}$ (which is satisfied for δ small enough)

$$\begin{aligned} \sum_{i \in \mathcal{I}} \int_0^R x^\alpha \tau_i^\eta(x) N_i^{\eta, \delta}(x) dx &\leq \sum_{i \in \mathcal{I}} \int_0^{\hat{x}} x^\alpha \tau_i^\eta(x) N_i^{\eta, \delta}(x) dx + \sum_{i \in \mathcal{I}} \int_{\hat{x}}^R x^{\alpha+1} \gamma_i^\eta(x) N_i^{\eta, \delta}(x) dx, \\ &\leq D_0 \frac{\hat{x}^{\alpha+1}}{\alpha+1} + B_{\alpha+1} := C_\alpha, \end{aligned}$$

hence a $l^1(\mathcal{I}; L^1)$ -bound for $(x^\alpha \tau^\eta N^\eta)_{\eta, \delta}$:

$$\forall \alpha > -1, \exists C_\alpha : \sum_{i \in \mathcal{I}} \int_0^R x^\alpha \tau_i^\eta(x) N_i^{\eta, \delta}(x) dx \leq C_\alpha, \quad \delta, \eta > 0. \quad (34)$$

To conclude to a $l^1(\mathcal{I}; W^{1,1})$ bound we need an estimate on the derivative. Relying on equation $(\mathbf{S}+\mathbf{V}_{\eta, \delta})$ again and (29), we get: for all $\alpha > 0$,

$$\begin{aligned} \sum_{i \in \mathcal{I}} \int_0^R \left| \frac{\partial}{\partial x} (x^\alpha \tau_i^\eta(x) N_i^{\eta, \delta}(x)) \right| dx &\leq \alpha \sum_{i \in \mathcal{I}} \int_0^R \left(x^{\alpha-1} \tau_i^\eta(x) N_i^{\eta, \delta}(x) + x^\alpha \left| \frac{\partial}{\partial x} (\tau_i^\eta N_i^{\eta, \delta})(x) \right| \right) dx, \\ &\leq \alpha \left(C_{\alpha-1} + \sum_{i \in \mathcal{I}} \int_0^R x^\alpha (\lambda^{\eta, \delta} + \gamma_i^\eta(x)) N_i^{\eta, \delta}(x) dx + \frac{1}{2^{\alpha-1}} \sum_{i \in \mathcal{I}} \int_0^R x^\alpha \gamma_i^\eta(x) N_i^{\eta, \delta}(x) dx \right), \\ &\leq \alpha \left(C_{\alpha-1} + \sum_{i \in \mathcal{I}} \int_0^R x^{\alpha-1} \tau_i^\eta(x) N_i^{\eta, \delta}(x) dx + \frac{1}{2^\alpha} \sum_{i \in \mathcal{I}} \int_0^R x^\alpha \gamma_i^\eta(x) N_i^{\eta, \delta}(x) dx \right), \\ &\leq \alpha (2C_{\alpha-1} + 2^{-\alpha} B_\alpha), \end{aligned}$$

and a similar control holds when α is zero according the direct equation of $(\mathbf{S}+\mathbf{V}_{\eta, \delta})$, so that at the end, together with (34), we have:

$$\forall \alpha \geq 0, \quad (x^\alpha \tau^\eta N^\eta)_{\eta, \delta > 0} \text{ bounded in } \ell^1(\mathcal{I}; W^{1,1}(\mathbb{R}_+)). \quad (35)$$

We deduce that $(x^\alpha \tau^\eta N^\eta)_{\delta > 0}$, for every $\alpha \geq 0$, belongs to a compact set of $\ell^1(\mathcal{I}; L^1(\mathbb{R}_+))$. Indeed by a diagonal argument we can extract from it (after M successive extractions) a subsequence (denoted identically) such that for every $i \in \mathcal{I}$ and all positive X $(x^\alpha \tau_i^\eta N_i^{\eta, \delta})_{\delta > 0}$ converges strongly in $L^1([0, X])$. Denote by $H_i^\eta \in \ell^1(\mathcal{I}; L^1(\mathbb{R}_+))$ the limit when $\alpha = 0$, then

$$\begin{aligned} \sum_{i \in \mathcal{I}} \int_0^\infty |x^\alpha \tau_i^\eta N_i^{\eta, \delta} - x^\alpha H_i^\eta| &\leq \sum_{i \in \mathcal{I}} \left(\int_0^X x^\alpha |\tau_i^\eta N_i^{\eta, \delta} - H_i^\eta| + \int_X^\infty x^\alpha \tau_i^\eta N_i^{\eta, \delta} + \int_X^\infty x^\alpha H_i^\eta \right) \\ &\leq \|x^\alpha (\tau^\eta N^\eta)_{\delta > 0} - H_i^\eta\|_{\ell^1(\mathcal{I}; L^1([0, X]))} + \frac{2}{X} C_{\alpha+1}, \end{aligned}$$

where $\int_X^\infty x^\alpha H^\eta$ has been bounded by $\frac{1}{X}C_{\alpha+1}$ thanks to Fatou's lemma. Thus, for any $\varepsilon > 0$ we can find X big enough such that the last term of the right-hand side is less than $\frac{\varepsilon}{2}$. The first term will be as well for all small δ since convergence is strong in $\ell^1(\mathcal{I}; L^1([0, X]))$ and we conclude to the strong $\ell^1(\mathcal{I}; L^1(\mathbb{R}_+))$ -convergence.

We have now everything gathered to pass to the limit, first in $\delta \rightarrow 0$ and $\eta > 0$ fixed to get rid of the positive boundary condition. The same argument holds for $((1+x^\alpha)\tau^\eta N^{\eta,\delta})_{\delta>0}$: we can extract a subsequence that converges component-wise in $L^1(\mathbb{R}_+)$ to some $H^{\eta,\alpha}$. However, for some $\omega \geq 0$ and all $i \in \mathcal{I}$, $x \mapsto (1+x^\omega)\tau_i^\eta(x)$ is bounded from below by a positive constant (from τ_i^η being equal to $\eta > 0$ on $[0, \eta]$ and $(\mathbf{H}\tau_{pos})$ and $(\mathbf{H}\tau_\infty)$) and therefore we deduce

$$N_i^{\eta,\delta} \xrightarrow[\delta \rightarrow 0]{L^1} N_i^\eta := \frac{H_i^{\eta,\omega}}{(1+x^\omega)\tau_i^\eta}, \quad \forall i \in \mathcal{I}.$$

Passing to the limit in the equation $(\mathbf{S}+\mathbf{V}_{\eta,\delta})$ on $N^{\eta,\delta}$ we find: $\forall i \in \mathcal{I}, x \in (0, +\infty)$,

$$\begin{cases} \frac{\partial}{\partial x} [\tau_i^\eta(x)N_i^\eta(x)] + (\lambda^\eta + \gamma_i^\eta(x))N_i^\eta(x) = 4 \sum_{j \in \mathcal{I}} \left(\gamma_j^\eta(2x)N_j^\eta(2x)\kappa_{ji} \right), \\ N_i^\eta(0) = 0, \quad \sum_{j \in \mathcal{I}} \left(\int_0^R N_j^\eta(s) ds \right) = 1, \quad N_i^\eta(x) \geq 0. \end{cases} \quad (\mathbf{S}+\mathbf{V}_\eta)$$

Limit as $\eta \rightarrow 0$ for N^η . All estimates (29)-(35) remain true for delta $\delta = 0$. If they ensure that $(x^\alpha \tau_i^\eta N_i^\eta)_{\eta>0}$, for all $i \in \mathcal{I}$, belongs to a compact set of $L^1(\mathbb{R}_+)$ (same arguments than for $(x^\alpha \tau_i^\eta N_i^{\eta,\delta})_{\delta>0}$) not necessarily $(N_i^\eta)_{\eta>0}$ anymore, since the limit τ_i of $(\tau_i^\eta)_{\eta>0}$ can vanish at zero. Nevertheless, combined to $(\mathbf{H}\tau_{pos})$ and $(\mathbf{H}\tau_\infty)$, we have that $(x^\alpha N_i^\eta)_{\eta>0}$ converges strongly (and thus weakly) in $L^1([\varepsilon, +\infty))$, for any $\varepsilon > 0, \alpha \geq 0$.

We first prove that the weak convergence of $(N_i^\eta)_{\eta>0}$ holds in L_0^1 , which is equivalent by Dunford-Pettis theorem to prove that $(N_i^\eta)_{\eta>0}$ is equi-integrable around 0. To do so we establish $L^\infty(\mathbb{R}_+)$ -bounds for $(\sum_i x^\alpha \tau_i^\eta N_i^\eta)_\eta$, and thus $(x^\alpha \tau_i^\eta N_i^\eta)_\eta$ for all $i \in \mathcal{I}$ and $\alpha \geq -\nu_0$, with $\nu_0 \geq 0$ defined by $(\mathbf{H}\tau_0)$.

- *Equi-integrability around 0.* It is enough to prove a bound around zero for $\alpha = -\nu_0$. Integrating $(\mathbf{S}+\mathbf{V}_\eta)$ on $(0, x')$, for $x' \leq x \in \mathbb{R}_+$, and summing on $i \in \mathcal{I}$ yields:

$$\sum_{i \in \mathcal{I}} \tau_i^\eta(x')N_i^\eta(x') \leq 2 \sum_{i \in \mathcal{I}} \int_0^{2x'} \gamma_i^\eta(s)N_i^\eta(s) ds \leq 2 \sum_{i \in \mathcal{I}} \int_0^{2x} \gamma_i^\eta(s)N_i^\eta(s) ds \leq 2B_0,$$

We introduce $f^\eta: x \mapsto \|\tau^\eta N^\eta\|_{\ell^1(\mathcal{I}; L^\infty(0,x))}$. For $x \in (0, \frac{x_\rho}{2})$ (x_ρ defined by (31)), we find

$$\begin{aligned} f^\eta(x) &\leq 2 \sum_{i \in \mathcal{I}} \left(\int_0^x \beta(s)\tau_i^\eta(s)N_i^\eta(s) ds + (2x)^{\nu_0} \int_x^{2x} s^{-\nu_0}\beta(s)\tau_i^\eta(s)N_i^\eta(s) ds \right), \\ &\leq 2\rho f^\eta(x) + 2^{\nu_0+1}x^{\nu_0} \int_x^{x_\rho} s^{-\nu_0}\beta(s)f^\eta(s) ds. \end{aligned}$$

Therefore, for x in $(0, \frac{x_\rho}{2})$ and for $c_0 := \frac{2^{\nu_0+1}}{1-2\rho} > 0$,

$$x^{-\nu_0} f^\eta(x) \leq c_0 \int_x^{x_\rho} \beta(s)s^{-\nu_0} f^\eta(s) ds := c_0 F^\eta(x),$$

and applying Grönwall's lemma to $x \mapsto x^{-\nu_0} f^\eta(x)$ finally brings: for $x \in (0, \frac{x_\rho}{2})$,

$$x^{-\nu_0} \sum_{i \in \mathcal{I}} \tau_i^\eta(x)N_i^\eta(x) \leq c_0 F^\eta\left(\frac{x_\rho}{2}\right) e^{c_0 \int_x^{\frac{x_\rho}{2}} \beta(s) ds} \leq c_0 \frac{2^{\nu_0+1}B_0\rho}{x_\rho^{\nu_0}} e^{c_0\rho} := C.$$

Therefore, noticing that for all $i \in \mathcal{I}$, and for $x > 0$,

$$\frac{x^{\nu_0}}{\tau_i^\eta(x)} \leq \max\left(1, \frac{x^{\nu_0}}{\tau_i(x)}\right) := f_i(x),$$

we find that for all i in \mathcal{I} , N_i^η is controlled on $(0, \frac{x_\rho}{2})$ by Cf_i that is L_0^1 and independent of η . We conclude that $(N_i^\eta)_{\eta>0}$, $i \in \mathcal{I}$, is equi-integrable around $x = 0$ as well.

By Dunford-Pettis theorem we deduce that for all $\alpha \geq -\nu_0$, $(x^\alpha N^\eta)_{\eta>0}$ converges (after M successive extractions) to some $x^\alpha N$ component-wise weakly in $L^1(\mathbb{R}_+)$. However for all $i \in \mathcal{I}$, $(x^\alpha N_i^\eta)_{\eta>0}$ converges strongly in $L^1([\varepsilon, +\infty))$ thus we have:

$$\begin{aligned} \int_0^{+\infty} |N_i^\eta(x) - N_i(x)| dx &= \int_0^\varepsilon |N_i^\eta(x) - N_i(x)| dx + \int_\varepsilon^{+\infty} |N_i^\eta(x) - N_i(x)| dx, \\ &\leq 2C \int_0^\varepsilon f_i(x) dx + \int_\varepsilon^{+\infty} |N_i^\eta(x) - N_i(x)| dx, \\ &\xrightarrow{\eta \rightarrow 0} 2C \int_0^\varepsilon f_i(x) dx, \end{aligned}$$

with the right-hand side arbitrarily small for ε, η small since f_i belongs to L_0^1 . We conclude that for all i in \mathcal{I} , $(N_i^\eta)_{\eta>0}$, converges to N_i strongly in $L^1(\mathbb{R}_+)$, and then, passing to the limit in the weak formulation of $(\mathbf{S}+\mathbf{V}_\eta)$, that N satisfies the direct problem of $(\mathbf{S}+\mathbf{V})$.

Step vi). Limit as $\eta, \delta \rightarrow 0$ for $\phi^{\eta, \delta}$. We want to derive for some positive k and all $i \in \mathcal{I}$, a $L^\infty(\mathbb{R}_+, \frac{dx}{1+x^k})$ -bound for $(\phi_i^{\eta, \delta})_{\eta, \delta > 0}$. Let fix η, δ positive such as to guarantee (25). We start by controlling $(\phi_i^{\eta, \delta})_{\eta, \delta > 0}$ on any interval $[0, A]$, for any positive A :

- *Bound on $[0, x_\rho]$, with x_ρ defined by (31).* Consider equation $(\mathbf{S}+\mathbf{V}_{\eta, \delta})$ on $\phi_i^{\eta, \delta}$, divide by τ_i^η and integrate on $[x, x_\rho]$: for all $i \in \mathcal{I}$, and for $x \in [0, x_\rho]$,

$$\begin{aligned} \phi_i^{\eta, \delta}(x) &\leq \phi_i^{\eta, \delta}(x_\rho) + 2 \int_x^{x_\rho} \beta(s) \left(\sum_{j \in \mathcal{I}} \phi_j^{\eta, \delta} \left(\frac{s}{2} \right) \kappa_{ij} \right) ds + \frac{\delta x_\rho}{\mu} \phi_i^{\eta, \delta}(0), \\ &\leq \max_{j \in \mathcal{I}} (\phi_j^{\eta, \delta}(x_\rho)) + \left(2\rho + \frac{x_\rho}{2MR} \right) \max_{j \in \mathcal{I}} \left(\|\phi_j^{\eta, \delta}\|_{L^\infty(0, x_\rho)} \right), \end{aligned}$$

and thus for $R = R(\eta, \delta)$ greater than $R_0 := \frac{x_\rho}{2M(1-2\rho)}$, i.e. for δ and η small enough we have: for all $i \in \mathcal{I}$, for $x \in [0, x_\rho]$,

$$\phi_i^{\eta, \delta}(x) \leq C(x_\rho) \|\phi^{\eta, \delta}(x_\rho)\|_{\ell^\infty(\mathcal{I})}, \quad C(x_\rho) := \left(1 - 2\rho - \frac{x_\rho}{2MR_0} \right)^{-1}.$$

- *Bound on $[x_\rho, A]$.* The map $G_i^{\eta, \delta} : x \mapsto e^{-\int_{x_\rho}^x \frac{\lambda^{\eta, \delta} + \gamma_i^\eta}{\tau_i^\eta} ds} \phi_i^{\eta, \delta}(x)$, $i \in \mathcal{I}$, decays on \mathbb{R}_+ from $(\mathbf{S}+\mathbf{V}_{\eta, \delta})$. So for any $A > x_\rho > \eta$, we can find $C(A) > 0$ (independent from η, δ) s.t.

$$\phi_i^{\eta, \delta}(x) \leq e^{\int_{x_\rho}^x \frac{\lambda^{\eta, \delta} + \gamma_i^\eta(s)}{\tau_i^\eta(s)} ds} \phi_i^{\eta, \delta}(x_\rho) \leq C(A) \|\phi^{\eta, \delta}(x_\rho)\|_{\ell^\infty(\mathcal{I})}, \quad \forall i \in \mathcal{I}, x \in [x_\rho, A].$$

- *Uniform bound on $[0, A]$.* To conclude we need a uniform bound for $\phi^{\eta, \delta}(x_\rho)$. Using the decay of $G_i^{\eta, \delta}$ and the normalization condition on $\phi^{\eta, \delta}$ we get: $\forall i \in \mathcal{I}, x \in [0, x_\rho]$,

$$\begin{aligned} 1 &\geq \sum_{i \in \mathcal{I}} \left(\int_0^{x_\rho} \phi_i^{\eta, \delta}(x) N_i^{\eta, \delta}(x) dx \right) \geq \sum_{i \in \mathcal{I}} \left(\phi_i^{\eta, \delta}(x_\rho) \int_0^{x_\rho} e^{-\int_x^{x_\rho} \frac{\lambda^{\eta, \delta} + \gamma_i^\eta(s)}{\tau_i^\eta(s)} ds} N_i^{\eta, \delta}(x) dx \right), \\ &\geq \sum_{i \in \mathcal{I}} \left(\phi_i^{\eta, \delta}(x_\rho) e^{-\rho} \int_{\frac{x_\rho}{2}}^{x_\rho} e^{-\int_x^{x_\rho} \frac{2\lambda}{\tau_i(s)} ds} N_i^{\eta, \delta}(x) dx \right), \end{aligned}$$

where the integral term converges towards a positive quantity as δ, η go to zero since $x_\rho > b$ and N_i is positive on $(\frac{b}{2}, +\infty)$ as proved in proposition (1). Thus $\|\phi^{\eta, \delta}(x_\rho)\|_{\ell^\infty(\mathcal{I})}$ is bounded and we can conclude

$$\forall A > 0, \exists C_0(A) : \quad \|\phi^{\eta, \delta}\|_{\ell^\infty(\mathcal{I}; L^\infty(0, A))} \leq C_0(A) \quad \eta, \delta > 0. \quad (36)$$

It remains to bound $\phi^{\eta, \delta}$ uniformly by $C(1 + x^k)$ on $\mathcal{V} \times [A, +\infty)$. One prove that the adjoint problem of $(\mathbf{S} + \mathbf{V}_{\eta, \delta})$ satisfies a maximum principle of the form of [24, Lemma 4.], the proof follows the same steps. Therefore, building for some $A_0 > 0$ a supersolution $\bar{\phi}$ of any problem $(\mathbf{S} + \mathbf{V}_{\eta, \delta})$, $\eta, \delta > 0$, on $\mathcal{V} \times [A_0, +\infty)$, greater than $\phi^{\eta, \delta}$ on $\mathcal{V} \times [0, A_0]$ and positive at $x = R(\eta, \delta)$, yields $\phi^{\eta, \delta} \leq \bar{\phi}$ everywhere. We look for a supersolution of the form $\bar{\varphi}_i : x \mapsto x^k + \theta$, $i \in \mathcal{I}$, with positive k, θ to be determined. It must satisfy on $[A_0, R]$

$$-\tau_i^\eta(x) \frac{\partial}{\partial x} \bar{\varphi}_i(x) + (\lambda^{\eta, \delta} + \gamma_i^\eta(x)) \bar{\varphi}_i(x) \geq 2\gamma_i^\eta(x) \sum_{j \in \mathcal{I}} \left(\bar{\varphi}_j \left(\frac{x}{2} \right) \kappa_{ij} \right) + \delta \phi_i^{\eta, \delta}(0), \quad i \in \mathcal{I}. \quad (37)$$

Since $\phi_i^{\eta, \delta}(0) \leq C_0(1)$ for $\eta, \delta > 0$ (see (36)) it is enough to find k and $A_0 \geq \eta$ such that

$$-k\tau_i(x)x^{k-1} + (\lambda^{\eta, \delta} + \gamma_i(x))(x^k + \theta) \geq 2\gamma_i(x) \left(\theta + \frac{x^k}{2} \right) + \delta C_0(1), \quad i \in \mathcal{I},$$

holds on $[A_0, +\infty)$. Dividing by $x^{k-1}\tau_i(x)$ we find that if

$$\left(1 - \frac{1}{2^{k-1}} \right) x\beta(x) \geq k + \frac{\theta\gamma_i(x)}{x^{k-1}\tau_i(x)} + \frac{\delta C_0(1)}{x^{k-1}\tau_i(x)} \quad (38)$$

is satisfied on $[A_0, +\infty)$ then (37) holds. Since $x \mapsto x\beta(x)$ goes to infinity as x does and both τ_i, γ_i belong to \mathcal{P}_∞ (assumptions $(\mathbf{H}\beta_\infty)$ and $(\mathbf{H}\tau_\infty)$), there exists $k > 0$ such that for any $\theta > 0$, there is a $A_0 > 0$ for which (38) holds true on $[A_0, +\infty)$. We can apply the maximal principle to $\bar{\phi} := \frac{A_0}{\theta} \bar{\varphi}$ (that satisfies $\bar{\phi} \geq \phi_i^{\eta, \delta}$ on $\mathcal{V} \times [0, A_0]$, (37) on $\mathcal{V} \times [A_0, R]$ and $\bar{\phi}_i(R) > 0$, uniformly in η, δ) to finally conclude:

$$\exists k, C, \theta > 0 : \forall \eta, \delta > 0 \text{ small}, \quad \phi_i^{\eta, \delta}(x) \leq Cx^k + \theta \quad \forall i \in \mathcal{I}, x \in [0, +\infty).$$

We obtained that $\phi_i^{\eta, \delta}$, for all $i \in \mathcal{I}$, is uniformly bounded in $L_{loc}^\infty(\mathbb{R}_+)$, therefore $\tau_i^\eta \partial_x \phi_i^{\eta, \delta}$ is uniformly bounded in $L_{loc}^\infty(\mathbb{R}_+^*)$ (from $(\mathbf{S} + \mathbf{V}_{\eta, \delta})$ and $\gamma \in L_{loc}^\infty(\mathbb{R}_+^*)$ and so is $(\partial_x \phi_i^{\eta, \delta})_{\delta, \eta}$ thanks to $(\mathbf{H}\tau_{pos})$). We can thus extract (again after M successive diagonal extractions) a subsequence still denoted $(\phi^{\eta, \delta})_{\eta, \delta > 0}$ converging in $\mathcal{C}^0(\mathbb{R}_+^*)^M$ towards some ϕ and such that for every $i \in \mathcal{I}$, $(\frac{\partial}{\partial x} \phi_i^{\eta, \delta})_{\eta, \delta}$ converges $L_{loc}^\infty(\mathbb{R}_+^*)$ -weakly* towards $\frac{\partial}{\partial x} \phi_i \in L_{loc}^\infty(\mathbb{R}_+^*)$ (and $\frac{\phi_i^{\eta, \delta}}{1+x^k}$ towards $\frac{\phi_i}{1+x^k}$ strongly in $L^1(\mathbb{R}_+)$).

It remains to check that ϕ satisfies the adjoint equation of $(\mathbf{S} + \mathbf{V})$ in $L_{loc}^1(\mathbb{R}_+^*)$. It is clear from $(\mathbf{H}\gamma)$ that all the terms in $\phi^{\eta, \delta}$ converge to the expected limit in $L_{loc}^1(\mathbb{R}_+^*)$. As for the derivative term, we can use the weak* convergence to derive: for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+^*)$, with $\text{supp}(\varphi) = K$ and $\eta \leq \min(K)$:

$$\int_K \left| \tau^\eta \frac{\partial}{\partial x} \phi_i^{\eta, \delta} - \tau \frac{\partial}{\partial x} \phi_i \right| \varphi \leq \int_K \left(\frac{\partial}{\partial x} \phi_i^{\eta, \delta} - \frac{\partial}{\partial x} \phi_i \right) \tilde{\varphi} \xrightarrow{\eta, \delta \rightarrow 0} 0,$$

for $\tilde{\varphi} = \tau \text{sgn} \left(\frac{\partial}{\partial x} \phi_i^{\eta, \delta} - \frac{\partial}{\partial x} \phi_i \right) \varphi \in L^1(K)$, so that the convergence holds in $L_{loc}^1(\mathbb{R}_+^*)$ as well.

At the end, the normalization condition holds as a consequence of the $L^\infty - L^1$ convergence written as:

$$1 = \sum_{i \in \mathcal{I}} \int_0^\infty \frac{\phi_i^{\eta, \delta}}{1+x^k} (1+x^k) N_i^{\eta, \delta} \xrightarrow{\eta, \delta \rightarrow 0} \sum_{i \in \mathcal{I}} \int_0^\infty \frac{\phi_i}{1+x^k} (1+x^k) N_i = \sum_{i \in \mathcal{I}} \int_0^\infty \phi_i N_i. \quad \square$$

4.2 Proof of Proposition 5

Proof. Eigenvalue. Consider $(\lambda_1, N^{(1)}, \phi^{(1)})$ and $(\lambda_2, N^{(2)}, \phi^{(2)})$ two solutions to $(\mathbf{S}+\mathbf{V})$. We have by duality that:

$$\lambda_1 \langle N^{(1)}, \phi^{(2)} \rangle = \langle -\frac{d}{dx}(\tau N^{(1)}) + \mathcal{F}N^{(1)}, \phi^{(2)} \rangle = \langle N^{(1)}, \tau \frac{d}{dx} \phi^{(2)} + \mathcal{F}^* \phi^{(2)} \rangle = \lambda_2 \langle N^{(1)}, \phi^{(2)} \rangle,$$

with $\langle N^{(1)}, \phi^{(2)} \rangle$ positive thanks to Proposition 1, so that $\lambda_1 = \lambda_2 := \lambda$.

Direct eigenvector. Now, we find that the entropy of $n := N^{(1)} e^{\lambda t}$ with respect to $p := N^{(2)} e^{\lambda t}$, written for $\psi := \phi^{(1)} e^{-\lambda t}$ and some strictly convex H is independent on time so that for all $t > 0$, $D^H[n|p](t) = D^H[N^{(1)}|N^{(2)}] = 0$, and we deduce with Lemma 2 that $N^{(1)} = CN^{(2)}$, for some $C > 0$. Thanks to the normalizing condition on $N^{(1)}$ and $N^{(2)}$ we conclude to $C = 1$.

Adjoint eigenvector. Let x_0 be any positive real number and define $\phi := C(\phi^{(2)} - \phi^{(1)})$ with $C := -\text{sgn}(\phi^{(2)}(x_0) - \phi^{(1)}(x_0))$. By linearity, ϕ satisfies the adjoint equation of $(\mathbf{S}+\mathbf{V})$. Thus for every $i \in \mathcal{I}$, the map

$$x \mapsto \phi_i(x) e^{-\Lambda_i(x)}, \quad \Lambda_i(x) := \int_{x_0}^x \frac{\lambda + \gamma_i(s)}{\tau_i(s)} ds$$

is decreasing on $[x_0, +\infty)$ (see computations of the proof of Proposition 1) which brings

$$\phi(x) \leq \phi(x_0) e^{\Lambda_i(x)} = -|\phi(x_0)| e^{\Lambda_i(x)} \leq 0, \quad \text{a.e. } x \in [x_0, +\infty).$$

Now from the normalization condition satisfied by $\phi^{(1)}$ and $\phi^{(2)}$ we have that $\sum_i \int N_i \phi_i$ is zero and thus:

$$\sum_{i \in \mathcal{I}} \int_0^\infty N |\phi| = \sum_{i \in \mathcal{I}} \left(\int_0^{x_0} N |\phi| - \int_{x_0}^\infty N \phi \right) \xrightarrow{x_0 \rightarrow 0} 0,$$

which allows to conclude that ϕ is zero almost everywhere on $\text{supp } N$, i.e. $\phi^{(2)} \equiv \phi^{(1)}$ on $\mathcal{V} \times [\frac{b}{2}, +\infty)$. If $b > 0$, $\beta \equiv 0$ on $[0, b]$ thus for $j = 1, 2$:

$$\phi_i^{(j)}(x) = \phi_i^{(j)}\left(\frac{b}{2}\right) e^{-\int_x^{b/2} \frac{\lambda}{\tau_i(s)} ds}, \quad \forall i \in \mathcal{I}, x \in [0, \frac{b}{2}],$$

and we have $\phi^{(2)} \equiv \phi^{(1)}$ as well on $\mathcal{V} \times [0, \frac{b}{2}]$. \square

A APPENDIX

A.1 Proof of Theorem 5 – Kreĭn-Rutman

The proof relies on Kreĭn-Rutman theorem which generalizes Peron-Frobenius theorem for matrices to the infinite dimension. We refer to [48, Section 6.6] for a similar application, [38] for the original paper and [17, Chapter VIII, Appendix] for a more recent proof and applications.

Proof. Let η, δ and ε be fixed positive real numbers and omit from notations the truncation parameters η and δ . We work on the Banach space

$$(X_R, \|\cdot\|_{X_R}) := (\mathcal{C}([0, R])^M, \|\cdot\|_{\ell^\infty})$$

whose positive cone $\{f \in X_R \mid f_i \geq 0, \forall i \in \{1, \dots, M\}\}$ has non-empty interior. We recall that $\|\cdot\|_{\ell^\infty}$ is defined by (21).

Direct problem. In the following we consider equivalently the quantity $u^\varepsilon = \tau^\varepsilon N^\varepsilon$ to hide the growth rate in the transport term. The objective is to apply Kreĭn-Rutman theorem to the linear operator $\mathcal{G}_\varepsilon: f \mapsto u$ defined on X_R , for $\alpha > 0$, as solution to

$$\begin{cases} \frac{\partial}{\partial x} u_i(x) + \frac{\alpha + \gamma_i^\varepsilon(x)}{\tau_i^\varepsilon(x)} u_i(x) = 4 \sum_{j \in \mathcal{I}} \left(\frac{\gamma_j^\varepsilon(2x)}{\tau_j^\varepsilon(2x)} u_j(2x) \kappa_{ji} \right) \mathbf{1}_{[0,R]}^\varepsilon(2x) + \frac{f_i(x)}{\tau_i^\varepsilon(x)}, \\ u_i(0) = \delta \sum_{j \in \mathcal{I}} \left(\int_0^R \frac{u_j(x)}{\tau_j^\varepsilon(x)} dx \right), \end{cases} \quad (39)$$

with semi-explicit expression:

$$\begin{aligned} u_i(x) = & u_i(0) e^{-\int_0^x \frac{\alpha + \gamma_i^\varepsilon(y)}{\tau_i^\varepsilon(y)} dy} + 2 \int_0^{2x} e^{-\int_{\frac{x}{2}}^x \frac{\alpha + \gamma_i^\varepsilon(y)}{\tau_i^\varepsilon(y)} dy} \beta^\varepsilon(s) \sum_{j \in \mathcal{I}} (u_j(s) \kappa_{ji}) \mathbf{1}_{[0,R]}^\varepsilon(s) ds \\ & + \int_0^x e^{-\int_s^x \frac{\alpha + \gamma_i^\varepsilon(y)}{\tau_i^\varepsilon(y)} dy} \frac{f_i(s)}{\tau_i^\varepsilon(s)} ds, \quad \forall i \in \mathcal{I}, \forall x \in [0, R], \end{aligned} \quad (40)$$

Let us verify that \mathcal{G}_ε fulfills the assumptions of the theorem.

i) \mathcal{G}_ε is well defined from X_R into itself. We aim at proving that for any f in X_R , (39) admits a solution in X_R . Relying on Banach-Picard fix-point theorem (that also provides uniqueness), we define for f in X_R , the operator T_f such that for w in X_R , $u = T_f(w)$ is the (explicit) solution to: $\forall i \in \mathcal{I}, \forall x \in (0, R)$,

$$\begin{cases} \frac{\partial}{\partial x} u_i(x) + \frac{\alpha + \gamma_i^\varepsilon(x)}{\tau_i^\varepsilon(x)} u_i(x) = 4\beta^\varepsilon(2x) \sum_{j \in \mathcal{I}} (w_j(2x) \kappa_{ji}) \mathbf{1}_{[0,R]}^\varepsilon(2x) + \frac{f_i(x)}{\tau_i^\varepsilon(x)}, \\ u_i(0) = \delta \sum_{j \in \mathcal{I}} \left(\int_0^R \frac{w_j(s)}{\tau_j^\varepsilon(s)} ds \right), \end{cases}$$

and prove that T_f is a contraction from X_R to X_R .

Let w^1, w^2 be two elements of X_R . We define $u^i = T_f(w^i)$, $i \in \{1, 2\}$, and $w = w^2 - w^1$, $u = u^2 - u^1$. Then u satisfies: $\forall i \in \mathcal{I}, \forall x \in (0, R)$,

$$\begin{cases} \frac{\partial}{\partial x} u_i(x) + \frac{\alpha + \gamma_i^\varepsilon(x)}{\tau_i^\varepsilon(x)} u_i(x) = 4\beta^\varepsilon(2x) \sum_{j \in \mathcal{I}} (w_j(2x) \kappa_{ji}) \mathbf{1}_{[0,R]}^\varepsilon(2x), \\ u_i(0) = \delta \sum_{j \in \mathcal{I}} \left(\int_0^R \frac{w_j(s)}{\tau_j^\varepsilon(s)} ds \right), \end{cases}$$

whose explicit solution is given, for all i in \mathcal{I} , all x in $[0, R]$, by:

$$u_i(x) = u_i(0) e^{-\int_0^x \frac{\alpha + \gamma_i^\varepsilon(y)}{\tau_i^\varepsilon(y)} dy} + 4 \int_0^x e^{-\int_s^x \frac{\alpha + \gamma_i^\varepsilon(y)}{\tau_i^\varepsilon(y)} dy} \beta^\varepsilon(2s) \sum_{j \in \mathcal{I}} (w_j(2s) \kappa_{ji}) \mathbf{1}_{[0,R]}^\varepsilon(2s) ds.$$

A change of variables and rough estimates, using (23) to bound τ^ε , bring: $\forall i \in \mathcal{I}, \forall x \in [0, R]$

$$\begin{aligned} |u_i(x)| & \leq \delta \sum_{j \in \mathcal{I}} \left(\int_0^R \frac{|w_j(s)|}{\tau_j^\varepsilon(s)} ds \right) + 4 \int_0^x e^{-\int_s^x \frac{\alpha + \gamma_i^\varepsilon(y)}{\tau_i^\varepsilon(y)} dy} \beta^\varepsilon(2s) \sum_{j \in \mathcal{I}} (|w_j(2s)| \kappa_{ji}) \mathbf{1}_{[0,R]}^\varepsilon(2s) ds, \\ & \leq \underbrace{\left(\frac{\delta MR}{\mu} + 2 \max_{i \in \mathcal{I}} \left(\sum_{j \in \mathcal{I}} \kappa_{ji} \right) \int_0^{2x} e^{-\frac{\alpha(x-\frac{s}{2})}{\|\tau^\eta\|_{L^\infty}}} \beta^\varepsilon(s) \mathbf{1}_{[0,R]}^\varepsilon(s) ds \right)}_{:=k_\varepsilon(x)} \|w\|_{X_R}. \end{aligned} \quad (41)$$

We need $k_\varepsilon(x)$ to be strictly less than 1 uniformly in x . However, for later purpose we prove a bound uniform in ε as well, which requires a little finer work.

First, let us impose $\delta MR < \mu$, say $2\delta MR \leq \mu$. Then, let us consider some small $\tilde{\varepsilon} > 0$ to be fixed, and focus on the integral part of k_ε : first for x in $[0, \frac{\tilde{\varepsilon}}{2}]$

$$I_\varepsilon(x) := \int_0^{2x} e^{-\frac{\alpha(x-\frac{s}{2})}{\|\tau^\eta\|_{L^\infty}}} \beta^\varepsilon(s) \mathbf{1}_{[0,R]}^\varepsilon(s) ds \leq \int_0^{\tilde{\varepsilon}} \beta^\varepsilon(s) ds = \|\beta^\varepsilon\|_{L^1(0,\tilde{\varepsilon})} \leq 2 \|\beta\|_{L^1(0,\tilde{\varepsilon})}$$

(using that β^ε converges to β in $L^1(0,R)$), and then for x in $[\frac{\tilde{\varepsilon}}{2}, R]$

$$\begin{aligned} I_\varepsilon(x) &\leq \int_0^{\min(2x,R)-\tilde{\varepsilon}} e^{-\frac{\alpha(x-\frac{s}{2})}{\|\tau^\eta\|_{L^\infty}}} \beta^\varepsilon(s) ds + \int_{\min(2x,R)-\tilde{\varepsilon}}^{\min(2x,R)} \beta^\varepsilon(s) ds, \\ &\leq e^{-\frac{\alpha}{\|\tau^\eta\|_{L^\infty}} \left(x - \frac{\min(2x,R)-\tilde{\varepsilon}}{2}\right)} \|\beta^\varepsilon\|_{L^1(0,R)} + \|\beta^\varepsilon\|_{L^1(\min(2x,R)-\tilde{\varepsilon}, \min(2x,R))}, \end{aligned}$$

which brings, after distinguishing whether x is smaller or larger than $\frac{R}{2}$ and using properties (23) of the convolution product, that for any ε small enough and x in $[\frac{\tilde{\varepsilon}}{2}, R]$:

$$I_\varepsilon(x) \leq 2 \left(e^{-\frac{\alpha}{\|\tau^\eta\|_{L^\infty}} \tilde{\varepsilon}} \|\beta\|_{L^1(0,R)} + \sup_{s \in [\tilde{\varepsilon}, R]} (\|\beta\|_{L^1(s-\tilde{\varepsilon}, s)}) \right) := I(\alpha, \tilde{\varepsilon}),$$

Therefore, β being $L^1_{loc}([0, +\infty))$ (see Assumption $(\mathbf{H}\beta_0)$), we can fix $\tilde{\varepsilon} = \tilde{\varepsilon}_0$ such that

$$\sup_{s \in [0, R]} (\|\beta\|_{L^1(s, s+\tilde{\varepsilon}_0)}) < \frac{1}{4} \left(4 \max_{i \in \mathcal{I}} \left(\sum_{j \in \mathcal{I}} \kappa_{ji} \right) \right)^{-1}.$$

This way, for any small ε , k_ε is bounded by $\frac{3}{4}$ on $[0, \frac{\tilde{\varepsilon}_0}{2}]$ and to have k_ε strictly less than 1 on $[\frac{\tilde{\varepsilon}_0}{2}, R]$ as well, it only remains to fix $\alpha = \alpha_0$ such that

$$e^{-\frac{\alpha_0}{\|\tau^\eta\|_{L^\infty}} \tilde{\varepsilon}_0} \|\beta\|_{L^1(0,R)} < \frac{1}{4} \left(4 \max_{i \in \mathcal{I}} \left(\sum_{j \in \mathcal{I}} \kappa_{ji} \right) \right)^{-1}.$$

Going back to (41), we are now able to conclude:

$$\|u\|_{X_R} \leq k \|w\|_{X_R}, \quad k := \frac{\delta MR}{\mu} + 4 \max_{i \in \mathcal{I}} \left(\sum_{j \in \mathcal{I}} \kappa_{ji} \right) I(\alpha_0, \tilde{\varepsilon}_0) < 1,$$

for $2\delta MR \leq \mu$ and $I, \alpha_0, \tilde{\varepsilon}_0$ independent of ε , defined by the three previous equations. As strict contraction, T_f thus admits a unique fix-point that is precisely the solution to (39).

To ensure $k < 1$ we impose from now one to the end of the proof that

$$2\delta MR \leq \mu, \quad \alpha_0 \leq \alpha. \quad (42)$$

ii) \mathcal{G}_ε is continuous. For any i in \mathcal{I} , x in $[0, R]$, taking the absolute value in (40) we obtain inequality (41) with at the right-hand side: terms in u instead of w , similarly controlled by $k \|u\|_{X_R}$, and an additional term in f , handled as follows:

$$\int_0^x e^{-\int_s^x \frac{\alpha + \gamma_i^\varepsilon(y)}{\tau_i^\varepsilon(y)} dy} \frac{|f_i(s)|}{\tau_i^\varepsilon(s)} ds \leq \left[\frac{1}{\alpha} e^{-\int_s^x \frac{\alpha}{\tau_i^\varepsilon(y)} dy} \right]_0^x \|f\|_{X_R} \leq \frac{1}{\alpha} \|f\|_{X_R}.$$

This provides $|u_i(x)|$ with a bound uniform in i and x allowing to conclude to

$$\|u\|_{X_R} \leq \frac{1}{(1-k)\alpha} \|f\|_{X_R}. \quad (43)$$

iii) \mathcal{G}_ε is strongly positive. Let f be a non-negative function in X_R , we first want to show that $u = \mathcal{G}_\varepsilon(f)$ is non-negative as well. It is easy to check that for any non-negative w , $T_f(w)$ is also non-negative. Therefore, recalling that u is defined as the fix-point of T_f , we have u non-negative. Combined with (40) we thus have:

$$u_i(x) \geq u_i(0) e^{-\int_0^x \frac{\alpha + \gamma_i^\varepsilon(y)}{\tau_i^\varepsilon(y)} dy} + \int_0^x e^{-\int_s^x \frac{\alpha + \gamma_i^\varepsilon(y)}{\tau_i^\varepsilon(y)} dy} \frac{|f_i(s)|}{\tau_i^\varepsilon(s)} ds$$

with

$$u_i(0) = \delta \sum_{j \in \mathcal{I}} \left(\int_0^R \frac{u_j(s)}{\tau_j^\varepsilon(s)} ds \right),$$

and we deduce that if additionally f is not the null function, then u is positive on \mathcal{S}_R .

iv) \mathcal{G}_ε is compact. We aim at showing that the image A by \mathcal{G}_ε of the unit ball \mathcal{B}_{X_R} of X_R is relatively compact in X_R . From *ii*), inequality (43), we know that for any f in \mathcal{B}_{X_R} , $u = \mathcal{G}_\varepsilon(f)$ is bounded in X_R , uniformly in f , thus so is $\frac{\partial}{\partial x} u$ from the equation: $\forall i \in \mathcal{I}$

$$\frac{\partial}{\partial x} u_i(x) = -\frac{\alpha + \gamma_i^\varepsilon(x)}{\tau_i^\varepsilon(x)} u_i(x) + 4\beta^\varepsilon(2x) \sum_{j \in \mathcal{I}} (u_j(2x) \kappa_{ji}) \mathbf{1}_{[0,R]}^\varepsilon(2x) + \frac{f_i(x)}{\tau_i^\varepsilon(x)} \quad (44)$$

Therefore, by Ascoli-Arzelà theorem, $A_i := \{u_i \mid u = \mathcal{G}_\varepsilon(f), f \in \mathcal{B}_X\}$, $i \in \mathcal{I}$, is relatively compact in $\mathcal{C}([0, R])$ for the supremum norm. Thus from any sequence $(u^n)_{n \in \mathbb{N}}$ in A we can extract, by M successive extractions, a subsequence $(u^{\varphi(n)})_{n \in \mathbb{N}}$ such that every sequence $(u_i^{\varphi(n)})_{n \in \mathbb{N}}$ of A_i , i in \mathcal{I} , converges in $\mathcal{C}([0, R])$. Then $(u^{\varphi(n)})_{n \in \mathbb{N}}$ converges in X_R and we have proved that A is relatively compact in X_R .

We can finally apply Kreĭn-Rutman theorem that gives us the existence and uniqueness of a positive eigenvalue α^ε and a *positive* eigenvector u^ε in X_R solution to $\mathcal{G}_\varepsilon(u^\varepsilon) = \alpha^\varepsilon u^\varepsilon$. Denoting $\frac{u^\varepsilon}{\tau^\varepsilon}$ the X_R vector with components $\frac{u_i^\varepsilon}{\tau_i^\varepsilon}$, $i \in \mathcal{I}$, we got the existence of

$$\lambda_{di}^\varepsilon := \alpha - \frac{1}{\alpha^\varepsilon} < \alpha, \quad N^\varepsilon = \left(\sum_{i \in \mathcal{I}} \left(\int_0^R \frac{|u_i^\varepsilon(x)|}{\tau_i^\varepsilon(x)} dx \right) \right)^{-1} \frac{u^\varepsilon}{\tau^\varepsilon}, \quad (45)$$

solution in $\mathbb{R} \times X_R$ to the direct problem of $(\mathbf{S} + \mathbf{V}_\varepsilon)$. The $\mathcal{C}^1([0, R])$ continuity of the N_i^ε , i in \mathcal{I} , is a direct consequence of (44) and the continuity of $\tau_i^\varepsilon > 0$ and γ_i^ε .

Adjoint problem. A function ϕ^ε is solution to the adjoint equation of $(\mathbf{S} + \mathbf{V}_\varepsilon)$ if and only if for all i in \mathcal{I} , $\hat{\phi}_i^\varepsilon: x \mapsto \phi_i^\varepsilon(R - x)$ satisfies for all x in $(0, R)$

$$\begin{cases} \hat{\tau}_i^\varepsilon(x) \frac{\partial}{\partial x} \hat{\phi}_i^\varepsilon(x) + (\lambda^\varepsilon + \hat{\gamma}_i^\varepsilon(x)) \hat{\phi}_i^\varepsilon(x) = 2\hat{\gamma}_i^\varepsilon(x) \sum_{j \in \mathcal{I}} \left(\hat{\phi}_j^\varepsilon\left(\frac{x}{2}\right) \kappa_{ij} \right) + \delta \hat{\phi}_i^\varepsilon(R), \\ \hat{\phi}_i^\varepsilon(0) = 0, \quad \sum_{j \in \mathcal{I}} \left(\int_0^R N_j^\varepsilon(s) \hat{\phi}_j^\varepsilon(R - s) ds \right) = 1, \quad \hat{\phi}_i^\varepsilon(x) > 0, \end{cases} \quad (46)$$

where $\hat{\tau}_i^\varepsilon: x \mapsto \tau_i^\varepsilon(R - x)$ and $\hat{\gamma}_i^\varepsilon: x \mapsto \gamma_i^\varepsilon(R - x)$. This brings us to a problem similar to the direct problem. By the same method we obtain the existence of $(\lambda_{ad}^\varepsilon, \hat{\phi}^\varepsilon)$ solution to (46) and thus a solution to the adjoint problem of $(\mathbf{S} + \mathbf{V}_\varepsilon)$.

Complete problem. We have proven the existence of $(\lambda_{di}^\varepsilon, N^\varepsilon)$ and $(\lambda_{ad}^\varepsilon, \hat{\phi}^\varepsilon)$ solution to the direct and adjoint equations of $(\mathbf{S} + \mathbf{V}_\varepsilon)$. It remains to check that $\lambda_{di}^\varepsilon = \lambda_{ad}^\varepsilon$. This is straightforward if we integrate the direct equation against the adjoint eigenvector:

$$\langle \mathcal{G}_\varepsilon N^\varepsilon, \hat{\phi}^\varepsilon \rangle = \langle N^\varepsilon, \mathcal{G}_\varepsilon^* \hat{\phi}^\varepsilon \rangle, \quad \text{i.e.} \quad \lambda_{di}^\varepsilon \langle N^\varepsilon, \hat{\phi}^\varepsilon \rangle = \lambda_{ad}^\varepsilon \langle N^\varepsilon, \hat{\phi}^\varepsilon \rangle,$$

with compact bracket notation $\langle f, g \rangle := \sum_i (\int f_i g_i)$. We conclude thanks to the normalization condition $\langle N^\varepsilon, \phi^\varepsilon \rangle = 1$. \square

Relying on estimate (43) and on the expression of λ^ε given by (45) we derive bounds for the family $(\lambda^\varepsilon, N^\varepsilon, \phi^\varepsilon)_{\varepsilon>0}$.

Proof (Lemma 3). Let us fix $\varepsilon > 0$ and take $\alpha = \alpha_0$, which do not contradict (42).

Uniform upper bound for λ_ε . From (43) applied to $f = u^\varepsilon > 0$ we have $\alpha^\varepsilon \leq \frac{1}{(1-k)\alpha_0}$. The bound then comes directly from the expression of λ^ε given by (45):

$$\lambda^\varepsilon \leq k\alpha_0 \leq \alpha_0 = \lambda_{up}.$$

Null lower bound for λ_ε . We assume by contradiction that λ^ε is non-positive. Integrating for $i \in \mathcal{I}$ and x in $(0, R]$, the direct equation of $(\mathbf{S}+\mathbf{V}_\varepsilon)$ brings with a change of variables:

$$\begin{aligned} \sum_{i \in \mathcal{I}} \left(\tau_i^\varepsilon(x) N_i^\varepsilon(x) \right) - \delta M + \lambda^\varepsilon \sum_{i \in \mathcal{I}} \int_0^x N_i^\varepsilon(s) ds + \sum_{i \in \mathcal{I}} \int_0^x \gamma_i^\varepsilon(s) N_i^\varepsilon(s) ds \\ = 2 \sum_{i \in \mathcal{I}} \int_0^{2x} \gamma_i^\varepsilon(s) N_i^\varepsilon(s) \mathbb{1}_{[0, R]}^\varepsilon(s) ds. \end{aligned} \quad (47)$$

Then, λ^ε non-positive would imply: $\forall x \in [0, R - \varepsilon]$,

$$\sum_{i \in \mathcal{I}} \left(\tau_i^\varepsilon(x) N_i^\varepsilon(x) \right) \geq \delta M + \int_0^x \beta^\varepsilon(s) \sum_{i \in \mathcal{I}} \left(\tau_i^\varepsilon(s) N_i^\varepsilon(s) \right) ds.$$

We can apply Grönwall's lemma to $x \mapsto \sum_{i \in \mathcal{I}} (\tau_i^\varepsilon(x) N_i^\varepsilon(x))$, that is continuous together with β^ε , and get: $\forall x \in [0, R - \varepsilon]$,

$$\sum_{i \in \mathcal{I}} \left(\tau_i^\varepsilon(x) N_i^\varepsilon(x) \right) \geq \delta M e^{\int_0^x \beta^\varepsilon(s) ds}.$$

However $(\mathbf{H}\beta_\infty)$ and $(\mathbf{H}\tau_\infty)$ (inherited by β^ε and τ^ε , resp.) successively guarantee: $\forall i \in \mathcal{I}$

- the existence for all $k \in \mathbb{N}$ of $A > 0$ such that $\beta^\varepsilon(s) \geq \frac{k}{s}$ for $s \geq A$, which implies

$$\sum_{i \in \mathcal{I}} \left(\tau_i^\varepsilon(x) N_i^\varepsilon(x) \right) \geq \delta M e^{\int_0^x \beta^\varepsilon(s) ds} \geq \delta M e^{\int_A^x \frac{k}{s} ds} = \delta M \left(\frac{x}{A} \right)^k, \quad x \in [A, R],$$

- and providing k big enough, the existence of $B > A$ s.t. $x^{-k} \tau_i^\varepsilon(x) \leq \frac{\delta M}{A^k}$, for $x \geq B$.

Together with the normalization condition of $(\mathbf{S}+\mathbf{V}_\varepsilon)$, we find:

$$1 = \sum_{i \in \mathcal{I}} \int_0^R N_i^\varepsilon(x) dx \geq \frac{A^k}{\delta M} \sum_{i \in \mathcal{I}} \int_B^{R-\varepsilon} x^{-k} \tau_i^\varepsilon(x) N_i^\varepsilon(x) dx \geq \int_B^{R-\varepsilon} dx = R - \varepsilon - B,$$

which is contradictory as soon as R gets big enough. Thus λ^ε is positive for R big enough.

Uniform bounds for N^ε . We fix i arbitrarily in \mathcal{I} , multiply the direct equation of $(\mathbf{S}+\mathbf{V}_\varepsilon)$ by $e^{-\int_0^x (\lambda^\varepsilon + \gamma_i^\varepsilon)/\tau_i^\varepsilon}$ and integrate in size: $\forall i \in \mathcal{I}, \forall x \in [0, R], \forall \varepsilon_0 > 0$

$$\begin{aligned} \tau_i^\varepsilon(x) N_i^\varepsilon(x) &= \delta e^{-\int_0^x \frac{\lambda^\varepsilon + \gamma_i^\varepsilon(y)}{\tau_i^\varepsilon(y)} dy} + 4 \int_0^x e^{-\int_s^x \frac{\lambda^\varepsilon + \gamma_i^\varepsilon(y)}{\tau_i^\varepsilon(y)} dy} \sum_{j \in \mathcal{I}} \left(\gamma_j^\varepsilon(2s) N_j^\varepsilon(2s) \kappa_{ji} \right) \mathbb{1}_{[0, R]}^\varepsilon(2s) ds \\ &\leq \delta + 2 \max_{i,j} (\kappa_{ji}) \left(\|\beta^\varepsilon\|_{L^1([0, \varepsilon_0])} \|\tau^\varepsilon N^\varepsilon\|_{\ell^\infty(\mathcal{I}; L^\infty([0, \varepsilon_0]))} + \|\gamma^\varepsilon\|_{\ell^\infty(\mathcal{I}; L^\infty(\varepsilon_0, R))} \right) \end{aligned}$$

Using (23), the fact that N^ε is positive for the lower bound, and that for some positive ε_0 , $\|\beta\|_{L^1([0,\varepsilon_0])}$ is less than $(8 \max_{i,j}(\kappa_{ji}))^{-1}$ such that on $[0, \varepsilon_0]$ $\tau_i^\varepsilon N_i^\varepsilon$ is less than $K := 2(\delta + 2 \max_{i,j}(\kappa_{ji}) \|\gamma^\varepsilon\|_{\ell^\infty(\mathcal{I}; L^\infty(\varepsilon_0, R))})$ for the upper bound, we conclude to what expected:

$$N_{low}(\delta, \eta) := \frac{\delta}{\|\tau^\eta\|_{L^\infty}} e^{-\left(\frac{R\lambda_{up}}{\mu} + \|\beta\|_{L^1(0,R)}\right)} > 0, \quad N_{up}(\delta, \eta) := \frac{\delta + \frac{K}{2} + 2 \max_{i,j}(\kappa_{ji}) \|\gamma^\varepsilon\|_{\ell^\infty(\mathcal{I}; L^\infty(\varepsilon_0, R))}}{\mu}.$$

As for the ϕ^ε , $\varepsilon > 0$, we can proceed the same way –using the adjoint equation of $(\mathbf{S} + \mathbf{V}_\varepsilon)$ to derive a $l^\infty(\mathcal{I}; L^\infty(0, R))$ -bound uniform in ε as well. \square

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