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Elementary Integral Series for Heun Functions

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Heun differential equations are the most general second order Fuchsian equations with four regular singularities. An explicit integral series representation of Heun functions involving only elementary integrands has hitherto been unknown and noted as an important open problem in a recent review. We provide such representations of the solutions of all equations of the Heun class: general, confluent, bi-confluent, doubly-confluent and triconfluent. All the series are illustrated with concrete examples of use and Python implementations are available for download. We demonstrate the utility of the integral series by providing the first representation of the solution to the Teukolsky radial equation governing the metric perturbations of rotating black holes that is convergent everywhere from the black hole horizon up to spatial infinity.

* Web page: http://www-lmpa.univ-littoral.fr/~plgiscard/; P.-L. G. is supported by the Agence Nationale de la Recherche young researcher grant No. ANR-19-CE40-0006.

I. INTRODUCTION

The study of Heun equations has generated significant interest in physics lately due to their proliferation in various contexts including quantum optics [1], condensed matter physics [2], quantum computing [3], two-state problems [4], black-hole theory [5, 6] and more [7].

From a mathematical standpoint, recent research has focused on relations and transformations involving Heun functions and other transcendent functions. These include relations with hypergeometric [8] and elliptic functions [9] and with solutions of Painlevé equations [10].

Yet few studies [7] have specifically focused on determining their properties most relevant to physical applications. For example, the lack of integral expansions of these functions involving only elementary integrands has been clearly identified as a major obstacle when extracting physical meaning in areas of black holes astrophysics [7], yet remains unaddressed in the mathematical literature. The present tackles this issue by determining a novel integral representation of the Heun equations involving only elementary functions that is tailored to physical applications. In particular, we demonstrate the applicability of the novel integral representation to the Teukolsky equation [11] that governs the metric perturbations of rotating black holes. In addition, we provide asymptotic behaviour near singular points to demonstrate uniform convergence of our solutions from horizon to infinity, by means of a single series. This stands in contrast to the state-of-the-art MST formalism [12], that uses *two* hypergeometric series (one convergent at the horizon and the other at infinity) that must then be matched after an analytic continuation procedure. This last step requires the introduction of an auxiliary parameter lacking physical correspondence, at the very least obscuring the physical picture. The convergence of the integral series over the entire domain from the black hole horizon up to spatial infinity therefore alleviates the need for such parameters lacking physical correspondence when calculating solutions of the Teukolsky radial equation. These solutions are of primary importance for computing quantities of physical interest such as gravitational wave fluxes [13] and quasinormal modes [6]. The present progress in integral representation is enabled by the method of path-sum [14], which generates the linear Volterra integral equation of the second kind satisfied by any function involved a system of coupled linear differential equations with variable coefficients.

This paper is organised as follows. In Section II we give the minimal necessary background on Heun equations and existing integral representations with their drawbacks. Section III is a self-contained presentation of the novel, elementary integral representations of all functions of Heun class, illustrated with concrete examples. Then, in Section IV we give the elementary integral series representation of the solution to the Teukolsky radial equation. This representation is the first one to be convergent from the black hole horizon up to spatial infinity. We conclude in §V with a brief discussion of the novel integral series and future prospects of the method of path-sum from which they stem for solving the coupled system of Teukolsky angular and radial equations.

II. HEUN DIFFERENTIAL EQUATIONS

A. General Heun Equation

In its canonical form, the General Heun Equation (GHE) is given as follows:

$$\frac{d^2y(z)}{dz^2}+\left[\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\varepsilon}{z-t}\right]\frac{dy(z)}{dz}+\frac{\alpha\beta z-q}{z(z-1)(z-t)}y(z)=0, \tag{1}$$

where $q\in\mathbb{C}$ is called the *accessory parameter*. The parameters satisfy the Fuch's condition: $1+\alpha+\beta=\gamma+\delta+\varepsilon$ The GHE has four singular points at $z=0,1,t,\infty$. Concerning its solutions, Maier–completing a task initiated by Heun [15] himself–has shown that there are 192 solutions can be generated using the symmetries of D_4 [16].

For specific parameter values the Heun equation reduces to other well-known equations of importance: e.g. setting $\varepsilon=0,\gamma=\delta=1/2$ yields the Mathieu equation [17].

B. Confluent Heun Equations

The GHE contains 4 regular singularities but we may apply a confluence procedure to two of its singularities such that we get an irregular singularity. We call the resultant equation a confluent Heun equation (CHE). The CHE contains at least one irregular singular point besides the regular singular points. We can construct local solutions in the vicinity of this irregular singular points by the means of *generally divergent* Thomé series [18]. By applying the confluence procedure laid out in [18] to the singularities at $z=t$ and $z=\infty$ in equation 2, we get:

$$\frac{d^2y(z)}{dz^2}+\left[\frac{\gamma}{z}+\frac{\delta}{z-1}+\varepsilon\right]\frac{dy(z)}{dz}+\frac{\alpha z-q}{z(z-1)}y(z)=0. \tag{2}$$

The number of parameters in the CHE as compared to the GHE is therefore reduced by one. By continuing application of the confluence procedure, we obtain the bi-confluent Heun equation

$$\frac{d^2y(z)}{dz^2}+\left[\frac{\gamma}{z}+\delta+\varepsilon z\right]\frac{dy(z)}{dz}+\frac{\alpha z-q}{z}y(z)=0, \tag{3}$$

and related doubly-confluent Heun equation

$$\frac{d^2y(z)}{dz^2}+\left[\frac{\delta}{z^2}+\frac{\gamma}{z}+1\right]\frac{dy(z)}{dz}+\frac{\alpha z-q}{z^2}y(z)=0, \tag{4}$$

and finally the triconfluent Heun equation

$$\frac{d^2y(z)}{dz^2}+\left[\gamma+\delta z+\varepsilon z^2\right]\frac{dy(z)}{dz}+(\alpha z-q)y(z)=0. \tag{5}$$

We refer the reader to [18] for further general informations on these equations.

C. Integral representations of Heun functions

Erdélyi was the first to give an integral equation relating the values taken at two points by a general Heun function [19]. His equation, a Fredholm integral equation, involves an hypergeometric kernel and can be used to obtain a series representation of Heun functions as sums of hypergeometric functions with coefficients determined via recurrence relations. Applications of this result in the special cases of Mathieu and Lamé equations were discussed by Sleeman [20]. Naturally, since Erdélyi's breakthrough many mathematical works on Heun equations were concerned with integral transformations involving Heun functions and higher transcendental functions [9, 21] This brief list of contributions is far from exhaustive and we refer the reader to the recent review [7] on this subject for more details.

The common feature of all of these integral transforms is that they contain higher transcendental functions which makes them physically opaque and of limited use for practical calculations. In addition, the resulting series representations for the Heun functions have insufficient radiuses of convergence causing difficulties for black hole perturbation theory (see Section IV). These issues were noted in the recent review [7] on Heun's functions, the current state of research on this being described as follows :

"No example has been given of a solution of Heun's equation expressed in the form of a definite integral or contour integral involving only functions which are, in some sense, simpler.[...] This statement does not exclude the possibility of having an infinite series of integrals with 'simpler' integrands".

In this work, we constructively prove the existence of such a representation for all types of Heun's functions and for all parameters, allowing them to take complex values too, in the form of infinite series of integrals whose integrands involve only rational functions and exponentials of polynomials. Furthermore, we show that the series converges everywhere except at the singular points of the Heun function. We show that any Heun function, general or (bi-, doubly-, tri-)confluent, is a sum of exactly two functions each of which satisfy a linear Volterra equation of the second kind with explicitely identified elementary kernels. In particular, any Heun function $H(z)$ itself satisfies a linear integral Volterra equation of the second kind with such an elementary kernel if either there is at least one non-singular point z_0 where $H(z_0)=H'(z_0)$ or there is a point where $H(z_0)=0$.

III. ELEMENTARY INTEGRAL SERIES FOR ALL TYPES OF HEUN FUNCTIONS

Owing to the emphasis of the present work on concrete results and a physical application, all the technical mathematical proofs are deferred to Appendix A.

A. Notation and convergence

The $*$ notation is useful to denote iterated integrals. Let $K(z,z_0)$ be a function of two variables that is continuous over $]z_0,z[$. We denote $K(z,z_0)=K^{*1}(z,z_0)$ and, for any integer $n>1$,

$$K^{*n}(z,z_0)=\int_{z_0}^{z}K^{*(n-1)}(z,\zeta_1)K(\zeta_1,z_0)d\zeta.$$

In other terms K^{*n} is the Volterra composition [22] of K with itself n-times. The only type of integral series that is required to present all results of this section is the following, called a Neumann series, $G(z,z_0) := \sum_{n=1}^{\infty} K^{*n}(z,z_0)$, i.e.

$$G(z,z_0) = K(z,z_0) + \int_{z_0}^{z} K(z,\zeta_1)K(\zeta_1,z_0)d\zeta_1 + \int_{z_0}^{z}\int_{\zeta_1}^{z} K(z,\zeta_2)K(\zeta_2,\zeta_1)K(\zeta_1,z_0)d\zeta_2 d\zeta_1 + \cdots,$$

see also Appendix A.A B. Iterated integral series as above are well known to converge provided the kernel $K(z,z_0)$ is bounded for all $z \geq z_0$ in an interval of interest $I \subseteq \mathbb{R}$ [23]. Indeed, supposing this condition is met then there exists a finite $M_K := \sup_{\substack{z,z_0 \in I \\ z \geq z_0}} |K(z,z_0)|$. This implies that, for all $z \geq z_0$, $z, z_0 \in I$,

$$\begin{aligned} \left|K^{*n}(z,z_0)\right| &\leq \int_{z_0}^{z}\int_{\zeta_1}^{z}\cdots\int_{\zeta_{n-2}}^{z} |K(z',\zeta_{n-1})|\cdots|K(\zeta_2,\zeta_1)|\,|K(\zeta_1,z)|d\zeta_{n-1}\cdots d\zeta_2 d\zeta_1, \\ &\leq M_K^n \int_{z_0}^{z}\int_{\zeta_1}^{z_0}\cdots\int_{\zeta_{n-2}}^{z} 1 d\zeta_{n-1}\cdots d\zeta_2 d\zeta_1 = M_K^n \frac{(z-z_0)^n}{n!}. \end{aligned} \tag{6}$$

This result translates into *unconditional convergence of the integral series representation of Heun functions on singularity-free intervals*. Indeed we show below that Heun functions are given by integral series on two kernels, K_1 and K_2, which depend on the function but are bounded on singularity-free intervals.

Last but not least, the function $G(z,z_0)$ defined above, is solution to the linear Volterra integral equation of the second kind

$$G(z,z_0) = K(z,z_0) + \int_{z_0}^{z} K(z,\zeta)G(\zeta,z_0)d\zeta, \tag{7}$$

or, in $*$ notation, $G = K + K * G$. Thus, the function G can either be evaluated from the integral series or by solving the above Volterra equation. This not only leads to fast reliable numerical implementations of the results presented here [24] available for download, but also to novel Volterra integral equations satisfied by Heun functions, see Section III C.

B. Representations

We emphasize that all results stated remain valid for complex parameter values. This is crucial notably when forming solutions of the Teukolsky equation in the study of quasinormal modes, for which the frequency parameter takes complex values (see the general setting as given in Eq. (1) and Eq. (22) for a concrete example with complex parameters).

Corollary III.1 (General Heun Equation)**.** *Let $H_G(z)$ be solution of the General Heun Equation,*

$$\frac{d^2H_G(z)}{dz^2} + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-t}\right]\frac{dH_G(z)}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-t)}H_G(z) = 0,$$

with initial conditions $H_G(z_0) = H_0$ and $\dot{H}_G(z_0) = H_0'$, assuming that $z_0 \in \mathbb{R}$ is not a singular point of H_G. Denote I the largest real interval that contains z_0 and does not contain any singular point of H_G. Then, for any $z \in I$,

$$H_G(z) = H_0 + H_0\int_{z_0}^{z} G_1(\zeta,z_0)d\zeta + (H_0' - H_0)\left(e^{z-z_0} - 1 + \int_{z_0}^{z}(e^{z-\zeta}-1)G_2(\zeta,z_0)d\zeta\right),$$

where $G_i = \sum_{n=1}^{\infty} K_i^{*n}$ and

$$\begin{aligned} K_1(z,z_0) &= 1 + e^{-z}\int_{z_0}^{z}\frac{\zeta_1^{\gamma}(\zeta_1-1)^{\delta}(t-\zeta_1)^{\varepsilon}}{z^{\gamma}(z-1)^{\delta}(t-z)^{\varepsilon}}e^{\zeta_1}\left(\frac{q-\alpha\beta\zeta_1}{(\zeta_1-1)\,\zeta_1\,(\zeta_1-t)} - \frac{\varepsilon}{t-\zeta_1} - \frac{\gamma}{\zeta_1} - \frac{\delta}{\zeta_1-1} - 1\right)d\zeta_1, \\ K_2(z,z_0) &= \left(\frac{q-\alpha\beta z}{(z-1)z(z-t)} - \frac{\varepsilon}{t-z} - \frac{\gamma}{z} - \frac{\delta}{z-1} - 1\right)e^{z-z_0} - \frac{q-\alpha\beta z}{(z-1)z(z-t)}, \end{aligned}$$

and both K_1 and K_2 are bounded over I, implying convergence of the Neumann series for G_1 and G_2 for all $z \in I$.

Example III.1 (Elementary integral series converging to a general Heun function)**.** *In order to illustrate concretely the above corollary, consider the following General Heun equation (here with arbitrary parameters),*

$$\frac{d^2H_G(z)}{dz^2} + \left[\frac{2}{z} + \frac{7}{z-1} + \frac{(-1)}{z-4}\right]\frac{dH_G(z)}{dz} + \frac{(3/2)z-1}{z(z-1)(z-4)}H_G(z) = 0, \tag{8}$$

with initial conditions $H_G(6) = H'_G(6) = 1$. Here, the largest real interval containing 6 and none of the singular points 0, 1 and $t = 4$ is $I =]4, +\infty[$. Thus Corollary III.1 indicates that for any $z \in]4, +\infty[$,

$$\begin{aligned} H_G(z) &= 1 + \int_{z_0}^{z} G_1(\zeta, z_0) d\zeta, \\ &= 1 + \sum_{n=1}^{\infty} \int_6^z K_1^{*n}(\zeta, 6) d\zeta = 1 + \int_6^z K_1(\zeta, 6) d\zeta + \int_6^z \int_6^{\zeta} K_1(\zeta, \zeta_1) K_1(\zeta_1, 6) d\zeta_1 d\zeta + \cdots, \end{aligned}$$

with the kernel K_1 given by

$$K_1(z, z_0) = 1 - e^{-z} \frac{(z-4)}{z^2 (z-1)^7} \int_{z_0}^{z} e^{\zeta_1} \frac{\zeta_1 (\zeta_1 - 1)^6}{2(\zeta_1 - 4)^2} \left(2\zeta_1^3 + 10\zeta_1^2 - 67\zeta_1 + 14\right) d\zeta_1.$$

In Fig. (1), we show a purely numerical evaluation of $H_G(z)$ together with analytical estimates based on the first few orders of the above series, i.e. we give $H_G^{(m)}(z) := 1 + \sum_{n=1}^{m} \int_6^z K_1^{*n}(\zeta, 6) d\zeta$, with $m = 1, 2, 3$ and $m = 6$. This exhibits the convergence of the Neumann series representation of the path-sum formulation of a general Heun function, as predicted by the theory.

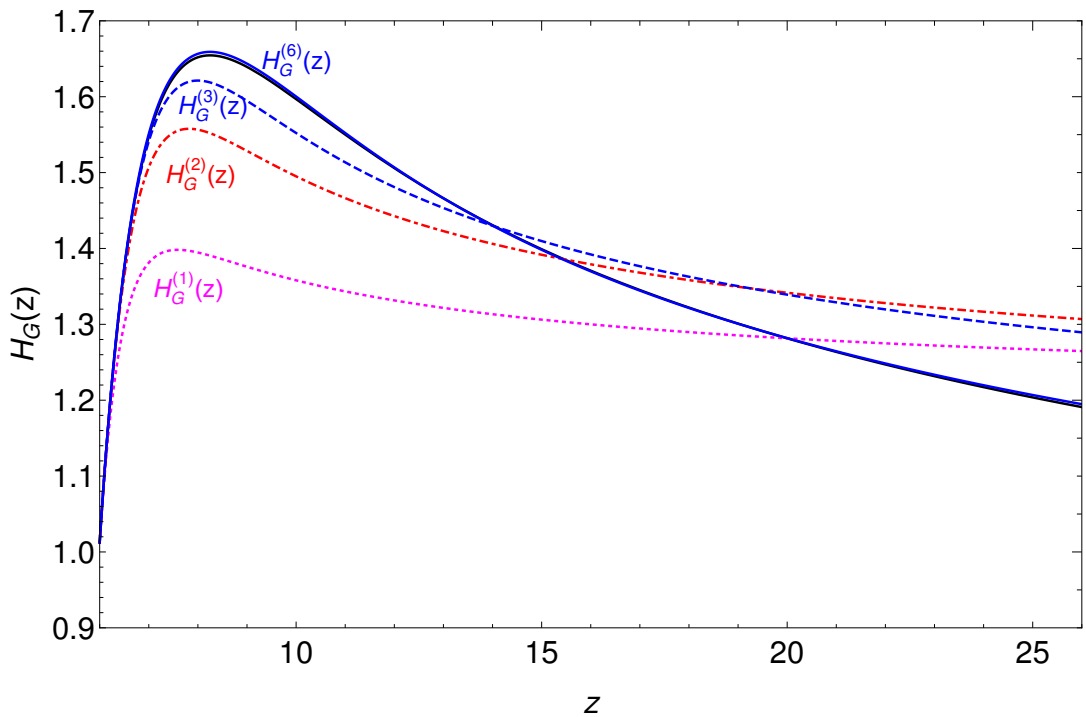

FIG. 1. **Convergence to a general Heun function with elementary integrals.** Numerical evaluation of the general Heun function solution of Eq. (8) (solid black line), together with the first integral approximands of it: $H_G^{(1)}(z)$ (dotted magenta line), $H_G^{(2)}(z)$ (dot-dashed red line), $H_G^{(3)}(z)$ (dashed blue line) and $H_G^{(6)}(z)$ (solid blue line, very close to the numerical solution). For orders $m \geq 9$, we reach the numerical solution to within machine precision. Note that the integral series given here is convergent on $z \in]4, +\infty[$ but we show only the interval $z \in [6, 26]$ for illustration purposes.

The results above continue to hold should e.g. $z_0 = 3$, in which case $I =]1, 4[$; $z_0 = 1/2$ implying $I =]0, 1[$; or $z_0 = -20$ giving $I =]-\infty, 0[$. In other terms, the integral representation given for the General Heun function is valid everywhere on $z \in \mathbb{R} \backslash \{0, 1, t = 4\}$ but can only be used in an interval I where initial conditions for H_G are available.

Corollary III.2 (Confluent Heun Equation)**.** *Let $H_C(z)$ be solution of the Confluent Heun Equation,*

$$\frac{d^2 H_C(z)}{dz^2} + \left[\frac{\gamma}{z} + \frac{\delta}{z-1} + \varepsilon\right] \frac{dH_C(z)}{dz} + \frac{\alpha z - q}{z(z-1)} H_C(z) = 0,$$

with initial conditions $H_C(z_0) = H_0$ and $\dot{H}_C(z_0) = H'_0$, assuming that $z_0 \neq 0$ and $z_0 \neq 1$. If $z_0 < 0$, let $I =]-\infty, 0[$, if $0 < z_0 < 1$

let $I =]0,1[$, and else for $z_0 > 1$ let $I =]1,+\infty[$. Then, for any $z \in I$,

$$H_C(z) = H_0 + H_0 \int_{z_0}^{z} G_1(\zeta, z_0) d\zeta + (H_0' - H_0) \left(e^{z-z_0} - 1 + \int_{z_0}^{z} (e^{z-\zeta} - 1) G_2(\zeta, z_0) d\zeta \right),$$

where $G_i = \sum_{n=1}^{\infty} K_i^{*n}$, $i = 1,2$, and

$$K_1(z,z_0) = 1 + e^{-z} \int_{z_0}^{z} \left\{ \frac{e^{\zeta\varepsilon} \zeta^{\gamma} (\zeta-1)^{\delta}}{e^{z\varepsilon} z^{\gamma} (z-1)^{\delta}} e^{\zeta} \left(\frac{q-\alpha\zeta}{(\zeta-1)\zeta} - \frac{\gamma}{\zeta} - \frac{\delta}{\zeta-1} - \varepsilon - 1 \right) \right\} d\zeta,$$
$$K_2(z,z_0) = \left(\frac{q-\alpha z}{(z-1)z} - \frac{\gamma}{z} - \frac{\delta}{z-1} - \varepsilon - 1 \right) e^{z-z_0} - \frac{q-\alpha z}{(z-1)z},$$

and both K_1 and K_2 are bounded over I, implying convergence of the Neumann series for G_1 and G_2 for all $z \in I$.

Example III.2 (Convergence to a Confluent Heun function)**.** *Let us now consider the following Confluent Heun function $H_C(z)$ satisfying*

$$\frac{d^2 H_C(z)}{dz^2} + \left[\frac{3}{z} + \frac{(2/3)}{z-1} + 4 \right] \frac{dH_C(z)}{dz} + \frac{5z-1}{z(z-1)} H_C(z) = 0 \tag{9}$$

with initial conditions $H_C(-5) = 0$ and $H_C'(-5) = 1$. Suppose that we wish to evaluate H_C on the interval $z \in]-\infty, 0[$, i.e. on both sides $z < z_0$ and $z > z_0$ of the conditions at $z_0 = -5$. Then Corollary III.2 indicates that, for any $z \in]-\infty, 0[$, we have

$$H_C(z) = e^{z+5} - 1 + \int_{-5}^{z} (e^{z-\zeta} - 1) G_2(\zeta, -5) d\zeta,$$

with $G_2 = \sum_{n=1}^{\infty} K_2^{*n}$ and

$$K_2(z,z_0) = \frac{3(5z-1) - e^{z-z_0}(3z+4)(5z-3)}{3(z-1)z}.$$

We emphasize that these results hold for all $z \in]-\infty, 0[$ since this interval is divergence free, more precisely K_2 is bounded continuous on any compact subinterval of $]-\infty, 0[$ and the integral series for G_2 is thus guaranteed to converge on this entire domain (this is shown in the appendix). Note that when considering $z < z_0$, all integrals remain the same as for $z > z_0$.

In Fig. (2) below, we show a purely numerical evaluation of $H_C(z)$ together with the truncated integral series approximations

$$\begin{aligned} H_C^{(m)}(z) &:= e^{z+5} - 1 + \int_{-5}^{z} (e^{z-\zeta} - 1) \Big(1 + \sum_{n=1}^{m} K_2^{*n}(\zeta, -5) \Big) d\zeta, \\ &= e^{z+5} - 1 + \int_{-5}^{z} (e^{z-\zeta} - 1) \Big(1 + K_2(\zeta, -5) + \int_{-5}^{\zeta} K_2(\zeta, \zeta_1) K_2(\zeta_1, -5) d\zeta_1 + \cdots \Big) d\zeta. \end{aligned} \tag{10}$$

Since kernel K_2 is singular at $z = 0$ just as H_C is, we expect the convergence speed of the integral series to slow down when approaching the singular point, as predicted by the bound of Eq.(6) presented in the appendix. This does not preclude analytically obtaining the correct asymptotic behavior for $H_C(z)$ as $z \to 0^-$. Indeed this follows from the behavior of K_2 under the same limit. We demonstrate such a procedure in Section IV D on the Teukolsky radial function, where we recover the asymptotic behavior of a Heun function from its integral series representation.

Corollary III.3 (Biconfluent Heun Equation)**.** *Let $H_B(z)$ be solution of the Biconfluent Heun Equation,*

$$\frac{d^2 H_B(z)}{dz^2} + \left[\frac{\gamma}{z} + \delta + \varepsilon z \right] \frac{dH_B(z)}{dz} + \frac{\alpha z - q}{z} H_B(z) = 0,$$

with initial conditions $H_B(z_0) = H_0$ and $\dot{H}_B(z_0) = H_0'$, assuming that $z_0 \neq 0$. If $z_0 > 0$, denote $I =]0,+\infty[$ otherwise let $I =]-\infty, 0[$. Then, for any $z \in I$,

$$H_B(z) = H_0 + H_0 \int_{z_0}^{z} G_1(\zeta, z_0) d\zeta + (H_0' - H_0) \left(e^{z-z_0} - 1 + \int_{z_0}^{z} (e^{z-\zeta} - 1) G_2(\zeta, z_0) d\zeta \right),$$

where $G_i = \sum_{n=1}^{\infty} K_i^{*n}$, $i = 1,2$, and

$$K_1(z,z_0) = 1 + e^{-z} \int_{z_0}^{z} \frac{\zeta_1^{\gamma}}{z^{\gamma}} e^{\zeta_1 - \frac{1}{2}(z-\zeta_1)(2\delta + \varepsilon(\zeta_1 + z))} \Big(\frac{q-\alpha\zeta_1}{\zeta_1} - \frac{\gamma}{\zeta_1} - \delta - \zeta_1 \varepsilon - 1 \Big) d\zeta_1,$$
$$K_2(z,z_0) = \Big(\frac{q-\alpha z}{z} - \frac{\gamma}{z} - \delta - z\varepsilon - 1 \Big) e^{z-z_0} - \frac{q-\alpha z}{z},$$

and both K_1 and K_2 are bounded over I, implying convergence of the Neumann series for G_1 and G_2 for all $z \in I$.

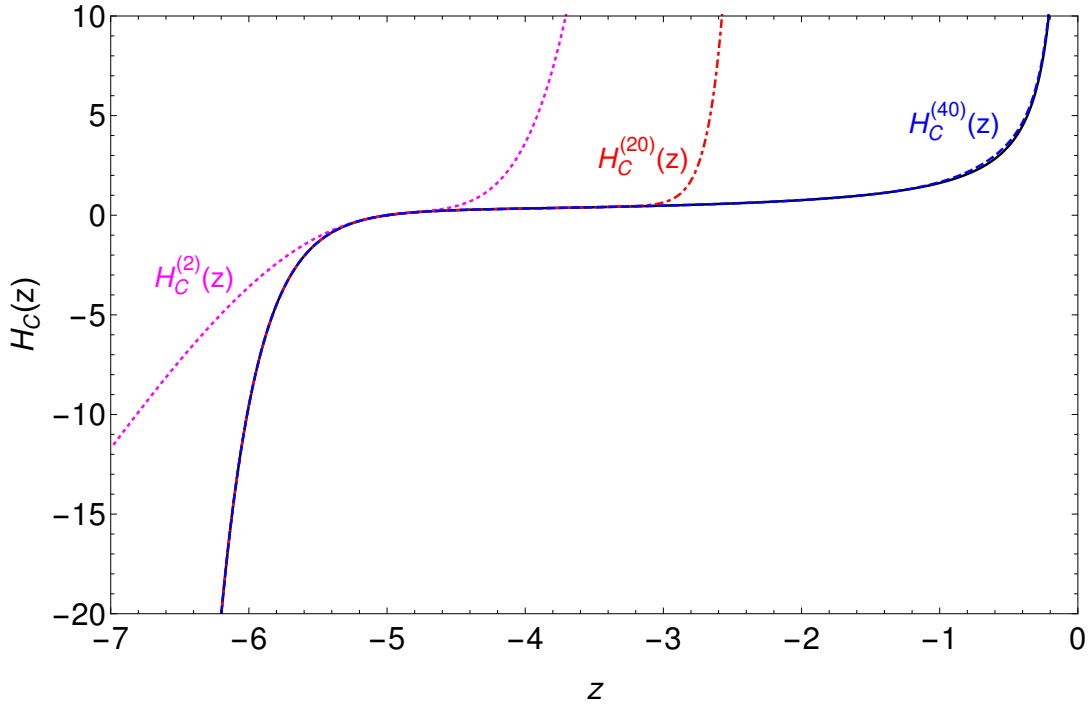

FIG. 2. **Convergence to a Confluent Heun function with elementary integrals over the interval** $]-\infty,0[$**.** Numerical solution of the Eq. (9) (solid black line) with conditions $H_C(-5)=0$, $H_C'(-5)=1$, together with its integral approximands as per Eq. (10), $H_C^{(2)}(z)$ (dotted magenta line), $H_C^{(20)}(z)$ (dot-dashed red line) and $H_C^{(40)}(z)$ (dashed blue line, very close to the numerical solution). Convergence near $z=0$ is slowed down due to K_2 being singular at $z=0$ just as H_C is. Still, the integral series is convergent over the entire domain $z\in]-\infty,0[$, a crucial property for perturbative black hole theory that is *unique* to the present approach. Here as in subsequent examples we plot the various functions over smaller intervals for z, for illustration purposes.

Example III.3 (Evaluating a Biconfluent Heun function via Volterra equations)**.** *Let us now consider the following Biconfluent Heun function $H_B(z)$ satisfying*

$$\frac{d^2H_B(z)}{dz^2}+\left[\frac{(1/10)}{z}+1+6z\right]\frac{dH_B(z)}{dz}+\frac{(-1)z-2}{z}H_B(z)=0, \tag{11}$$

with initial conditions $H_B(2/3)=0$ and $H_B'(2/3)=-4$. Then Corollary III.3 indicates that for $z>0$,

$$H_B(z)=2+2\int_{2/3}^{z}G_1(\zeta,2/3)d\zeta-6\left(e^{z-2/3}-1+\int_{2/3}^{z}(e^{z-\zeta}-1)G_2(\zeta,2/3)\right)d\zeta, \tag{12}$$

with $G_i=\sum_{n=1}^{\infty}K_i^{*n}$ for $i=1,2$, and

$$K_1(z,z_0)=1+\int_{z_0}^{z}\frac{(19-10\zeta(6\zeta+1))e^{-(z-\zeta)(3\zeta+3z+2)}}{10\zeta^{9/10}z^{1/10}}d\zeta,$$
$$K_2(z,z_0)=\frac{\big(19-10z(6z+1)\big)e^{z-z_0}-10(z+2)}{10z}.$$

Instead of evaluating functions G_1 and G_2 from their integral series, it is preferable to solve the linear integral Volterra equations Eq. (7) that they satisfy. Such equations are very well behaved and numerically easy to solve [24], so that we can evaluate H_B thanks to Eq. (12) with high numerical accuracy, see Fig. (3) below.

Corollary III.4 (Doubly-confluent Heun Equation)**.** *Let $H_D(z)$ be solution of the Doubly-confluent Heun Equation,*

$$\frac{d^2H_D(z)}{dz^2}+\left[\frac{\delta}{z^2}+\frac{\gamma}{z}+1\right]\frac{dH_D(z)}{dz}+\frac{\alpha z-q}{z^2}H_D(z)=0$$

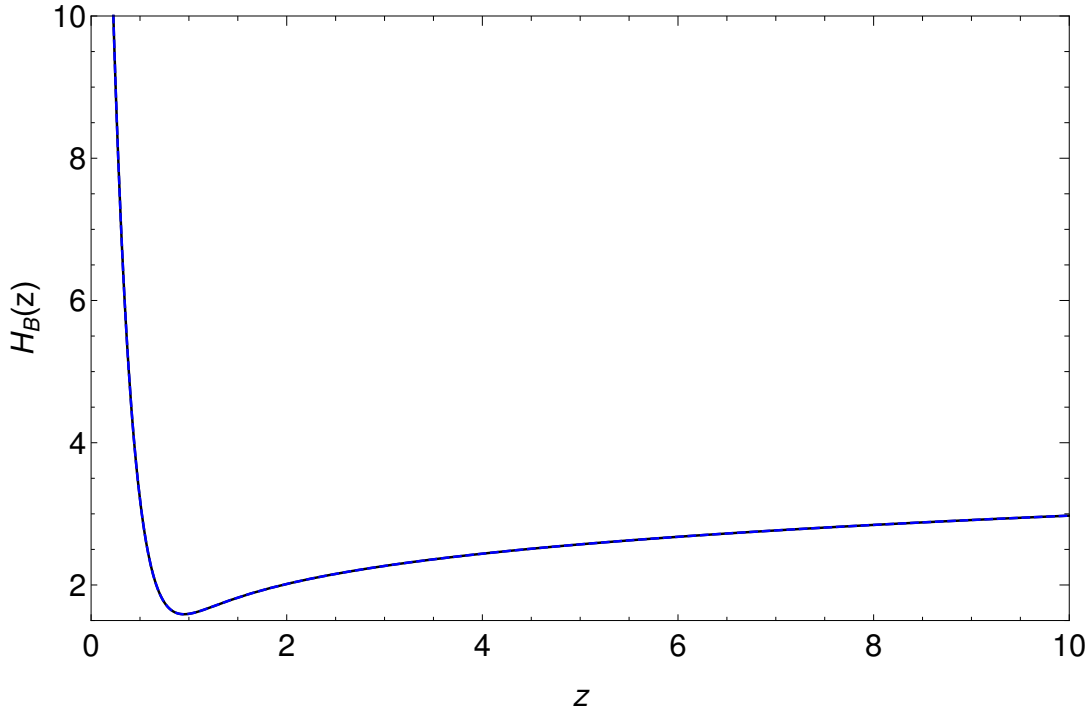

FIG. 3. **Biconfluent Heun function from Volterra equations.** Numerical solution of the Eq. (9) (solid black line), together with the function predicted by Eq.(12) (dashed blue line). The two are indistinguishable. The integral representation of Eq. (12) is valid for $z \in]0,+\infty[$, we here show only $z \in]0,10[$ for illustration purposes.

with initial conditions $H_D(z_0) = H_0$ and $\dot{H}_D(z_0) = H_0'$, assuming that $z_0 \neq 0$. If $z_0 > 0$, denote $I =]0,+\infty[$ otherwise let $I =]-\infty,0[$. Then, for any $z \in I$,

$$H_D(z) = H_0 + H_0 \int_{z_0}^{z} G_1(\zeta, z_0) d\zeta + (H_0' - H_0) \left(e^{z-z_0} - 1 + \int_{z_0}^{z} (e^{z-\zeta} - 1) G_2(\zeta, z_0) d\zeta \right),$$

where $G_i = \sum_{n=1}^{\infty} K_i^{*n}$, $i = 1,2$, and

$$K_1(z, z_0) = 1 + e^{-z} \int_{z_0}^{z} \left\{ \frac{\zeta_1^{\gamma}}{z^{\gamma}} e^{-\frac{\delta}{\zeta_1} + 2\zeta_1 + \frac{\delta}{z} - z} \left(\frac{q - \alpha\zeta_1}{\zeta_1^2} - \frac{\gamma}{\zeta_1} - \frac{\delta}{\zeta_1^2} - 2 \right) \right\} d\zeta_1,$$
$$K_2(z, z_0) = \left(\frac{q - \alpha z}{z^2} - \frac{\delta}{z^2} - \frac{\gamma}{z} - 2 \right) e^{z-z_0} - \frac{q - \alpha z}{z^2},$$

and both K_1 and K_2 are bounded over I, implying convergence of the Neumann series for G_1 and G_2 for all $z \in I$.

Example III.4 (Convergence to a Doubly-Confluent Heun function)**.** *Let us now consider the following Doubly-Confluent Heun equation, once again with arbitrarily chosen parameters for the example,*

$$\frac{d^2 H_D(z)}{dz^2} + \left[\frac{(-2)}{z^2} + \frac{1}{z} + 1 \right] \frac{dH_D(z)}{dz} + \frac{10z - (-1)}{z^2} H_D(z) = 0 \tag{13}$$

with initial conditions $H_D(1) = H_D'(1) = 1/2$. Then Corollary III.4 indicates that for $z \in]0,+\infty[$,

$$H_D(z) = \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \int_1^z K_1^{*n}(\zeta, 1) d\zeta,$$

with

$$K_1(z, z_0) = 1 + \frac{e^{-z-\frac{1}{z}}}{z} \int_{z_0}^{z} e^{2\zeta_1 + \frac{2}{\zeta_1}} \frac{1}{\zeta_1} (1 - 2\zeta_1^2 + 11\zeta_1) d\zeta_1$$

In Fig. (4) below, we show a purely numerical evaluation of $H_D(z)$ together with analytical approximations based on the first few orders of the above series, i.e. we give $H_D^{(m)}(z) := 1+\sum_{n=1}^{m}\int_1^z K_1^{*n}(\zeta,1)d\zeta$, with $m=3,5,8$. This demonstrates again the convergence of the Neumann series representation of the path-sum formulation of a general Heun function, as predicted by the theory. Here the exact $H_D(z)$ and $H_D^{(m)}(z)$ become indistinguishable for $m \geq 9$.

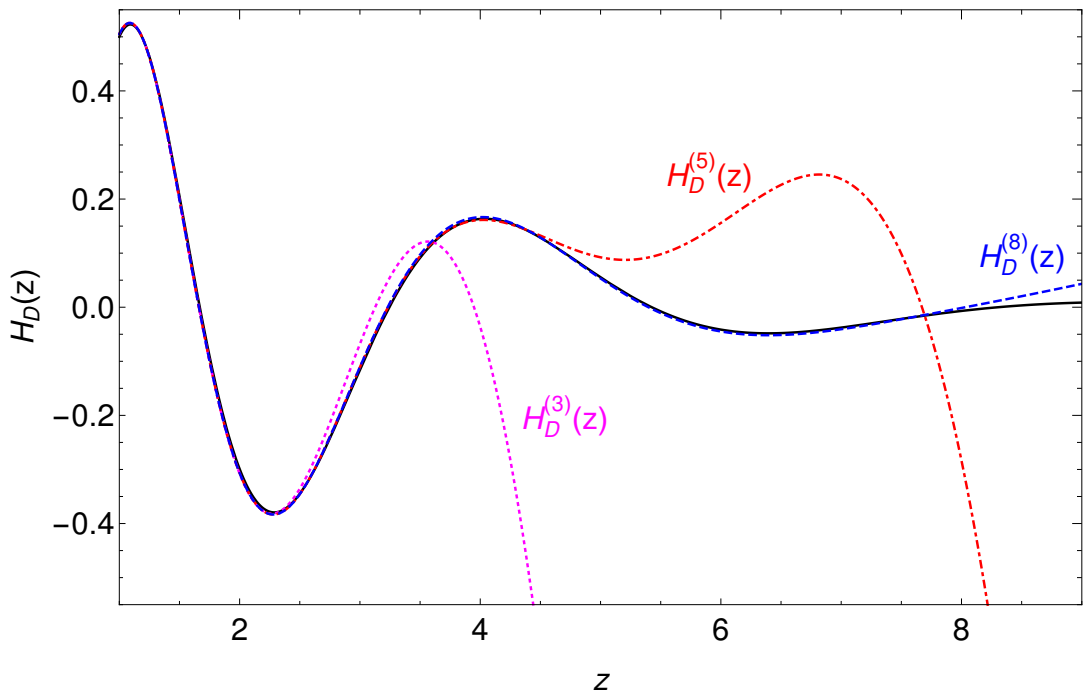

FIG. 4. **Convergence to Doubly-Confluent Heun function with elementary integrals.** Numerical solution of the Eq. (13) (solid black line), together with its integral approximands $H_D^{(3)}(z)$ (dotted magenta line), $H_D^{(5)}(z)$ (dot-dashed red line) and $H_D^{(8)}(z)$ (dashed blue line). Note the integral series provided here is convergent for $z \in]0,+\infty[$ and show only the interval $z \in [1,9]$ for illustration purposes.

Corollary III.5 (Triconfluent Heun Equation)**.** *Let $H_T(z)$ be solution of the Triconfluent Heun Equation,*

$$\frac{d^2H_T(z)}{dz^2}+\left[\gamma+\delta z+\varepsilon z^2\right]\frac{dH_T(z)}{dz}+(\alpha z-q)H_T(z)=0,$$

with initial conditions $H_T(z_0)=H_0$ and $\dot{H}_T(z_0)=H_0'$. Then, for any $z\in\mathbb{R}$,

$$y(z)=H_0+H_0\int_{z_0}^{z}G_1(\zeta,z_0)d\zeta+(H_0'-H_0)\left(e^{z-z_0}-1+\int_{z_0}^{z}(e^{z-\zeta}-1)G_2(\zeta,z_0)d\zeta\right),$$

where $G_i=\sum_{n=1}^{\infty}K_i^{*n}$, $i=1,2$ and

$$K_1(z,z_0)=1-e^{-\frac{1}{6}z\left(6\gamma+2z^2\varepsilon+3\delta z+6\right)}\int_{z_0}^{z}e^{\frac{1}{6}\zeta\left(6\gamma+3\delta\zeta+2\zeta^2\varepsilon+6\right)}\left(\zeta(\alpha+\delta+\zeta\varepsilon)+\gamma-q+1\right)d\zeta,$$

$$K_2(z,z_0)=-\left(\varepsilon z^2+(\alpha+\delta)z-q+\gamma+1\right)e^{z-z_0}-(q-z\alpha),$$

and both K_1 and K_2 are bounded over $\mathbb{R}$, implying convergence of the Neumann series for G_1 and G_2 for all $z\in\mathbb{R}$.

Example III.5 (Convergence to a complex-valued Triconfluent Heun function)**.** *Consider the Triconfluent Heun function defined as the solution to*

$$\frac{d^2H_T(z)}{dz^2}+\left[2-z+7z^2\right]\frac{dH_T(z)}{dz}+(z-(2+i))H_T(z)=0, \tag{14}$$

where $i^2=-1$, and with initial conditions $H_T(-5)=H_T'(-5)=2$. Corollary III.5 indicates that for $z\in\mathbb{R}$,

$$H_D(z)=2+2\int_{z_0}^{z}G_1(\zeta,-10)d\zeta,$$

with $G_1 = \sum_{n=1}^{\infty} K_1^{*n}$ *and*

$$K_1(z,z_0) = 1 - e^{\frac{1}{6}(3-14z)z^2-3z} \int_{z_0}^{z} e^{-\frac{1}{6}(3-14\zeta)\zeta^2+3\zeta}(7\zeta^2+1-i)\,d\zeta.$$

We show in Fig. (5) convergence to the complex-valued triconfluent Heun function by the integral series

$$\begin{aligned} H_T^{(m)}(z) &:= 2 + 2\int_{z_0}^{z} \sum_{n=1}^{m} K_1^{*n}(\zeta,-10)d\zeta, \\ &= 2 + 2\int_{z_0}^{z} K_1(\zeta,-10)d\zeta + \int_{z_0}^{z}\int_{z_0}^{z} K_1(z,\zeta_1)K_1(\zeta_1,-10)d\zeta_1 d\zeta + \cdots \end{aligned}$$

With this example, we emphasize that all the integrals representations obtained here remain valid for complex-valued Heun functions.

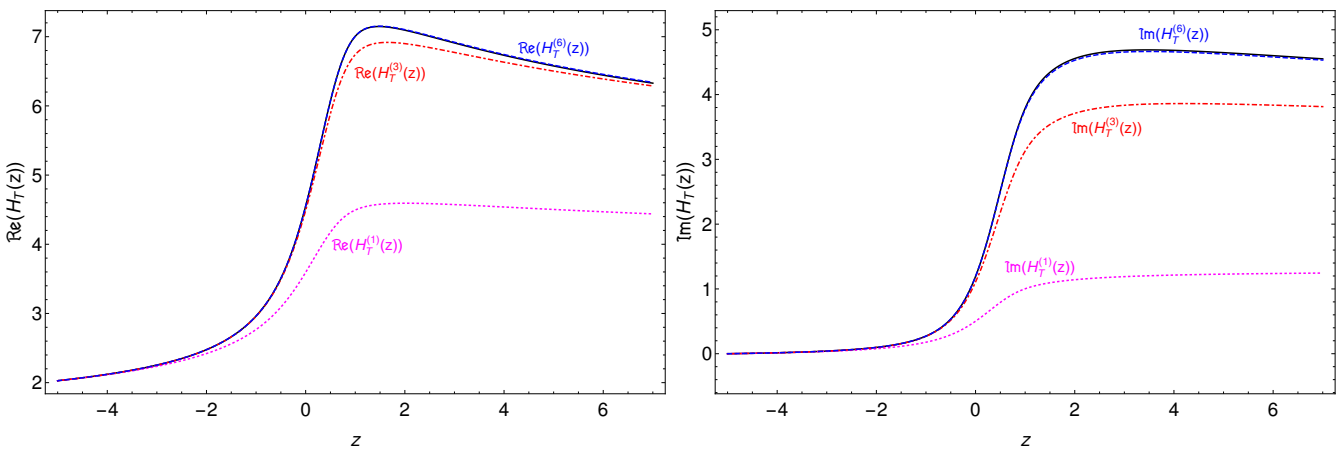

FIG. 5. **Convergence to a complex-valued Triconfluent Heun function with elementary integrals.** Numerical solution of the Eq. (14) (solid black line), together with its integral approximands $H_T^{(1)}(z)$ (dotted magenta line), $H_T^{(3)}(z)$ (dot-dashed red line) and $H_T^{(6)}(z)$ (dashed blue line). Left figure: real parts of these quantities. Right figure: imaginary parts of these quantities. Note that the integral series provided is convergent over the entire real line, we here show only the interval $z \in [-5,7]$ for illustration purposes.

Remark III.1 (On singularities and complex integration)**.** *The integral series representation are convergent on any interval not crossing a singularity as shown in Section. III A. In particular, this allows us to evade the issue of relating the behaviour of the Heun function at one singular point to its behaviour at another so long as there is a singularity-free open interval between these points. As the representations given here fail to be defined at singularities, if there is the need to pass through one, one must resort to contour integration in the complex plane* $z \in \mathbb{C}$ *to bypass the singularity. In this situation, attention must be paid to branch cuts, path-dependency may arise and the Stoke phenomenon can be an issue, see [24, 25] for a discussion.*

C. Volterra equations for Heun functions

If one needs to numerically evaluate a Heun functions using one of the integral series above, it is preferable to do so without evaluating the various orders of the series. Rather, it is much easier and faster to numerically solve the Volterra integral equations Eq. (7) satisfied by G_1 and G_2. This directly and reliably gives access to these functions, i.e. to what the integral series converges to, see [24] for a downloadable Python implementation. This observation also imply that novel linear Volterra integral equations of the second kind are satisfied by Heun functions, depending on the Cauchy conditions:

Theorem III.1. *Let* $H(z)$ *be a function of the Heun type. If there is a point* z_0 *where initial conditions are such that* $H_0 = H_0'$*, then the derivative* $\dot{H}(z)$ *of* $H(z)$ *satisfies the Volterra equation*

$$\dot{H}(z) = K_1(z,z_0) + \int_{z_0}^{z} K_1(z,\zeta)\dot{H}(\zeta)d\zeta.$$

Proof. If there is a point z_0 where initial conditions are such that $H_0 = H_0'$, then by any of the above corollaries, the Heun function $H(z)$ is equal to $H_0 + H_0 \int_{z_0}^{z} G_1(\zeta, z_0) d\zeta$. By Eq. (7) or Theorem A.1 this implies that the derivative the Heun function, being proportional to G_1, satisfies a linear Volterra integral equation of the second kind with kernel K_1, given in the theorem. This is the first known integral equation satisfied by Heun functions in terms of elementary functions. □

Theorem III.2. *Let $H(z)$ be a function of the Heun type and let $\ddot{H}(z)$ be its second derivative. If there is a point z_0 where initial conditions are such that $H_0 = 0$, then $\ddot{H}(z) - \dot{H}(z)$ satisfies the linear Volterra equation with kernel K_2*

$$\ddot{H}(z) - \dot{H}(z) = K_2(z, z_0) + \int_{z_0}^{z} K_2(z, \zeta)\big(\ddot{H}(\zeta) - \dot{H}(\zeta)\big) d\zeta.$$

Proof. Entirely similar to the proof of the previous theorem, this time with G_2 and K_2. □

IV. APPLICATION TO BLACK-HOLE PERTURBATION THEORY

A. Motivations

The theory of perturbations of Kerr black holes is governed by the Teukolsky equation [11]. This equation provides the basic mathematical framework to study the stability of black holes [26] and yields physical insights in the broader field of gravitational wave astrophysics [27].

In the frequency domain, the Teukolsky equation can be decoupled into radial and angular components [11]. Determining the analytical solutions of the radial equation has been an active area of research since the first formulation of the equations [12, 27]. To this end, state-of-the-art approaches all rely on the same strategy: i) obtain two series expansions of the solution, one convergent near the black hole horizon the other at spatial infinity; and ii) match both expansions at some intermediate point using a matching parameter. The standard implementation of this strategy, due to Mano, Suzuki and Takasugi (MST) [12], relies on a series of hypergeometric functions at the black hole horizon and of Coulomb wave functions at spatial infinity. Matching both expansions requires the introduction of an auxiliary parameter ν. We stress that this parameter is *not* part of the original parameters of the Teukolsky equation. Rather ν is a mathematical checkpost introduced to establish the convergence and matching of the hypergeometric and Coulomb series [13] and is an unavoidable mathematical step in matched asymptotic expansions of confluent heun solutions [28]. The MST strategy successfully yields accurate numerical data for studying gravitational wave radiation from Kerr black holes [13, 27]. It is *"the only existing method that can be used to calculate the gravitational waves emitted to infinity to an arbitrarily high post-Newtonian order in principle."* [27]. At the same time, it has been explicitly recognised that the mathematical complexity of the formalism obscures physical insights into the problem [27]. In particular, the auxiliary parameter ν, which has been called "renormalised angular momentum" to make it more palatable, has limited correspondence to physical phenomenon, if any.

More recently, explicit, analytic solutions to the Teukolsky equation have been established in terms of confluent Heun functions [5] and general Heun functions [29]. Yet, Cook and Zalutskiy [6] note that in order to extract physical quantities of interest out of this approach, one is forced to revert to Leaver's formalism [30] because *"the series solution around z = 1 has a radius of convergence no larger than 1, far short of infinity"*. Thus, just as for the MST formalism the problem is, in essence, that we are lacking a single representation of the solution to the Teukolsky radial equation that is convergent from the black hole horizon up to spatial infinity. The integral series provided in this work addresses this issue completely since it converges on this entire domain, thereby retaining the crucial features of the MST formalism that lead to its widespread applicability, while also not requiring any auxiliary, unphysical parameter. In a similar vein, we can assert that our formalism is suited for practical numerical and even analytical, calculations since the integral series are rapidly convergent, and their asymptotic behavior is analytically available. We may therefore also hope that the integral series representation will help solve the well-recognised computational difficulties that emerge from the MST formalism when applied to gravitational wave physics

For completeness, we begin with a brief discussion of the theory of the Teukolsky equation and its reduction to Heun form. We then give the series representation of its solution. Finally, we establish its asymptotics at both the black hole horizon ($z \to 1^+$) and spatial infinity ($z \to +\infty$).

B. The Teukolsky Equation : background

The Teukolsky Equation [11] is a gauge invariant equation that governs the curvature perturbations of the Kerr black hole [31].The so-called "master" equation for the spin (s) weighted scalar wave function ${}_s\psi$ in Boyer-Lindquist co-ordinates

$\{t,r,\theta,\phi\}$ [32] is written as:

$$\left[\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right]\frac{\partial_s^2\psi}{\partial t^2}+\left(\frac{4Mar}{\Delta}\right)\frac{\partial_s^2\psi}{\partial t\partial\phi}+\left[\frac{a^2}{\Delta}-\frac{1}{sin^2\theta}\right]\frac{\partial_s^2\psi}{\partial\phi^2}$$
$$-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial_s\psi}{\partial r}\right)-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial_s\psi}{\partial\theta}\right)-2s\left[\frac{a(r-M)}{\Delta}+\frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial_s\psi}{\partial\phi}$$
$$-2s\left[\frac{M(r^2-a^2)}{\Delta}-r-ia\cos\theta\right]\frac{\partial}{\partial t}+(s^2\cot^2\theta-s)_s\psi=4\pi\Sigma T \tag{15}$$

where the auxiliary variables are given by:

$$\Sigma\equiv r^2+a^2\cos^2\theta,\Delta\equiv r^2-2Mr+a^2 \tag{16}$$

Here, M is the mass of the black hole, a is its angular momentum (per unit mass), T is the source term built from the energy-momentum tensor [11] and the spin parameter $s=0,\pm1/2,\pm1,\pm2\pm3/2$ for scalar, neutrino, electromagnetic and gravitational fields respectively.

The equation 15 can be separated in time and frequency domain [11]. The latter can be performed for the vacuum case ($T=0$) by the following separation ansatz:

$${}_s\psi(t,r,\theta,\phi)=e^{-i\omega t}e^{im\phi}S(\theta)R(r). \tag{17}$$

For the radial function $R(r)$ we obtain the Teukolsky Radial Equation (TRE):

$$\Delta^{-s}\frac{d}{dr}\left[\Delta^{s+1}\frac{dR(r)}{dr}\right]+\left[\frac{K^2-2is(r-M)K}{\Delta}+4is\omega r-\lambda\right]R(r)=0, \tag{18}$$

where,

$$K\equiv(r^2+a^2)\omega-am,\quad \lambda\equiv{}_sA_{lm}(a\omega)+a^2\omega^2-2am\omega. \tag{19}$$

For the angular equation, we make $x\equiv\cos\theta$. Now the function $S(\theta)={}_sS_{lm}(x;a\omega)$ is the spin weighted spheroidal function [33] which gives the solution for the Teukolsky Angular Equation (TAE):

$$\partial_x\left[(1-x^2)\partial_x[{}_sS_{lm}(x;c)]\right]+\left[(cx)^2-2csx+s+{}_sA_{lm}(c)-\frac{(m+sc)^2}{1-x^2}\right]{}_sS_{lm}(x;c)=0, \tag{20}$$

where $c=a\omega$ is the oblateness parameter, m is the azimuthal separation constant and ${}_sA_{lm}(c)$ is the angular separation constant. The equations 18 and 20 are coupled equations which require simultaneous evaluation of the parameters ω and ${}_sA_{lm}(c)$. Given a value for ${}_sA_{lm}(c)$, we can solve 18 for the complex frequency ω and given the latter, we can solve 20 as an eigenvalue problem for ${}_sA_{lm}(c)$.

C. Teukolsky Radial Equation in Heun Form

We now reduce the Teukolsky Radial Equation to the non-symmetrical Heun form, which allows us to represent its solution with the results of Section. III. There is one small consideration to be noted: depending on the sign of the spin s we wish to operate in, certain parameters of the CHE form of the TAE and TRE flip their signs as given in [5]. However this is not of relevance for our purposes since our main aim is to work with the CHE *form* of the equations that obviously remains irrespective of the sign of the spin parameter.

The radial function $R(r)$ solution to Eq. 18 has three singularities: an irregular singular point at $r=\infty$ and two regular singular points corresponding to the roots of $\Delta=0$, which are $r_\pm=M\pm\sqrt{M^2-a^2}$. The values $r_\pm$ correspond to the event and Cauchy horizon respectively (for an in-depth introduction to the notation and terminology on black-hole mathematics, we refer the reader to [31]). Having identified these, we may now map the Teukolsky Radial Equation into an Heun equation. We close following the standard treatment [6]. We begin by letting the radial function R be of the form

$$R(r)=(r-r_+)^\xi(r-r_-)^\eta e^{\zeta r}H(r), \tag{21}$$

where the parameters ζ,ξ,η are given by

$$\zeta=\pm i\omega\equiv\zeta_\pm,\quad \xi=\frac{-2\pm(s+2i\sigma_+)}{2}\equiv\xi_\pm,\quad \eta=\frac{-s\pm(s-2i\sigma_-)}{2}\equiv\eta_\pm,\quad \sigma_\pm=\frac{2\omega Mr_\pm-ma}{r_+-r_-}. \tag{22}$$

With the dimensionless variables $\bar{r} \equiv r/M$, $\bar{a} \equiv a/M$, $\bar{\omega} \equiv M\omega$, $\bar{\zeta} \equiv M\zeta$, we transform the radial coordinate r into the dimensionless variable z defined by $z = (r - r_-)/(r_+ - r_-) = (\bar{r} - \bar{r}_-)/(\bar{r}_+ - \bar{r}_-)$. Now, any of the eight possible combinations of the parameters $\{\zeta, \xi, \eta\}$ given in Eqs. (22) will reduce the Teukolsky Radial Equation (18) into the following equation for the auxiliary function H,

$$\frac{d^2H(z)}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + 4p\right)\frac{dH(z)}{dz} + \frac{4\alpha pz - \sigma}{z(z-1)}H(z) = 0, \tag{23}$$

which is a Confluent Heun equation. Here, the following variables have been introduced to clarify the equation,

$$p = (\bar{r}_+ - \bar{r}_-)\frac{\bar{\zeta}}{2}, \quad \alpha = 1 + s + \xi + \eta - 2\bar{\zeta} + s\frac{i\bar{\omega}}{\bar{\zeta}}, \quad \gamma = 1 + s + 2\eta, \quad \delta = 1 + s + 2\xi, \tag{24}$$
$$\sigma = {}_sA_{lm}(\bar{a}\bar{\omega}) + \bar{a}^2\bar{\omega}^2 - 8\bar{\omega}^2 + p(2\alpha + \gamma - \delta) + \left(1 + s - \frac{\gamma+\delta}{2}\right)\left(s + \frac{\gamma+\delta}{2}\right).$$

Furthermore, the local solutions at the singularities have the exact same form for all eight combinations of the parameters $\{\zeta, \xi, \eta\}$ given in Eqs. (22) [6]. BFrom there, expressions for quantities of physical interest such as Quasinormal Modes and Totally Transmitting Modes have been obtained [6]. The equation can be solved by various methods such as Frobenius series about the singular points [5] and continued fractions [30].

D. Teukolsky radial function: everywhere convergent representation

1. *Elementary integral series*

The solution of the Confluent Heun equation (23) satisfied by the auxiliary function $H(z)$ is described by Corollary (III.2). Since the singular points are located at $z = 0, 1, +\infty$, given any initial conditions for $H(z_0)$ and $\dot{H}(z_0)$ at $z_0 \in]1, +\infty[$, the integral series representation of $H(z)$ is guaranteed to converge on the entire domain $]1, +\infty[$, that is *from the black hole horizon up to spatial infinity*. This crucial property stands in stark contrast with the hypergeometric and Coulomb series, which converge close to 1 and to $+\infty$, respectively. Because of this, we do not need to introduce the unphysical parameter ν.

Recall that the Teukolsky radial function R and auxiliary function H are related by Eq. (21). The auxiliary function is a confluent Heun function given by the following integral series representation, convergent for any $z \in]1, +\infty[$,

$$H(z) = H_0 + H_0\int_{z_0}^{z} G_1(\zeta, z_0)d\zeta + (H_0' - H_0)\left(e^{z-z_0} - 1 + \int_{z_0}^{z}(e^{z-\zeta} - 1)G_2(\zeta, z_0)d\zeta\right),$$

where $G_i = \sum_{n=1}^{\infty} K_i^{*n}$, $i = 1, 2$, and

$$K_1(z, z_0) = 1 + e^{-(1+4p)z}z^{-\gamma}(z-1)^{-\delta}\int_{z_0}^{z} e^{(1+4p)\zeta}\zeta^{\gamma}(\zeta-1)^{\delta}\left(\frac{\sigma - 4\alpha p\zeta}{(\zeta-1)\zeta} - \frac{\gamma}{\zeta} - \frac{\delta}{\zeta-1} - 4p - 1\right)d\zeta,$$
$$K_2(z, z_0) = \left(\frac{\sigma - 4\alpha pz}{(z-1)z} - \frac{\gamma}{z} - \frac{\delta}{z-1} - 4p - 1\right)e^{z-z_0} - \frac{\sigma - 4\alpha pz}{(z-1)z}.$$

Here we assumed $z_0 \in]1, +\infty[$ then $H_0 := H(z_0)$, $H_0' := \dot{H}(z_0)$ and all parameters are given by Eq. (24).

We now demonstrate convergence of the integral series in two physically relevant yet numerically challenging situations. We reproduce the solution of the Teukolsky radial equation Eq. (23) near the singularity at $z = 1$, i.e. $r = r_+$, where it tends to oscilate wildly. We set all signs to $+$ in Eq. (22), let $M = 1$, $s = -2$, $m = 2$, and pick two realistic sets of values for a, ω and ${}_sA_{lm}(\bar{a}\bar{\omega})$. The values of M, s, m are consistent with solving for higher order modes of gravitational QNMs while we have substituted artificial values of ω and ${}_sA_{lm}$ so as to illustrate the validity of our method for the kind of parameters that occur in QNM computation. We show the real parts of the solution in Figs. (6) and (7), demonstrating convergence of the increasing orders in this challenging situation.[47] We emphasize that in practice we do not evaluate increasingly high orders of the integral series as shown here to illustrate convergence, instead we directly evaluate the result of the entire integral series from Volterra equations, see [24], with concrete Python implementations available for download.

2. *Asymptotic behavior for $z \to +\infty$*

Further witnessing to the convergent for all $z \in]1, +\infty[$ we here recover the asymptotic behavior of $H(z)$ in both limits $z \to 1^+$ and $z \to +\infty$ from the integral series. We emphasize that this is not possible with any other series representation of $H(z)$, which

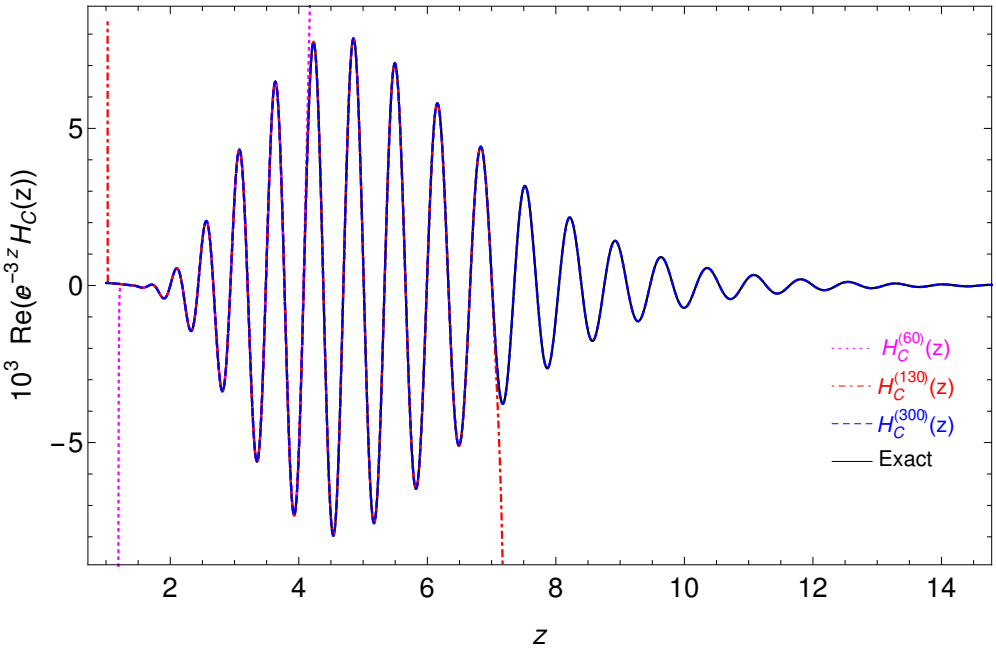

FIG. 6. **Solution of the Teukolsky radial equation near a black hole horizon.** In black, real part of the solution of radial Teukolsky equation, rescaled as $10^3 e^{-3z} H_C(z)$ for better visualization. Also shown are increasing orders of the integral series representation, $H_C^{(m)}$ denoting the sum of the first m terms of the integral series. Note in particular the limit value at the singularity $\lim_{z\to 1} H_C(z) = 0.00158589 - i0.00281538$ is correctly determined from high enough orders of the integral series. However in practice it is much faster to solve the Volterra equations satisfied by the integral series rather than compute high orders of the Neumann series as done here for illustration purposes. Parameters: $a = 0.2$, $\omega = 2 + i0.5$, ${}_sA_{lm}(\bar{a}\bar{\omega}) = 1 + i0.75$ with $z_0 = 2$, $H_0 = 0$, $H_0' = 3$.

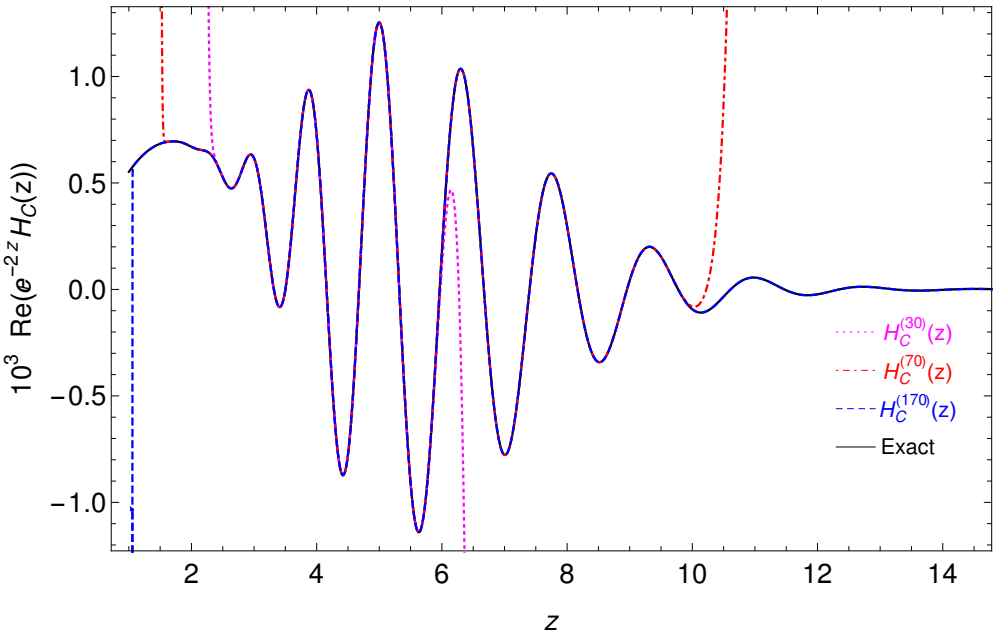

FIG. 7. **Solution of the Teukolsky radial equation near a black hole horizon.** In black, real part of the solution of radial Teukolsky equation, rescaled as $10^3 e^{-2z} H_C(z)$ for better visualization. Also shown are increasing orders of the integral series representation, $H_C^{(m)}$ denoting the sum of the first m terms of the integral series. The limit value at the singularity $\lim_{z\to 1} H_C(z) = 0.00408817 - i0.00195016$ is recovered from high enough orders of the series. Parameters: $a = 0.95, \omega = 2.2 + i0.6, {}_sA_{lm}(\bar{a}\bar{\omega}) = 1 + i0.75$ and $z_0 = 3.5$, $H_0 = 0$ and $H_0' = 2 + 3i$.

converges either in the vicinity of 1^+ or of $+\infty$. This analysis is also crucial in confirming that the solution of the Heun equation

determined here is exact as it correctly captures the behavior of the function near its singular points. This cannot be done from eikonal solutions, which are only close to the exact solutions away from the singularities [34].

From now on, we write $F(z) \sim_{a.e.}$ to present the leading term of the asymptotic expansion of the function $F(z)$, disregarding constant factors. For example, we would write $1+2/z \sim_{a.e} z^{-1}$ as $z \to 0$.

We begin by determining the asymptotic behavior of $K_1(z,z_0)$ for $z \gg 1$. This depends on two cases: $p=0$ and $p \neq 0$. We suppose first that $p=0$ and assume that $\delta+\gamma>0$. In this situation, the confluent Heun function becomes a well understood hypergeometric function [35] for which we will nonetheless show that we recover the correct asymptotic behavior. Setting $p=0$ we get, as $z \to +\infty$,

$$K_1(z,z_0) \sim_{a.e} 1+e^{-(1+4p)z}z^{-\gamma}(z-1)^{-\delta}\left(-e^{z}z^{\gamma+\delta}+e^{z_0}z_0^{\gamma+\delta}\right) \sim_{a.e} e^{-z}z^{-\delta-\gamma}e^{z_0}z_0^{\gamma+\delta}.$$

Then $K_1(z,z_0)$ is asymptotically the product of a function depending only on z and of a function depending only on z_0. This property is sufficient to determine the asymptotic behavior of G_1 in closed-form [48]

$$G_1(z,z_0) \sim_{a.e} e^{-z}z^{-\delta-\gamma}e^{z_0}z_0^{\gamma+\delta}e^{\int_{z_0}^{z} e^{-\zeta}\zeta^{-\delta-\gamma}e^{\zeta}\zeta^{\gamma+\delta}d\zeta} = (z_0/z)^{\delta+\gamma}$$

implying that $\int_{z_0}^{z} G_1(\zeta,z_0)d\zeta \sim_{a.e.} z^{1-\delta-\gamma}$ for $z \to +\infty$. Analyzing K_2 and G_2 yields the same results. Indeed, with $p=0$, we have

$$K_2(z,z_0) \sim_{a.e.} \left(\frac{-1}{z}(\gamma+\delta)-1\right)e^{z-z_0},$$

which is the product of a function of z and a function z_0 so we determine

$$G_2(z,z_0) \sim_{a.e.} \left(\frac{-1}{z}(\gamma+\delta)-1\right)e^{z-z_0}e^{-(z-z_0)}(z_0/z)^{\gamma+\delta},$$

that is $G_2 \sim_{a.e.} z^{-\gamma-\delta}$. From there $e^{z-z_0}-1+\int_{z_0}^{z}(e^{z-\zeta}-1)G_2(\zeta,z_0)d\zeta \sim_{a.e.} z^{1-\gamma-\delta}$. Thus for $p=0$ and $\delta+\gamma>0$, we get $H(z) \sim_{a.e.} z^{1-\delta-\gamma}$ regardless of the conditions at z_0 and provided $\delta+\gamma>0$, as expected [35]. Further cases arise for $\delta+\gamma \leq 0$ but we do not discuss these here as they correspond to well known hypergeometric results.

Let us now suppose that $p \neq 0$. Then, since $e^{4p\zeta}\zeta^{\gamma}(\zeta-1)^{\delta}f(\zeta) = -e^{4p\zeta}\zeta^{\gamma+\delta}\left(\frac{4\alpha p+\gamma+\delta}{\zeta}+4p+O(1/\zeta^2)\right)$, we have, asymptotically for $z \to +\infty$,

$$K_1(z,z_0) \sim_{a.e.} 1-e^{-4pz}z^{-\gamma}z^{-\delta} \times z^{\gamma+\delta}\left(e^{4pz}-4\alpha p E_{-\gamma-\delta+1}(-4pz)\right).$$

where $E_n(z)$ is the exponential integral function, with asymptotic expansion $E_n(x) \sim_{a.e.} e^{-x}/x$ as $x \to +\infty$. This result greatly simplifies K_1, reducing it to

$$K_1(z,z_0) \sim_{a.e} -\frac{\alpha}{z}, \text{ as } z \to +\infty.$$

This allows us to determine the asymptotic behavior of G_1 straightforwardly as

$$G_1(z,z_0) \sim_{a.e} \frac{-\alpha}{z}e^{\int_{z_0}^{z} -\alpha/\zeta\, d\zeta} = -\alpha z^{-1-\alpha},$$

and therefore $\int_{z_0}^{z} G_1(\zeta,z_0)d\zeta \sim_{a.e.} z^{-\alpha}$ for $z \to +\infty$.

We proceed similarly for K_2 and G_2. We have $K_2(z,z_0) = f(z)e^{z-z_0}+O(1/z)$, so that asymptotically $K_2(z,z_0) \sim_{a.e.} f(z)e^{z}e^{-z_0}$ for $z \to +\infty$. Then $K_2(z,z_0)$ is asymptotically the product of a function depending only on z and of a function depending only on z_0. We therefore obtain

$$G_2(z,z_0) \sim_{a.e.} e^{z-z_0}f(z)e^{\int_{z_0}^{z} f(\zeta)d\zeta}, \text{ as } z \to +\infty.$$

The right hand-side is

$$\begin{aligned} e^{z-z_0}f(z)e^{\int_{z_0}^{z} f(\zeta)d\zeta} &= e^{-4p(z-z_0)}\left(\frac{z_0-1}{z-1}\right)^{\delta}\left(\frac{z_0}{z}\right)^{\gamma+\sigma}\left(\frac{1-z_0}{1-z}\right)^{4\alpha p-\sigma} \\ &\times \frac{1}{z(z-1)}(\gamma-z(\gamma+\delta+4p(\alpha+z-1)+z-1)+\sigma) \end{aligned}$$

which yields the asymptotic result,

$$G_2(z,z_0) \sim_{a.e.} e^{-4pz} z^{-\delta-\gamma-4\alpha p}, \text{ as } z \to +\infty.$$

This implies that

$$\left(e^{z-z_0} - 1 + \int_{z_0}^{z} (e^{z-\zeta} - 1) G_2(\zeta, z_0) d\zeta\right) \sim_{a.e.} e^{-4pz} z^{-4\alpha p-\delta-\gamma}.$$

Gathering our results, we conclude that when $p \neq 0$,

$$H(z) \sim_{a.e.} z^{-\alpha} \quad \text{or} \quad H(z) \sim_{a.e.} e^{-4pz} z^{-4\alpha p-\delta-\gamma}, \text{ as } z \to +\infty,$$

which gives the same asymptotic behavior as obtained from series designed to converge when $z \to +\infty$ [6, 37]. The result for $p = 0$ yields the correct asymptotics of the hypergeometric function obtained in this case.

3. *Asymptotic behavior for $z \to 1^+$*

In this situation, we begin with

$$K_1(z,z_0) \sim_{a.e} 1 + e^{-(1+4p)z}(z-1)^{-\delta} \int_{z_0}^{z} \left\{ e^{(1+4p)\zeta} (\zeta-1)^{\delta-1} \left(c + c'(\zeta-1)\right) \right\} d\zeta,$$

where $c = \sigma - 4\alpha p - \delta$ and $c' = \gamma + 4p + 1$. In order to progress without presenting cumbersome equations, denote $F(\zeta)$ the following indefinite integral

$$F_\delta(\zeta) := \int e^{(1+4p)\zeta} (\zeta-1)^{\delta} d\zeta = -e^{4p+1}(\zeta-1)^{\delta+1} E_{-\delta}\big(-(1+4p)(\zeta-1)\big),$$

where $E_n(x)$ is the exponential integral function. In particular $E_n(x) \sim_{a.e.} x^{n-1}c_1 + c_2$ as $x \to 0^+$ and where c_1 and c_2 are non-zero real constants that are irrelevant here. This implies $F_\delta(\zeta) \sim_{a.e.} (\zeta-1)^{1+\delta}$. Now given that

$$K_1(z,z_0) = 1 + e^{-(1+4p)z}(z-1)^{-\delta}\big(cF_{\delta-1}(z) - cF_{\delta-1}(z_0) + c'F_\delta(z) - c'F_\delta(z_0)\big).$$

then $K_1(z,z_0) \sim_{a.e.} 1$, as $z \to 1^+$. This implies that $G_1(z,z_0) \sim_{a.e} e^{z-z_0}$ and therefore $\int_{z_0}^{z} G_1(\zeta,z_0)d\zeta \sim_{a.e.} 1$ as $z \to 1^+$.

For K_2 and G_2 we begin by noting that for z close to 1,

$$K_2(z,z_0) \sim_{a.e.} \frac{4\alpha p}{z-1}(1 - e^{z-z_0}) - \frac{\delta}{z-1}, \text{ as } z \to 1^+.$$

from which it follows that $G_2(z,z_0) \sim_{a.e.} (z-1)^{-\delta}$ for $z \to 1^+$, and therefore

$$\left(e^{z-z_0} - 1 + \int_{z_0}^{z} (e^{z-\zeta} - 1) G_2(\zeta, z_0) d\zeta\right) \sim_{a.e.} (z-1)^{1-\delta}, \text{ as } z \to 1^+.$$

Note that this assumes that $\delta > 0$. If this is not the case, then the asymptotics is $O(1)$.

Gathering our results, we get that

$$H(z) \sim_{a.e} 1 \text{ or } (z-1)^{1-\delta}, \text{ as } z \to 1^+.$$

which gives the same asymptotic behavior as obtained from series representations of $H(z)$ for z close to 1 [6, 37].

E. Remarks on the Teukolsky Angular Equation

The Teukolsky angular equation 20 has two regular singular points at $x = \pm 1$ and an irregular singular point at infinity. Just like the radial equation, we can transform it to either the Bocher symmetrical form [6] or the non-symmetric canonical form of the confluent Heun equation [5]. It follows that any solution to the angular equation has an integral series representation as described in this work.

V. CONCLUSION

In this work, we present novel integral series representations for all functions of Heun class. The major advantage of these representations is that 1) they involve only elementary integrands (rational and exponential functions); 2) they are unconditionally convergent everywhere except at the singular points of the Heun function being studied; and 3) they demonstrate that all functions of Heun class either satisfy linear Volterra integral equations of the second kind, or are given by the sum of two functions satisfying such equations. This allows for fast and reliable evaluations of the Heun functions without computing increasing orders of their integral series representations. For physics applications, in particular in gravitational physics [38], point 2) above is crucial to calculate solutions of the homogenous Teukolsky radial equation by means of representation that is convergent from the black hole horizon up to spatial infinity.

This paves the way to alternatives to the MST formalism in black hole studies, alleviates the need for introducing the auxiliary parameter ν. For these reasons integral series representations also seem well suited to solving the Teukolsky equations in the extremal spinning case, for which the MST formalism requires a dedicated expansion in terms of special functions [39]. At the opposite, in the Heun formalism the Teukolsky equations simply become double confluent Heun equations which are immediately available through Corollary III.4. Although a complete review of the literature is beyond the scope of this work, we can mention the burgeoning of novel mathematical techniques for specific questions in black hole studies (in particular for calculating quasinormal mode frequencies), see e.g. [40] and [41]. In this respect, the approach presented here is more of a 'general-purpose' nature since its foremost aim is to represent Heun functions with no convergence issues. We think provides a good starting framework for problem-specific calculations in gravitational wave physics.

Finally, we stress that our novel mathematical results were obtained by applying the method of path-sum to Heun's equation. This method, relying on the algebraic combinatorics of walks on graphs, was originally designed to solve systems of coupled differential equations and compute matrix functions. While it already proved successful in the fields of quantum dynamics [42], matrix theory [43] and more [44], we think that this work opens new venues for its use in ordinary differential equations and general relativity.

Data availability The algorithms and data that supports the findings of this study are available within the article. The pseudocode necessary to reproduce the examples is available for download at `http://www-lmpa.univ-littoral.fr/~plgiscard/`, Python implementation available at `https://github.com/tbirkandan/integralseries_heun`.

A. APPENDIX: PROOF OF THE RESULTS

The method of proof is as follows: we map the Heun equation onto a system of two coupled linear first order differential equations with variable coefficients. The solution of such systems is given by a formal object called a path-ordered exponential, which we present below. Then we use the path-sum method to evaluate this path-ordered exponential. Finally we extract the desired Heun function from the path-sum solution.

A. Path-ordered exponentials

All the results are corollaries of the general purpose method of path-sum, which permits the exact calculation of path-ordered exponentials of finite variable matrices. The path-ordered exponential $\mathsf{U}(z)$ of a variable matrix $\mathsf{M}(z)$ is the unique matrix solution to the system of coupled first order ordinary linear differential equations with variable coefficients encoded by $\mathsf{M}(z)$, i.e.

$$\frac{d}{dz}\mathsf{U}(z,z_0) = \mathsf{M}(z).\mathsf{U}(z,z_0), \tag{25}$$

and such that for all z_0, $\mathsf{U}(z_0,z_0) = \mathsf{Id}$ is the identity matrix of relevant dimension. The solution of Eq. (25) is the path-ordered exponential $\mathsf{U}(z,z_0)$ of M, denoted

$$\mathsf{U}(z,z_0) = \mathscr{P}e^{\int_{z_0}^{z}\mathsf{M}(\zeta)d\zeta} = \mathsf{Id} + \int_{z_0}^{z}\mathsf{M}(\zeta_1)d\zeta_1 + \frac{1}{2}\int_{z_0}^{z}\int_{z_0}^{z}\mathscr{P}\{\mathsf{M}(\zeta_2)\mathsf{M}(\zeta_1)\}d\zeta_2 d\zeta_1 + \cdots$$

where $\mathscr{P}$ the path-ordering operator,

$$\mathscr{P}\big\{\mathsf{M}(\zeta_2)\mathsf{M}(\zeta_1)\big\} = \begin{cases}\mathsf{M}(\zeta_2)\mathsf{M}(\zeta_1), & \text{if } \zeta_2 \geq \zeta_1\\ \mathsf{M}(\zeta_1)\mathsf{M}(\zeta_2), & \text{otherwise.}\end{cases}$$

We refer the reader to [45] for the origins of this notation.

Although used primarily to gain analytical understanding into the dynamics of quantum systems driven by time-dependent forces, path-sum relies solely on the algebraic combinatorics of walks on graphs that is valid irrespectively of the nature or size of the matrix M. It is also only distantly related to the famous Feynman's path-integrals. The interest here is that when calculating path-ordered exponentials, the method natively generates integral representations of the solutions. The strategy thus consists in calculating the ordered exponential of a matrix $\mathsf{M}(z)$ designed so that the solution of Eq. (25) should involve the desired Heun's function.

In order to recover an integral representation for all of Heun's functions, remark that Eqs. (1)–5) all take the form

$$y''(z) - B_1(z)y'(z) - B_2(z)y(z) = 0, \tag{26}$$

We thus focus on obtaining the integral representation of the solution of Eq. (26) in terms of integrals involving B_1 and B_2, irrespectively of what these functions are. To this end, we begin by exihibiting a matrix $\mathsf{M}(z)$ whose path-ordered exponential involves a function solution to Eq. (26).

Proposition A.1. *Let $y(z)$ be a solution of Eq. (26) with initial conditions $y(z_0) = y_0$ and $y'(z_0) = y'_0$. Let*

$$\mathsf{M}(z) = \begin{pmatrix} 1 & 1 \\ B_1(z) + B_2(z) - 1 & B_1(z) - 1 \end{pmatrix}, \tag{27}$$

and let $\mathsf{U}(z, z_0) := \mathscr{P} e^{\int_{z_0}^{z} \mathsf{M}(\zeta) d\zeta}$ be the path-ordered exponential of M. Then

$$y(z) = y_0 \mathsf{U}_{11}(z, z_0) + (y'_0 - y_0)\mathsf{U}_{12}(z, z_0).$$

Proof. By direct differentiation. Let $\psi(z) = (\psi_1(z), \psi_2(z))^{\mathrm{T}}$ such that $\dot{\psi}(z) = \mathsf{M}(z).\psi(z)$. This implies $\dot{\psi}_1 = \psi_1 + \psi_2$, $\quad \dot{\psi}_2 = (B_1 + B_2 - 1)\psi_1 + (B_1 - 1)\psi_2$, where we omitted the (z) arguments to alleviate the notation. Then

$$\ddot{\psi}_1 = \dot{\psi}_1 + (B_1 + B_2 - 1)\psi_1 + (B_1 - 1)(\dot{\psi}_1 - \psi_1),$$

which is $\ddot{\psi}_1 - (B_1 - 1 + 1)\dot{\psi}_1 - (B_1 + B_2 - 1 - B_1 + 1)\psi_1 = 0$, that is $\ddot{\psi}_1 - B_1\dot{\psi}_1 - B_2\psi_1 = 0$. This is precisely Eq. (26). Now, since $\psi_1(z_0) = y_0$ is the desired initial condition, and since $\dot{\psi}(z_0) = \mathsf{M}(z_0).\psi(z_0)$, then to get $\dot{\psi}_1(z_0) = y'_0$ we must have $\psi_2(z_0) = y'_0 - y_0$. From there and given that $\psi(z) = \mathsf{U}(z, z_0).\psi(z_0)$, we obtain

$$\psi_1(z) = y_0 \mathsf{U}_{11}(z, z_0) + (y'_0 - y_0)\mathsf{U}_{12}(z, z_0),$$

which completes the proof. □

B. Path-sum formulation

We may now use the method of path-sum to calculate the path-ordered exponential of M to recover the desired integral representations. We first state and prove the general result concerning Eq. (26) before giving its corollaries in the specific cases of the general Heun, confluent, biconfluent, doubly-confluent and triconfluent Heun functions.

Theorem A.1. *Let $\mathsf{M}(z)$ be given as in Eq. (27), let $\mathsf{U}(z, z_0)$ be its path-ordered exponential. Then*

$$\mathsf{U}_{11}(z, z_0) = 1 + \int_{z_0}^{z} G_1(\zeta, z_0)\, d\zeta,$$
$$\mathsf{U}_{12}(z, z_0) = \int_{z_0}^{z} (e^{z-\zeta} - 1)\big(1 + G_2(\zeta, z_0)\big)\, d\zeta,$$

where $G_i(z, z_0)$, $i = 1, 2$, satisfy the linear integral Volterra equation of the second kind

$$G_i(z, z_0) = K_i(z, z_0) + \int_{z_0}^{z} K_i(z, \zeta) G_i(\zeta, z_0) d\zeta,$$

with kernels

$$K_1(z, z_0) = 1 + e^{-z} \int_{z_0}^{z} \Big\{ e^{\zeta}\, e^{\int_{\zeta}^{z} B_1(\zeta') d\zeta'} (B_1(\zeta) + B_2(\zeta) - 1) \Big\} d\zeta,$$
$$K_2(z, z_0) = (B_1(z) + B_2(z) - 1) e^{z - z_0} - B_2(z).$$

Given that a linear Volterra integral equations of the second kind always has an explicit solution in the form of a Neumann series of the kernel obtained from Picard iteration, we present below the ensuing elementary integral series representations for the solution $y_0\mathsf{U}_{11}(z,z_0)+(y_0'-y_0)\mathsf{U}_{12}(z,z_0)$ of Eq. (26). This will be greatly facilitated by Volterra compositions, presented in the proof of the Theorem.

Proof. The central mathematical concept enabling the path-sum formulation of path-ordered exponentials is the $*$-product. This product is defined on a large class of distributions [46], however for the present work only its definition on smooth functions of two variables is required. For such functions the $*$-product reduces to the Volterra composition, a product between functions first expounded by Volterra and Pérès in the 1920s [22] and which had largely fallen out of use by the early 1950s for a reason that appears, restrospectively, to be the lack of a mathematical theory of distributions. The Volterra composition of two smooth functions of two variables $f(z,z_0)$ and $g(z,z_0)$ is

$$\big(f*g\big)(z,z_0)=\int_{z_0}^{z}f(z,\zeta)g(\zeta,z_0)d\zeta\,\Theta(z-z_0),$$

with $\Theta(.)$ the Heaviside theta function under the convention that $\Theta(0)=1$. This extends to functions of less than two variables, for example if $h(z)$ is a smooth function of one variable, then

$$\big(h*g\big)(z,z_0)=h(t')\int_{z_0}^{z}g(\zeta,z_0)d\zeta\,\Theta(z-z_0),\quad \big(g*h\big)(z,z_0)=\int_{z_0}^{z}g(z,\zeta)h(\zeta)d\zeta\,\Theta(z-z_0).$$

That is, the variable of $h(z)$ is always treated as the left variable of a function of two variables.

The identity element for the $*$-product is the Dirac distribution, denoted $1_*\equiv\delta(z-z_0)$, an observation which we here accept without proof as it would require presenting the full theory of the $*$-product [46]. Similarly we accept without proof that for any bounded function $f(z,z_0)$ of two variables, $f^{*0}=1_*$, while $f^{*1}=f$ and $f^{*n+1}=f*f^{*n}=f^{*n}*f$ [22]. Furthermore, if f is bounded the Neumann series $\sum_{n=0}^{\infty}\big(f^{*n}\big)(z,z_0)$ converges superexponentially and thus unconditionally [23] to an object, called the $*$-resolvent R_f of f, given by

$$\begin{aligned}R_f(z,z_0)=\sum_{n=0}^{\infty}\big(f^{*n}\big)(z,z_0)&=\delta(z-z_0)+f(z,z_0)\Theta(z-z_0)+\int_{z_0}^{z}f(z,\zeta_1)f(\zeta_1,z_0)d\zeta_1\Theta(z-z_0)\\&+\int_{z_0}^{z}\int_{\zeta_1}^{z}f(z,\zeta_2)f(\zeta_2,\zeta_1)f(\zeta_1,z_0)d\zeta_2d\zeta_1\Theta(z-z_0)+\cdots.\end{aligned}$$

Seeing this as steming from a Picard iteration entails an additional property of $*$-resolvents, namely that they solve the Volterra equation of the second kind with kernel f, $R_f=1_*+f*R_f$. Thus we have $R_f*(1_*-f)=1_*$ and are therefore justified in writing $R_f=\big(1_*-f\big)^{*-1}$. In order to avoid distributions altogether, it is more convenient to define $G_f:=R_f-1_*$ and rewite the Volterra equation satisfied by R_f as $G_f=f+f*G_f$, which is an ordinary linear integral Volterra equation of the second kind. The Neumann integral series obtained from Picard iterations for G_f as above is now $G_f=\sum_{n=1}^{\infty}f^{*n}$, the convergence of which is guaranteed provided f is bounded over the interval of interest as discussed in the main text. In this case, truncating the series at order m, yields a relative error of at most $\big|G_f(z,z_0)-\sum_{n=1}^{m}f^{*n}(z,z_0)\big|\leq\frac{M_f^m}{m!}$ with $M_f:=\sup_{\zeta,\zeta'\in]z_0,z[:\zeta\geq\zeta'}|f(\zeta,\zeta')|$.

Path-sum expresses the path-ordered exponential of any finite variable matrix in terms of a finite number of Volterra compositions and $*$-resolvents. The path-sum formulation of the path-ordered exponential of the 2×2 matrix $\mathsf{M}(z)$ is $\mathsf{U}_{11}(z)=1*R_1$, and $R_1=\Big(1_*-\mathsf{M}_{11}-\mathsf{M}_{12}*\big(1_*-\mathsf{M}_{22}\big)^{*-1}*\mathsf{M}_{21}\Big)^{*-1}$, where the $*$-multiplication by 1 on the left is a short-hand notation for an integral with respect to the left variable, since for any f smooth, $(1*f)(z,z_0)=\int_{z_0}^{z}f(\zeta,z_0)d\zeta\Theta(z-z_0)$. Furthermore, since M_{22} depends on a single variable, its $*$-resolvent can be shown to be

$$\big(1_*-\mathsf{M}_{22}\big)^{*-1}=1_*+\mathsf{M}_{22}e^{1*\mathsf{M}_{22}}\Rightarrow\big(1_*-\mathsf{M}_{22}\big)^{*-1}(z,z_0)=\delta(z-z_0)+\mathsf{M}_{22}(z)e^{\int_{z_0}^{z}\mathsf{M}_{22}(\zeta)d\zeta}\Theta(z-z_0).$$

Now the form of U_{11} as claimed in the theorem follows upon writing the $*$-products as explicit integrals with M given by Proposition. (A.1). For U_{12}, the path-sum formulation reads

$$\mathsf{U}_{12}=1*\big(1_*-\mathsf{M}_{11}\big)^{*-1}*\mathsf{M}_{12}*R_2,\qquad R_2=\Big(1_*-\mathsf{M}_{22}-\mathsf{M}_{21}*\big(1_*-\mathsf{M}_{11}\big)^{*-1}*\mathsf{M}_{12}\Big)^{*-1},$$

and the theorem result for U_{12} follows upon writing the $*$-products as explicit integrals with M given by Proposition. (A.1) □

Since R_1 and R_2 are $*$-resolvents, we may express them as the unconditionally convergent Neumann series involving the corresponding kernels K_1 and K_2, i.e. $R_i=1_*+\sum_{n=1}^{\infty}K_i^{*n}$ or equivalently $G_i=\sum_{n=1}^{\infty}K_i^{*n}$, $i=1,2$. This yields an explicit representation for the solution of Eq. (26) as series of elementary integrals:

Theorem A.2. *Let $y(z)$ be the unique solution of*

$$y''(z) - B_1(z)y'(z) - B_2(z)y(z) = 0,$$

such that $y(z_0) = y_0$ and $y'(z_0) = y'_0$. Then

$$y(z) = y_0 + y_0 \int_{z_0}^{z} G_1(\zeta, z_0) d\zeta + (y'_0 - y_0) \int_{z_0}^{z} (e^{z-\zeta} - 1)\big(1 + G_2(\zeta, z_0)\big) d\zeta,$$

where G_1 and G_2 satisfy linear Volterra integral equations of the second kind with kernels respectively given by

$$K_1(z, z_0) = 1 + e^{-z} \int_{z_0}^{z} \Big\{ e^{\int_\zeta^z B_1(\zeta') d\zeta'} e^{\zeta} \big(B_1(\zeta) + B_2(\zeta) - 1\big) \Big\} d\zeta,$$
$$K_2(z, z_0) = (B_1(z) + B_2(z) - 1) e^{z - z_0} - B_2(z).$$

In consequence, G_1 and G_2 have the following representation as integral series $G_i(z, z_0) = \sum_{n=1}^{\infty} K_i^{*n}(z, z_0)$,

$$G_i(z, z_0) = K_i(z, z_0) + \int_{z_0}^{z} K_i(z, \zeta_1) K_i(\zeta_1, z_0) d\zeta_1 + \int_{z_0}^{z} \int_{\zeta_1}^{z} K_i(z, \zeta_2) K_i(\zeta_2, \zeta_1) K_i(\zeta_1, z_0) d\zeta_2 d\zeta_1 + \cdots$$

for $i = 1, 2$. The series representation is guaranteed to converge to G_i everywhere except at the singular points of K_i. More precisely, let $]z_0, z_1[$ be an open interval over which K_i is divergent free and let $M_{K_i} := \sup_{\zeta, \zeta' \in]z_0, z_1[: \zeta \geq \zeta'} |K_i(\zeta, \zeta')|$. Then $|G_i(z, z_0) - \sum_{n=1}^{m} K_i^{*n}(z, z_0)| \leq M_{K_i}^m / m!$.

This immediately provides the Corollaries of the main text for the general Heun, confluent, biconfluent, doubly-confluent and triconfluent Heun's functions upon replacing B_1 and B_2 appearing in Eq. (26) and Theorem A.2 with their values as dictated by Eqs. (1–5).

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[1] Xie Q, Hai W. 2010 Analytical results for a monochromatically driven two-level system. *Phys. Rev. A.* **82**, 032117.
[2] Crampé N, Nepomechie R, Vinet L. 2019 Free-Fermion entanglement and orthogonal polynomials. *J. Stat. Mech.* **2019**, 093101.
[3] Giorgadze G. 2009 Monodromic Quantum Computing. *Int. j. comput. sci.* **15**, 259–294.
[4] Shahverdyan T, Ishkhanyan T, Grigoryan A, Ishkhanyan A. 2015 Analytic solutions of the quantum two-state problem in terms of the double bi- and triconfluent Heun functions. *J. Contemp. Physics (Armenian Ac. Sci.)* **50**, 211–226.
[5] Borissov R, Fiziev P. 2010 Exact Solutions of Teukolsky Master Equation with Continuous Spectrum. *Bulg. J. Phys.* **48**, 065–089.
[6] Cook G, Zalutskiy M. 2014 Gravitational perturbations of the Kerr geometry: High-accuracy study. *Phys. Rev. D.* **90**, 124021.
[7] Hortaçsu M. 2018 Heun Functions and Some of Their Applications in Physics. *Adv. High Energy Phys.* **2018**, 8621573.
[8] Maier R. 2004 On reducing the Heun equation to hypergeometric equation. *J. Diff. Equ.* **213**, 171–203.
[9] Takemura K. 2017 Integral transformation of Heun equations and some applications. *J. Math. Soc. Japan* **69**, 849–891.
[10] Slavyanov S. 2018 Painlevé equations as classical analogues of Heun equation. *J. Phys. A.* **29**, 7329–7335.
[11] Teukolsky S. 1973 Perturbations of a Rotating Black Hole. I. Fundamental Equations for Gravitational, Electromagnetic, and Neutrino-Field Perturbations. *Astroph. J.* **185**, 635–648.
[12] Mano S, Suzuki H, Takasugi E. 1996 Analytical Solutions of the Teukolsky Equation and Their Low Frequency Expansions. *Prog. Theor. Phys.* **95**, 1079–1096.
[13] Fujita R, Tagoshi H. 2004 New Numerical Methods to Evaluate Homogeneous Solutions of the Teukolsky Equation. *Prog. Theor. Phys.* **112**, 1079–1096.
[14] Giscard PL, Lui K, Thwaite SJ, Jaksch D. 2015 An exact formulation of the time-ordered exponential using path-sums. *J. Math. Phys.* **56**, 053503.
[15] Heun K. 1888 Zur Theorie der Riemann'chen Functionen Zweiter Ordnung mit Verzweigungspunkten. *Math. Ann.* **33**, 161–179.
[16] Maier R. 2007 The 192 Solutions of the Heun equation. *Math. Comp.* **76**, 811–843.
[17] Daniel D. 2020 Exact solutions of Mathieu equations. *Prog. Theor. Exp. Phys.* **4**, 043A01.
[18] Slavyanov S, Lay W. 2000 *Special Functions: A Unified Theory Based on Singularities*. Oxford University Press, New York.
[19] Erdélyi A. 1942 Integral Equations for Heun Functions. *Q. J. Math.* **13**, 107–112.
[20] Sleeman B. 1969 Non-linear Integral Equations For Heun Functions. *Proc. Edinb. Math. Soc.* **16**, 281–289.
[21] Valent G. 1986 An Integral Transform Involving Heun Functions and a Related Eigenvalue Problem. *SIAM J. Math. Anal.* **17**, 688–703.
[22] Volterra V, Pérès J. 1924 *Leçons sur la composition et les fonctions permutables*. Éditions Jacques Gabay.
[23] Linz P. 1985 *Analytical and Numerical Methods for Volterra Equations*. Society for Industrial and Applied Mathematics.
[24] Birkandan T, Giscard PL, Tamar A. 2021 Computations of general Heun functions from their integral series representations. *Preprint* pp. 1–6. arXiv:2106.13729.
[25] Motygin OV. 2015 On numerical evaluation of the Heun functions. *2015 Days on Diffraction (DD)* pp. 1–6.

[26] Costa R. 2020 Mode stability for the Teukolsky equation on extremal and subextremal Kerr spacetimes. *Comm. Math. Phys.* **378**, 705–781.

[27] Sasaki M, Tagoshi H. 2003 Analytic Black Hole Perturbation Approach to Gravitational Radiation. *Living Rev. Relativ.* **6**, 6.

[28] El-Jaick L, Figueiredo B. 2008 Solutions for confluent and double-confluent Heun equations. *J. Math. Phys.* **49**, 083508.

[29] Hatsuda Y. 2021 Quasinormal modes of Kerr-de Sitter black holes via the Heun function. *Class. Quantum Gravity* **38**, 025015.

[30] Leaver E. 1985 An analytic representation for the quasi-normal modes of Kerr black holes. *Proc. R. Soc. Lond. A* **402**, 285–298.

[31] Misner C, Thorne K, Wheeler. 1973 *Gravitation*. W.H. Freeman and Company, San Francisco, USA.

[32] Boyer R, Lindquist R. 1967 Maximal Analytic Extension of the Kerr Metric. *Jour. Math. Phys.* **8**, 265.

[33] Breuer RA, Ryan MP, Waller S. 1977 Some properties of spin-weighted spheroidal harmonics. *Proc. R. Soc. Lond. A* **358**, 71–86.

[34] Birkandan T, Hortaçsu M. 2017 Heun-type solutions for Schwarzschild metric with electromagnetic fields. *EPL* **119**, 20002.

[35] Erdélyi A, Magnus W, Oberhettinger F, Tricomi FG. 1955 *Higher Transcendental Functions (Vol. 3)*. McGraw Hill.

[36] Giscard PL. 2020 On the solutions of linear Volterra equations of the second kind with sum kernels. *J. Integral Equations Applications* **32**, 429–455.

[37] Ronveaux A, Arscott FM, Slavyanov SY, D. S, G. W, P. M, A. D. 1995 *Heun's Differential Equations*. Oxford University Press.

[38] Motohashi H, Noda S. 2021 Exact solution for wave scattering from black holes: formulation. *Preprint*. arXiv:2103.10802.

[39] Casals M, Longo Micchi LF. 2019 Spectroscopy of extremal and near-extremal Kerr black holes. *Phys. Rev. D* **99**, 084047.

[40] Aminov G, Grassi A, Hatsuda Y. 2020 Black Hole Quasinormal Modes and Seiberg-Witten Theory. *Preprint*. arXiv:2006.06111.

[41] Carneiro da Cunha B, Cavalcante JaP. 2020 Confluent conformal blocks and the Teukolsky master equation. *Phys. Rev. D* **102**, 105013.

[42] Giscard PL, Bonhomme C. 2020 Dynamics of quantum systems driven by time-varying Hamiltonians: Solution for the Bloch-Siegert Hamiltonian and applications to NMR. *Phys. Rev. Research* **2**, 023081.

[43] Giscard PL, Thwaite SJ, Jaksch D. 2013 Evaluating Matrix Functions by Resummations on Graphs: The Method of Path-Sums. *SIAM Journal on Matrix Analysis and Applications* **34**, 445–469.

[44] Giscard PL, Choo Z, Thwaite SJ, Jaksch D. 2016 Exact Inference on Gaussian Graphical Models of Arbitrary Topology using Path-Sums. *Journal of Machine Learning Research* **17**, 1–19.

[45] Dyson FJ. 1952 Divergence of Perturbation Theory in Quantum Electrodynamics. *Phys. Rev.* **85**, 631–632.

[46] Giscard PL, Pozza S. 2021 Tridiagonalization of systems of coupled linear differential equations with variable coefficients by a Lanczos-like method. *Linear Algebra Its Appl.* **624**, 153–173.

[47] Very similar behaviours in terms of convergence and oscillations are observed for the imaginary parts, which are not shown owing to length concerns.

[48] This is because the solution of a linear Volterra integral equation of the second kind with kernel $K_1(z,z_0) = k(z)l(z_0)$ is known exactly [36].